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FOREWORD

ON THE 16TH OF APRIL 1823, a number of fairies, summoned by Ganesha, the god of mathematical wisdom, assembled in Berlin at the cradle of an infant, to grant boons and bestow blessings. This was the first-born of a not too prosperous businessman who had married in June of the previous year; both parents were Jewish but had been baptized into the evangelical faith. Alas, as in all fairy tales, one old witch managed to creep in and resolved to undo, if she could, the work of the fairies.

"He will have genius, said the first fairy, and will be a worthy successor of Gauss, Dirichlet, Jacobi. —His life will be short and unhappy, said the witch. —He will have many brothers and sisters, said the next fairy, and will be tenderly attached to them, while remaining his mother's favorite. —He will lose them all, said the witch; seventeen years from now he will see the last one, a beloved small sister, die at the age of seven. —He will have brilliant teachers at the Gymnasium and will make giant strides in his mathematical studies. —But first, said the witch, his parents will misguidedly send him for four years to a private school whose rigid discipline will almost break his already fragile health and make him a nervous wreck for the rest of his life. —In his first year as a student at the University of Berlin, he will attract the attention of Humboldt, the grand old man of German science, and of Crelle, the editor of the leading mathematical journal of his time, and will have more than twenty papers accepted by Crelle that same year. —Maybe, said the witch; but first, for his support at the University, his mother will have to accept a paltry sum from the royal "indigent fund."—So what? said one big fairy with a strong American accent. Soon Humboldt will get him a yearly grant of 250 dollars from the RSF,[1] and will get it renewed when needed. —O.K., retorted the witch; but uncertainties about the payment and renewal of this stipend will

[1] Perhaps she means "Royal Science Foundation" (an obvious anachronism). By "dollar," of course, she means "thaler."

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plague and humiliate him for the rest of his life. —No matter, said the next fairy. Gauss, one of the hardest men to please in the mathematical world, will invite him, still a first-year student, to a visit in Göttingen, and from then on will take the deepest interest, not only in his work but also in his well-being. Jacobi, intent upon making him a "privatdozent" and anxious to cut bureaucratic red tape, will arrange for him to receive an honorary doctorate at the hands of Kummer in Breslau: surely an unheard-of favor to a second-year student! Gauss, while proposing Dirichlet for a coveted distinction (the order *"Pour le mérite"*), will let it be known that he has "almost hesitated" between him and young Eisenstein. —Much good this will do him! exclaimed the witch with a sneer. It will so enrage Jacobi that he will practically accuse your darling of plagiarism, in a wholly unmotivated footnote in Crelle's Journal. —Who knows not what the easily inflamed Jacobi can do once his temper is aroused? said another good fairy. This incident will deeply distress the young man for a while but will do him no further harm. Soon he will be a privatdozent, and the great Riemann will be one of his students. —Not for long! Riemann will migrate to Göttingen and forget whatever number-theory his young teacher thinks he has taught him. In the meanwhile, I will have seen to it that Kronecker and Heine leave Berlin; the young man will remain isolated without any congenial friend or companion." One fairy thought that Gauss' name had such virtue that it would silence the malevolent creature: "In 1847, the great Gauss will write a highly flattering foreword for a collection of his protégé's papers. —Hardly anyone will read them," replied the witch with utter contempt. "Then he will be so beaten up by the Prussian soldiery, during the revolutionary upheavals of 1848, that he will have to keep to his bed for a week; for two years he will publish nothing, and the reputation of being "red" will stick to him and threaten to jeopardize his stipend and his career. —He will not remain idle during those two years, in spite of all discouragement and ill health; his publications of the year 1850 will show him at the peak of his powers. Dirichlet, with Jacobi's concurrence, will propose him for membership in the Berlin Academy, and he will finally be elected in 1852, as Jacobi's successor, a young man of not yet 29 years of age. —And then he will die, said the witch triumphantly. —But his name

will survive, said a tiny fairy. —Hardly so, said the hag. Following academic usage, Dirichlet will read to the Academy a beautiful and moving eulogy of Jacobi, and Kummer will perform the same service for Dirichlet. But no member of the Academy will ever bother about the memory of the melancholy young man who had died in 1852. Still less will it occur to them to provide for the publication of his works, while voting ample funds for those of Jacobi, Dirichlet, Steiner, and later for Weierstrass and Kronecker. He will be forgotten, once and for all. —It is lucky, said one last fairy in a small voice, that you remind me of Kronecker. For many years, your curse will indeed prevent him from remembering the companion of his youth. But I will cause him to rediscover his friend's work before it it too late, and he will make it the theme of the main lecture to be given at the inauguration ceremonies of the German Mathematical Society." The witch laughed loudly. "It will be too late! I will kill Kronecker's wife, and he will cancel his lecture. —He will offer to write it up. —Before he does, I will kill him too, and then that name, which I do not want even to utter, will sink into final oblivion." There were no more fairies; but Ganesha had the last word. "You forget, he said, that all your curses are of limited duration; one hundred and fifty years from today, their force will be spent."

And so it has been. Now, at long last, we have the complete works of Eisenstein in two handsome, well-printed and well-bound volumes, of which the Chelsea Publishing Company may be proud. That every number-theorist will wish to possess them, along with those of Kummer and those of Hecke, should go almost without saying; but it is the reviewer's task to justify that statement by a brief description of the contents of these two volumes. First something should be said about Eisenstein's attitude towards mathematics.

That attitude is already summed up in the fascinating youthful autobiography (vol. II, pp. 882–898) which he submitted in 1843, instead of the more usual curriculum, when applying for the "Abitur" (roughly the equivalent of a graduation diploma from the Gymnasium, entitling him to enter the University). It is again described in his beautiful short address (vol. II, pp. 762–764) at his reception into the Academy in 1852. But he expressed it more concisely at Kronecker's examination for the doctorate in 1845. Following a medieval tradition which still survives in some countries, part of the examination consisted in "defending," not only the main thesis which had to be a substantial piece of work, but also a number of propositions of varying degrees of seriousness, against "opponents" chosen, partly by the Faculty, partly by the candidate among his own friends. One of Kronecker's propositions was: *"Mathesis et ars et scientia dicenda"* (mathematics is both art and science). Eisenstein objected that "mathematics is only art." Kronecker's answer has not been preserved; perhaps he felt much like Eisenstein.

As Eisenstein explains in greater detail, at first in his autobiography (pp. 892–895) and later in his academic address, it is this intense feeling for mathematical beauty which led him to devote himself wholly to number-theory, especially after his early study of Gauss, Dirichlet and Jacobi had persuaded him that, in their upper reaches (*in ihren "höchsten und feinsten Partien"*), number-theory and function-theory were becoming inseparable. The same view is expressed by Gauss in his foreword (reprinted here in vol. II, pp. 917–918) to Eisenstein's *Abhandlungen* of 1847. By function-theory, in this context, they mean primarily the theory of elliptic and related functions, and accessorily, perhaps, Dirichlet's use of "Dirichlet series." If one enlarges this to include abelian and related automorphic functions, there is still (or perhaps there is again) no more timely topic in the mathematics of today.

Within Eisenstein's production (an astonishingly abundant one, in view of the short time that fate allotted to him), one can clearly distinguish several periods. At first we see him getting into his stride by giving to Crelle more than twenty short papers (pp. 1–166 of vol. 1), many of them bearing upon the laws of quadratic, cubic and biquadratic reciprocity and upon Gaussian sums ("cyclotomy"); the series opens with a rather intriguing paper ([**1**, pp.1–5]; cf. also [**3**] and [**4**]) on binary cubic forms, with results related to the divisibility of the class-number of binary quadratic forms by 3; this is a topic which has again attracted some attention recently. Then, still in his first year, Eisenstein ventures into his first major undertaking [**24**, pp. 167–286]; but here he is unlucky. The example of Gauss' theory of binary quadratic forms had misled his immediate successors into believing that the key to algebraic number-fields was to be found in the study of forms of degree n in n variables over $\mathbf{Z}$, decomposable into n linear factors; this seems to have been Jacobi's and Dirichlet's view; perhaps it was Gauss' opinion. Eisenstein seeks to treat, from this point of view, the theory of cubic cyclic extensions of $\mathbf{Q}$, not without a considerable measure of success; but this was soon to be made obsolete by Kummer's theory of ideals, and is now no more than a historical curiosity.

Next comes the impressive series of papers on elliptic functions and their application to the higher reciprocity laws which fills up most of the remainder of the first volume (pp. 291–482). To us those proofs of the reciprocity laws are perhaps no more than a historically interesting example of complex multiplication; to Kummer, they were the apex of Eisenstein's arithmetical work. Pehaps Kummer, to whom elliptic functions always remained a sealed book, overvalued such proofs, while undervaluing Eisenstein's later work on the reciprocity laws. But the series ends up with a great paper [**28** f, pp. 357–478], the *Genaue Untersuchung* of 1847, which excited Kronecker's enthusiasm when he discovered it late in life, and which still deserves ours; it is nothing less than the sketch of a complete theory of elliptic and modular functions, based on principles essentially distinct from those of Jacobi and from those of Weierstrass. Not only does it go, without any use of function-theory, well beyond Weierstrass (while anticipating him by nearly 15 years), but, as I have more amply demonstrated elsewhere, its principles can be profitably applied to important current problems. It may not be superfluous to point out that those same principles had already been clearly adumbrated by Eisenstein in two of his early productions [**6**, pp. 28–34], and [**9**, pp. 55–58].

With the *Genaue Untersuchung,* Eisenstein's genius has reached full maturity, and, as Dirichlet was to say in 1849, "he has learnt the art of self-criticism, in which he had been lacking before." Perhaps this, even more than discouragement or ill health, is why, for the next three years, there appears only one brief "research announcement" [**32**, vol. II, p. 505] and one relatively minor paper [**33**, vol. II, pp. 506–535] under his name.

After that, we have only masterpieces, filling up more than half of the second volume. True, his theory of quadratic forms in 3 or more variables, which he had initiated as early as 1846 (cf. his letter to Gauss of April 1846, vol. II, pp. 838–843, and [**30**, vol. I, pp. 483–502]) was destined to remain a mere torso; but it is such an imposing one (see the papers [**32**], [**35**], [**40**], [**43**] in vol. II) that no history of the subject could be written without giving it a prominent place. The same can be said of his memorable note on the coefficients of the expansions of algebraic functions [**42**] and of his work on "cyclotomy," i.e., on the Gaussian sums ([**33**], [**39**]) and on the higher laws of reciprocity or rather on the local norm-residue symbol ([**36**], [**38**]); their value, in fact, could hardly have been fully appreciated until rather recently. But special mention should be made of the great paper [**34**, pp. 536–619] on the lemniscatic functions. Already in the *Disquisitiones,* Gauss had obscurely hinted at the analogies between the division of the circle and the division of the lemniscate; presumably he had nothing more in mind at that time than the fact that they generate abelian extensions of **Q** and of **Q**(i), respectively. Kummer, from 1845 on, had constructed the arithmetical theory of the cyclotomic fields, at least those generated by pth roots of unity when p is a prime; in connection with his need for data about the class-numbers of such fields, he had gone far into the investigation of the p-adic properties of exponential, circular and logarithmic functions and of Gaussian sums. At first, as we have noticed, Eisenstein had been skeptical about Kummer's researches and had rather followed Gauss and Dirichlet by investigating decomposable forms. By 1850, however, he is not only converted to Kummer's ideal-theory; he has extended it at any rate to the extensions of **Q**(i) generated by the division of the lemniscate, if not futher; he has acquired the general concept of an algebraic integer (which had remained foreign to Dirichlet and always remained foreign to Kummer), discovering for the first time that such integers make up a ring. But this is only a small part of his paper, and is quickly disposed of; the bulk of the paper is devoted to a p-adic investigation of lemniscatic functions, extending to them much of what Kummer had done for exponential functions. On this subject, it is not at all clear that we have even caught up with him.

The remainder of volume II consists of a number of *Eisensteiniana,* all of them of fascinating interest. Firstly, we have his letters to the Göttingen mathematician M. A. Stern, reprinted from the publication of Hurwitz and Rudio; this is a deeply moving human document, with valuable sidelights on Eisenstein's work. Then follows, as a special boon to the acquirers of these volumes, the series of his yearly letters to Gauss, preserved at the University library of Göttingen and hitherto unpublished and unknown; not only do they illuminate the touching relationship between him and his venerated patron, but nearly everyone of them describes, *in statu nascendi,* his latest ideas and discoveries; we see him develop from a bright beginner into a mature mathematician. Next we have his autobiography, already mentioned, followed by a rarity: Eisenstein's testimony on the incidents of March 1848, and on his scandalous mistreatment at the hands of the soldiery (not mitigated by the fact, on which he insists, that others suffered even more); then his preface to a posthumous publication of a friend, Gauss' foreword to his *Abhandlungen,* and finally K. R. Biermann's useful biographical notice on Eisenstein, reprinted from Crelle's Journal. None of this is superfluous, all of it is welcome.

It is fervently to be hoped that not only libraries, but many young mathematicians will be able to acquire these volumes and profit from them. Eisenstein tells us that his love for mathematics came from studying first Euler and Lagrange, then Gauss; studying the great work of the past is still the best education.

ANDRÉ WEIL