Midterm 2 Duration: 45 minutes

*This test has **5 questions** on **10 pages**, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Signature: ____

Question:	1	2	3	4	5	Total
Points:	10	9	5	7	9	40
Score:						

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 2 marks (for Q1) or 3 marks (for Q2), but not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

2 marks 1. (a) What is the value of $\arccos\left(\cos\left(\frac{5\pi}{3}\right)\right)$

Answer: $\frac{\pi}{3}$

Solution: $\arccos\left(\cos\left(\frac{5\pi}{3}\right)\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

2 marks (b) Find $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

Answer: $\frac{4}{5}$

Solution: $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sqrt{1-\cos^2\left(\arccos\left(-\frac{3}{5}\right)\right)} = \sqrt{1-\left(-\frac{3}{5}\right)^2} = \frac{4}{5}$.

2 marks (c) Let $f(x) = \ln\left(\frac{xe^x}{\sqrt{x^2+1}}\right)$ for $x > 0$ (ln denotes the natural logarithm). Find $f'(x)$.

Answer: $\frac{1}{x} + 1 - \frac{x}{x^2+1} = \frac{x^3+x+1}{x(x^2+1)}$

Solution: We first write using the properties of ln and exp

$$f(x) = \ln\left(\frac{xe^x}{\sqrt{x^2+1}}\right) = \ln(x) + x - \frac{1}{2}\ln\left(x^2+1\right).$$

Then the derivative is

$$f'(x) = \frac{1}{x} + 1 - \frac{x}{x^2+1}.$$

2 marks (d) Let y be such that $\frac{1}{\cos(y)} = x$. Assume $y \in \left(-\frac{\pi}{2}, 0\right)$. Find $\frac{dy}{dx}$ and express it in terms of x only.

Answer: $\frac{1}{x\sqrt{x^2-1}}$

Solution: Since $y \in \left(-\frac{\pi}{2}, 0\right)$, we have $\sin(y) = -\sqrt{1-\cos^2(y))}$. Rewrite the equation as

$$\frac{1}{x} = \cos(y)$$

and differentiate in x to get

$$-\frac{1}{x^2} = -\frac{dy}{dx}\sin(y).$$

Transforming sin into cos using the Pythagorean Theorem we get

$$-\frac{1}{x^2} = -\frac{dy}{dx}\sqrt{1-\cos^2(y))}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}.$$

2 marks (e) Let y be such that $x^3y + e^y = e$. Find the value of y'' at the point $x = 0$ (recall that e is the Euler constant, $e \simeq 2.7182818$).

Answer: $y''(0) = 0$

Solution: Differentiating once with respect to x we get

$$3x^2y + x^3y' + y'e^y = 0$$

which at $x = 0$ gives

$$y'(0)e^{y(0)} = 0,$$

hence $y'(0) = 0$. Differentiating a second time, we obtain

$$6xy + 6x^2y' + x^3y'' + y''e^y + (y')^2e^y = 0,$$

which at $x = 0$ gives

$$y''(0)e^{y(0)} = 0,$$

hence $y''(0) = 0$.

3 marks 2. (a) The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$f(t) = \frac{a}{1 + be^{-t}}$$

where t is measured in hours. At time $t = 0$, the population is 20 cells and is increasing at a rate of 10 cells/hour. Find the values of a and b.

Answer: $a = 40$, $b = 1$

Solution: First $f'(t) = \dfrac{abe^{-t}}{(1 + be^{-t})^2}$. At $t = 0$, we have

$$f(0) = 20 = \frac{a}{1 + b}, \quad f'(0) = 10 = \frac{ab}{(1 + b)^2}.$$

Therefore,

$$\frac{f(0)}{f'(0)} = 2 = \frac{1 + b}{b}$$

and $b = 1$. Replacing in the equation for $f(0)$ we get $a = 40$.

3 marks

(b) At noon, two ships A and B leave the same harbour. The ship A sails West at a constant speed of 15 km/h. The ship B sails North at a constant speed of 20 km/h. How fast is the distance between the two ships changing at 2:00 pm ?

Answer: 25 km/h

Solution: Call x the distance from ship A to the harbour P and y the distance from ship B to the harbour. Call z the distance between the ships. They are linked by the Pythagorean Theorem:

$$z^2 = x^2 + y^2.$$

Differentiation with respect to time gives

$$2\frac{dz}{dt}z = 2\frac{dx}{dt}x + 2\frac{dy}{dt}y.$$

At 2:00 pm, $x = 30$ and $y = 40$. Therefore $z = \sqrt{30^2 + 40^2} = 50$ and

$$\frac{dz}{dt} = \frac{1}{z}\left(\frac{dx}{dt}x + \frac{dy}{dt}y\right) = \frac{1}{50}(15 \times 30 + 20 \times 40) = 25.$$

3 marks

(c) Robert is preparing a roasted turkey. When he takes the turkey out of the oven, its temperature is 220°C. After 10 minutes, it is down to 120°C. The temperature of the kitchen is 20°C. Assume that the turkey is cooling according to Newton's law of cooling. Give the expression of the temperature of the turkey as a function of time (all parameters have to be explicitly computed, you can use $\ln(2) \simeq 0.69$).

Answer: $T(t) = 200e^{-0.069t} + 20$

Solution: Call T the temperature of the turkey in function of the time t in minutes. According to Newton's law of cooling,

$$\frac{dT}{dt} = k(T - T_0),$$

where k is an unknown factor and $T_0 = 20$ is the surrounding temperature. Solving for T, we find

$$T(t) = (T(0) - T_0)e^{kt} + T_0.$$

We already know that $T(0) = 220$. Therefore

$$T(t) = 200e^{kt} + 20.$$

To find k, we use

$$T(10) = 200e^{10k} + 20 = 120,$$

therefore $k = \frac{1}{10}\ln\left(\frac{1}{2}\right) = -0.069$

Full-Solution Problems. In questions 3–5, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

5 marks 3. Let $A(a,b)$ and $B(c,d)$ be two points on an ellipse of equation

$$x^2 + 2y^2 = 1.$$

Let L_A and L_B be the tangent lines to the ellipse at the points A and B.

If L_A and L_B have the same slope, then what relationship must there be between a and b ? What relationship is there between c and d ?

Solution: The slope of the tangent at a point of the ellipse (if it exists) is given by $\frac{dy}{dx}$. Differentiating implicitly, we get

$$2x + 4\frac{dy}{dx}y = 0,$$

that is $\frac{dy}{dx} = -\frac{x}{2y}$.

The slopes s_A and s_B of L_A and L_B are

$$s_A = -\frac{a}{2b}, \quad s_B = -\frac{c}{2d}.$$

To have equal slopes, we require that

$$-\frac{a}{2b} = -\frac{c}{2d}, \qquad (*)$$

That implies, using the equation of the ellipse,

$$\frac{a^2}{1-a^2} = \frac{c^2}{1-c^2}.$$

This gives

$$a^2 = c^2,$$

which is possible only if $a = c$ or $a = -c$.

If $a = c$, then by (*) we must have $b = d$ and A and B coincide.

If $a = -c$, then by (*) we must have $b = -d$ and A and B are on the opposite sides of a segment passing through the center of the ellipse.

3 marks 4. (a) Use linear approximation to estimate $\ln(1.1)$.

Answer: $\ln(1.1) \simeq 0.1$

Solution: We use linear approximation for $f(x) = \ln(x)$ at the point $a = 1$. We have $f'(x) = \frac{1}{x}$. The linear approximation L of f at $a = 1$ is

$$L(x) = f(a) + f'(a)(x - a) = x - 1.$$

Therefore

$$\ln(1.1) \simeq 1.1 - 1 = 0.1.$$

3 marks (b) Give an estimate of the error in your approximation.

Answer: $|\ln(1.1) - 0.1| \leq 0.005$

Solution: By Taylor-Lagrange formula, there exists c between 1 and 1.1 such that

$$f(x) = L(x) + \frac{f''(c)}{2}(1.1 - 1)^2.$$

Therefore

$$|f(x) - L(x)| \leq |f''(c)| 0.005.$$

Since $f''(c) = \frac{-1}{c^2}$ and $c \in [1, 1.1]$, we have

$$|f''(c)| \leq 1,$$

and therefore

$$|f(x) - L(x)| \leq 0.005.$$

1 mark (c) Is your approximation an overestimate or an underestimate ?

Answer: overestimate

Solution: Solution 1
From the previous answer, one has

$$f(x) = L(x) + \frac{f''(c)}{2}(1.1-1)^2.$$

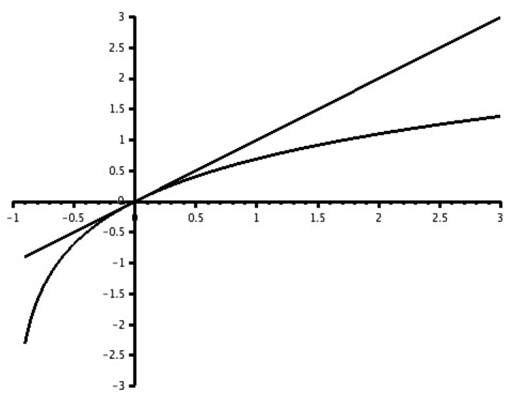
and $f''(c) = -\frac{1}{c^2} < 0$. Therefore

$$f(x) - L(x) < 0$$

and we have an overestimate.

Solution 2

One can see from the graph of f that the graph is always below the tangent. Hence the linear approximations are always overestimates.



5. Consider the function given by the expression $f(x) = \arccos\left(\sqrt{1-x^2}\right) + \frac{\pi}{2}$.

1 mark (a) Where is f defined ? Justify.

Answer: $[-1, 1]$

Solution: We want $1 - x^2 \geq 0$, that is $x \in [-1; 1]$. Then $\sqrt{1-x^2} \in [-1, 1]$ is a consequence of $x \in [-1; 1]$. So the function f is well defined on $[-1, 1]$.

1 mark (b) Where is f continuous ? Justify.

Answer: $[-1, 1]$

Solution: The function f is a composite function of continuous classical functions, hence it is itself continuous on its domain of definition.

2 marks (c) Where is f differentiable ? Justify

Answer: $(-1, 0) \cup (0, 1)$

Solution: For differentiability, we have to take into account the fact that $\sqrt{\cdot}$ is not differentiable at 0, that is for $x = \pm 1$, and arccos is not differentiable at ± 1, that is for $x = 0$. Therefore f is differentiable on $(-1, 0) \cup (0, 1)$.

2 marks

(d) Calculate the derivative of f (simplify your answer).

Solution: Applying the chain rule, we get

$$f'(x) = -\frac{-\frac{-2x}{2\sqrt{1-x^2}}}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} = \begin{cases} \dfrac{1}{\sqrt{1-x^2}} & \text{for } x > 0, \\ -\dfrac{1}{\sqrt{1-x^2}} & \text{for } x < 0. \end{cases}$$

3 marks

(e) For $x > 0$, recognize in the derivative of f the derivative of a classical inverse trigonometric function. Express f using this function.

Solution: We remark that $f' = \arcsin'$. As a consequence, f and arcsin differ only by a constant: there exists $c \in \mathbb{R}$ such that for $x \in (0, 1)$,

$$f(x) = \arcsin(x) + c.$$

To find c, we compute f at a point, for example $x = \frac{\sqrt{2}}{2}$. We have

$$f\left(\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}, \quad \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

Therefore $c = \frac{\pi}{2}$.