YOUNG TABLEAUX AND HOPF ALGEBRAS

Études tableaux

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We discuss a braided monoidal category $\mathfrak{Y}$ whose objects are semistandard Young tableaux, and a tensor structure comes from the classical Knuth's product of tableaux.

The category $\mathfrak{Y}$ descends by means of the RSK correspondence from the universal braided monoidal category $\mathfrak{CM}$ introduced in [KS].

This is work in progress, a complement to [KS].

NOMINA

Karl Pearson, FRS (1857 - 1936)

(+ **William Sealy Gosset (Student)** (1876 – 1937), statistician and brewer, Head Brewer of Guinness,

and

Sir Ronald Aylmer Fisher FRS (1890 – 1962))

Gilbert de Beauregard Robinson (1906 - 1992)

His mother, Esther Toutant Beauregard, was a French girl, a grandnephew of the Confederate general Pierre Gustave Toutant Beauregard (1818-1893).

His thesis advisor was **Alfred Young** (1873 - 1940)

Craige Schensted (1927 – 2021)

Donald Ervin Knuth (b. 1938)

Robert Steinberg (1922 - 2014)

§1. Robinson - Shensted - Knuth correspondence

1.1. Contingency tables and generalized permutations. Recall (cf. [P]) that a contingency table is a rectangular matrix $A = (a_{ij})$ with integer nonnegative coefficients, cf. also [DK] where they appear under the name *arrays*.

If such A has size, say, $n \times m$, then we remark with Knuth, cf. [Kn], that it is the same as a "generalized permutation" : a two line array of integers

$$Perm(A) = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} u_1 & \dots & u_k \\ v_1 & \dots & v_k \end{pmatrix} \tag{1.1.1}$$

with

$$1 \leq u_1 \leq \dots \leq u_k$$

and if $i \leq j, u_i = u_j$ then $v_i \leq v_j$, i.e. the columns are arranged lexicografically, with

a_{ij} = number of occurences of a column $\begin{matrix} i \\ j \end{matrix}$ in (1.1.1).

Thus

the number of i's among u_p's $= \Sigma_i A := \sum_j a_{ij}$,

(i-th horizontal margin), so

$$k = \Sigma A := \sum_{i,j} a_{ij}.$$

Concatenation

Let CM_{nm} denote the set of contingency tables with n rows and m columns. We have an associative operation

$$CM_{nm} \times CM_{n'm} \longrightarrow CM_{n+n',m},\ (A, B) \mapsto \begin{pmatrix} A \\ B \end{pmatrix}.$$

On the other hand we have an obvious operation of concatenation for generalized permutations generalizing embeddings of symmetric groups

$$S_n \times S_{n'} \longrightarrow S_{n+n'}$$

Evidently

$$Perm \begin{pmatrix} A \\ B \end{pmatrix} = Perm(A)Perm(B) \tag{1.1.2}$$

1.2. Insertion. Recall the notions of SYT and SSYT.

A SYT may be defined inductively, using a growth procedure.

We have a basic operation of *insertion* of a natural number v to a SSYT T, to be denoted

$$I_v(T)$$

for example

$$I_v(\emptyset) = \boxed{\mathrm{u}}$$

Each SSYT may be obtained by a sequence of insertions from the empty tableau.

However such a representation is not unique.

1.2.1. Example of a SSYT (in fact it is standard):

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 8 & 9 \\ \hline 3 & 5 & 7 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array}$$

We have

$$T = I_9 I_8 I_4 I_2 I_1 I_7 I_5 I_3 I_6(\emptyset)$$

(we go downstairs by the rows and from right to the left in each row; this is a distinguished way).

1.3. RSK correspondence. To a generalized permutation $Perm(A)$ one associates a couple of semistandard Young tableaux $(P(A), Q(A))$ of the same shape with $N = \Sigma A$ cells, with

$$P(A) = I_{\mathbf{v}}(\emptyset) = I_{v_N} \ldots I_{v_1}(\emptyset)$$

and Q encodes the order of adding cells to P, cf. [S].

We have

$$Q(A) = P(A^t),$$

cf. [Kn], Thm. 3.

Each SSYT T is $P(A)$ for some $A \in CM$. The number of cells in $T = \Sigma A$.

1.4. A geometric interpretation of RSK (for standard Young tableaux) was given by R.Steinberg, [Ste]. It is based on the following fact.

Let V be an n-dimensional vector space over an infinite field. Any unipotent automorphism u of V defines a partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$$

of n where λ_i are the sizes of the Jordan blocks for u, and therefore a Young diagram T_λ.

Let $\mathcal{F}$ be the variety of full flags in V, and

$$\mathcal{F}_\lambda = \{(V_1 \subset V_2 \subset \ldots \subset V_n = V | \ \forall i \ u(V_i) = V_i\} \subset \mathcal{F}$$

Then

the set of irreducible components of $\mathcal{F}_\lambda$ is in bijection with the set of SYT with table T_λ.

1.5. Knuth describes the fibers of the map

$$P: \ CM \longrightarrow Y$$

by means of certain equivalence relation on CM, see [Kn], Th. 6.

Standard Young tableaux correspond to permutation matrices. The corresponding equivalence relation appeared in [KL] who in turn refer to Vogan, Jantzen and Joseph, see [KL], §5, and is defined for any Weil group W; the equivalence classes are called *left cells*.

§2. Knuth multiplication and a braided structure

2.1. Multiplication of tableaux and Knuth's theorem. More generally, Knuth introduces an associative operation of multiplication for SSYT.

If $T = P(A) = I_{\mathbf{v}}(\emptyset)$ as above and T' is another SSYT then

$$T' \cdot T = I_{\mathbf{v}}(T') = I_{v_N} \ldots I_{v_1}(T') \tag{2.1.1}$$

If $T' = Perm(B)$ then

$$T' \cdot T = Perm \begin{pmatrix} A \\ B \end{pmatrix}, \tag{2.1.2}$$

cf. (1.1.2) and [Kn], Corollary of Thm. 6, [Kn2], 5.1.4.

2.2. Braided category $\mathfrak{CM}$. On the other hand we know from [KS] that the concatenation of contingency tables is a part of certain braided structure. Namely, one defines an additive *braided tensor category* $\mathfrak{CM}$ whose objects are all contingency tables (matrices), and morphisms are generated by certain "fusions" of matrices (called "contractions" in *op. cit.*), subject to some relations.

More precisely, one introduces two operations called vertical and horizontal fusions on the set CM where the vertical (resp. horizontal) fusion does not change the number of rows (resp. columns); both operations do not change ΣM.

They give rise to two partial orders on CM denoted by $\leq_v, \leq_h$.

The arrows in $\mathfrak{CM}$ are generated by:

$h_{M'M}: \ M' \longrightarrow M$ if $M' \leq_h M$, and

$h_{MM''}: \ M \longrightarrow M''$ if $M'' \leq_v M$.

They are subject to the transitivity relations, and to the mixed relation:

$$h_{AB}v_{CA} = \sum_{D: B \leq_v D, C \leq_h D} v_{DB}h_{CD}$$

A fusion of a matrix M is called *anodyne* if it does not change the set of nonzero elements of M. We require that the anodyne fusions become invertible arrows in $\mathfrak{CM}$.

$\mathfrak{CM}$ is generated as a tensor category by the collection of 1×1 contingency tables $A_n = (n)$ whose tensor products form an $\mathbb{N}$-graided braided bialgebra $\mathfrak{a}$ in $\mathfrak{CM}$.

We have an orthogonal decomposition

$$\mathfrak{CM} = \oplus_{n \geq 0} \mathfrak{CM}_n$$

where $\mathfrak{CM}_n$ is the full subcategory of contingency tables A with $\Sigma A = n$.

If $\mathbf{k}$ is a field then the category of additive functors $Funct(\mathfrak{CM}_n; Vect^f(\mathbf{k}))$ is equivalent to the category $Perv(Sym^n(\mathbb{C}), \mathcal{S})$ of perverse sheaves on $Sym^n(\mathbb{C}) = \mathbb{C}^n/\Sigma_n$ smooth along the diagonal stratification $\mathcal{S}$.

2.3. Example: $\mathfrak{CM}_2$.

This part corresponds to

$$A_2 \underset{\mu'}{\overset{\Delta}{\rightleftarrows}} A_1 \otimes A_1$$

Relation:

$$\Delta\mu' = 1 + R$$

where

$$R: \ A_1 \otimes A_1 \xrightarrow{\sim} A_1 \otimes A_1.$$

A picture in $\mathfrak{CM}_2$:

$$\begin{array}{ccccc}
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \overset{\beta}{\underset{}{\overset{\sim}{\longleftarrow}}} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \overset{\alpha}{\overset{\sim}{\longrightarrow}} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\delta\downarrow & & \mu\downarrow & & \downarrow\gamma \\
(1\ 1) & \overset{\nu}{\longleftarrow} & (2) & \overset{\nu}{\longrightarrow} & (1\ 1) \\
\gamma\uparrow & & \mu\uparrow & & \uparrow\delta \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \overset{\alpha}{\overset{\sim}{\longleftarrow}} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \overset{\beta}{\overset{\sim}{\longrightarrow}} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{array} \qquad (a)$$

The maps $\alpha, \beta, \gamma, \delta$ are anodyne, whence invertible.

Relation:

$$\nu\mu = \gamma\alpha + \delta\beta,$$

or

$$\nu\mu\alpha^{-1}\gamma^{-1} = 1 + \delta\beta\alpha^{-1}\gamma^{-1}$$

which is the same as

$$\Delta\mu' = 1_{A_1 \otimes A_1} + R,$$

comme il faut.

Corresponding diagram of generalized permutations:

$$\begin{array}{ccccc}
\begin{matrix} 1\ 2 \\ 1\ 2 \end{matrix} & \overset{\sim}{\longleftarrow} & \begin{matrix} 1\ 2 \\ 1\ 1 \end{matrix} & \overset{\sim}{\longrightarrow} & \begin{matrix} 1\ 2 \\ 2\ 1 \end{matrix} \\
\downarrow & & \mu\downarrow & & \downarrow \\
\begin{matrix} 1\ 1 \\ 1\ 2 \end{matrix} & \longleftarrow & \begin{matrix} 1\ 1 \\ 1\ 1 \end{matrix} & \overset{\nu}{\longrightarrow} & \begin{matrix} 1\ 1 \\ 1\ 2 \end{matrix} \\
\uparrow & & \uparrow & & \uparrow \\
\begin{matrix} 1\ 2 \\ 2\ 1 \end{matrix} & \overset{\sim}{\longleftarrow} & \begin{matrix} 1\ 2 \\ 1\ 1 \end{matrix} & \overset{\sim}{\longrightarrow} & \begin{matrix} 1\ 2 \\ 1\ 2 \end{matrix}
\end{array} \qquad (b)$$

Corresponding diagram of semistandard Young tableaux, $\mathfrak{Y}_2$:

$$
\begin{array}{ccccc}
\begin{array}{|c|c|}\hline 1&2\\\hline\end{array} & \overset{\beta}{\underset{\sim}{\longleftarrow}} & \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} & \overset{\alpha}{\underset{\sim}{\longrightarrow}} & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \\
\delta\downarrow & & \mu\downarrow & & \downarrow\gamma \\
\begin{array}{|c|c|}\hline 1&2\\\hline\end{array} & \overset{\nu}{\longleftarrow} & \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} & \overset{\nu}{\longrightarrow} & \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \\
\gamma\uparrow & & \mu\uparrow & & \uparrow\delta \\
\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \overset{\alpha}{\underset{\sim}{\longleftarrow}} & \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} & \overset{\beta}{\underset{\sim}{\longrightarrow}} & \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}
\end{array}
\qquad (c)
$$

Note that here $\mu = 1_{\begin{array}{|c|c|}\hline 1&1\\\hline\end{array}}$, and $\delta = 1_{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}}$

Relation:

$$\nu = \gamma\alpha + \delta\beta = \gamma\alpha + \beta,$$

or

$$\nu\alpha^{-1}\gamma^{-1} = 1_{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}} + \beta\alpha^{-1}\gamma^{-1}$$

So $\mathfrak{Y}_2$ has three objects which are all SSYT with two cells with contents $\{1\}$ or $\{1, 2\}$.

Warning: all objects of $\mathfrak{Y}_2$ are isomorphic but $\mathfrak{Y}_2$ is not a groupoid, it is an additive category.

2.4. Example: $\mathfrak{CM}_3$ and $\mathfrak{Y}_3$. (a) A part of $\mathfrak{CM}_3$ corresponding to

$$A_3 \overset{\leftarrow}{\longrightarrow} A_2 \otimes A_1 \oplus A_1 \otimes A_2$$

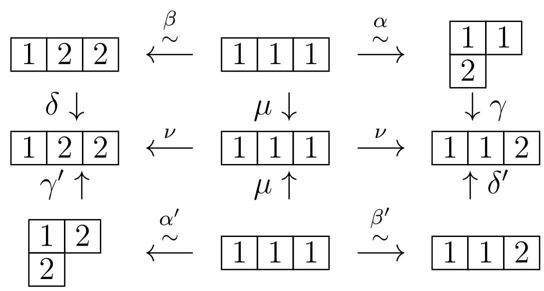
will be

$$
\begin{array}{ccccc}
\begin{pmatrix}1&0\\0&2\end{pmatrix} & \overset{\beta}{\underset{\sim}{\longleftarrow}} & \begin{pmatrix}1\\2\end{pmatrix} & \overset{\alpha}{\underset{\sim}{\longrightarrow}} & \begin{pmatrix}0&1\\2&0\end{pmatrix} \\
\delta\downarrow & & \mu\downarrow & & \downarrow\gamma \\
(1\ 2) & \overset{\nu}{\longleftarrow} & (3) & \overset{\nu}{\longrightarrow} & (2\ 1) \\
\gamma'\uparrow & & \mu'\uparrow & & \uparrow\delta' \\
\begin{pmatrix}0&2\\1&0\end{pmatrix} & \overset{\alpha'}{\underset{\sim}{\longleftarrow}} & \begin{pmatrix}2\\1\end{pmatrix} & \overset{\beta'}{\underset{\sim}{\longrightarrow}} & \begin{pmatrix}2&0\\0&1\end{pmatrix}
\end{array}
\qquad (a)
$$

(c) A part of $\mathfrak{Y}_3$ corresponding to

$$Y_3 \overset{\leftarrow}{\longrightarrow} Y_2 \otimes Y_1 \oplus Y_1 \otimes Y_2$$

will be



(c)

Note that here

$$\mu = 1_{\boxed{1|1|1}},\ \delta = 1_{\boxed{1|2|2}},\ \delta' = 1_{\boxed{1|1|2}}$$

Objects of $\mathfrak{Y}_3$ are all SSYT with 3 cells and contents $\{1\}$, $\{1,2\}$ or $\{1,2,3\}$.

The absence of gaps in the contents corresponds to the absence of rows with all zeros in the contingency tables.

For a SSYT Y let $\max(Y)$ denote the maximal number in the contents of Y.

There are $5 = 1 + 4$ objects in the "$A_1 \otimes A_2$" local system over

$$X_2 = Sym_2\mathbb{C} \setminus \Delta,$$

they are all SSYT Y with 3 cells, no gaps, and $\max(Y) \leq 2$.

There are $9 = 1 + 4 + 4$ objects in the "$A_1^{\otimes 3}$" local system over

$$X_3 = Sym_3\mathbb{C} \setminus \cup(\text{diagonals}),$$

they are all SSYT Y with 3 cells, no gaps, and $\max(Y) \leq 3$, see below §3.

2.5. The general case seems similar. The objects of $\mathfrak{Y}_n$ are all SSYT Y with n cells, no gaps, and $\max(Y) \leq n$.

For each partition of n,

$$\mathbf{n} = (n_1, \ldots, n_p),\ \sum n_i = n,\ n_1 \geq \ldots \geq n_p$$

we have a local system "$A_{\mathbf{n}}$", or

$$"A_{n_1} \otimes \ldots \otimes A_{n_p}"$$

over

$$X_n = Sym_n\mathbb{C} \setminus \cup(\text{diagonals}),$$

given by a groupoid whose objects are all SSYT Y with n cells, no gaps, and $\max(Y) \leq n$.

2.6. Let Y_n denote the row tableaux with all 1's. The tensor product $Y_n \otimes Y_m$ is a row tableau with n 1's followed by m 2's, etc.

2.6.1. Conjecture. $\mathfrak{Y} = \oplus_{n \geq 0} \mathfrak{Y}_n$ is equivalent to a free $\mathbb{N}$-graded braided monoidal category with $\mathfrak{Y}_0 = \{\mathbf{1}\}$ and one generator $y = \boxed{1} \in \mathfrak{Y}_1$.

2.7. Relation to the plactic monoid? In [F], 2.1 Fulton defines a *plactic monoid*

$$M = F/R$$

and says that the monoid of tableaux is isomorphic to M.

§3. Details for $\mathfrak{CM}_3$ and $\mathfrak{Y}_3$

3.1. Here is the 3×3 master square $\mathcal{A}_3$ for $\mathfrak{CM}_3$:

$$\begin{array}{ccccc}
\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} & \longleftarrow & \left\{\begin{matrix}\begin{pmatrix}0&1\\1&0\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\\0&1\end{pmatrix}\\ \begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix}\\ \begin{pmatrix}1&0\\1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&1\\1&0\end{pmatrix}\end{matrix}\right\} & \longleftarrow & \left\{\begin{matrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\\ \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\1&0&0\end{pmatrix}\\ \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\end{matrix}\right\} \\
\downarrow & & \downarrow & & \downarrow \\
\left\{\begin{matrix}\begin{pmatrix}2\\1\end{pmatrix}\\ \begin{pmatrix}1\\2\end{pmatrix}\end{matrix}\right\} & \longleftarrow & \left\{\begin{matrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}\\ \begin{pmatrix}0&1\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}\\ \begin{pmatrix}2&0\\0&1\end{pmatrix}, \begin{pmatrix}0&2\\1&0\end{pmatrix}\\ \begin{pmatrix}0&1\\2&0\end{pmatrix}, \begin{pmatrix}1&0\\0&2\end{pmatrix}\end{matrix}\right\} & \longleftarrow & \left\{\begin{matrix}\begin{pmatrix}0&1&1\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1\\0&1&0\end{pmatrix}\\ \begin{pmatrix}1&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&1\end{pmatrix}\\ \begin{pmatrix}0&1&0\\1&0&1\end{pmatrix} \begin{pmatrix}0&0&1\\1&1&0\end{pmatrix}\end{matrix}\right\} \\
\downarrow & & \downarrow & & \downarrow \\
\{(3)\} & \longleftarrow & \{(2\ \ 1), (1\ \ 2)\} & \longleftarrow & \{(1\ \ 1\ \ 1)\}
\end{array}$$

3.2. The cardinal matrix :

$$A_3 = |\mathcal{A}_3| = \begin{pmatrix} 1 & 6 & 6 \\ 2 & 8 & 6 \\ 1 & 2 & 1 \end{pmatrix}$$

A finer structure on A_3:

$$A_3 = \begin{pmatrix} 1 & 6 & 6 \\ 2 & 4+4 & 6 \\ 1 & 2 & 1 \end{pmatrix}$$

Here $8 = 4 + 4$ means that the set $\mathcal{A}_3(2,2)$ contains 4 matrices with contents $\{1,2\}$ and 4 matrices with contents $\{1,1,1\}$.

3.3. Metamatrix of weight $n = 3$.

3.3.1. First line:

m_{11}:

$$(3); \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}; \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \|$$

m_{12}, 2 elements:

$$\begin{pmatrix}2 & 1\end{pmatrix}; \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 2\end{pmatrix}; \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \mid \begin{pmatrix}1 & 2\end{pmatrix}; \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 2\end{pmatrix}; \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}$$

m_{13}:

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}; \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\end{pmatrix}; \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}$$

3.3.2. Second line:

m_{21}, 2 elements:

$$\begin{pmatrix}1\\ 2\end{pmatrix}; \begin{pmatrix}1 & 2 & 2\\ 1 & 1 & 1\end{pmatrix}; \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}$$

$$\begin{pmatrix}2\\ 1\end{pmatrix}; \begin{pmatrix}1 & 1 & 2\\ 1 & 1 & 1\end{pmatrix}; \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}$$

m_{22}, the central element, consisting of $8 = 4 + 4$ elements:

(a) 4 of contents $\{1, 1, 1\}$:

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}; \begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 1\end{pmatrix}; \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \| \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}; \begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 2\end{pmatrix}; \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}$$

$$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}; \begin{pmatrix}1 & 2 & 2\\ 1 & 1 & 2\end{pmatrix}; \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array} \| \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}; \begin{pmatrix}1 & 2 & 2\\ 2 & 1 & 2\end{pmatrix}; \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$$

(b) and 4 of contents $\{1, 2\}$:

$$\begin{pmatrix}0&1\\2&0\end{pmatrix};\begin{pmatrix}1&2&2\\2&1&1\end{pmatrix};\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\Big| \quad \begin{pmatrix}0&2\\1&0\end{pmatrix};\begin{pmatrix}1&1&2\\2&2&1\end{pmatrix};\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}$$

$$\begin{pmatrix}2&0\\0&1\end{pmatrix};\begin{pmatrix}1&1&2\\1&1&2\end{pmatrix};\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}\Big| \quad \begin{pmatrix}1&0\\0&2\end{pmatrix};\begin{pmatrix}1&2&2\\1&2&2\end{pmatrix};\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}$$

m_{23}, $6 = 3 + 3$ elements:

(a)

$$\begin{pmatrix}1&0&0\\0&1&1\end{pmatrix};\begin{pmatrix}1&2&2\\1&2&3\end{pmatrix};\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}$$

$$\begin{pmatrix}0&1&0\\1&0&1\end{pmatrix};\begin{pmatrix}1&2&2\\2&1&3\end{pmatrix};\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\Big| \quad \begin{pmatrix}0&0&1\\1&1&0\end{pmatrix};\begin{pmatrix}1&2&2\\3&1&2\end{pmatrix};\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$$

(b)

$$\begin{pmatrix}0&1&1\\1&0&0\end{pmatrix};\begin{pmatrix}1&1&2\\2&3&1\end{pmatrix};\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\Big| \quad \begin{pmatrix}1&0&1\\0&1&0\end{pmatrix};\begin{pmatrix}1&1&2\\1&3&2\end{pmatrix};\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}$$

$$\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix};\begin{pmatrix}1&1&2\\1&2&3\end{pmatrix};\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}$$

We see that the first tableau is standard, whereas the second one is not.

3.3.3. Third line:

m_{31}:

$$\begin{pmatrix}1\\1\\1\end{pmatrix};\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix};\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

m_{32}, $6 = 3 + 3$ elements:

$$\begin{pmatrix}0&1\\1&0\\1&0\end{pmatrix};\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix};\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

$$\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix};\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix};\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$\begin{pmatrix}1&0\\1&0\\0&1\end{pmatrix};\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix};\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

and

$$\begin{pmatrix}1&0\\0&1\\0&1\end{pmatrix};\begin{pmatrix}1&2&3\\1&2&2\end{pmatrix};\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

$$\begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix};\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix};\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

$$\begin{pmatrix}0&1\\0&1\\1&0\end{pmatrix};\begin{pmatrix}1&2&3\\2&2&1\end{pmatrix};\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

The corner,

m_{33}, $6=3+3=2+2+2$ elements (all permutation matrices):

We have 3 shapes, and 4 standard tableaux:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array},$$

Correspondingly,

$$3! = 1^2 + 2^2 + 1^2$$

Elements:

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix};\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix};\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix};\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix};\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

***

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix};\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix};\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix};\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix};\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

***

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix};\boxed{1\,|\,2\,|\,3},\boxed{1\,|\,2\,|\,3}$$

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix};\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix};\begin{smallmatrix}1&3\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\end{smallmatrix}$$

That's it for $n = 3$.

3.4. Young metamatrix, $n = 3$**.**

3.4.1. We list here both tableaux P, Q.

For a Young tableau T, let

$|T|$ = the biggest number in T.

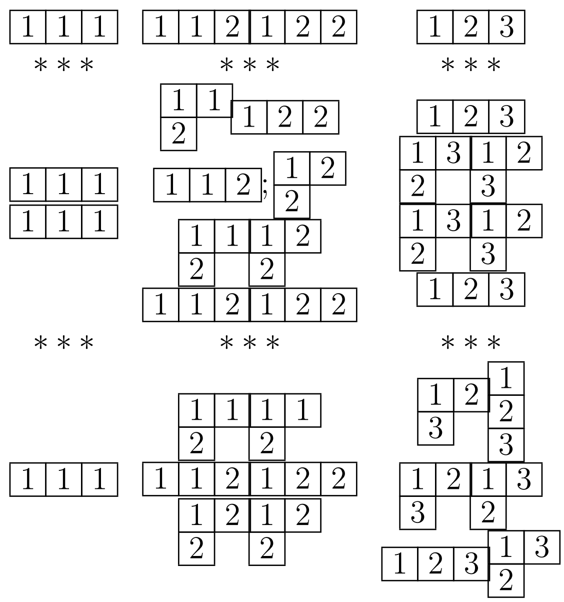
In our $n \times n$ metamatrix $M = (M_{ij})$ there will be all couples (T', T'') of SSYT on n boxes where T', T'' have the same shape.

$$M_{pq} = \{(T', T'')|\ |T'| = p, |T''| = q\}$$

Matrix:

$\begin{smallmatrix}1&1&1\end{smallmatrix},\begin{smallmatrix}1&1&1\end{smallmatrix}$	$\begin{smallmatrix}1&1&2\end{smallmatrix},\begin{smallmatrix}1&1&1\end{smallmatrix};\begin{smallmatrix}1&2&2\end{smallmatrix},\begin{smallmatrix}1&1&1\end{smallmatrix}$	$\begin{smallmatrix}1&2&3\end{smallmatrix},\begin{smallmatrix}1&1&1\end{smallmatrix}$
* * *	* * *	* * *
$\begin{smallmatrix}1&1&1\end{smallmatrix},\begin{smallmatrix}1&2&2\end{smallmatrix}$ $\begin{smallmatrix}1&1&1\end{smallmatrix},\begin{smallmatrix}1&1&2\end{smallmatrix}$	$\begin{smallmatrix}1&1\\2&\end{smallmatrix},\begin{smallmatrix}1&1\\2&\end{smallmatrix};\begin{smallmatrix}1&2&2\end{smallmatrix},\begin{smallmatrix}1&1&2\end{smallmatrix}$ $\begin{smallmatrix}1&1&2\end{smallmatrix},\begin{smallmatrix}1&2&2\end{smallmatrix};\begin{smallmatrix}1&2\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\2&\end{smallmatrix}$ $\begin{smallmatrix}1&1\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\2&\end{smallmatrix};\begin{smallmatrix}1&2\\2&\end{smallmatrix},\begin{smallmatrix}1&1\\2&\end{smallmatrix}$ $\begin{smallmatrix}1&1&2\end{smallmatrix},\begin{smallmatrix}1&1&2\end{smallmatrix};\begin{smallmatrix}1&2&2\end{smallmatrix},\begin{smallmatrix}1&2&2\end{smallmatrix}$	$\begin{smallmatrix}1&2&3\end{smallmatrix},\begin{smallmatrix}1&2&2\end{smallmatrix}$ $\begin{smallmatrix}1&3\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\2&\end{smallmatrix};\begin{smallmatrix}1&2\\3&\end{smallmatrix},\begin{smallmatrix}1&2\\2&\end{smallmatrix}$ $\begin{smallmatrix}1&3\\2&\end{smallmatrix},\begin{smallmatrix}1&1\\2&\end{smallmatrix};\begin{smallmatrix}1&2\\3&\end{smallmatrix},\begin{smallmatrix}1&1\\2&\end{smallmatrix}$ $\begin{smallmatrix}1&2&3\end{smallmatrix},\begin{smallmatrix}1&1&2\end{smallmatrix}$
* * *	* * *	* * *
$\begin{smallmatrix}1&1&1\end{smallmatrix},\begin{smallmatrix}1&2&3\end{smallmatrix}$	$\begin{smallmatrix}1&1\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\end{smallmatrix};\begin{smallmatrix}1&1\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\end{smallmatrix}$ $\begin{smallmatrix}1&1&2\end{smallmatrix},\begin{smallmatrix}1&2&3\end{smallmatrix};\begin{smallmatrix}1&2&2\end{smallmatrix},\begin{smallmatrix}1&2&3\end{smallmatrix}$ $\begin{smallmatrix}1&2\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\end{smallmatrix};\begin{smallmatrix}1&2\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\end{smallmatrix}$	$\begin{smallmatrix}1&2\\3&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\end{smallmatrix};\begin{smallmatrix}1\\2\\3\end{smallmatrix},\begin{smallmatrix}1\\2\\3\end{smallmatrix}$ $\begin{smallmatrix}1&2\\3&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\end{smallmatrix};\begin{smallmatrix}1&3\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\end{smallmatrix}$ $\begin{smallmatrix}1&2&3\end{smallmatrix},\begin{smallmatrix}1&2&3\end{smallmatrix};\begin{smallmatrix}1&3\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\end{smallmatrix}$

3.4.2. Only tableaux P left:



3.5. The category $\mathfrak{Y}_3$ has 9 objects:

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array},$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},$$

and

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}.$$

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