FLOPS AND SCHOBERS

Grothendieck resolutions and the web of parabolics

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PLAN

1. Perverse sheaves on the disc, vanishing cycles, cohomology.
2. Atiyah flop and $\mathfrak{sl}_2$; flober.
3. Parabolic Grothendieck resolutions. $\mathfrak{sl}_3$ and spaces of triangles.

This is a report on a joint work with Alexei Bondal and Mikhail Kapranov, see [BKS].

§1. Vanishing cycles and perverse sheaves

1.1. What is the vanishing cycles? Let $\mathcal{D}(\mathbb{A}^1; 0)$ denote the bounded derived category of complexes $\mathcal{F}$ of sheaves over $\mathbb{A}^1 = \mathbb{C}$ (in the usual topology) with values in vector spaces over a fixed field k; we require the cohomology of these complexes to be locally constant over $U = \mathbb{A}^1 \setminus \{0\}$, and of finte type over k.

In other words $H^*(\mathcal{F}) \in Constr(\mathbb{A}^1, 0)$.

VARIANT: one could take $\mathcal{D}^b(Constr(\mathbb{A}^1, 0))$.

We have

$$\mathcal{F}_0 \overset{\sim}{=} R\Gamma(\mathbb{A}^1, \mathcal{F}) \in \mathcal{D}(*)$$

We define

$$\Phi(\mathcal{F}) := Cone(\mathcal{F}_0 = R\Gamma(\mathbb{A}^1, \mathcal{F}) \longrightarrow R\Gamma(U_1, \mathcal{F}) = \mathcal{F}_1)$$

where $U_1 = D(1, \epsilon)$ - small disc with center at 1;

$$\Psi(\mathcal{F}) = \mathcal{F}_1.$$

Thus we have a canonical map

$$u : \Psi(\mathcal{F}) \longrightarrow \Phi(\mathcal{F}).$$

Duality theorem. *The functors Φ, Ψ commute with (Verdier) duality.*

Corollary. We define the *variation* map

$$v(\mathcal{F}) = u(\mathcal{F}^*)^* : \Phi(\mathcal{F}) \longrightarrow \Psi(\mathcal{F}).$$

Unravelling the definitions,

$$vu = 1 - T$$

where

$$T : \Psi \xrightarrow{\sim} \Psi$$

is the monodromy.

It follows that

$$R\Gamma(\mathbb{A}^1; \mathcal{F}) = Cone(u : \Psi(\mathcal{F}) \longrightarrow \Phi(\mathcal{F}))[-1]$$

Dually,

$$R\Gamma_c(\mathbb{A}^1; \mathcal{F}) = Cone(v : \Phi(\mathcal{F}) \longrightarrow \Psi(\mathcal{F}))[???]$$

1.2. What is a perverse sheaf?

Definition. *$\mathcal{F}$ is called a perverse sheaf if $\Psi(\mathcal{F}), \Phi(\mathcal{F}) \in Constr(\mathbb{A}^1, 0)$.*

The full subcategory

$$Perv(\mathbb{A}^1, 0) \subset \mathcal{D}(\mathbb{A}^1; 0)$$

whose objects are perverse sheaves, is an abelian category.

Let $Hyp'(\mathbb{A}^1, 0)$ denote an abelian category whose objects ("hyperbolic sheaves") are collections

$$E = (\Phi, \Psi, v : \Phi \longrightarrow \Psi, u : \Psi \longrightarrow \Phi)$$

where $\Phi, \Psi \in Vect^f(k)$, u, v are k-linear maps such that

$$T_\psi : \ 1 - vu \qquad (Inv)$$

is invertible.

Lemma. (Inv) is equivalent to

$$T_\phi : \ 1 - uv \qquad (Inv)'$$

is invertible.

Theorem (Kashiwara, Malgrange, Beilinson, ...). The above functors induce an equivalence of categories

$$Perv(\mathbb{A}^1, 0) \xrightarrow{\sim} Hyp'(\mathbb{A}^1, 0)$$

1.3. DIRAC VERSION

1.3.1. For $\mathcal{F} \in \mathcal{D}(\mathbb{A}^1, 0)$ we define

$$E_\pm(\mathcal{F}) = \mathcal{F}_{\pm 1} = R\Gamma(\mathcal{F}; U_i) \in \mathcal{D}(k),\ i = \pm 1$$

where $U(a) = D(a; \epsilon)$ - small disc with center a.

$$E_0(\mathcal{F}) = Cone(R\Gamma(\mathbb{A}^1; \mathcal{F}) \longrightarrow R\Gamma(\mathcal{F}; U_1 \cup U_{-1}))$$

Thus we have canonical maps

$$\delta_\pm : E_\pm(\mathcal{F}) \longrightarrow E_0(\mathcal{F})$$

1.3.2. Duality. The functors $E_\pm, E_0$ commute with Verdier duality.

As a corollary we get maps

$$\gamma_\pm(\mathcal{F}) := \delta_\pm(\mathcal{F}^*)^* : E_0(\mathcal{F}) \longrightarrow E_\pm(\mathcal{F})$$

The compositions

$$T_+ = \gamma_-\delta_+ :\ \mathcal{F}_1 \longrightarrow \mathcal{F}_{-1},\ T_- = \gamma_+\delta_- :\ \mathcal{F}_{-1} \longrightarrow \mathcal{F}_1$$

are (upper, lower) half-monodromies.

1.3.3.

$$R\Gamma(\mathbb{A}^1; \mathcal{F}) = [E_+(\mathcal{F}) \oplus E_-(\mathcal{F}) \xrightarrow{\delta} E_0(\mathcal{F})],$$

in horizontal degrees $0, 1$;

$$R\Gamma_c(\mathbb{A}^1; \mathcal{F}) = [E_0(\mathcal{F}) \xrightarrow{\gamma} E_+(\mathcal{F}) \oplus E_-(\mathcal{F})],$$

in horizontal degrees $1, 2$ (NON STANDARD NORMALIZATION)

1.3.4. A complex $\mathcal{F} \in \mathcal{D}(\mathbb{A}^1, 0)$ belongs to $Perv(\mathbb{A}^1, 0)$ iff $E_*(\mathcal{F}) \in Vect^f(k) \subset \mathcal{D}(k), * = 0, \pm$.

Let us denote $\mathcal{S}$ a stratification of $\mathbb{R}$ into 3 strata:

$$C_0 = \{0\},\ C_+ = \mathbb{R}_{>0},\ C_- = \mathbb{R}_{<0},$$

and by $Hyp(\mathcal{S})$ a category whose objects are collections

$$E_0, E_\pm \in Vect^f(k),\ \gamma_\pm : E_0 \longrightarrow E_\pm, \delta_\pm : E_\pm \longrightarrow E_0$$

such that:

(a) $\gamma_\pm\delta_\pm = \mathrm{Id}$;

(b) The maps $\gamma_\mp\delta_\pm : E_\pm \longrightarrow E_\mp$ are isomorphisms.

Theorem [KS] (a). The above functors induce an equivalence of categories

$$E : Perv(\mathbb{A}^1, 0) \xrightarrow{\sim} Hyp(\mathcal{S})$$

1.4. CATEGORICAL VERSIONS: SPHERICAL FUNCTORS AND SPHERICAL PAIRS

Fig. Schober.

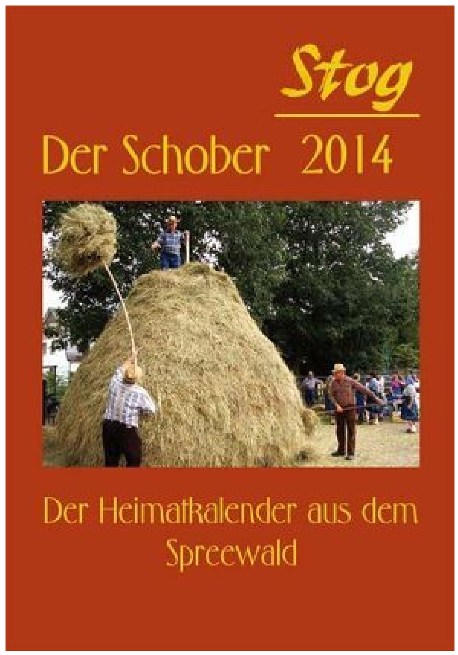

Fig. Another Schober.

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§2. Grothendieck resolution for $\mathfrak{sl}_2$ and the Atiyah flop

2.1. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra of diagonal matrices. The Weyl group $W = \{1, s\}$ acts on $\mathfrak{h}$, $sh = -h$.

$$ch: \ \mathfrak{g} \longrightarrow \mathfrak{h}/W, \ p(A) = -\det A = -ad + bc = \lambda^2$$

where $\operatorname{Spec}(A) = \{\lambda, -\lambda\}$.

$$\mathcal{F}\ell = G/B = \{0 = V_0 \subset V_1 \subset V_2 = V = \mathbb{C}^2\} \stackrel{\sim}{=} \mathbb{P}^1$$

- the variety of flags.

We denote by $\tilde{\mathfrak{g}}$ the variety

$$\tilde{\mathfrak{g}} = \{(A \in \mathfrak{g}, \mathcal{F} \in \mathcal{F}\ell) | \ A(V_1) \subset V_1\}$$

We have an obvious projection $\tilde{\mathfrak{g}} \longrightarrow \mathcal{F}\ell$ which identifies $\tilde{\mathfrak{g}}$ with the cotangent bundle $T^*\mathcal{F}\ell$.

A map

$$\tilde{\mathfrak{g}} \longrightarrow \mathfrak{h}, \ (A, \mathcal{F}) \mapsto A|_{V_1} \in \mathbb{C}$$

Another obvious projection

$$\pi : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$$

is nonramified two-fold covering over the open subvariety $\mathfrak{g}^{rss}$ of matrices A with $\lambda(A) \neq 0$. Its complement

$$\mathcal{N} = \{A | \lambda(A) = 0\} = \{A | \ \det A = 0\}$$

is the subvariety of nilpotent matrices, a quadratic cone.

For $A \in \mathcal{N} \setminus \{0\}$ $\pi^{-1}(A)$ consists of 1 element; $\pi^{-1}(0) = G/B = \mathbb{P}^1$.

We have a commutatice square

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \\ \mathfrak{h} & \longrightarrow & \mathfrak{h}/W \end{array}$$

2.2. Atiyah flop. We define

$$Z = \mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{g}$$

Explicitely, a point of Z is a couple (A, λ), where A is a matrix from $\mathfrak{sl}_2$ and λ is a square root of its determinant:

$$-a^2 - bc = \lambda^2.$$

In other words, Z is a quadratic cone in $\mathbb{C}^4$.

Thus, we have canonical maps

$$\tilde{\mathfrak{g}} \xrightarrow{\pi_1} Z \xrightarrow{\pi_2} \mathfrak{g}$$

In fact, (3.2) is the *Stein decomposition* of π:

$$Z = \Gamma(\tilde{\mathfrak{g}}; \mathcal{O}_{\tilde{\mathfrak{g}}}),$$

and π_1 is the canonical map (EXPLAIN)

π_2 is a ramified covering, whereas the fibers of π_1 are connected.

π_1 is a blowing down of a curve $C \cong \mathbb{P}^1$; it is a small resolution of the isolated singularity $0 \in Z$.

We denote

$$\pi_+ = \pi_1 : \; X_+ = \tilde{\mathfrak{g}} \longrightarrow Z.$$

Let $s : \mathfrak{h} \longrightarrow \mathfrak{h}$ be the Weyl reflection, $s(\lambda) = -\lambda$ on $\mathfrak{h}$.

We define $X_- := s^*\tilde{\mathfrak{g}}$, i.e. it fits into the Cartesian square

$$\begin{array}{ccc} X_- & \longrightarrow & \tilde{\mathfrak{g}} \\ \downarrow & & \downarrow \\ \mathfrak{h} & \xrightarrow{s} & \mathfrak{h} \end{array}$$

We have a canonical map

$$\pi_- : \; X_- \longrightarrow Z.$$

which is a small resolution.

Finally, we define

$$X_0 := X_- \times_Z X_+$$

it is the blowing up of the singularity $0 \in Z$.

The diagram

$$X_- \xleftarrow{p_-} X_0 \xrightarrow{p_+} X_+ \qquad (At)$$

is an example of an Atiyah flop. The maps $p_\pm$ are proper.

2.3. Atiyah - Grothendieck flober. For a variety X let $\mathcal{D}(X)$ denote the bounded derived category of coherent sheaves on X, and $Perf(X)$ the trangulated category of perfect complexes; if X is smooth these categories are equivalent.

The diagram (At) induces two diagrams functors between triangulated categories

$$\mathcal{D}(X_-) \xleftarrow{p_{-*}} \mathcal{D}(X_0) \xrightarrow{p_{+*}} \mathcal{D}(X_+) \qquad (At_*)$$

and

$$\mathcal{D}(X_-) \xrightarrow{p_-^*} \mathcal{D}(X_0) \xleftarrow{p_+^*} \mathcal{D}(X_+) \qquad (At^*)$$

which is a categorical analog of a hyperbolic shea f over $\mathbb{A}^1$, in the Dirac form.

This means that it satisfies the properties:

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Let us denote it $\mathcal{AG}$.

2.4. $R\Gamma$ and $R\Gamma_c$ for a Schober.

Definition.

$$H^0(\mathbb{A}^1, \mathcal{AG}) = \mathrm{holim}(At^*),$$

this is the homotopy kernel of a couple of arrows;

$$H_c^2(\mathbb{A}^1, \mathcal{AG}) = \mathrm{hocolim}(At_*),$$

this is the homotopy cokernel of a couple of arrows.

Theorem. We have equivalences of stable categories

$$Perf(Z) \overset{\sim}{=} H^0(\mathbb{A}^1, \mathcal{AG});\ \mathcal{D}(Z) \overset{\sim}{=} H_c^2(\mathbb{A}^1, \mathcal{AG}).$$

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§3. Parabolic Grothendieck resolutions: the case of $\mathfrak{sl}_3$

3.1. Levis, parabolics, complex and real strata. Let $L_0 \subset G = GL_n(\mathbb{R})$ be the subgroup of diagonal matrices, the minimal Levi subgroup, $\mathfrak{h} = Lie(L_0) = \mathbb{R}^n$, with coordinates $x_1, \ldots, x_n$.

In $\mathfrak{h}$ consider the root arrangement consisting of hyperplanes $x_i = x_j$. Let $\mathcal{S}$ (resp. $\mathcal{S}_{\mathbb{C}}$) denote the corresponding stratification of $\mathfrak{h}$ (resp. the corresponding complex stratification of $\mathfrak{h}_{\mathbb{C}}$).

We have a canonical map

$$\mathcal{S} \longrightarrow \mathcal{S}_{\mathbb{C}}. \qquad (3.1.1)$$

We have bijections

$$\mathcal{S}_{\mathbb{C}} \xrightarrow{\sim} \{\textit{Levi subgroups } L \supset L_0\}$$

Given a Levi $L \supset L_0$, the corresponding complex stratum is $Lie(Z(L))$.

$$\mathcal{S} \xrightarrow{\sim} \{\textit{Parabolic subgroups } P \supset L_0\}$$

The map (3.1.1) corresponds to associating to a parabolic its Levi factor.

Example. $n = 3$. (we list the closures of strata).

L_0 corresponds to $\mathfrak{h}_{\mathbb{C}}$. 6 real chambers in $\mathfrak{h}$ are in bijection with 6 parabolics P_{ijk} where (ijk) is a permutation of (123) and P_{ijk} consists of matrices respecting the flag $V_i \mathbb{R} e_i \subset V_i \oplus V_j$.

There are 3 Levi's L_{ij} corresponding to three complex lines $\ell_{ij,\mathbb{C}} : x_i = x_j$,

$$L_{ij} = GL(V_i \oplus V_j) \times GL(V_k).$$

Each L_{ij} is contained in 2 parabolics $P_{ij}^{\pm}$ corresponding to two rays of the real line ℓ_{ij}.

For example:

$$L_{12} = \{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \}, Z(L_{12}) = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \}$$

$$P_{12}^{+} = \{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \},\ P_{12}^{-} = \{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \}.$$

We have 6 one-dimensional real strata.

Finally, G corresponds to the smallest stratum $x_1 = x_2 = x_3$.

3.2. Parabolic Grothendieck resolutions. Let $P \subset G$ be a parabolic, so

$$\mathcal{F}\ell_P = G/P = \{P^x := xPx^{-1}\}$$

is a partial flag variety.

By definition

$$\tilde{\mathfrak{g}}_P = \{(A, P'),\ P' \in G/P,\ A \in \mathfrak{p}' = Lie(P')\}$$

Thus $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_B$, whereas $\mathfrak{g} = \tilde{\mathfrak{g}}_G$.

For $P \subset P'$ we have a commutative square

$$\begin{array}{ccc} \tilde{\mathfrak{g}}_P & \longrightarrow & \tilde{\mathfrak{g}}_{P'} \\ \downarrow & & \downarrow \\ G/P & \longrightarrow & G/P' \end{array}$$

3.3. SEVERAL DEFINITIONS OF SINGULAR VARIETIES Z_P

(i) **Stein factorization**

$$\tilde{\mathfrak{g}}_P \xrightarrow{\pi_1} Z_P = \mathrm{Spec}(\tilde{\mathfrak{g}}_P, \mathcal{O}_{\tilde{\mathfrak{g}}_P}) \xrightarrow{\pi_2} \tilde{\mathfrak{g}}_G = \mathfrak{g}$$

where π_2 is finite, and π_1 has connected fibers and is birational.

(ii) Let $\mathfrak{p} = Lie(P)$, $\mathfrak{n}_\mathfrak{p} \subset \mathfrak{p}$ its nilpotent radical, $\mathfrak{l}_\mathfrak{p} = \mathfrak{p}/\mathfrak{n}_\mathfrak{p}$ the Levi quotient, $\mathfrak{m} = \mathfrak{l}/Z(\mathfrak{l})$.

Let

$$\tilde{\mathfrak{l}} \longrightarrow Z(\mathfrak{l}) \longrightarrow \mathfrak{l}$$

be the Grothendieck resolution and its affinization. We define

$$Z(\mathfrak{p}) = \mathfrak{p} \times_\mathfrak{l} Z(\mathfrak{l}),$$

and varying P we get the unversal family

$$Z_P = G \times_P Z(\mathfrak{p}) := (G \times Z(\mathfrak{p}))/P \longrightarrow \mathcal{F}\ell_P = G/P$$

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3.4. Triangle and its flags. We consider the case of $\mathfrak{sl}_3$.

We have a map

$$\tilde{\mathfrak{g}} \longrightarrow \mathcal{F}\ell$$

whose fiber over $B \in \mathcal{F}\ell$, or over a flag

$$F: \ 0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3 \tag{3.4.2}$$

is $\mathfrak{b} = Lie(B)$, or the space of matrices $A \in \mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ such that $A(V_i) \subset V_i$, $i = 1, 2$.

Or we can consider the flag F as a pair

$$\text{point } * = \mathbb{P}(V_1) \subset \text{ straight line } \mathbb{P}(V_2) \cong \mathbb{P}^1 \subset \mathbb{P}^2 = \mathbb{P}(V_3)$$

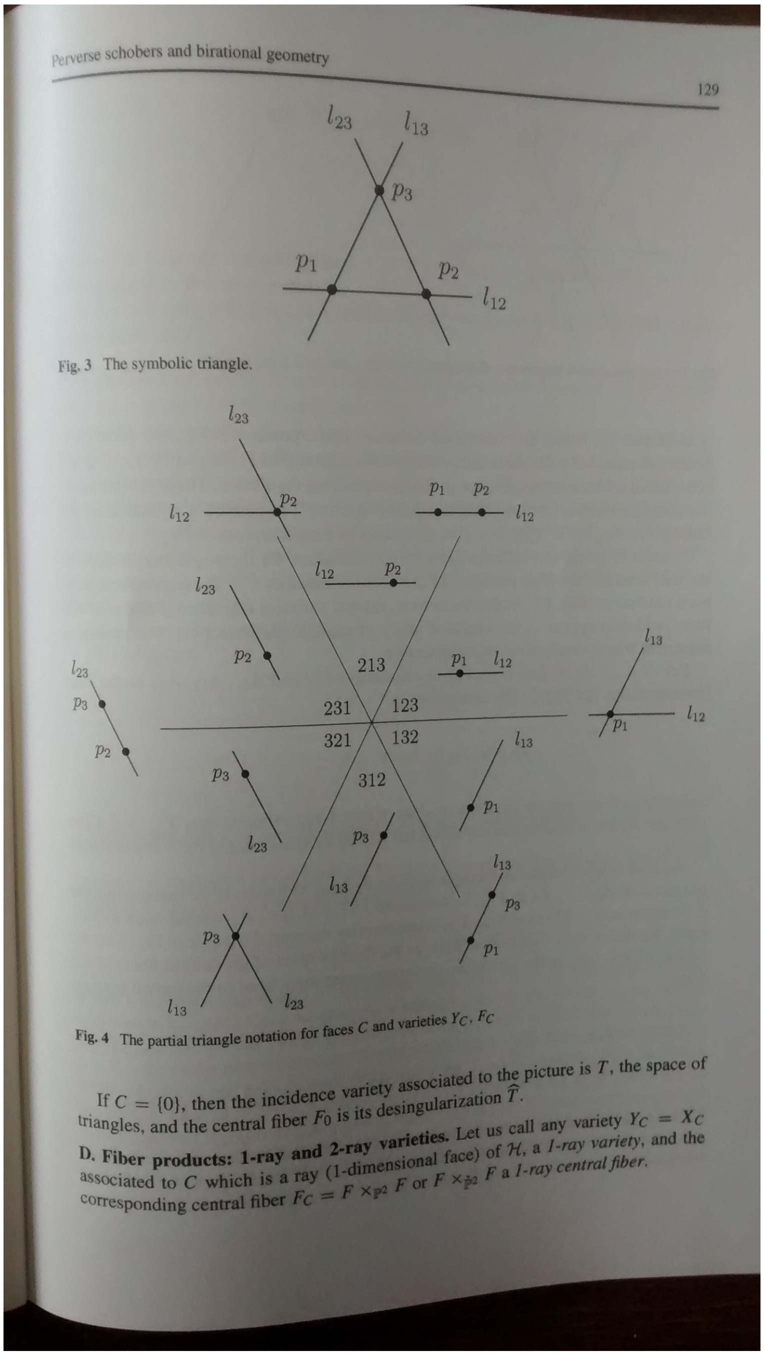

Fig. 3 The symbolic triangle.

Fig. 4 The partial triangle notation for faces C and varieties Y_C, F_C

If $C = \{0\}$, then the incidence variety associated to the picture is T, the space of triangles, and the central fiber F_0 is its desingularization $\widehat{T}$.

D. Fiber products: 1-ray and 2-ray varieties. Let us call any variety $Y_C = X_C$ associated to C which is a ray (1-dimensional face) of $\mathcal{H}$, a *1-ray variety*, and the corresponding central fiber $F_C = F \times_{\mathbb{P}^2} F$ or $F \times_{\check{\mathbb{P}}^2} F$ a *1-ray central fiber*.

Fig. Triangle.

Consider a triangle

$$\Delta = \cup_{1\leq i,j\leq 3}\ell_{ij} \subset \mathbb{P}(V),\ \ell_{ij} = \ell_{ji}$$

as on Fig. above, with vertices

$$p_1 = \ell_{12}\cap\ell_{13},\ p_2 = \ell_{12}\cap\ell_{23},\ p_3 = \ell_{13}\cap\ell_{23}$$

To Δ we associate a Cartan subalgebra

$$\mathfrak{h}(\Delta) = \{A\in\mathfrak{g}|\ Ap_i\subset p_i\}$$

(where p_i is considered as a line in V).

To Δ there correspond 13 *elements* which are in bijection with the cells of the root stratification $\mathcal{S}(A_2)$ on $\mathbb{R}^2$, and with parabolics containing $\mathfrak{h}(\Delta)$.

(a) Let us call a 0-element a flag $F = (p\ell)$ in $\mathbb{P} = \mathbb{P}(V)$.

To each $F\in\Delta$ there corresponds a Borel subalgebra $\mathfrak{b}(F)\subset\mathfrak{g}$ as above.

We denote

$$\mathfrak{p}(F) = \mathfrak{q}(F) = \mathfrak{b}(F)$$

We have $\dim\mathfrak{b}(F) = 5$;

The space of flags

$$Flags = \text{Elements}_0$$

has dimension 3.

The borels $\mathfrak{b}(F)$ form a 2-dimensional vector bundle over $Flags$, whose total space is nothing but the 8-dimensional Grothendieck resolution $\tilde{\mathfrak{g}}$.

0-elements belonging to a given triangle Δ are in bijection with 6 chambers of $\mathcal{S}(A_2)$.

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(b) By definition, 1-elements are of two kinds:

(b1) A 1-element of the first kind is a pair of distinct straight lines $E = (\ell, \ell')$ in $\mathbb{P}$. Let $p = \ell\cap\ell'$.

The element E contains 2 flags: $F = (p\subset\ell)$ and $F' = (p\subset\ell')$. We write $F\in E$.

Define two Lie subalgebras

$$\mathfrak{p}(E) = \mathfrak{b}(F)\cup\mathfrak{b}(F')$$

it is a parabolic; and

$$\mathfrak{q}(E) = \mathfrak{b}(F) \cap \mathfrak{b}(F'), \dim \mathfrak{q}(E) = 4.$$

The space of 1-elements of the first kind is an open subspace

$$\text{Elements}'_1 \subset \mathbb{P} \times \mathbb{P},\ \dim \text{Elements}'_1 = 4.$$

(b2) A 1-element of the second kind is a 1-element of the first kind in the dual projective plane $P^\vee$.

Explicitely, it is a pair of distinct points $E' = (p, p')$ in $\mathbb{P}$. Let ℓ be the straight line through p, p'.

Two flags belong to this element $F = (p \subset \ell)$ and $F' = (p' \subset \ell)$.

Define two Lie subalgebras

$$\mathfrak{p}(E') = \mathfrak{b}(F) \cup \mathfrak{b}(F'), \dim \mathfrak{p}(E) = 6.$$

it is a parabolic; and

$$\mathfrak{q}(E) = \mathfrak{b}(F) \cap \mathfrak{b}(F'),\ \dim \mathfrak{q}(E) = 4.$$

The space of 1-elements of the second kind is an open subspace

$$\text{Elements}''_1 \subset \mathbb{P}^\vee \times \mathbb{P}^\vee,\ \dim \text{Elements}''_1 = 4.$$

$3+3$ elements belonging to a fixed triangle Δ are in bijection with $3+3$ 1-cells of $\mathcal{S}(A_2)$, see Fig. ??? below.

FIGURE: TRIANGLE AND ITS 1-ELEMENTS

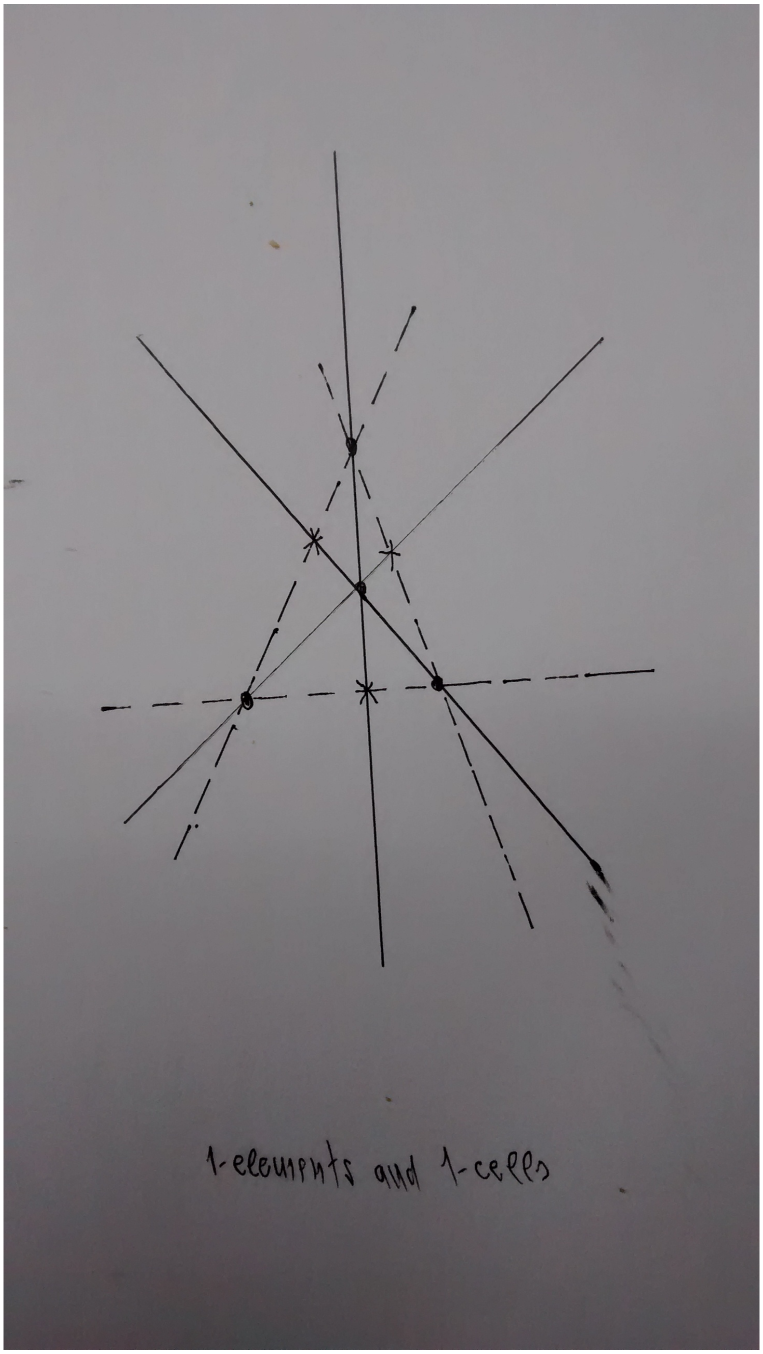

Fig. ???. 1-elements and 1-cells.

(c) A 2-element is a triple of distinct points p_1, p_2, p_3 in $\mathbb{P}$, i.e. a triangle Δ. It corresponds to the unique 0-cell in $\mathcal{S}(A_2)$.

There are 6 flags $F : p_i \subset \ell_{ij}$ in Δ; we write this as $F \in \Delta$.

We define two Lie subalgebras

$$\mathfrak{p}(\Delta) = \cup_{F\in\Delta}\ \mathfrak{b}(F) = G,$$

and

$$\mathfrak{q}(\Delta) = \cap_{F\in\Delta}\ \mathfrak{b}(F) = \mathfrak{h}(\Delta),\ \dim(\mathfrak{q}(\Delta)) = 2.$$

The space of triangles

$$Triangles = \text{Elements}_2 \subset (\mathbb{P}^2)^3$$

has dimension 6.

It carries a vector bundle whose fiber over Δ is $\mathfrak{q}(\Delta)$.

The total space of this bundle has dimension 8 and is birational with $\mathfrak{g}$.

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TO RECUPERATE:

Let E be an element ($=$ a triangle element), and $Cell(E)$ the corresponding cell of $\mathcal{S}(A_2)$.

The flags $F \in E$ are in bijection with chambers adjacent to $Cell(E)$.

The parabolic corresponding to E is

$$\mathfrak{p}(E) = \sum_{F\in Fl(E)} \mathfrak{b}(F).$$

On the other hand

$$\mathfrak{q}(E) = \cap_{F\in Fl(E)}\ \mathfrak{b}(F).$$

We call Lie algebras $\mathfrak{q}(E)$ **carabolic** ones, for Cartan, indicating that they lie between a Cartan $\mathfrak{q}(\Delta) = \mathfrak{h}(\Delta)$ and a Borel.

The carabolics (resp. parabolics) containing a given Cartan are in bijection with $\mathcal{S}(A_2)$.

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COMPACTIFICATIONS AND DESINGULARIZATIONS

3.5. Origin: the Schubert variety. We have an embedding

$$i:\ Triangles \hookrightarrow \mathbb{P}(V)^3 \times \mathbb{P}(V)^{\vee 3}, i(\Delta) = (p_1, p_2, p_3; \ell_{12}, \ell_{13}, \ell_{23}). \qquad (3.5.1)$$

Let Tr denote the Zarisky closure of $i(Triangles)$.

SCHUBERT DESINGULARIZATION

For $T = (p_i) \in Triangles$ quadrics $q \in S^2(V^*)$ circumscribed around T, i.e. such that

$$q(p_1) = q(p_2) = q(p_3) = 0$$

form a 3-dimensional linear subspace of $S^2(V^*)$, whence an embedding

$$Triangles \hookrightarrow \mathbb{P}(V)^3 \times \mathbb{P}(V)^{\vee 3} \times \mathrm{Gr}(3, S^2(V^*)).$$

By definition Tr^{Sch} is the closure of its image, cf. [Sch], [Se], [KM]; according to *loc. cit.* it is nonsingular.

It comes together with an obvous map

$$Tr^{Sch} \longrightarrow Tr$$

which is an isomorphism over an open $Triangles \subset Tr$, and is therefore a desingularization of the compact variety Tr.

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ANOTHER REALIZATION OF THE SCHUBERT VARIETY; THE CARTAN VECTOR BUNDLE ON IT

Variety of reductions

Let R^o denote the variety of Cartan subalgebras in $\mathfrak{g}$. We have an embedding

$$R^o \hookrightarrow \mathrm{Gr}(2, \mathfrak{g});$$

let R denote its closure, cf. [IM]. R carries a tautological rank 2 vector bundle ???

We have an embedding

$$\hat{i}: \ Triangles \longrightarrow \mathbb{P}^3 \times \mathbb{P}^{\vee 3} \times R,$$

with

$$\hat{i}(\Delta) = (i(\Delta), \mathfrak{h}(\Delta)).$$

We define $\widehat{Tr}$ as the Zarisky closure of the image of $\hat{i}$.

Proposition.

$$\widehat{Tr} \overset{\sim}{=} Tr^{Sch}$$

Therefore we have over Tr^{Sch} the tautological 2-dimensional fiber bundle; denote its total space X_0.

3.6. **1-rays.** Define two open 8-dimensional 1-element variety: Y_1' (resp. Y_1'') as the total space of a 4-dimensional fiber bundle over the 4-dimensional space of 1-elements $Elements_1'$ (resp. $Elements_1''$).

The fiber of Y_1' (resp. of Y_1'') over an element $E' = (p, p', \ell)$ (resp. over $E'' = (p, \ell, \ell'))$ is the corresponding carabolic subalgebra: interesection of two borels

$$\mathfrak{q}(E) = \cap_{F \in E} \mathfrak{b}(F).$$

We compactify $Elements_1'$ as follows: we have an open embedding

$$Elements_1' \hookrightarrow Flags \times_{\mathbb{P}^\vee} Flags,$$

and we set

$$El_1' := Flags \times_{\mathbb{P}^\vee} Flags.$$

Similarly we set

$$El_1'' := Flags \times_{\mathbb{P}} Flags.$$

The carabolic fiber bundles Y_1', Y_1'' may be extended to the compactfied spaces.

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This may be proved by constructing them as fiber products, similarly to Atiyah case.

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We have an embedding

$$Y_1' \hookrightarrow X_w \times X_{w'} = \tilde{\mathfrak{g}}_w \times \tilde{\mathfrak{g}}_{w'}$$

(resp. $Y_1'' \hookrightarrow X_w \times X_{w'}$) corresponding to two chambers neighboring a wall. We define X_1' (resp. X_1'') as the closure of its image.

3.7. Résumé. We have constructed a web of 13 smooth projective varieties $X(C),\ C \in \mathcal{S}(A_2)$, and proper morphisms

$$X(C) \longrightarrow X(C'),\ C \leq C'$$

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