Midterm 1 Duration: 45 minutes

*This test has **5 questions** on **8 pages**, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ________________ Last Name: ________________

Student-No: ________________ Section: ________________

Signature: ________________

Question:	1	2	3	4	5	Total
Points:	9	12	6	8	5	40
Score:						

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 3 marks, but not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

9 marks

1. Determine whether each of the following limits exists, and find the value if they do. If a limit below does not exist, determine whether it "equals" ∞, $-\infty$, or neither.

(a) $\lim_{t\to 1}\left(\frac{1}{t-1}-\frac{3}{(t-1)(t+2)}\right)$

Answer: $\frac{1}{3}$

Solution:

$$\begin{aligned}
\lim_{t\to 1}\left(\frac{1}{t-1}-\frac{3}{(t-1)(t+2)}\right) &= \lim_{t\to 1}\left(\frac{t+2}{(t-1)(t+2)}-\frac{3}{(t-1)(t+2)}\right) && \text{same denominator}\\
&= \lim_{t\to 1}\frac{t-1}{(t-1)(t+2)} && \text{add fractions}\\
&= \lim_{t\to 1}\frac{1}{t+2} && \text{cancel}\\
&= \frac{1}{3} && \text{lim rational function}
\end{aligned}$$

(b) $\lim_{x\to 3}\frac{x-3}{\sqrt{x^2+7}-4}$

Answer: $\frac{4}{3}$

Solution: When $x\to 3$ both numerator and denominator go to zero — something cancels.

$$\begin{aligned}
\frac{x-3}{\sqrt{x^2+7}-4} &= \frac{x-3}{\sqrt{x^2+7}-4}\cdot\frac{\sqrt{x^2+7}+4}{\sqrt{x^2+7}+4} && \text{multiply by conj}\\
&= \frac{(x-3)(\sqrt{x^2+7}+4)}{x^2+7-16} = \frac{(x-3)(\sqrt{x^2+7}+4)}{x^2-9} && \text{clean up}\\
&= \frac{(x-3)(\sqrt{x^2+7}+4)}{(x-3)(x+3)} && \text{factor}\\
&= \frac{\sqrt{x^2+7}+4}{x+3} && \text{cancel}
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{x\to 3}\frac{x-3}{\sqrt{x^2+7}-4} &= \lim_{x\to 3}\frac{\sqrt{x^2+7}+4}{x+3}\\
&= \frac{\sqrt{9+7}+4}{6} = \frac{\sqrt{16}+4}{6}\\
&= \frac{4}{3}
\end{aligned}$$

(c) $\lim_{t \to 1} \frac{\sqrt{t^2 - 2t + 1}}{t - 1}$

Answer: Does not exist

Solution: We have

$$\sqrt{t^2 - 2t + 1} = \sqrt{(t-1)^2} = |t - 1|.$$

The left and right limits are different:

$$\lim_{t \to 1^+} \frac{\sqrt{t^2 - 2t + 1}}{t - 1} = \lim_{t \to 1^+} \frac{|t-1|}{t-1} = \lim_{t \to 1^+} \frac{t-1}{t-1} \qquad \text{from right}$$
$$= 1$$
$$\lim_{t \to 1^-} \frac{\sqrt{t^2 - 2t + 1}}{t - 1} = \lim_{t \to 1^-} \frac{|t-1|}{t-1} = \lim_{t \to 1^-} \frac{-(t-1)}{t-1} \qquad \text{from left}$$
$$= -1$$

Thus the limit does not exist.

12 marks 2. (a) If $f(x) = 4x^4 + e^x + cx$, where c is a constant, find $f'(x)$.

Answer: $f'(x) = 16x^3 + e^x + c$

Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}$, $\frac{\mathrm{d}}{\mathrm{d}x} cx = c\frac{\mathrm{d}}{\mathrm{d}x} x$, and $x^0 = 1$:

$$f'(x) = 16x^3 + e^x + c$$

(b) Let $f(t) = \frac{\sqrt{t}}{t^2 - 1}$. Determine where $f(t)$ is differentiable.

Answer: f is differentiable on $(0, +\infty) \setminus \{1\}$

Solution:

- The ratio of differentiable functions is also differentiable except where the denominator is zero.
- The numerator is differentiable on $(0, +\infty)$ and the denominator is differentiable on $\mathbb{R} \setminus \{-1, 1\}$.
- Hence f is differentiable on $(0, +\infty) \cap \mathbb{R} \setminus \{-1, 1\} = (0, +\infty) \setminus \{1\}$

(c) Suppose the tangent line to the curve $y = f(x)$ at $x = 1$ passes through the points $(2, 3)$ and $(0, 1)$. Find $f(1)$ and $f'(1)$.

Answer: $f(1) = 2$ and $f'(1) = 1$

Solution:

- The line passes through $(0, 1)$ and $(2, 3)$, so its gradient is $(3 - 1)/(2 - 0) = 1$.
- Its y-intercept is 1 (it passes through $(0, 1)$, its equation is $y = x + 1$.
- Hence when $x = 1$ we have $y = 2$.

The tangent line must touch the curve and 1 and have the same gradient, thus $f(1) = 2$ and $f'(1) = 1$.

(d) Let f be a function such that $f(1) = 1$ and $f'(1) = 2$ and define

$$g(x) = f(x) \cdot e^x + \frac{f(x)}{e^x}.$$

Compute $g'(1)$.

Answer: $g'(1) = 3e + \frac{1}{e}$

Solution: We use the product and quotient rules

$$g'(x) = f'(x)e^x + f(x)e^x + \frac{f'(x)e^x - f(x)e^x}{(e^x)2}$$

$$g'(1) = 2e + e + \frac{2e - e}{e^2} = 3e + \frac{1}{e}.$$

Full-Solution Problems. In questions 3–5, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

6 marks 3. Show that the following equation has at least two different solutions:

$$\frac{x^4 + x^2 - 1}{x^2 + 1} = 0$$

Solution: We use the intermediate value theorem.

- The function $f(x) = \frac{x^4+x^2-1}{x^2+1}$ is a rational function, since $x^2 + 1 \geq 1 > 0$ for all $x \in \mathbb{R}$, it is defined on $\mathbb{R}$, so it is continuous on $\mathbb{R}$.
- At $x = 0$, $f(0) = -1$.
- At $x = 1$, $f(1) = \frac{1}{2}$
- At $x = -1$, $f(-1) = \frac{1}{2}$
- By the IVT since the function is continuous and is negative at $x = 0$ and positive at $x = 1$, there exists some $0 < c_1 < 1$ so that $f(c_1) = 0$.
- By the IVT since the function is continuous and is negative at $x = 0$ and positive at $x = -1$, there exists some $-1 < c_2 < 0$ so that $f(c_2) = 0$.
- Since $c_2 < 0 < c_1$ there exist at least two points at which the function is zero.

4. Let $f(x) = \dfrac{\pi|x|^3 + x^2 + 4}{x^3+1}$

4 marks (a) Determine the horizontal asymptotes of the graph $y = f(x)$

Solution: Since we want to compute limit as $x \to +\infty$ we may assume that $x > 0$. Then $|x| = x$ and

$$\begin{aligned}\frac{\pi|x|^3 + x^2 + 4}{x^3+1} &= \frac{\pi x^3 + x^2 + 4}{x^3+1} \\ &= \frac{x^3(\pi + x^2/x^3 + 4/x^3}{x^3(1 + 1/x^3)} \\ &= \frac{\pi + x^2/x^3 + 4/x^3}{1 + 1/x^3}\end{aligned}$$

so as $x \to +\infty$

$$\lim_{x\to+\infty} f(x) = \frac{\pi + 0 + 0}{1+0} = \pi$$

Since we want to compute limit as $x \to -\infty$ we may assume that $x < 0$. Then $|x| = -|x|$ and so

$$\begin{aligned}\frac{\pi|x|^3 + x^2 + 4}{x^3+1} &= \frac{-\pi x^3 + x^2 + 4}{x^3+1} \\ &= \frac{x^3(-\pi + x^2/x^3 + 4/x^3}{x^3(1 + 1/x^3)} \\ &= \frac{-\pi + x^2/x^3 + 4/x^3}{1 + 1/x^3}\end{aligned}$$

so as $x \to -\infty$

$$\lim_{x\to-\infty} f(x) = \frac{-\pi + 0 + 0}{1+0} = -\pi$$

4 marks (b) Determine the vertical asymptote(s) of the graph $y = f(x)$. For each vertical asymptote $x = a$, determine whether each of the one-sided limits "equals" $+\infty$ or $-\infty$ as x approaches a.

Solution:

- Since the numerator and denominator are continuous functions for all x, the only possible asymptote occurs when the denominator is zero.
- We have $x^3 + 1 = (x+1)(x^2 - x + 1)$, and $x^2 - x + 1$ has no real roots (discriminant$= -3 < 0$), hence $x = -1$ is the only solution to $x^3 + 1 = 0$.
- As $x \to -1$ the numerator goes to $\pi + 1 + 4 = \pi + 5$.

- As $x \to -1^+$ the denominator goes to zero but is positive. Hence

$$\lim_{x \to -1^+} f(x) = +\infty$$

- As $x \to -1^-$ the denominator goes to zero but is negative. Hence

$$\lim_{x \to -1^-} f(x) = -\infty$$

5 marks 5. Let

$$f(x) = \begin{cases} c + (x-1)^2 \cos\left(\frac{1}{x-1}\right) & x < 1 \\ 2 & x = 1 \\ \frac{x^2-1}{x-1} & x > 1 \end{cases}$$

where c is a constant. Find, if it exists, the value of c that makes $f(x)$ continuous at $x = 1$.

Solution: In order for f to be continuous at $x = 1$, the left and right limits must agree and be equal to the value of the function $f(1)$.

- At $x = 1$, $f(1) = 2$.
- The right-limit is

$$\begin{aligned} \lim_{x\to1^+} f(x) &= \lim_{x\to1^+} \frac{x^2-1}{x-1} \\ &= \lim_{x\to1^+} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to1^+}(x+1) \\ &= 2 \end{aligned}$$

- The other limit takes a little more work and we use the squeeze / sandwich theorem. When $x < 1$ we have

$$-(x-1)^2 \leq (x-1)^2 \cos\left(\frac{1}{x-1}\right) \leq (x-1)^2 \qquad \text{since } -1 \leq \cos(\theta) \leq 1$$

As $x \to 1^-$ we have

$$\lim_{x\to1^-} -(x-1)^2 = \lim_{x\to1^-} (x-1)^2 = 0$$

Hence, by the squeeze theorem

$$\lim_{x\to1^-} (x-1)^2 \cos\left(\frac{1}{x-1}\right) = 0.$$

Therefore,

$$\lim_{x\to1^-} c + (x-1)^2 \cos\left(\frac{1}{x-1}\right) = c.$$

- In order to have a continuous function we therefore need $c = 2$.