

Midterm 2 Duration: 45 minutes

This test has 5 questions on 10 pages, for a total of 40 points.

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	Total
Points:	10	9	5	7	9	40
Score:						

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 2 marks (for Q1) or 3 marks (for Q2), but not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

2 marks

1. (a) What is the value of $\arccos\left(\cos\left(\frac{5\pi}{3}\right)\right)$ Answer: $\frac{\pi}{3}$

$$\text{Solution: } \arccos\left(\cos\left(\frac{5\pi}{3}\right)\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

2 marks

(b) Find $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$ Answer: $\frac{4}{5}$

$$\text{Solution: } \sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sqrt{1 - \cos^2\left(\arccos\left(-\frac{3}{5}\right)\right)} = \sqrt{1 - \left(-\frac{3}{5}\right)^2} = \frac{4}{5}.$$

2 marks

- (c) Let $f(x) = \ln\left(\frac{xe^x}{\sqrt{x^2+1}}\right)$ for $x > 0$ (\ln denotes the natural logarithm). Find $f'(x)$.

$$\text{Answer: } \frac{1}{x} + 1 - \frac{x}{x^2+1} = \frac{x^3+x+1}{x(x^2+1)}$$

Solution: We first write using the properties of \ln and \exp

$$f(x) = \ln\left(\frac{xe^x}{\sqrt{x^2+1}}\right) = \ln(x) + x - \frac{1}{2}\ln(x^2+1).$$

Then the derivative is

$$f'(x) = \frac{1}{x} + 1 - \frac{x}{x^2+1}.$$

2 marks

- (d) Let y be such that $\frac{1}{\cos(y)} = x$. Assume $y \in \left(-\frac{\pi}{2}, 0\right)$. Find $\frac{dy}{dx}$ and express it in terms of x only.

$$\text{Answer: } \frac{1}{x\sqrt{x^2-1}}$$

Solution: Since $y \in \left(-\frac{\pi}{2}, 0\right)$, we have $\sin(y) = -\sqrt{1-\cos^2(y)}$. Rewrite the equation as

$$\frac{1}{x} = \cos(y)$$

and differentiate in x to get

$$-\frac{1}{x^2} = -\frac{dy}{dx}\sin(y).$$

Transforming \sin into \cos using the Pythagorean Theorem we get

$$-\frac{1}{x^2} = -\frac{dy}{dx}\sqrt{1-\cos^2(y)}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}.$$

2 marks

- (e) Let y be such that $x^3y + e^y = e$. Find the value of y'' at the point $x = 0$ (recall that e is the Euler constant, $e \simeq 2.7182818$).

$$\text{Answer: } y''(0) = 0$$

Solution: Differentiating once with respect to x we get

$$3x^2y + x^3y' + y'e^y = 0$$

which at $x = 0$ gives

$$y'(0)e^{y(0)} = 0,$$

hence $y'(0) = 0$. Differentiating a second time, we obtain

$$6xy + 6x^2y' + x^3y'' + y''e^y + (y')^2e^y = 0,$$

which at $x = 0$ gives

$$y''(0)e^{y(0)} = 0,$$

hence $y''(0) = 0$.

3 marks

2. (a) The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$f(t) = \frac{a}{1 + be^{-t}}$$

where t is measured in hours. At time $t = 0$, the population is 20 cells and is increasing at a rate of 10 cells/hour. Find the values of a and b .

$$\text{Answer: } a = 40, b = 1$$

Solution: First $f'(t) = \frac{abe^{-t}}{(1 + be^{-t})^2}$. At $t = 0$, we have

$$f(0) = 20 = \frac{a}{1 + b}, \quad f'(0) = 10 = \frac{ab}{(1 + b)^2}.$$

Therefore,

$$\frac{f(0)}{f'(0)} = 2 = \frac{1 + b}{b}$$

and $b = 1$. Replacing in the equation for $f(0)$ we get $a = 40$.

3 marks

- (b) At noon, two ships A and B leave the same harbour. The ship A sails West at a constant speed of 15 km/h. The ship B sails North at a constant speed of 20 km/h. How fast is the distance between the two ships changing at 2:00 pm ?

Answer: 25 km/h

Solution: Call x the distance from ship A to the harbour P and y the distance from ship B to the harbour. Call z the distance between the ships. They are linked by the Pythagorean Theorem:

$$z^2 = x^2 + y^2.$$

Differentiation with respect to time gives

$$2\frac{dz}{dt}z = 2\frac{dx}{dt}x + 2\frac{dy}{dt}y.$$

At 2:00 pm, $x = 30$ and $y = 40$. Therefore $z = \sqrt{30^2 + 40^2} = 50$ and

$$\frac{dz}{dt} = \frac{1}{z} \left(\frac{dx}{dt}x + \frac{dy}{dt}y \right) = \frac{1}{50} (15 \times 30 + 20 \times 40) = 25.$$

3 marks

- (c) Robert is preparing a roasted turkey. When he takes the turkey out of the oven, its temperature is 220°C . After 10 minutes, it is down to 120°C . The temperature of the kitchen is 20°C . Assume that the turkey is cooling according to Newton's law of cooling. Give the expression of the temperature of the turkey as a function of time (all parameters have to be explicitly computed, you can use $\ln(2) \simeq 0.69$).

Answer: $T(t) = 200e^{-0.069t} + 20$

Solution: Call T the temperature of the turkey in function of the time t in minutes. According to Newton's law of cooling,

$$\frac{dT}{dt} = k(T - T_0),$$

where k is an unknown factor and $T_0 = 20$ is the surrounding temperature. Solving for T , we find

$$T(t) = (T(0) - T_0)e^{kt} + T_0.$$

We already know that $T(0) = 220$. Therefore

$$T(t) = 200e^{kt} + 20.$$

To find k , we use

$$T(10) = 200e^{10k} + 20 = 120,$$

therefore $k = \frac{1}{10} \ln\left(\frac{1}{2}\right) = -0.069$

Full-Solution Problems. In questions 3–5, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

5 marks

3. Let $A(a, b)$ and $B(c, d)$ be two points on an ellipse of equation

$$x^2 + 2y^2 = 1.$$

Let L_A and L_B be the tangent lines to the ellipse at the points A and B .

If L_A and L_B have the same slope, then what relationship must there be between a and b ? What relationship is there between c and d ?

Solution: The slope of the tangent at a point of the ellipse (if it exists) is given by $\frac{dy}{dx}$. Differentiating implicitly, we get

$$2x + 4\frac{dy}{dx}y = 0,$$

that is $\frac{dy}{dx} = -\frac{x}{2y}$.

The slopes s_A and s_B of L_A and L_B are

$$s_A = -\frac{a}{2b}, \quad s_B = -\frac{c}{2d}.$$

To have equal slopes, we require that

$$-\frac{a}{2b} = -\frac{c}{2d}, \quad (*)$$

That implies, using the equation of the ellipse,

$$\frac{a^2}{1-a^2} = \frac{c^2}{1-c^2}.$$

This gives

$$a^2 = c^2,$$

which is possible only if $a = c$ or $a = -c$.

If $a = c$, then by (*) we must have $b = d$ and A and B coincide.

If $a = -c$, then by (*) we must have $b = -d$ and A and B are on the opposite sides of a segment passing through the center of the ellipse.

3 marks

4. (a) Use linear approximation to estimate
- $\ln(1.1)$
- .

Answer: $\ln(1.1) \simeq 0.1$

Solution: We use linear approximation for $f(x) = \ln(x)$ at the point $a = 1$. We have $f'(x) = \frac{1}{x}$. The linear approximation L of f at $a = 1$ is

$$L(x) = f(a) + f'(a)(x - a) = x - 1.$$

Therefore

$$\ln(1.1) \simeq 1.1 - 1 = 0.1.$$

3 marks

- (b) Give an estimate of the error in your approximation.

Answer: $|\ln(1.1) - 0.1| \leq 0.005$

Solution: By Taylor-Lagrange formula, there exists c between 1 and 1.1 such that

$$f(x) = L(x) + \frac{f''(c)}{2}(1.1 - 1)^2.$$

Therefore

$$|f(x) - L(x)| \leq |f''(c)|0.005.$$

Since $f''(c) = \frac{-1}{c^2}$ and $c \in [1, 1.1]$, we have

$$|f''(c)| \leq 1,$$

and therefore

$$|f(x) - L(x)| \leq 0.005.$$

1 mark

(c) Is your approximation an overestimate or an underestimate ?

Answer: overestimate

Solution: Solution 1

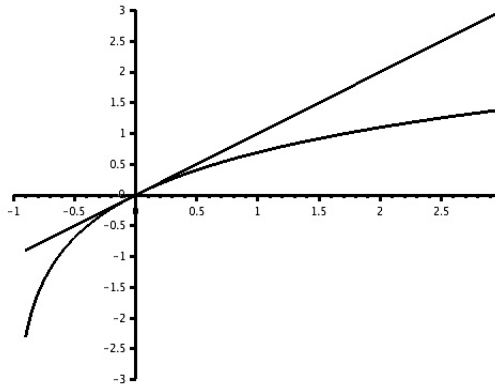
From the previous answer, one has

$$f(x) = L(x) + \frac{f''(c)}{2}(1.1 - 1)^2.$$

and $f''(c) = -\frac{1}{c^2} < 0$. Therefore

$$f(x) - L(x) < 0$$

and we have an overestimate.

Solution 2One can see from the graph of f that the graph is always below the tangent. Hence the linear approximations are always overestimates.5. Consider the function given by the expression $f(x) = \arccos(\sqrt{1-x^2}) + \frac{\pi}{2}$.

1 mark

(a) Where is f defined ? Justify.Answer: $[-1, 1]$

Solution: We want $1 - x^2 \geq 0$, that is $x \in [-1; 1]$. Then $\sqrt{1 - x^2} \in [-1, 1]$ is a consequence of $x \in [-1; 1]$. So the function f is well defined on $[-1, 1]$.

1 mark

(b) Where is f continuous ? Justify.Answer: $[-1, 1]$

Solution: The function f is a composite function of continuous classical functions, hence it is itself continuous on its domain of definition.

2 marks

(c) Where is f differentiable ? JustifyAnswer: $(-1, 0) \cup (0, 1)$

Solution: For differentiability, we have to take into account the fact that $\sqrt{\cdot}$ is not differentiable at 0, that is for $x = \pm 1$, and \arccos is not differentiable at ± 1 , that is for $x = 0$. Therefore f is differentiable on $(-1, 0) \cup (0, 1)$.

2 marks

- (d) Calculate the derivative of
- f
- (simplify your answer).

Solution: Applying the chain rule, we get

$$f'(x) = -\frac{-\frac{2x}{2\sqrt{1-x^2}}}{\sqrt{1 - (\sqrt{1-x^2})^2}} = \begin{cases} \frac{1}{\sqrt{1-x^2}} & \text{for } x > 0, \\ -\frac{1}{\sqrt{1-x^2}} & \text{for } x < 0. \end{cases}$$

3 marks

- (e) For
- $x > 0$
- , recognize in the derivative of
- f
- the derivative of a classical inverse trigonometric function. Express
- f
- using this function.

Solution: We remark that $f' = \arcsin'$. As a consequence, f and \arcsin differ only by a constant: there exists $c \in \mathbb{R}$ such that for $x \in (0, 1)$,

$$f(x) = \arcsin(x) + c.$$

To find c , we compute f at a point, for example $x = \frac{\sqrt{2}}{2}$. We have

$$f\left(\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}, \quad \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

Therefore $c = \frac{\pi}{2}$.