

Midterm 1 Duration: 45 minutes

This test has 5 questions on 8 pages, for a total of 40 points.

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	Total
Points:	9	12	6	8	5	40
Score:						

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 3 marks, but not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

9 marks

1. Determine whether each of the following limits exists, and find the value if they do. If a limit below does not exist, determine whether it “equals” ∞ , $-\infty$, or neither.

(a) $\lim_{t \rightarrow 1} \left(\frac{1}{t-1} - \frac{3}{(t-1)(t+2)} \right)$

Answer: $\frac{1}{3}$

Solution:

$$\begin{aligned} \lim_{t \rightarrow 1} \left(\frac{1}{t-1} - \frac{3}{(t-1)(t+2)} \right) &= \lim_{t \rightarrow 1} \left(\frac{t+2}{(t-1)(t+2)} - \frac{3}{(t-1)(t+2)} \right) && \text{same denominator} \\ &= \lim_{t \rightarrow 1} \frac{t-1}{(t-1)(t+2)} && \text{add fractions} \\ &= \lim_{t \rightarrow 1} \frac{1}{t+2} && \text{cancel} \\ &= \frac{1}{3} && \text{lim rational function} \end{aligned}$$

(b) $\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x^2+7}-4}$

Answer: $\frac{4}{3}$

Solution: When $x \rightarrow 3$ both numerator and denominator go to zero — something cancels.

$$\begin{aligned} \frac{x-3}{\sqrt{x^2+7}-4} &= \frac{x-3}{\sqrt{x^2+7}-4} \cdot \frac{\sqrt{x^2+7}+4}{\sqrt{x^2+7}+4} && \text{multiply by conj} \\ &= \frac{(x-3)(\sqrt{x^2+7}+4)}{x^2+7-16} = \frac{(x-3)(\sqrt{x^2+7}+4)}{x^2-9} && \text{clean up} \\ &= \frac{(x-3)(\sqrt{x^2+7}+4)}{(x-3)(x+3)} && \text{factor} \\ &= \frac{\sqrt{x^2+7}+4}{x+3} && \text{cancel} \end{aligned}$$

Hence

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x^2+7}-4} &= \lim_{x \rightarrow 3} \frac{\sqrt{x^2+7}+4}{x+3} \\ &= \frac{\sqrt{9+7}+4}{6} = \frac{\sqrt{16}+4}{6} \\ &= \frac{4}{3} \end{aligned}$$

$$(c) \lim_{t \rightarrow 1} \frac{\sqrt{t^2 - 2t + 1}}{t - 1}$$

Answer: Does not exist

Solution: We have

$$\sqrt{t^2 - 2t + 1} = \sqrt{(t - 1)^2} = |t - 1|.$$

The left and right limits are different:

$$\lim_{t \rightarrow 1^+} \frac{\sqrt{t^2 - 2t + 1}}{t - 1} = \lim_{t \rightarrow 1^+} \frac{|t - 1|}{t - 1} = \lim_{t \rightarrow 1^+} \frac{t - 1}{t - 1} \quad \text{from right}$$

$$= 1$$

$$\lim_{t \rightarrow 1^-} \frac{\sqrt{t^2 - 2t + 1}}{t - 1} = \lim_{t \rightarrow 1^-} \frac{|t - 1|}{t - 1} = \lim_{t \rightarrow 1^-} \frac{-(t - 1)}{t - 1} \quad \text{from left}$$

$$= -1$$

Thus the limit does not exist.

12 marks

2. (a) If $f(x) = 4x^4 + e^x + cx$, where c is a constant, find $f'(x)$.

Answer: $f'(x) = 16x^3 + e^x + c$

Solution: Since $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}cx = c\frac{d}{dx}x$, and $x^0 = 1$:

$$f'(x) = 16x^3 + e^x + c$$

(b) Let $f(t) = \frac{\sqrt{t}}{t^2 - 1}$. Determine where $f(t)$ is differentiable.

Answer: f is differentiable on $(0, +\infty) \setminus \{1\}$

Solution:

- The ratio of differentiable functions is also differentiable except where the denominator is zero.
- The numerator is differentiable on $(0, +\infty)$ and the denominator is differentiable on $\mathbb{R} \setminus \{-1, 1\}$.
- Hence f is differentiable on $(0, +\infty) \cap \mathbb{R} \setminus \{-1, 1\} = (0, +\infty) \setminus \{1\}$

- (c) Suppose the tangent line to the curve $y = f(x)$ at $x = 1$ passes through the points $(2, 3)$ and $(0, 1)$. Find $f(1)$ and $f'(1)$.

Answer: $f(1) = 2$ and $f'(1) = 1$

Solution:

- The line passes through $(0, 1)$ and $(2, 3)$, so its gradient is $(3 - 1)/(2 - 0) = 1$.
- Its y -intercept is 1 (it passes through $(0, 1)$, its equation is $y = x + 1$).
- Hence when $x = 1$ we have $y = 2$.

The tangent line must touch the curve at $x = 1$ and have the same gradient, thus $f(1) = 2$ and $f'(1) = 1$.

- (d) Let f be a function such that $f(1) = 1$ and $f'(1) = 2$ and define

$$g(x) = f(x) \cdot e^x + \frac{f(x)}{e^x}.$$

Compute $g'(1)$.

Answer: $g'(1) = 3e + \frac{1}{e}$

Solution: We use the product and quotient rules

$$g'(x) = f'(x)e^x + f(x)e^x + \frac{f'(x)e^x - f(x)e^x}{(e^x)^2}$$
$$g'(1) = 2e + e + \frac{2e - e}{e^2} = 3e + \frac{1}{e}.$$

Full-Solution Problems. In questions 3–5, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

6 marks

3. Show that the following equation has at least two different solutions:

$$\frac{x^4 + x^2 - 1}{x^2 + 1} = 0$$

Solution: We use the intermediate value theorem.

- The function $f(x) = \frac{x^4+x^2-1}{x^2+1}$ is a rational function, since $x^2 + 1 \geq 1 > 0$ for all $x \in \mathbb{R}$, it is defined on \mathbb{R} , so it is continuous on \mathbb{R} .
- At $x = 0$, $f(0) = -1$.
- At $x = 1$, $f(1) = \frac{1}{2}$
- At $x = -1$, $f(-1) = \frac{1}{2}$
- By the IVT since the function is continuous and is negative at $x = 0$ and positive at $x = 1$, there exists some $0 < c_1 < 1$ so that $f(c_1) = 0$.
- By the IVT since the function is continuous and is negative at $x = 0$ and positive at $x = -1$, there exists some $-1 < c_2 < 0$ so that $f(c_2) = 0$.
- Since $c_2 < 0 < c_1$ there exist at least two points at which the function is zero.

4. Let $f(x) = \frac{\pi|x|^3 + x^2 + 4}{x^3 + 1}$

4 marks

(a) Determine the horizontal asymptotes of the graph $y = f(x)$

Solution: Since we want to compute limit as $x \rightarrow +\infty$ we may assume that $x > 0$. Then $|x| = x$ and

$$\begin{aligned} \frac{\pi|x|^3 + x^2 + 4}{x^3 + 1} &= \frac{\pi x^3 + x^2 + 4}{x^3 + 1} \\ &= \frac{x^3(\pi + x^2/x^3 + 4/x^3)}{x^3(1 + 1/x^3)} \\ &= \frac{\pi + x^2/x^3 + 4/x^3}{1 + 1/x^3} \end{aligned}$$

so as $x \rightarrow +\infty$

$$\lim_{x \rightarrow +\infty} f(x) = \frac{\pi + 0 + 0}{1 + 0} = \pi$$

Since we want to compute limit as $x \rightarrow -\infty$ we may assume that $x < 0$. Then $|x| = -x$ and so

$$\begin{aligned} \frac{\pi|x|^3 + x^2 + 4}{x^3 + 1} &= \frac{-\pi x^3 + x^2 + 4}{x^3 + 1} \\ &= \frac{x^3(-\pi + x^2/x^3 + 4/x^3)}{x^3(1 + 1/x^3)} \\ &= \frac{-\pi + x^2/x^3 + 4/x^3}{1 + 1/x^3} \end{aligned}$$

so as $x \rightarrow -\infty$

$$\lim_{x \rightarrow -\infty} f(x) = \frac{-\pi + 0 + 0}{1 + 0} = -\pi$$

4 marks

(b) Determine the vertical asymptote(s) of the graph $y = f(x)$. For each vertical asymptote $x = a$, determine whether each of the one-sided limits “equals” $+\infty$ or $-\infty$ as x approaches a .

Solution:

- Since the numerator and denominator are continuous functions for all x , the only possible asymptote occurs when the denominator is zero.
- We have $x^3 + 1 = (x + 1)(x^2 - x + 1)$, and $x^2 - x + 1$ has no real roots (discriminant = $-3 < 0$), hence $x = -1$ is the only solution to $x^3 + 1 = 0$.
- As $x \rightarrow -1$ the numerator goes to $\pi + 1 + 4 = \pi + 5$.

- As $x \rightarrow -1^+$ the denominator goes to zero but is positive. Hence

$$\lim_{x \rightarrow -1^+} f(x) = +\infty$$

- As $x \rightarrow -1^-$ the denominator goes to zero but is negative. Hence

$$\lim_{x \rightarrow -1^-} f(x) = -\infty$$

5 marks 5. Let

$$f(x) = \begin{cases} c + (x - 1)^2 \cos\left(\frac{1}{x - 1}\right) & x < 1 \\ 2 & x = 1 \\ \frac{x^2 - 1}{x - 1} & x > 1 \end{cases}$$

where c is a constant. Find, if it exists, the value of c that makes $f(x)$ continuous at $x = 1$.

Solution: In order for f to be continuous at $x = 1$, the left and right limits must agree and be equal to the value of the function $f(1)$.

- At $x = 1$, $f(1) = 2$.
- The right-limit is

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \rightarrow 1^+} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \rightarrow 1^+} (x + 1) \\ &= 2 \end{aligned}$$

- The other limit takes a little more work and we use the squeeze / sandwich theorem. When $x < 1$ we have

$$-(x - 1)^2 \leq (x - 1)^2 \cos\left(\frac{1}{x - 1}\right) \leq (x - 1)^2 \quad \text{since } -1 \leq \cos(\theta) \leq 1$$

As $x \rightarrow 1^-$ we have

$$\lim_{x \rightarrow 1^-} -(x - 1)^2 = \lim_{x \rightarrow 1^-} (x - 1)^2 = 0$$

Hence, by the squeeze theorem

$$\lim_{x \rightarrow 1^-} (x - 1)^2 \cos\left(\frac{1}{x - 1}\right) = 0.$$

Therefore,

$$\lim_{x \rightarrow 1^-} c + (x - 1)^2 \cos\left(\frac{1}{x - 1}\right) = c.$$

- In order to have a continuous function we therefore need $c = 2$.