MCKAY CORRESPONDENCE

Vadim Schechtman

Colloquium talk, Institut de Mathématiques de Toulouse

September 2014

Plan

§1. Classical

1.1. Platonic solids.

1.2. Finite subgroups of $SU(2)$.

1.3. Kleinian singularities and graphs A, D, E.

1.4. Root systems, Cartan matrices and Dynkin graphs.

1.5. Root systems, simple and affine Lie algebras.

1.6. Representations of finite groups.

1.7. McKay correspondence : two formulations.

§2. Derived categories

2.1. Gonzalez-Sprinberg - Verdier.

2.2. Kapranov - Vasserot.

2.3. Bezrukavnikov - Kaledin

§3. Perron - Frobenius and Gamma products

John McKay (born 1939) is a dual British/Canadian citizen, a mathematician at Concordia University, known for his discovery of monstrous moonshine, his joint construction of some sporadic simple groups, and for the McKay correspondence relating certain finite groups to Lie groups.

§1. Classical

1.1. Platonic solids and their symmetry groups.

Platon, dialogue $T\iota\mu\alpha\iota o\varsigma$, 360 AD

Tetraèdre

Cube — Octaèdre

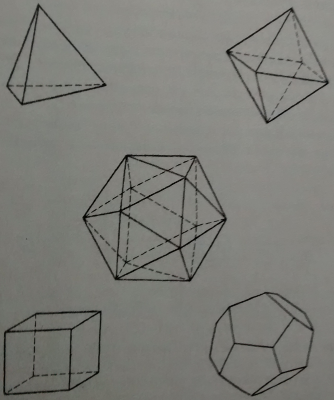
Icosaèdre — Dodecaèdre

Their symmetry groups :

$$G_{tetr} = A_4, G_{oct} = S_4, G_{icos} = A_5 \subset SO(3).$$

ОТ РЕДАКТОРА ПЕРЕВОДА (ВМЕСТО ПРЕДИСЛОВИЯ)

Среди пяти платоновых тел икосаэдр занимает «живое» место: в природе кристаллов в форме дра нет, но есть живые организмы (радиолярии



Самая древняя рукотворная модель икосаэдра — иг- ьная кость эпохи Птоломеев, найденная в Египте,— нится ныне в Британском музее. Новейшая модель —

3

1.2. Finite subgroups of $SL(2)$. We can consider the groups G above as subgroups

$$G \subset \mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) = PGL(2,\mathbb{C}) = PSL(2,\mathbb{C}).$$

We have an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow SL(2,\mathbb{C}) \xrightarrow{p} PSL(2,\mathbb{C}) \longrightarrow 1.$$

Exercise. Construct a double covering $SU(2) \longrightarrow SO(3)$. (Use quaternions ; note that $SU(2) = \{x \in Q |\ N(x) = 1\} = S^3$.)

Finite subgroups of $SL(2,\mathbb{C})$ (or of $SU(2)$).

Cyclic $G(A_n)$, of order $n+1$;

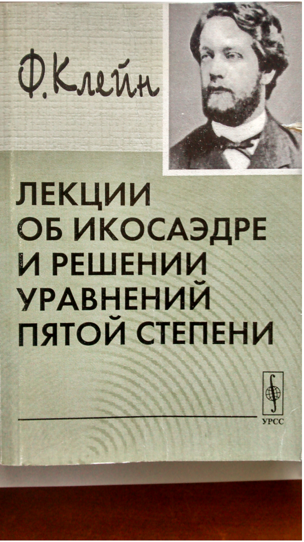
Binary dihedral $G(D_n)$, of order $4(n-2)$;

Binary tetrahedral, $G(E_6) = \pi^{-1}(G_{tetr})$, of order 24 ;

Binary octahedral, $G(E_7) = \pi^{-1}(G_{cube})$, of order 48 ;

Binary icosahedral, $G(E_8) = \pi^{-1}(G_{icos})$, of order 120.

Felix Klein, 25 avril 1849 [Düsseldorf] - 22 juin 1925 [Göttingen]



Leçons sur l'icosahèdre et sur la solution des équations de 5me degré.

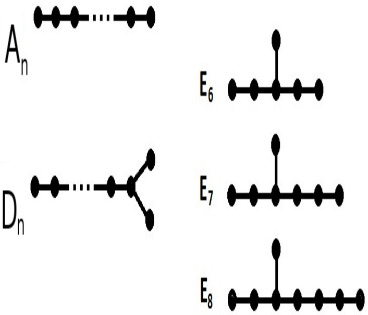
1.3. Kleinian singularities and Dynkin graphs. A finite $G \subset SL(2,\mathbb{C})$ acts on $\mathbb{C}^2$; let $X = \mathbb{C}^2/G$.

It is a singular surface; let $p : Y \longrightarrow X$ be its minimal desingularization.

The exceptional divizor

$$Y_0 = p^{-1}(0) = D_1 \cup \ldots D_r, \;\; D_i \overset{\sim}{=} \mathbb{P}^1(\mathbb{C}).$$

The incidence graph $Inc(\Gamma)$ is one of the simply laced Dynkin graphs A_n, D_n, E_6, E_7, E_8, cf. [Brie], see the picture.



A
n
D
n
E
6
E
7
E
8

The intersection matrix

$$A(G) = (D_i \cdot D_j)_{i,j=1}^r$$

is the corresponding Cartan matrix, defined by

$$A(\Gamma) = 2I - Inc(\Gamma).$$

Example. Singularity E_8 :

$$X = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3,$$

H.Schwarz (1872).

1.4. Root systems. These graphs arise in the classification of *finite reflection groups, root systems, and simple Lie algebras* (Killing, Cartan, Coxeter).

Let $(V, (.,.))$ be a real Euclidean space. To a nonzero $\alpha \in V$ we can associate an orthogonal reflection $s_\alpha : V \xrightarrow{\sim} V$,

$$s_\alpha(x) = x - \frac{2(x, \alpha)}{2(\alpha, \alpha)}.$$

A reduced irreducible *root system* is a finite subset $R \subset V \setminus \{0\}$ which generates V such that for all $\alpha, \beta \in R$

(R1) $s_\alpha(R) \subset R$ for all $\alpha \in R$;

(R2) $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$;

(R3) (reduced) $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$.

(R4) (irreducible) $R \neq R' \oplus R''$ with nonempty root systems R', R''.

A base of simple roots : a base of V

$$\{\alpha_1, \ldots, \alpha_r\} \subset R$$

such that for all $\alpha \in R$, $\alpha = \sum n_i \alpha_i$ where the numbers n_i are all negative or all positive. Such bases exist ; fix one.

The Cartan matrix

$$A = \{a_{ij}) = (2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)) \in GL(r, \mathbb{Z}),$$

$a_{ii} = 2, a_{ij} \leq 0$ for $i \neq j$.

Dynkin graph Γ : Vertices $V(\Gamma) = \{1, \ldots, r\}$

The possible angles between α_i, α_j are : $\pi/2, 2\pi/3, 3\pi/4, 5\pi/6$; we join the corresponding vertices by $0, 1, 2, 3$ edges respectively.

(This does not interest us : if the number of edges is > 1, we draw an arrow from the shorter root to the longer one.)

Classification :

$$A_n,\, B_n,\, C_n,\, D_n,\, E_6,\, E_7,\, E_8,\, F_4,\, G_2.$$

Simply laced :

$$A_n,\, D_n,\, E_6,\, E_7,\, E_8.$$

These root systems are in one-to-one correspondence with *finite dimensional simple Lie algebras*.

Examples. $\mathfrak{g}(A_n) = \mathfrak{sl}(n+1)$;
$\mathfrak{g}(B_n) = \mathfrak{so}(2n+1)$; $\mathfrak{g}(C_n) = \mathfrak{sp}(2n)$; $\mathfrak{g}(D_n) = \mathfrak{so}(2n)$.

Example. A_n.

$$V = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum x_i = 0\}.$$

All roots :

$$R = \{\epsilon_i - \epsilon_j,\ 0 \leq i \neq j \leq n\}$$

where $\epsilon_i = (0, \ldots, 1, \ldots, 0)$ (1 on i-th place).

Simple roots :

$$\alpha_1 = \epsilon_1 - \epsilon_0, \alpha_2 = \epsilon_2 - \epsilon_1 \ldots, \alpha_n = \epsilon_n - \epsilon_{n-1}$$

$$\angle(\alpha_i, \alpha_{i+1}) = 2\pi/3$$

Example. E_8. Cartan matrix :

$$A(E_8) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Affine root systems and affine Cartan matrices

Let

$$\theta = \sum_{i=1}^{r} n_i \alpha_i$$

be the longest root. We set

$$\alpha_0 = -\theta$$

and define the extended Cartan matrix

$$\hat{A} = (a_{ij})_{i,j=0}^{r} \in Mat(r+1, \mathbb{Z})$$

and the extended Dynkin graph $\hat{\Gamma}$ as before. It has a distinguished vertex 0.

The matrix $\hat{A}$ is degenerate now ; it has a multiplicity one eigenvector with eigenvalue 0 :

$$\delta = (1, n_1, \ldots, n_r)$$

1.5. Representations of finite groups. Let G be a finite group. *A (complex, finite dimensional) representation* of G is a pair (V, ρ) where V is a finite dimensional complex vector space, and ρ is a homomorphism

$$\rho : G \longrightarrow GL(V).$$

They form a category $Rep(G)$.

One has two operations in $Rep(G)$: $\oplus$ and $\otimes$ which are commutative, associative and distrubutive.

Important fact : $Rep(G)$ is *semisimple*. There is a finite number of simple reps $V_1, ..., V_m$, and every $V \overset{\sim}{=} \oplus V_i^{n_i}$.

The Grothendieck ring $R(G) = K_0(Rep(G))$.

***

Characters. For $\rho \in Rep(G)$ define $\chi_\rho : G \longrightarrow \mathbb{C}$, $\chi_\rho(g) = tr\rho(g)$.

Then

$$\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}, \ \chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}.$$

A character is a *central* function : $\chi(xyx^{-1}) = \chi(y)$.

Denote by G/G the set of conjugacy classes of G, and consider the ring $Maps.centr(G, \mathbb{C}) = Maps(G/G, \mathbb{C})$.

Main Theorem. The mapping $\rho \mapsto \chi_\rho$ induces an isomorphism of $\mathbb{C}$-algebras

$$\chi : R(G)_{\mathbb{C}} \xrightarrow{\sim} Maps(G/G, \mathbb{C}).$$

A base of the vector space on the left : irreducible reps, $\rho_1, \ldots, \rho_m$. a possible base on the right — the delta faunctions of conjugacy classes, $C_i \subset G,\ 1 \leq i \leq m$.

Corollary. There are as many irreducibles as conjugacy classes :

$$|Irr(G)| = |G/G|$$

Table of characters of G is the matrix

$$X(G) = \{\chi_i(c_j)\}$$

It follows from the Main Thm that $X(G) \in GL_m(\mathbb{C})$.

The following immediate fact will be crucial.

Steinberg Lemma, (R.Steinberg), [St]. *Let $y \in R(G)$ be an arbitrary element. Consider the operator $M_y = (m_{ij})$ of multiplication by y in $R(G)$:*

$$y\chi_i = \sum_{i=1}^{m} m_{ij}\chi_j.$$

Then for all $g \in G$ the vector

$$(\chi_1(g), \ldots, \chi_m(g))$$

is an eigenvector of $M(y)$ with eigenvalue $\chi_y(g)$.

Equivalently,

$$X_G M_y X_G^{-1} = Diag(\chi_y(C_1), \ldots, \chi_y(C_m)).$$

$\square$

Let $\rho \in Rep(G)$ be a representation of dimension d ; we define, with Steinberg

$$A = A(\rho) = dI - M([\rho]).$$

Let us call $A(\rho)$ the **Steinberg matrix** of ρ.

1.6. Theorem (McKay correspondence), [Mc]. *Let $\rho : G = G(\Gamma) \hookrightarrow SU(2)$ be a finite subgroups as in 1.2, with $\Gamma = A_n, D_n, E_6, E_7, E_8$.*

There exist bijections

$$Irr(G) \overset{\sim}{=} Vert(\hat{\Gamma})),$$

with the trivial representation 1 *corresponding to the distinguished vertex of* $\hat{\Gamma}$*, and*

$$G/G \overset{\sim}{=} Vert(\hat{\Gamma})),$$

such that the Steinberg matrix $A(\rho)$ *coincides with the Cartan matrix of the extended graph* $\hat{\Gamma}$*.*

Corollary. The dimensions $n_i = \dim \rho_i$ coincide with the coefficients in the decomposition

$$\theta = \sum n_i \alpha_i.$$

Corollary. There is a bijection

$$G/G \overset{\sim}{=} Irr(G).$$

Proof. As was pointed out by Brylinski, [B], the vertices of Γ are in natural bijection with G/G : to a component D_i of the exceptional divizor Y_0 we associate a loop $\gamma(D_i)$ around D_i which gives a conjugacy class in $G = \pi_1(Y \setminus Y_0)$.

2.1. Let $\rho : G \hookrightarrow SL(2, \mathbb{C})$ as before; we can consider a twisted group ring $A_\rho[G]$, $A = \mathbb{C}[x, y]$.

Left $A_\rho[G]$-modules $=$ G-equivariant coherent sheaves on $V = \mathbb{C}^2 = \operatorname{Spec} A$.

We can assign to each $\pi \in Rep(G)$, $\pi : G \longrightarrow GL(E_\pi)$, an $A_\rho[G]$-module $V \otimes E_\pi$; this gives rise to a morphism

$$R(G) = K_0(Rep(G)) \longrightarrow K_0(Coh_G(V)).$$

Consider a Cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{p_1} & Y \\ p_2 \downarrow & & \downarrow \\ V & \longrightarrow & V/G = X \end{array}$$

Categorical McKay equivalence

Theorem (Kapranov - Vasserot), [KV]. (i) *A functor*

$$F: \ D^b(Coh_G(V)) \longrightarrow D^b(Coh(Y)),$$

$$F(M) = (p_{1*}p_2^* M)^G$$

is an equivalence of categories.

(ii) *For* $\pi \in Irr(G)$ *let* $\pi^! \in Coh_G(V)$ *denote the skyscraper sheaf whose fiber at* 0 *is* π*, and* 0 *everywhere else.*

Then

$$F(1^!) = \mathcal{O}_{Y_0}, \ F(\rho_i^!) = \mathcal{O}_{D_i}(-1)[1], \ 1 \leq i \leq r.$$

.

On the level of Grothendieck groups this was proven earlier by Gonzalez-Sprinberg - Verdier.

The following remarkable generalization was obtained by Bezrukavnikov and Kaledin. Let V be a finite-dimensional $\mathbb{C}$-vector space equipped with a non-degenerate skew-symmetric form ω, and let $G \subset Sp(V)$ be a finite subgroup. Suppose that there exists a resolution of singularities $p : X \longrightarrow V/G$ such that $\omega|_{X\setminus p^{-1}(0)}$ extends to a non-degenerate closed form on the whole X.

Theorem, [BK]. There exists an equivalence of triangulated categories

$$F : \ D^b(Coh_G(V)) \overset{\sim}{=} D^b(Coh(X)).$$

Remark. There is no doubt that the equivalence F may be upgraded to the level of "fine" derived categories (*dg-categories*, or *stable* ∞*-categories*), and hence induces the isomorphism of *all* Quillen groups K_i.

§3. Perron - Frobenius and Gamma products

3.1. Let $G \subset SL(2, \mathbb{C})$ as before correspond to a Dynkin graph Γ,

$$Irr(G) = \{1 = \rho_0, \rho_1, \ldots, \rho_r\} \overset{\sim}{=} Vert(\Gamma).$$

The vector

$$p = (n_0 = 1, n_1, \ldots, n_r) \in \mathbb{N}^{r+1}, \;\; n_i = \deg \rho_i$$

is a Perron - Frobenius eigenvector of the extended Cartan matrix $A(\hat{\Gamma})$ with eigenvalue 0. Let R be the finite root system corresponding to Γ;

$$Vert(\Gamma) \overset{\sim}{=} \{\alpha_1, \ldots, \alpha_r\}$$

(the set of simple roots). The number

$$h = \sum_{i=0}^{r} n_i$$

is called the Coxeter number of R; the longest root $\theta = \sum_{i=1}^{r} n_i \alpha_i$.

Set

$$k = \left(\prod_{j=1}^{r} n_j^{n_j} \right)^{1/h}.$$

The following formula has been discovered by physicists and proven in [CA].

Theorem, [CA]. For all $1 \leq i \leq r$

$$n_i = k \prod_{\alpha > 0} \gamma((\alpha, \rho)/h)^{-(\alpha, \alpha_i)}.$$

Here ρ is as usually the half-sum of positive roots, and $\gamma(x) := \Gamma(x)/\Gamma(1-x)$.

A similar formula exists for the Perron-Frobenius vector of a finite Cartan matrix (one should replace $\gamma(x)$ by $\Gamma(x)$), cf. [CAS].

It would be interesting to find a motivic interpretation of these formulas.

References

[BGP] J.Bernstein, I.Gelfand, V.Ponomarev, Coxeter functors and the Gabriel's theorem, *UMN* **XXVIII**, Issue 2(170) (1973), 19 - 33.

[BK] R.Bezrukavnikov, D.Kaledin, McKay equivalence for symplectic resolutions of singularities, arXiv :math/0401002.

[Br] T.Bridgeland, Stability conditions and Kleinian singularities, *IMRN* **21** (2009), 4142 - 4157.

[Brie] E.Brieskorn, Singular elements of semisimple algebraic groups, Proc ICM 1970 (Nice), t. 2, 279 - 284.

[B] J.-L.Brylinski, A correspondence dual to McKay's, arXiv :alg-geom/9612003.

[CA] V.Cohen - Aptel, Produits Gamma associés aux systèmes de racines, Thése, 2012, Toulouse.

[CAS] V.Cohen - Aptel, V.Schechtman, Vecteurs de Perron - Frobenius et produits Gamma, *CRAS*, Ser. I **250** (2012), 1003 - 1006.

[GSV] G.Gonzales-Sprinberg, J.L.Verdier, Construction géométrique de la correspondance de McKay, *Ann. Sci. ENS* **16** (1983), 409 - 449.

[KV] M.Kapranov, E.Vasserot, Kleinian singulatities, derived categories and Hall algebras, arXiv :math/9812016.

[Mc] J.McKay, Cartan matrices, finite groups of quaternions and Klein singularities, *Proc. AMS.*

[St] R.Steinberg, Finite subgroups of SU_2, Dynkin diagrams and affine Coxeter elements, *Pacific J. Math.* **118** (1985), 587 - 597.