**GINDIKIN - KARPELEVICH PATTERNS**

*in Automorphic forms and Conformal field theory*

*Moscow lectures, May 2013*

*Preliminary version*

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## ZEITTAFEL

Israel GELFAND (1913 - 2009)
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## INTRODUCTION

The classical Gindikin - Karpelevich formula expresses an integral over a nilpotent subgroup of a semisimple real Lie group as a ratio of products of Gamma functions.

We will show how some remarkable structures connected with this formula appear in different (but related) areas: the Langlands' theory of Eisenstein series, integrable models of Quantum field theory, the Knizhnik - Zamolodchikov equations.

The reader may consult [Karp] (a) for the history of this formula.

I have given lectures on these subjects in Higher School of Economics in Moscow on May 2013.

## Lecture 1. Eisenstein series

**1.1. Eisenstein series and scattering matrix.** Cf. [HS (a)]. Let $G = SL_2(\mathbb{R}) \supset \Gamma = SL_2(\mathbb{Z})$; $G = KAN$ where $K = SO(2)$, $N = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},\ n \in \mathbb{R}\}$,

$$A = \{a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t \in \mathbb{R}\}$$

For $x = kan \in G$ the elements $k, a, n$ are uniquely defined; we set $t = H(x)$ if $a = a(t)$.

Let $P = MAN = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $M = \{\pm 1\}$.

The *Eisenstein series*

$$E(\lambda; x) = \sum_{\gamma \in \Gamma/\Gamma \cap P} e^{(-\lambda-1)H(x\gamma)} = \frac{1}{2} \sum_{\gamma \in \Gamma/\Gamma \cap N} e^{(-\lambda-1)H(x\gamma)} =$$

$$\frac{1}{2} \sum_{\gamma \in \Gamma/\Gamma \cap N} |(x\gamma)_1|^{-\lambda-1} \tag{1.1.1}$$

where

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,\ x_1 = (a, c)^t,\ |x_1| = (a^2 + c^2)^{1/2},$$

$\lambda \in \mathbb{C}$, $Re\lambda > 1$. The series converges absolutely and uniformly for $x \in D \subset G$, $D$ compact and $\Re\lambda \geq 1 + \epsilon$.

Note that

$$|x_1| = |kx|_1,\ x \in G, k \in K.$$

It follows that $E(\lambda; x) = E(\lambda; kx)$, $k \in K$, so we can consider $E(\lambda; x)$ as a function of $z \in X = K\backslash G$.

*Constant term.*

$$\int_{N/\Gamma_\infty} E(\lambda; a(t)n)dn = e^{(-1-\lambda)t} + c(\lambda)e^{(-1+\lambda)t}$$

where $\Gamma_\infty = \Gamma \cap N$. Here

$$c(\lambda) = \pi^{-\lambda} \frac{\Gamma(\lambda/2)\zeta(\lambda)}{\Gamma(-\lambda/2)\zeta(-\lambda)}$$

(a) $c(\lambda)$ is meromorphic on $\mathbb{C}$;

$$c(\lambda)c(-\lambda) = 1.$$

If $\Re\lambda = 0$, $|c(\lambda)| = 1$.

(b) For all $x \in G$ $E(\lambda; x)$ may be meromorphically continued to $\mathbb{C}$ and

$$E(\lambda; x) = c(\lambda)E(-\lambda; x)$$

**1.2. Some more details: Fourier expansion.** Harish-Chandra prefers the Iwasawa decomposition $G = KAN$. But in this no. we pass to the "opposite" Iwasawa $G = NAK$.

The group $G$ acts on $X = \{z \in \mathbb{C} | \ \Im z > 0\}$ by $gz = (az + b)/(cz + d)$ which identifies $G/K \xrightarrow{\sim} X$, $K$ being the stabilizer of $i$.

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{y}{|cz+d|^2}$$

Rewriting the definition (1.1.1) for the Iwasawa decomposition $G = KAN$, we get a function $E(g; s)$ such that $E(gk; s) = E(kg; s),\ k \in K$, whence

$$E(z; s) = \frac{1}{2}\sum_{\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s = \sum_{\bar{\Gamma}_\infty\backslash\bar{\Gamma}} \Im(\gamma z)^s =$$

$$\frac{1}{2\zeta(2s)} \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{y^s}{|mz+n|^{2s}} = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2,(m,n)=1} \frac{y^s}{|mz+n|^{2s}}, \tag{1.2.1}$$

$z \in X$ where $\bar{\Gamma} = PSL_2(\mathbb{Z})$ and $\bar{\Gamma}_\infty$ is the image of $\Gamma_\infty$ in $\bar{\Gamma}$. Here the previous $\lambda + 1 = 2s$.

*Differential equation.* Laplacian: $\Delta = y^2(\partial_x^2 + \partial_y^2)$ where $z = x + iy$; it is a $G$-invariant differential operator on $X$. We have

$$\Delta y^s = s(s-1)y^s$$

and $\Delta$ commutes with $\gamma \in \Gamma$; hence each term in the sum is an eigenfunction of $\Delta$.

It follows that

$$\Delta E(z; s) = s(s-1)E(z; s)$$

*Fourier expansion*, cf. [G], Thm. 3.1.8:

$$E(z; s) = y^s + c(s)y^{1-s}+$$

$$\frac{2y^{1/2}}{\xi(2s)} \sum_{n\in\mathbb{Z},n\neq 0} \sigma_{1-2s}(n)|n|^{s-1/2}K_{s-1/2}(2\pi|n|y)e^{2\pi inx} \tag{1.2.2}$$

where

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),\ c(s) = \frac{\xi(2s-1)}{\xi(2s)},$$

$$\sigma_s(n) = \sum_{d|n} d^s,$$

and

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(u+u^{-1})/2} u^{s-1} du$$

The last function is the *Macdonald's Bessel function*; it is a unique solution of the Bessel equation

$$y^2 f''(y) + yf'(y) - (y^2 + s^2) f(y) = 0$$

rapidly decaying as $y \to \infty$.

The following is used for the proof of (1.2.2).

**Important integral:**

$$\int_{\mathbb{R}} (x^2+1)^{-s} e^{-2\pi i x y} dx = B(1/2, s-1/2) \text{ if } y = 0$$

$$= \frac{2\pi^s |y|^{s-1/2}}{\Gamma(s)} K_{s-1/2}(2\pi|y|) \text{ if } y \neq 0, \tag{1.2.3}$$

cf. [G] (1.3.9), [Bu], Ch. 1, (6.8). Cf. also instructive [Ha] (a) (3.29).

This formula simply means that

**The Fourier transform of a spherical vector is a Whittaker vector.**

*Functional equations:* The functional equation for $\zeta(s)$ is

$$\xi(s) = \xi(1-s)$$

which implies

$$c(s)c(1-s) = 1$$

Next, $K_s = K_{-s}$ implies

$$E(z; s) = c(s)E(z; 1-s)$$

which in turn is equivalent to

$$E^*(z; s) = E^*(z; 1-s)$$

for $E^*(z; s) = \xi(s)E(z; s)$.

**1.3. Where the Bessel function $K_s$ comes from: the Whittaker function.** Cf. [G], 3.4.

Notation: $n(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in N,\ u \in \mathbb{R}$.

Let $\psi:\ \mathbb{R} \longrightarrow S^1 = \{z \in \mathbb{C} |\ |z| = 1\}$ be a character, $\nu \in \mathbb{C}$. Let us call a $(\nu, \psi)$-*Whittaker function* a function $W:\ X \longrightarrow \mathbb{C}$ such that

$$\Delta W(z) = \nu(1-\nu)W(z) \tag{w1}$$

and

$$W(n(u)z) = \psi(u)W(z) \tag{w2}$$

It is easy to construct such functions. As a préparatoire, put $I_s(z) = y^s$. Then $I_\nu$ satisfies $(w1)$. Then apply averaging (moyennisation), i.e. set

$$W_{\nu,\psi}(z) = \int_{\mathbb{R}} I_\nu(n(u)z)\psi(-u)du.$$

This function will satisfy (w1) and (w2).

If $\psi_m(u) = e^{2\pi imu}$ then

$$W_{\nu,\psi}(z) = \frac{2\pi^\nu |m|^{\nu-1/2}}{\Gamma(\nu)} y^{1/2} K_{\nu-1/2}(2\pi|m|y)e^{2\pi imx} \tag{1.3.1}$$

**Multiplicity one theorem.** *If $\Psi(z)$ is a $(\nu,\psi)$-Whittaker function of rapid decay, i.e. for all $N > 0$ $|y^N\Psi(z)| \to 0$ as $y \to \infty$ and $\nu \neq 0, 1$ then there exists $a \in \mathbb{C}$ such that $\Psi(z) = aW_{\nu,\psi}(z)$.*

**Maass forms.** Passing to the global case, let us call a *Maass form* a smooth function $f : X \longrightarrow \mathbb{C}$ such that

$(M1)$ For all $z \in X, \gamma \in \Gamma$, $f(\gamma z) = f(z)$.

$(M2)$ $\Delta f(z) = (\nu^2 - 1/4)f(z)$ for some $\nu \in \mathbb{C}$.

$(M3)$ $f(z)$ decays rapidly as $y \to 0$.

The Eisenstein series is an example. Let $f(z)$ be such a form; then $f(z+1) = f(z)$ due to $(M1)$; consider its Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(y)e^{2\pi inx}$$

Each term of it satisfies $(M2)$.

Then for $n \neq 0$

$$a_n(y) = a_n y^{1/2} K_\nu(2\pi|n|y)$$

### 1.4. Digression: Kronecker series, Epstein zeta function, the Chowla - Selberg formula.

**1.5. Scattering matric for several cusps.** Cf. [I], 13.3.

For a more general discrete subsgroup $\Gamma \subset G$ ("a Fuchsian group of the first kind") $\Gamma$ has a finite number of "cusps" $a_i \in \partial\bar{X} = \mathbb{P}^1(\mathbb{R}),\ i \in I$.

The points $a \in \partial\bar{X}$ are in bijection with parabolic (Borel) subgroups $P_a \subset G$; set $\Gamma_i = \Gamma \cap P_{a_i}$. For each $i$ there exists $\sigma_i \in G$ such that $\sigma_i a_i = \infty$ and $\sigma_i^{-1}\Gamma_i\sigma_i = \Gamma_\infty$. Let

$$\gamma_i = \sigma_i \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma_i^{-1} \in \Gamma_i$$

Given a character $\chi : \Gamma \longrightarrow \mathbb{C}^*$ trivial on $\Gamma_i$, one defines the Eisenstein series

$$E_i(z; s, \chi) = \sum_{\gamma \in \Gamma_i \backslash \Gamma} \bar{\chi}(\gamma)(\Im(\sigma_i^{-1}\gamma z))^s$$

One has

$$E_i(\gamma z; s, \chi) = \chi(\gamma) E_i(z; s, \chi)$$

and

$$\Delta E_i(z; s, \chi) = s(s-1) E_i(z; s, \chi)$$

Thus we have a vector Eisenstein series $E = (E_i)_{i \in I}$. For $i, j \in I$,

$$E_i(\sigma_j z; s, \chi) = \delta_{ij} y^s + \phi_{ij}(s, \chi) y^{1-s} + \sum_{n \in \mathbb{Z}, n \neq 0} \phi_{ij}(n; s, \chi) W_s(nz)$$

where

$$W_s(nz) = 2|y|^{1/2} K_{s-1/2}(2\pi |y|) e^{2\pi i x}$$

(this is $\Gamma(s) W_{s,1}$ from (1.3.1)), and $\phi_{ij}(n; s, \chi)$ are certain Dirichlet series.

The scattering matrix $\Phi(s, \chi) = (\phi_{ij}(s, \chi))$ satisfies the functional equation

$$\Phi(s, \chi) = \Phi(1-s, \chi)$$

and is unitary on the critical line:

$$\Phi(s, \chi) \overline{\Phi(s, \chi)}^t = 1$$

The vector Eisenstein series $E(z; s, \chi)$ satisfies the functional equation

$$E(z; s, \chi) = \Phi(s, \chi) E(z; 1-s, \chi)$$

The series $E_i(z; s, \chi)$ has a pole at $s = 1$ iff $\chi$ equals the trivial character $\chi_0$. In that case the pole is simple with the residue

$$\operatorname{res}_{s=1} E_i(z; s, \chi) = \operatorname{vol}(\Gamma \backslash X)^{-1}.$$

**1.5.1. Example**, cf. [I], 13.5. For example let

$$\Gamma = \Gamma_0(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G |\ c \equiv 0 \mod N \}$$

An even prilmitive Dirichlet character $\chi :\ (\mathbb{Z}/N\mathbb{Z})^* = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \longrightarrow \mathbb{C}^*$ induces a character of $\Gamma_0(N)$ as above. To each decomposition $N = vw$ with $(v, w) = 1$ corresponds a couple of cusps; the corresponding matrix element of the scattering matrix has the form

$$\phi_{vw}(s, \chi) = N^{-s} \chi_v(w) \bar{\chi}_w(v) \pi^{1/2} \frac{\Gamma(s - 1/2) L(2s-1, \chi_w \bar{\chi}_v)}{\Gamma(s) L(2s, \chi_w \bar{\chi}_v)}$$

**1.6. Eisenstein series and scattering matrix for a semisimple group $G$ (Gelfand; Langlands).** Let $G = \mathfrak{G}(\mathbb{R}) \supset \Gamma = \mathfrak{G}(\mathbb{Z})$ where $\mathfrak{G}$ is a simply-connected Chevalley group; $G = KAN$, $\mathfrak{a} = Lie(A)$. For $x = kA(x)n$ we set $a(x) = \log A(x) \in \mathfrak{a}$.

For example if $\mathfrak{G} = SL_n$, $K = SO(n)$, $A = \mathrm{diag}(a_1, \ldots, a_n)$, $a_i > 0$, $N$ is the group of upper triangular matrices with 1's on the diagonal.

*Spherical functions*:

$$\phi_\lambda(x) = \int_K e^{(\lambda-\rho)(a(xk))} dk, \ x \in G,$$

$\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$. We have $\phi_\lambda = \phi_{w\lambda}, \ w \in W$.

*Eisenstein series*:

$$E(x;_\lambda) = \sum_{\gamma \in \Gamma/\Gamma\cap N} e^{(\lambda-\rho)(a(x\gamma))}$$

is an eigenfunction of $Z(U\mathfrak{g})$ where $\mathfrak{g} = Lie(G)$.

Constant term:

$$\int_{N/N\cap\Gamma} E(xn;\lambda) = \sum_{w \in W} c(w;\lambda) e^{(w\lambda-\rho)(a(x))}$$

where

$$c(w;\lambda) = \prod_{\alpha>0, w\alpha<0} \frac{\xi(\lambda(\alpha^\vee))}{\xi(1+\lambda(\alpha^\vee))},$$

cf. [L], Appendix 3. One has

$$c(ww';\lambda) = c(w;w'\lambda)c(w';\lambda)$$

and

$$E(x;\lambda) = c(w;\lambda)E(x;w\lambda)$$

**1.7. Adelic language.**

## Lecture 2. Harish-Chandra theory

Spherical functions (Calogero - Moser model) and their limits: Whittaker functions (Toda chain).

### 2.0. Rank 1 case: Hypergeometric function and scattering

**2.0.1.** The Gauss' hypergeometric function $F(a,b,c;x)$ is the solution of the differential equation

$$x(1-x)f''(x)+(c-(a+b+1)x)f'(x)-abf(x)=0 \qquad (Hyp.1.1)$$

which for $|x|<1$ has the form

$$F(x)=1+\frac{ab}{1!c}x+\frac{a(a+1)b(b+1)}{2!c(c+1)}x+\ldots$$

It can be continued to a single-valued analytic function in the domain

$$D=\mathbb{C}\setminus\mathbb{R}_{\geq 1}$$

*Functional equation* (cf. [WW], 14.51 as corrected by Bargmann [B], 10d):

$$F(a,b,c;-x)=x^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}F(a,1+a-c,1+a-b;-x^{-1})$$
$$+x^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}F(b,1+b-c,1+b-a;-x^{-1}),$$

$x>1, a-b\notin\mathbb{Z}$.

**2.0.2. Application to the spherical function.** Cf. [HS (b)], no. 13. Let $G=KAN$ be a real simple Lie group of split rank 1, $\mathfrak{a}=LieA$, $\dim_{\mathbb{R}}\mathfrak{a}=1$, $\Sigma=\{\alpha,2\alpha\}\subset\mathfrak{a}^*$ the positive roots, $p=\dim\mathfrak{g}_\alpha,\ q=\dim\mathfrak{g}_{2\alpha},\ \rho=(p+2q)\alpha/2$, $H\in\mathfrak{a}, H(\alpha)=1,\ \lambda\in\mathfrak{a}^*_{\mathbb{C}}$.

The spherical function

$$\phi_\lambda(x)=\int_K e^{\langle-\lambda-\rho,a(xk)\rangle}dk$$

is bi-$K$-invariant; $\phi_\lambda=\phi_{-\lambda}$, cf. [HS (b)], Cor. to Lemma 17.

Set $\psi_\lambda(t)=\phi_\lambda(\exp(tH))$.

Then $\psi_\lambda(t)$ satisfies a differential equation of type (Hyp.1.1) with $x=-\sinh^2 t$. More precisely,

$$\psi_\lambda(t)=F(a,b,c;-\sinh^2 t)$$

with

$$a = \frac{1}{4}(p + 2q - 2\lambda(H)),\ b = \frac{1}{4}(p + 2q + 2\lambda(H)),\ c = \frac{1}{2}(p + q + 1)$$

Asymptotics:

$$\lim_{t\to\infty}\left(e^{t\rho(H)}\psi_\lambda(t) - c(\lambda)e^{\lambda(H)t} - c(-\lambda)e^{-\lambda(H)t}\right) = 0$$

where

$$c(\lambda) = \frac{\Gamma(\lambda(H))\Gamma((p + q + 1)/2)}{\Gamma((p + 2q + 2\lambda(H))/4)\Gamma((p + 2 + 2\lambda(H))/4)}$$

We see here the "incoming wave" $c(-\lambda)e^{-\lambda(H)t}$ and the "outcoming wave" $c(\lambda)e^{\lambda(H)t}$.

**2.1. Real semisimple noncompact Lie groups: structure.** Cf. [Hel] (a), Ch. IX. Let $G$ be a real connected semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup, $\mathfrak{g} = Lie(G)$, $\mathfrak{k} = Lie(K)$. $\mathfrak{k}$ is the fixed subspace of the *Cartan involution* $\theta :\ \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$.

Let $\mathfrak{p} = \{x \in \mathfrak{g}|\ \theta(x) = -x\}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (*the Cartan decomposition*).

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. We have the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{\alpha\in R}\ \mathfrak{g}_\alpha),$$

where $R \subset \mathfrak{a}^*$ is called *the set of restricted roots*; set $m_\alpha = \dim \mathfrak{g}_\alpha$. If $G$ is split over $\mathbb{R}$ then $R$ is reduced and all $m_\alpha = 1$.

If $G$ is complex then all $m_\alpha = 2$ (in this case many formulas below simplify a lot).

We set

$$\rho = \frac{1}{2}\sum_{\alpha>0} m_\alpha \alpha.$$

If $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$,

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}.$$

Let $M$ (resp. $M'$) be the centralizer (resp. normalizer) of $\mathfrak{a}$ in $K$. The *Weyl group*: $W = M'/M$.

A *Weyl chamber* is a connected component of the set

$$\mathfrak{a}' = \{H \in \mathfrak{a}|\ \forall\alpha \in R\ \alpha(H) \neq 0\} \subset \mathfrak{a}\}$$

Fix a Weyl chamber $\mathfrak{a}_+$. A root $\alpha$ is called *positive* if $\alpha(H) > 0$ for all $H \in \mathfrak{a}_+$; a positive root is *simple* if it is not a sum of two positive roots. If

$$\{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{a}^*$$

is the set of all simple roots then

$$\mathfrak{a}_+ = \{H \in \mathfrak{a} |\ \alpha_i > 0, i = 1, \ldots, r\}$$

**Weyl denominator formula.** If $R$ is reduced,

$$\sum_{w \in W} \det w \cdot e^{w\rho} = \prod_{\alpha > 0} (e^{\alpha} - e^{-\alpha}). \tag{2.1.1}$$

The root lattice: $Q = \mathbb{Z}R \subset \mathfrak{a}^*$; the positive cone:

$$Q_+ = \{\mu = \sum n_i \alpha_i |\ n_i \in \mathbb{N}\}$$

We set

$$\mathfrak{n} = \oplus_{\alpha > 0}\ \mathfrak{g}_{\alpha}.$$

Then

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

Let $A, N \subset G$ be the connected Lie subgroups with the Lie algebras $\mathfrak{a}, \mathfrak{n}$ respectively. Then one has:

*Cartan decompostion*: $G = KAK$; $G = K\bar{A}_+K$ where $A_+ = \exp \mathfrak{a}_+$.

*Iwasawa decomposition*: $G = NAK$.

*Bruhat decomposition*: Let $\bar{N} = \theta(N)$. Then

$$\bar{N}MAN \subset G$$

is an open submanifold whose complement is of Haar measure 0.

**Symmetric space $X = G/K$.** If $\mathfrak{a}' \subset \mathfrak{a}$ denotes the subset of regular elements, $A' = \exp \mathfrak{a}'$, then $G' = KA'K$ is dense open in $G$ and $X' = G' \cdot o$ is dense open in $X$, and

$$K/M \times A^+ \xrightarrow{\sim} X' \tag{2.1.2}$$

The space $X$ is a Riemannian manifold with the metric induced from the Killing form on $\mathfrak{g}$.

**2.1.1. Example.** $G = SL_n(\mathbb{R})$, $K = SO(n)$, $N$ — upper triangular matrices with 1's on the diagonal, $A \subset G$ – diagonal matrices with positive entries,

$$A_+ = \{\mathrm{diag}(a_1, \ldots, a_n) |\ a_1 > \ldots > a_n > 0\}.$$

The maximal torus $T = MA$ where $M = \{\mathrm{diag}(\pm 1, \ldots, \pm 1)\}$.

**Exercice.** *Show that $X = G/K$ is isomorphic to the space of real $n \times n$ symmetric positively definite matrices with determinant* 1.

*What is an analogue of all this for $G = SL_n(\mathbb{C})$?*

**2.2. Invariant differential operators: Harish-Chandra isomorphism.** Set $X = G/K$ and let $\mathcal{D}_G(X)$ denote the algebra of $G$-invariant differential operators on $X$.

Recall the Iwasawa decomposition

$$G = NAK.$$

It allows to define, given a differential operator $D$ on $X$, its *$N$-radial part* $\Delta_N(D) \in \mathcal{D}(A)$.

Let $\mathcal{D}^c(A)$ be the algebra of $A$-invariant differential operators on $A$, i.e. operators with constant coefficients (so it is isomorphic to a polynomial algebra in $r$ generators, at least in the split case).

Let $\mathcal{D}^c_W(A) \subset \mathcal{D}^c(A)$ be the subalgebra of $W$-invariant operators (so it is also isomorphic to a polynomial algebra in $r$ variables).

**2.2.1. Theorem** (Harish-Chandra), cf. [Hel] (b), Ch. II, Corollary 5.19. *The mapping*

$$D \mapsto HC(D) := e^{-\rho}\Delta_N(D)e^{\rho}$$

*defines an algebra isomorphism*

$$HC :\ \mathcal{D}_G(X) \xrightarrow{\sim} \mathcal{D}^c_W(A).$$

**2.3. Examples: radial parts of Laplacians.** $X$ is equipped with a Riemannian metrics, so we can speak about the Laplacians.

(a) *Horospheric coordinates*, cf. [Hel] (b), Ch. II, 3.8. Let $N$ act on $X$ with the transversal part $A \cdot o$. Then

$$\Delta_N(L_X) = e^{\rho}L_A e^{-\rho} - \rho^2. \tag{2.3.1}$$

We see that

$$HC(L_X) = e^{-\rho}\Delta_N(L_X)e^{\rho} = L_A - \rho^2 \tag{2.3.2}$$

is an operator with constant coefficients, in accordance with 2.2.1.

(b) *Spherical coordiantes*, cf. [Hel] (b), Ch. II, 3.9. Let $K$ act on $X$ with the transversal part $A_+ \cdot o$. Then

$$\Delta_K(L_X) = L_A + \sum_{\alpha>0} m_\alpha(\coth\alpha)\partial_\alpha \tag{2.3.3}$$

This is basically the Schrödinger operator for the Calogero - Moser model.

If $G$ is complex then

$$\Delta_K(L_X) = \delta^{-1/2}(L_A - \rho^2)\delta^{1/2} \tag{2.3.4}$$

(cf. (2.3.1)) where

$$\delta^{1/2} = \sum_{w \in W} \det w \cdot e^{w\rho} = \prod_{\alpha>0} (e^{\alpha} - e^{-\alpha}),$$

cf. (2.2.1), [Hel] (b), Ch. II, Prop. 3.10.

**2.4. Spherical functions: Harish-Chandra theory.** A smooth function $\phi : G \longrightarrow \mathbb{C}$ is called a *spherical function* (also: "a zonal s.f.", "an elementary s.f.") if:

$\phi(1) = 1$; $\phi(kxk') = \phi(x)$ for all $k, k' \in K$; $\phi$ is an eigenfunction of all differential operators $D \in \mathcal{D}_G(X)$.

For $x \in G$ let $a(x) \in \mathfrak{a}$ denote $\log A(x)$ where $x = nA(x)k$ is the Iwasawa decomposition; it is correctly defined.

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ define

$$\phi_\lambda(x) = \int_K e^{(i\lambda+\rho)(a(xk))} dk$$

**2.4.1. Theorem.** *$\phi_\lambda$ is a spherical function.*

*All spherical functions are of the form $\phi_\lambda$.*

*$\phi_\lambda = \phi_\mu$ iff there exists $w \in W$ such that $\mu = w\lambda$.*

*Finally, the eigenvalues of operators $D \in \mathcal{D}_G(X)$ are given by*

$$D\phi_\lambda = HC(D)(i\lambda)\phi_\lambda. \tag{2.4.1}$$

The following "noncompact" expression of $\phi_\lambda$ is crucial for the calculation of the Harish - Chandra function.

Recall the opposite nilpotent subgroup $\bar{N} := \theta N$; normalize the Haar measure on it by

$$\int_{\bar{N}} e^{-2\rho(a(z))} dz = 1.$$

**2.4.2. Theorem.** *For $y \in A$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$,*

$$\phi_\lambda(y) = e^{(i\lambda-\rho)(y)} \int_{\bar{N}} e^{(i\lambda-\rho)(ya(z)y^{-1})-(i\lambda+\rho)(a(z))} dz.$$

Cf. [Hel] (b), Ch. IV, Prop. 6.3.

**2.5. Harish-Chandra decomposition.** Cf. [Hel] (b), Ch. IV, §5. Thus, the spherical functions form a space

$$A/W \cong \bar{A}_+$$

Our next aim will be to construct a family $\{\phi_w\}_{w\in W}$ of $W$ linearly independent solutions of (2.4.1) in the domain $A_+ \cdot o$.

Rewrite (2.3.2) as

$$\Delta_K(L_X) = L_A + 2\partial_\rho + \sum_{\alpha>0} m_\alpha(\coth\alpha - 1)\partial_\alpha \tag{2.5.1}$$

Denote by $\mathcal{R}$ the ring of functions on $A_+$ of the form

$$f(y) = \sum_{\mu\in Q_+} a_\mu e^{-\mu(\log y)},$$

and by $\mathcal{R}_+ \subset \mathcal{R}$ the ideal of $f$'s with $a_0 = 0$.

Note that

$$\coth\alpha - 1 = 2\sum_{n=1}^{\infty} e^{-2n\alpha} \in \mathcal{R}_+$$

We can consider (2.5.1) as a perturbation of the operator

$$\Delta_K(L_X)_0 = L_A + 2\partial_\rho$$

The function $\phi_\lambda$ is a solution of the differential equation

$$\Delta_K(L_X)\phi_\lambda = -(\lambda^2 + \rho^2)\phi_\lambda \tag{2.5.2}$$

Let look for a solution of (2.5.2) by perturbation theory. A solution of a nonperturbed equation:

$$\Delta_K(L_X)_0 e^{i\lambda-\rho} = -(\lambda^2+\rho^2)e^{i\lambda-\rho}$$

Let us look for a solution of (2.5.2) in the form

$$\psi_\lambda(\exp a\cdot o) = e^{(i\lambda-\rho)(a)}\sum_{\mu\in Q_+}\gamma_\lambda(\mu)e^{-\mu(a)},\ a\in\mathfrak{a}_+ \tag{2.5.3}$$

where we set $\gamma_\lambda(0) = 0$. We get a recurrence relation for the coefficients $\gamma_\lambda(\mu)$:

$$(\mu, \mu - 2i\lambda)\gamma_\lambda(\mu) =$$
$$\sum_{\alpha>0} m_\alpha \sum_{n\geq 1}(\alpha, -i\lambda - 2n\alpha + \mu + \rho)\gamma_\lambda(\mu - 2n\alpha) \tag{2.5.4}$$

The resulting series (2.5.3) converges absolutely and uniformly in each domain

$$\{a\in\mathfrak{a}_+ |\ \alpha_i(a)\geq\epsilon>0\}\subset\mathfrak{a}_+.$$

It follows from the commutativity of $\mathcal{D}_G(X)$ that $\psi_\lambda$ satisfies the differential equations

$$\Delta_K(D)\psi_\lambda = HC(D)(i\lambda)\psi_\lambda \tag{2.5.5}$$

for all $D\in\mathcal{D}_G(X)$.

Suppose that

$(w - w')\lambda \notin iQ$ for all $w \neq w' \in W$.

Then the functions $\psi_{w\lambda}$, $w \in W$, form a basis of $|W|$ linearly independent solutions of (2.5.2) (or equivalently of (2.5.5)).

It wollows:

**2.6. Theorem.** *Suppose that a weight $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ is such that*

$$\text{for all } w \neq w' \in W \ (w - w')\lambda \notin iQ.$$

*Then*

$$\phi_\lambda(a) = \sum_{w \in W} c(w\lambda)\psi_{w\lambda}(a)e^{(iw\lambda - \rho)(a)}, \ a \in \mathfrak{a}_+.$$

Thus for $a \in \mathfrak{a}_+$

$$\lim_{t\to\infty}(\phi_\lambda(ta) - \sum_{w\in W} c(w\lambda)e^{(iw\lambda-\rho)(ta)}) = 0.$$

**2.7. Gindikin - Karpelevich formula.** Cf. [Hel] (b), Ch. IV, §6; [HO], [Karp]. It remains to compute the function $c(\lambda)$.

The starting point is a noncompact *Harish-Chandra formula*:

$$c(\lambda) = \int_{\bar{N}} e^{-(i\lambda+\rho)(a(y))}dy$$

if $\Re(i\lambda) \in \mathfrak{a}^*_+$, cf. [Hel] (b), Ch. IV, §6, (7).

**2.7.1. Rank** 1 **case.** Cf. [Hel], Ch. IV, §6, no. 2. In this case $\bar{\mathfrak{n}} = \mathfrak{g}_\alpha + \mathfrak{g}_{-2\alpha}$. Let $m_i = \dim\ \mathfrak{g}_{i\alpha}$, $i = 1, 2$, Then

$$c_\alpha(\lambda) = c\int_{\mathfrak{g}_\alpha}\int_{\mathfrak{g}_{-2\alpha}}((1 + b|x|^2)^2 + 4b|y|^2)^{-(i\lambda+1)b'}dxdy$$

where $b = (4(m_1 + 4m_2))^{-1}$, $b' = (m_1 + 2m_2)/4$, $c$ - a constant independent of $\lambda$. The integral is a product of two Beta-functions. Using the duplication formula, one gets

$$c_\alpha(\lambda) = 2^{m_1/2+m_2-i\lambda\cdot\alpha'}\frac{\Gamma((m_1 + m_2 + 1)/2)\Gamma(i\lambda\cdot\alpha')}{\Gamma((m_1/2 + 1 + i\lambda\cdot\alpha')/2)\Gamma((m_1/2 + m_2 + i\lambda\cdot\alpha')/2)}$$

where $\alpha' = \alpha/\alpha^2$.

*A simpler formula for the split case*:

$$c_\alpha(\lambda) = B(1/2, i\lambda\cdot\alpha/2) = \frac{\pi^{1/2}\Gamma(i\lambda\cdot\alpha/2)}{\Gamma(1/2 + i\lambda\cdot\alpha/2)},$$

cf. [L] (b), p. 16.

**2.7.2. Gindikin - Karpelevich product formula.** Let $w \in W$; define

$$\bar{\mathfrak{n}}_w = \oplus_{\alpha \in R_+ \cap w R_-} \ \mathfrak{g}_{-\alpha}, \ \bar{N}_w = \exp \bar{\mathfrak{n}}_w.$$

**Theorem.**

$$c_w(\lambda) := \int_{\bar{N}_w} e^{-(i\lambda+\rho)a(y)} dy = \prod_{\alpha \in R_+ \cap w R_-} c_\alpha(\lambda_\alpha).$$

As a corollary one gets

**2.7.3. Theorem.** (i) *General case.*

$$c(\lambda) = c \prod_{\alpha > 0} \frac{2^{-i\lambda\cdot\alpha}\Gamma(i\lambda \cdot \alpha)}{\Gamma((m_\alpha/2 + 1 + i\lambda \cdot \alpha')/2)\Gamma((m_\alpha/2 + m_{2\alpha} + i\lambda \cdot \alpha')/2)}$$

*where the constant $c$ is defined by $c(-i\rho) = 1$.*

(ii) *If $G$ is split over $\mathbb{R}$, all $m_\alpha = 1$, all $m_{2\alpha} = 0$:*

$$c(\lambda) = \prod_{\alpha > 0} \frac{\pi^{1/2}\Gamma(i\lambda(\alpha^\vee)/2)}{\Gamma(1/2 + i\lambda(\alpha^\vee)/2)}$$

(iii) *If $G$ is complex, all $m_\alpha = 2$, all $m_{2\alpha} = 0$:*

$$c(\lambda) = \frac{\prod_{\alpha > 0} \rho \cdot \alpha}{\prod_{\alpha > 0} i\lambda \cdot \alpha}.$$

**2.8. Example: The case of $SL_2(\mathbb{R})$: spherical functions on the Lobachevsky half-plane.** Cf. [He] (b), Introduction, §4; (c), §4.

For $G = SL_2(\mathbb{R}) \supset K = SO(2)$

$$G/K = H := \{z | \ \Im z > 0\}$$

Another realization: the Poincaré disc

$$D = \{z \in \mathbb{C} | \ |z| < 1\} = SU(1,1)/SO(2),$$

where

$$SU(1,1) = \{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \ |a|^2 - |b|^2 = 1\} \overset{\sim}{=} SL_2(\mathbb{R})$$

acts on $D$ by

$$g \cdot z = \frac{az + b}{\bar{b}z + \bar{a}}.$$

The Cayley transformation

$$z \mapsto w = -i\frac{z + i}{z - i}$$

is a $G$-equivariant isomorphism of $D$ onto $H = \{w \in \mathbb{C} | \, \Im w > 0\}$, cf. [He] (a), Ch. I, Exercice G.

The spherical (*geodesic polar*) coordiantes on $D$ are $(r, \theta)$ where

$$z = \tanh r e^{i\theta} \in D, \;\; r = d(o, z).$$

The Laplacian is

$$\Delta = \partial_r^2 + 2\coth 2r \partial_r + 4\frac{1}{\sinh^2 2r}\partial_\theta^2.$$

A spherical function $\phi(r)$ is radial (depends only on $r$) and satisfies the equation

$$\phi'' + 2\coth 2r\phi' = (-\lambda^2 + 1)\phi$$

The Harish-Chandra spherical function (compact form):

$$\phi_\lambda(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cosh 2r - \sinh 2r\cos\theta)^{-(i\lambda+1)/2}d\theta$$

This is a Legendre function.

**2.8.1. Exercice.** Write down the Whittaker equation and the Whittaker function. Cf. [Ha] (a), §3.

The standard functions:

$$\psi_\lambda(r) = e^{(i\lambda-1)r}\sum_{n=0}^{\infty}\gamma_n(\lambda)e^{-nr}$$

where $\gamma_0(\lambda) = 1$ and

$$\gamma_n(\lambda) = \frac{1}{n(n-i\lambda)}\sum_{\ell=1}^{[n/2]}(2n - 2\ell - i\lambda + 1)\gamma_{n-2\ell}(\lambda)$$

The Harish-Chandra decomposition:

$$\phi_\lambda(r) = c(\lambda)\psi_\lambda(r) + c(-\lambda)\psi_{-\lambda}(r)$$

where

$$c(\lambda) = B(1/2, i\lambda/2),$$

cf. [Hel] (c), Thm. 4.6.

## Lecture 3. Toda quantum field theory

Dual symmetric spaces: $SU(2) = (SU(2)\times SU(2))/SU(2)$ and $SL_2(\mathbb{C})/SU(2)$. The correponsing $\sigma$-models: the $WZW$-models of central charge $k \in \mathbb{Z}_{\geq 1}$ and the $H^3_+$-WZW-model. The correlation functions satisfy the KZ equations.

Their corresponding reductions: the minimal models and the Liouville model. The correlation functions satisfy the BPZ equations.

Similarly for arbitrary compact Lie groups.

**3.1. Toda field theory.** Cf. [F]. Let $R \subset \mathfrak{a}^* \cong \mathbb{R}^r$ be a simply laced root system of rang $r$; $\alpha_i, i = 1, \ldots, r$ simple roots. We equip $\mathfrak{a}^*$ with a $W$-invariant scalar product, $\rho = (1/2)\sum_{\alpha>0}\alpha$.

It is a two-dimensional field theory with:

*Classical fields*: maps $\phi(z)$

$$\phi:\ \mathbb{C} \longrightarrow \mathfrak{a}^*,$$

Lagrangian

$$S_{b,\mu}(\phi) = \int\left(\frac{1}{8\pi^2}((\partial_x\phi)^2 + (\partial_y\phi)^2) + \mu\sum_{i=1}^r e^{b\alpha_i\cdot\phi}\right)dxdy$$

where $z = x + iy$, $\mu, b \in \mathbb{R}$. Set

$$\varsigma = (b + b^{-1})\rho$$

After quantization the theory is conformally invariant, with

$$\langle\phi_i(z), \phi_j(w)\rangle = -\log|z-w|^2\delta_{ij},$$

the holomorphic stress energy tensor

$$T(z) = -\frac{1}{2}(\partial_z\phi)^2 + \varsigma\cdot\partial_z^2\phi$$

and central charge

$$c = r + 12\varsigma^2$$

The theory possesses a $W(R)$-symmetry. The chiral algebra $W(R)$ contains $r$ holomorphic fields $W_j(z)$, with $W_2(z) = T(z)$. The field $W_j(z)$ has the spin equal to the corresponding exponent of the Lie algebra $\mathfrak{g} = \mathfrak{g}(R)$.

The exponential fields $V_\lambda(z) = e^{\lambda\cdot\phi(z)}$, $\lambda \in \mathfrak{a}^*$ are primary fields; this means that

$$W_{j,0}V_\lambda = c_j(\lambda)V_\lambda,\ W_{j,n}V_\lambda = 0,\ n > 0.$$

The conformal dimension of $V_\lambda$ is

$$\Delta(\lambda) = c_2(\lambda) = \frac{1}{2}(\varsigma^2 - (\lambda - \varsigma)^2) = \frac{\lambda \cdot (2\varsigma - \lambda)}{2}$$

**3.2. Weyl reflections.** Let us introduce a shifted Weyl group action on $\mathfrak{a}^*$ by setting

$$w_b(\lambda) = \varsigma + w(\lambda - \varsigma).$$

Then $c_j(\lambda) = c_j(w_b(\lambda))$.

**Reflection amplitudes**:

$$V_\lambda(z) = R_w(\lambda) V_{w_b(\lambda)}(z)$$

*Computation of these numbers* ("Gindikin - Karpelevich formula").

One introduces normalized fields

$$V_\lambda^n(z) = N(\lambda) V_\lambda(z)$$

in such a way that

$$\langle V_\lambda^n(z), V_\lambda^n(w) \rangle = \frac{1}{|z - w|^{4\Delta}},$$

cf. [KT], 6.14.

For these normalized fields

$$V_\lambda^n(z) = V_{w_b(\lambda)}^n(z)$$

It follows that

$$R_w(\lambda) = \frac{N(\lambda)}{N(w_b(\lambda))}.$$

The numbers $N(\lambda)$ are calculated as follows, [F].

Suppose that

$$2\lambda + \sum_{i=1}^{r} \ell_i \alpha_i = 0.$$

Then

$$N(\lambda)^2 = |z|^{4\Delta} \langle V_\lambda(z) V_\lambda(0) \prod_{i=1}^{r} \hat{Q}_i^{\ell_i} / \ell_i! \rangle$$

where

$$\hat{Q}_i = \int_{\mathbb{C}} e^{b\alpha_i(z)} dx dy$$

are the "screening operators". This is a "Coulomb gas" complex Selberg integral and the answer is given by

**3.3. Theorem.**

$$N(\lambda)^2 = (\pi\mu\gamma(b^2))^{-2\lambda\cdot\rho/b} \prod_{\alpha>0} \frac{\Gamma(1+\varsigma\cdot\alpha/b)\Gamma(1+\varsigma\cdot\alpha\cdot b)}{\Gamma(1-\varsigma\cdot\alpha/b)\Gamma(1-\varsigma\cdot\alpha\cdot b)} \times$$

$$\prod_{\alpha>0} \frac{\Gamma(1+(\lambda-\varsigma)\cdot\alpha/b)\Gamma(1+(\lambda-\varsigma)\cdot\alpha\cdot b)}{\Gamma(1-(\lambda-\varsigma)\cdot\alpha/b)\Gamma(1-(\lambda-\varsigma)\cdot\alpha\cdot b)}$$

It follows:

$$R_w(\lambda) = \frac{N(\lambda)}{N(w_b\lambda)} = \frac{A(w_b\lambda)}{A(\lambda)} \tag{3.3.1}$$

where

$$A(\lambda) = (\pi\mu\gamma(b^2))^{\lambda\cdot\rho/b} \prod_{\alpha>0} \Gamma(1-(\lambda-\varsigma)\cdot\alpha/b)\Gamma(1-(\lambda-\varsigma)\cdot\alpha\cdot b)$$

The functional equation for operators $V_\lambda(z)$ takes the form

$$A(\lambda)V_\lambda(z) = A(w_b\lambda)V_{w_b\lambda}(z)$$

**3.4. Quasiclassical ("mini-superspace") limit.** The Whittaker function $\Psi_\lambda(x)$, $\lambda \in \mathfrak{a}^*, x \in \mathfrak{a}$ is certain solution of the Schrödinger equation for the Toda system

$$\left(-\Delta_x + 2\pi\mu \sum_{i=1}^{r} e^{b\alpha_i(x)}\right)\Psi_\lambda(x) = \lambda^2\Psi_\lambda(x).$$

Let

$$\Lambda = \{x|\ \alpha_i(x) > 0,\ 1 \leq i \leq r\} \subset \mathfrak{a}$$

denote the fundamental Weyl chamber. The function $\Psi_\lambda(x)$ possesses in the opposite chamber $-\Lambda$ the asymptotics

$$\Psi_\lambda(x) \sim \sum_{w\in W} c_w(\lambda)e^{i\lambda(x)}$$

where

$$c_w(\lambda) = \prod_{\alpha>0} (\pi\mu/b^2)^{i(w\lambda-\lambda)\cdot\alpha/2b} \frac{\Gamma(-iw\lambda\cdot\alpha/b)}{\Gamma(-i\lambda\cdot\alpha/b)}, \tag{3.4.1}$$

cf. [F] (b) (5.2), [OP] §12, [Ha] (b) Thm (7.8). This is the Whittaker limit of Harish-Chandra - Gindikin - Karpelevich formula.

A remarkable fact is that

*the functions* (3.4.1) *are the* $b \to 0$ *limit of reflections amplitudes* (3.3.1).

## §4. Matsuo equations

**4.1. Heckman - Opdam hypergeometric functions.** Cf. [HO]. Let $\mathfrak{a}$ be a Euclidean space of dimension $r$ with inner product $x \cdot y$, $R \subset \mathfrak{a}^*$ a possibly non-reduced root system of rank $r$.

For example the type $BC_1 = \{\pm\alpha,\ \pm 2\alpha\}$ correponds to the case (2.7.1).

Let $P \subset \mathfrak{a}^*$ denote the weight lattice and $Q \subset P$ the root lattice. We fix a base of simple roots $\{\alpha_1, \ldots, \alpha_r\} \subset R$ and define the positive cone

$$Q_+ = \oplus_{i=1}^r \mathbb{N}\alpha_i \subset Q.$$

let $W$ be the Weyl group; the scalar product is $W$-invariant. We identify $\mathfrak{a}$ with $\mathfrak{a}^*$ using the scalar product.

Let

$$k:\ R \longrightarrow \mathbb{C}$$

be a function such that $k(\alpha) = k(w\alpha)$ for all $w \in W$. We set

$$\varrho = \frac{1}{2}\sum_{\alpha>0} k(\alpha)\alpha$$

Set $\mathfrak{h} := \mathfrak{a}_{\mathbb{C}}$. For $\alpha \in R$ set $\alpha^\perp = \{x \in \mathfrak{h} |\ \alpha(x) = 0,$

$$\mathfrak{h}_{reg} = \{x \in \mathfrak{h} |\ \forall \alpha \in R\ e^{\alpha(x)} \neq 1\}$$

For $\beta \in \mathfrak{h}$ we shall denote $\partial_\beta$ the derivation in the direction $\beta$.

Let $\mathcal{O}$ denote the algebra of functions $f:\ \mathfrak{h}_{reg} \longrightarrow \mathbb{C}$ generated by the functions $(1-e^\alpha)^{-1}$ and $\mathcal{R}$ the algebra of differential operators on $\mathfrak{h}$ with coefficients in $\mathcal{O}$ (one checks easily that the product of two elements from $\mathcal{R}$ belongs to $\mathcal{R}$).

The subalgebra $\mathcal{R}_0 \subset \mathcal{R}$ of operators with constant coefficients will be identified with the polynomial algebra $S\mathfrak{h}$ = the algebra of polynomial functions on $\mathfrak{h}^*$, and $\mathcal{R} = \mathcal{O} \otimes_{\mathbb{C}} S\mathfrak{h}$.

Fix an orthonormal base $\{x_i\}$ of $\mathfrak{a}$ and set $\partial_i := \partial_{x_i}$.

**4.1.1. Laplace operator**:

$$L = \sum_{i=1}^r \partial_i^2 + \sum_{\alpha>0} k(\alpha)\coth(\alpha/2)\partial_\alpha \in \mathcal{R}$$

since

$$\coth(\alpha/2) = -1 + \frac{2}{1-e^\alpha}$$

Using the expansion

$$\frac{1}{1-e^{\alpha}} = 1 + e^{\alpha} + e^{2\alpha} + \dots$$

we have for each $D \in \mathcal{R}$ an expansion

$$D = \sum_{\mu \in Q_+} e^{\mu} q_{\mu}, \;\; q_{\mu} \in \mathcal{R}_0$$

which converges in the negative Weyl chamber

$$\mathfrak{h}_- = \{x \in \mathfrak{h} | \forall \alpha > 0 \;\alpha(x) < 0\}.$$

For example

$$L = \sum_{i=1}^{r} \partial_i^2 - 2\partial_{\varrho} - 2\sum_{\alpha>0} k(\alpha) \sum_{j=1}^{\infty} e^{j\alpha}\partial_{\alpha}$$

**4.1.2. Harish-Chandra homomorphism.**

*Argument shift and conjugation by* $\delta$

Define

$$\delta = \prod_{\alpha>0} |2\sinh(\alpha/2)|^{k(\alpha)}$$

If

$$D = \sum_{\mu \in Q_+} e^{\mu} q_{\mu}, \;\; q_{\mu} \in \mathcal{R}_0$$

and

$$D' := \delta D \delta^{-1} = \sum_{\mu \in Q_+} e^{\mu} q'_{\mu}, \;\; q'_{\mu} \in \mathcal{R}_0$$

then

$$q'_0(\lambda) = q_0(\lambda + \varrho), \;\; \lambda \in \mathfrak{h}^*$$

Let us denote by $HC(D) \in S\mathfrak{h}$ the element

$$\lambda \in \mathfrak{h}^* \mapsto q_0(\lambda + \rho).$$

**4.1.3. Theorem** (Harish-Chandra isomorphism). *Consider the subspace:*

$$Z_{\mathcal{R}}(L) = \{D \in \mathcal{R} | \; [D, L] = 0, \forall w \in W \;\, w(D) = D \;\} \subset \mathcal{R};$$

*it is clear that this is a subalgebra. The map* $D \mapsto HC(D)$ *defines an isomorphism of algebras*

$$HC: \;\; Z_{\mathcal{R}}(L) \xrightarrow{\sim} S\mathfrak{h}^W.$$

Cf. [HO], 2.10: it is formulated as a conjecture there but must be proven since then. For the symmetric spaces it is due to Harish-Chandra.

In particular the algebra $Z_{\mathfrak{R}}(L)$ is commutative; we shall denote this algebra by $\mathbb{D}$.

We will be interested in the system of differential equations

$$D\phi_\lambda = HC(D)(\lambda)\phi_\lambda,\ D \in \mathbb{D} \qquad (HO)_\lambda$$

where $\lambda \in \mathfrak{h}^*_{reg}$ is a parameter. Its solutions are called the hypergeometric functions connected with the root system $R$ and the weight function $k$.

There are $|W|$ linearly independent solutions for $\lambda$ generic.

**4.2. Matsuo equations.** Cf. [M]. Define a homomorphism $\nu : W \longrightarrow GL(\mathbb{C}[W])$ by

$$\nu(s)(w) = sw.$$

For $\alpha \in R$ set

$$\sigma_\alpha = \nu(s_\alpha)$$

Define a diagonal matrix $\epsilon_\alpha \in End(\mathbb{C}[W])$ by

$$\epsilon_\alpha(w) = \pm w \text{ for } w^{-1}\alpha \in R_\pm$$

For $\lambda \in \mathfrak{h}^*,\ \xi \in \mathfrak{h}$ define a diagonal matrix $e_\xi(\lambda) \in End(\mathbb{C}[W])$ by

$$e_\xi(\lambda)(w) = \langle w\lambda, \xi\rangle w$$

Consider functions $\psi(u) = \sum_{w\in W} \psi_w(u)w :\ \mathfrak{h} \longrightarrow \mathbb{C}[W]$. The system of Matsuo equations:

$$\partial_\xi \psi(u) = A_{\xi,\lambda}(u)\psi(u) \qquad (M)_\lambda$$

where

$$A_{\xi,\lambda}(u) = \frac{1}{2}\sum_{\alpha\in R_+} k(\alpha)\alpha(\xi)(\coth(\alpha(u)/2)(\sigma_\alpha - 1) + \sigma_\alpha\epsilon_\alpha) + e_\xi(\lambda)$$

This is a system of $r = \dim \mathfrak{a}$ differential equations of the first order.

**4.2.1. Proposition.** *The operators $A_{\xi,\lambda}$ commute:*

$$[A_{\xi,\lambda}, A_{\eta,\lambda}] = 0 \qquad (4.2.1)$$

*and the system is integrable*:

$$[\partial_\xi - A_{\xi,\lambda}, \partial_\eta - A_{\eta,\lambda}] = 0$$

*for all $\xi, \eta \in \mathfrak{h}$.*

$W$-invariance:

**4.2.2. Proposition.**

$$\nu(w)A_{\xi(u),\lambda}\nu(w)^{-1} = A_{w\xi,\lambda}(wu)$$

**4.2.3. Corollary.** *If $\psi(u)$ is a solution of (M) then $\nu(w)\psi(w^{-1}u)$ is a solution as well.*

**4.2.4. Proposition.** *The dimension of space of local solutions of (M) is $|W|$.*

**4.3. The map $(M)_\lambda \longrightarrow (HO)_\lambda$.** Define the map

$$M: \ \mathbb{C}[W] \longrightarrow \mathbb{C}, \ M(\sum_{w\in W} a_w w) = \sum_{w\in W} a_w.$$

**Theorem.** *Fix $\lambda \in \mathfrak{h}^*$. If $\phi(u) = \sum_w \phi_w(u)w$ is a solution of the equations $(M)_\lambda$ then $M(\phi)$ is a solution of $(HO)_\lambda$.*

**4.4. A base of $|W|$ local solutions of $(M)_\lambda$.** Let us introduce exponential local coordinates on $\mathfrak{h}$

$$y_i(u) = e^{\alpha_i(u)}, \ u \in \mathfrak{h}, \ i = 1, \dots, r.$$

Then

$$\partial_{y_i} = \frac{\partial_{\alpha_i^\vee}}{y_i}.$$

Consider the constant parts of the operators $A_{\xi,\lambda}(u)$:

$$A_{\xi,\lambda}(u) = A^0_{\xi,\lambda} + \sum_{\mu\in Q_{++}} A^\mu_{\xi,\lambda} e^{\mu(u)}$$

where

$$A^0_{\xi,\lambda} = \frac{1}{2}\sum_{\alpha\in R_+} k(\alpha)\alpha(\xi)\sigma_\alpha(\sigma_\alpha - 1) + e_\xi(\lambda) + \varrho(\xi)$$

The $|W|\times|W|$-matrices $A^0_{\xi,\lambda}$ are triangular (with respect to the (partial) ordering by length in $W$), with the diagonal elements

$$(A^0_{\xi,\lambda})_{ww} = (w\lambda + \varrho)(\xi)$$

The matrices $A^0_{\xi,\lambda}$ commute with each other for different $\xi$, due to (4.2.1).

Let

$$\psi^0_{\lambda,w} = \sum_{w'\in W} a_{\lambda;ww'}w', \ a_{\lambda;ww} = 1, \tag{4.4.1}$$

denote the common eigenvector

$$A^0_{\xi,\lambda}\psi^0_{\lambda,w} = (w\lambda + \varrho)(\xi)\psi^0_{\lambda,w}$$

Since the matrices $A^0_{\xi,\lambda}$ are triangular, the matrix

$$(\psi^0_{\lambda,w})_{w\in W}$$

is triangular as well (cf. *Moebius inversion*).

In the new coordinates the equations $(M)$ have the form

$$\frac{\partial\phi}{\partial y_i} = \left(\frac{A^0_{\alpha_i^\vee}}{y_i} + O(1)\right)\phi$$

**4.5. Theorem.** *Suppose that for all $w \in W$ $w\lambda - \lambda \notin Q(R)$. Then the Matsuo system $(M)_\lambda$ admits a unique base of solutions $\{\psi_{\lambda,w}\}_{w\in W}$ of the form*

$$\psi_{\lambda,w}(u) = e^{\langle w\lambda+\varrho,u\rangle}\left(\psi^0_{\lambda,w} + \sum_{\mu\in Q_+}\psi^\mu_{\lambda,w}e^{\langle\mu,u\rangle}\right).$$

**4.6. Theorem (the first Matsuo product formula).**

$$M(\psi^0_{\lambda,w}) = \prod_{\alpha\in R_+\cap w^{-1}R_-}\frac{k(\alpha)+2k(2\alpha)+\langle\lambda,\alpha^\vee\rangle}{\langle\lambda,\alpha^\vee\rangle}.$$

**4.7. Harmonic polynomials.** Cf. [Hel] (b), Ch. III; [Chev]. Consider the symmetric algebra $S\mathfrak{h} = \mathbb{C}[\mathfrak{h}^*]$. The subalgebra of invariants $S\mathfrak{h}^W \subset S\mathfrak{h}$ is a polynomial algebra on generators $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ (Chevalley).

Let $D_i$ denote the Fourier transform of $f_i$: it is a differential operator with constant coefficients acting on $S\mathfrak{h}^*$. We identify $\mathfrak{h}^*$ with $\mathfrak{h}$ using the scalar product, so the operators $D_i$ act on $S\mathfrak{h}$.

*The space of harmonic polynomials*

$$\mathcal{H} = \{f \in S\mathfrak{h}|\ D_i f = 0,\ 1 \le i \le r\} \subset S\mathfrak{h}$$

It is graded by the degree

$$\mathcal{H} = \oplus_{i=0}^\infty \mathcal{H}^i$$

and the Poincaré polynomial is equal

$$P_{\mathcal{H}}(t) := \sum_{i=0}^\infty \dim \mathcal{H}^i t^i = \prod_{j=1}^r \frac{t^{d_i}-1}{t-1}$$

It coincides with

$$P_{\mathcal{H}}(t) = P_W(t) = \sum_{i=0}^\infty |W_i| t^i$$

where

$$W_i = \{w \in W|\ \ell(w) = i\}$$

It has dimension $|W|$ and

$$\mathcal{H} \xrightarrow{\sim} S\mathfrak{h}/I,\ S\mathfrak{h} \cong \mathcal{H} \otimes I$$

where $I = (S\mathfrak{h})^W \cdot S\mathfrak{h}$.

**4.8.** Since the matrices $A^0_{\xi,\lambda} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[W]), \xi \in \mathfrak{h}$, commute and the map

$$\xi \mapsto A^0_{\xi,\lambda}$$

is $\mathbb{C}$-linear, this map extends to a map

$$A^0_\lambda :\ S\mathfrak{h} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}[W]),\ f \mapsto A^0_{f,\lambda}.$$

Define a map

$$J^0_\lambda : S\mathfrak{h} \longrightarrow \mathbb{C}[W],\ f \mapsto J^0_{f,\lambda}$$

by the symmetrization of $A$:

$$(J^0_{f,\lambda})_w = \sum_{w' \in W} (A^0_{f,\lambda})_{ww'}$$

**4.9. Theorem (the second Matsuo product formula).** *Let $\{h_w\}_{w\in W}$ be a homogeneous base of $\mathcal{H}$. Then the determinant of the $W \times W$ matrix*

$$\det((J^0_{h_w,\lambda})_{w'}) = c \prod_{\alpha>0} (k(\alpha) + 2k(2\alpha) + \langle \lambda, \alpha^\vee \rangle)^{|W|/2}$$

**4.10. Main theorem.** *If $\phi = \sum_{w\in W} \phi_w \cdot w$ is a solution of Matsuo equations $(M)_\lambda$ then $M(\phi) = \sum_{w\in W} \phi_w$ is a solution of Heckman - Opdam equations $(HO)_\lambda$.*

*If for all $\alpha > 0$ $k(\alpha) + 2k(2\alpha) + \langle \lambda, \alpha^\vee \rangle \neq 0$, the Matsuo averaging map*

$$M : \ \mathrm{Sol}((M)_\lambda) \longrightarrow \mathrm{Sol}((HO)_\lambda)$$

*is an isomorphism.*

**4.11. Rational limit: Calogero system. Many-body gauge.** Cf. [FV].

## Lecture 5. Confluent KZ equations and integral formulas

**5.1. From Gauss hypergeometric function to Whittaker function.** Cf. [R].

*Hypergeometric equation*:

$$\Big\{\delta(\delta+c-1)-z(\delta+a)(\delta+b)\Big\}f(z)=0 \tag{5.1.1}$$

where $\delta = zd/dz$.

It has 3 regular singular points: $0, 1, \infty$.

Bases of solutions:

near 0:

$$f_1^0(z)=F(a,b,c;z),$$
$$f_2^0(z)=z^{1-c}F(1+a-c,1+b-c,2-c;z)$$

near 1:

$$f_1^1(z)=z^{-a}F(a,a+1,a+b-c+1;1-1/z),$$
$$f_2^1(z)=z^{a-c}(1-z)^{c-a-b}F(c-a,-a+1,-a-b+c+1;1-1/z)$$

near $\infty$:

$$f_0^\infty=e^{i\pi a}z^{-a}F(a,a-c+1,a-b+1;1/z),$$
$$f_1^\infty=e^{i\pi(-a+c)}z^{a-c}(1-z)^{-a-b+c}F(-a+1,-a+c,-a+b+1;1/z).$$

Here

$$F(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!},$$
$$(a)_n=\prod_{i=0}^{n-1}(a+i).$$

*Confluent hypergeometric equation*:

$$\Big\{\delta(\delta+c-1)-z(\delta+a)\Big\}f(z)=0 \tag{5.1.2}$$

It has two singular points: 0 which is regular and $\infty$ which is irregular.

Bases of solutions:

near 0:

$$f_1^0(z)={}_1F_1(a,c;z):=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n n!},$$
$$f_2^0(z)=z^{-c+1}\,{}_1F_1(a-c+1,-c+2;z);$$

near $\infty$:

$$f_1^\infty(z) = z^{-a}\ {}_2F_0(a, a-c+1; -1/z),$$

$$f_2^\infty(z) = z^{a-c}e^z\ {}_2F_0(-a+1, -a+c; 1/z).$$

*Confluence: from* (5.1.1) *to* (5.1.2).

One has to make a change of variable $z \mapsto z/b$ and set $b \longrightarrow \infty$.

**5.2. Integral formulas.**

A hypergeometric integral:

$$\int t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

A confluent hypergeometric integral:

$$\int t^{b-1}e^{-t}(1-zt)^{-a}dt$$

**5.3. FMTV equations.** Cf. [FMTV].

Let

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha>0}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})\right)$$

be a simple Lie algebra; we fix an invariant scalar product $(.,.)$ on $\mathfrak{g}$ and root vectors $e_\alpha \in \mathfrak{g}_\alpha$ in such a way that $(e_\alpha, e_{-\alpha}) = 1$.

The Casimir element

$$\Omega = \sum_{i=1}^{r} h_i \otimes h_i + \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha}$$

where $\{h_i\}$ is an orthonormal basis in $\mathfrak{h}$.

We consider two compatible systems of differential equations on a function

$$v(z;\mu) \in M = M_1 \otimes \ldots M_n$$

where $(z;\mu) = (z_1, \ldots, z_n; \mu_1, \ldots, \mu_r) \in \mathbb{C}^{n+r}$, $M_i$ being $\mathfrak{g}$-modules.

The first one is

$$\kappa\frac{\partial v}{\partial z_i} = \mu^{(i)}v + \sum_{j\neq i}\frac{\Omega^{(ij)}v}{z_i - z_j},\ 1 \le i \le n \tag{5.3.1}$$

where $\kappa \in \mathbb{C}^*$, $\mu = \sum \mu_i h_i \in \mathfrak{h}$ and $\mu^{(i)}$ denotes the action on the $i$-th tensor factor. This is a deformed KZ system.

The second one is

$$\kappa\frac{\partial v}{\partial \mu_s}=\sum_{i=1}^{n} z_i\mu^{(i)}v+\sum_{\alpha>0}\frac{(\alpha,h_s)}{(\alpha,\mu)}e_\alpha e_{-\alpha}v,\ 1\leq s\leq r \tag{5.3.2}$$

This system is sometimes called the *Casimir connection.*

All these $n+r$ differential operators commute with each other.

**5.4. Integral solutions.** Cf. [SV], [FMTV]. Suppose that for each $i$ $M_i$ is a highest weight module generated by a vacuum vector $x_i$ of weight $\Lambda_i$. Let $\{\alpha_s,\ s=1,\ldots,r\}$ be a base of simple roots and $f_s:=e_{-\alpha_s}$.

We set $\Lambda=\sum\Lambda_i$. We have the weight decomposition

$$M=\oplus_{\lambda\in Q_+}M_{\Lambda-\lambda}$$

where $Q_+=\{\sum n_s\alpha_s|\ (n_1,\ldots,n_r)\in\mathbb{N}^r\}$.

We shall describe solutions of (5.3.1), (5.3.2) with values in a fixed weight subspace $M_{\Lambda-\lambda},\ \lambda=\sum m_s\alpha_s$. They will have the form

$$\int_{\gamma(z)}\omega(z,t)\phi(\mu,z;t)^{1/\kappa}dt$$

Set $m=\sum m_s$. Let

$$u:\ [m]:=\{1,\ldots,m\}\longrightarrow[r] \tag{5.4.1}$$

be the non-decreasing map such that $|u^{-1}(s)|=m_s$ for $1\leq s\leq r$ (*an unfolding*). Define a function of $n+m$ variables $z_1,\ldots,\ z_n;t_1,\ldots,t_m$:

$$\phi(z;t)=\prod_{p<q}(t_p-t_q)^{(\alpha_{u(p)},\alpha_{u(q)})}\prod_{p,i}(t_p-z_i)^{(\alpha_{u(p)},\Lambda_i)}\prod_{i<j}(z_i-z_j)^{(\Lambda_i,\Lambda_j)}$$

Next, set

$$\phi_\mu(z;t)=e^{\sum(\mu,\Lambda_j)z_j-\sum(\mu,\alpha_{u(p)})t_p}\phi(z;t)$$

Now let us describe a logarithmic $m$-form $\omega(z;t)$ with values in $M_{\Lambda-\lambda}$. This weight space is generated by monomials of the form

$$f_{\bar I}v=f_{I_1}v_1\otimes\ldots f_{I_n}v_n$$

Here we denote

$$f_{I_j}=f_{i_{j1}}\ldots f_{i_{jq_j}},\ I_j=(i_{j1},\ldots,i_{jq_j})\in[r]^q,\ 1\leq j\leq n.$$

Here $\bar I$ runs through the set $\mathcal{P}(\lambda,n)$ of all $n$-tuples

$$\bar I=(I_1,\ldots,I_n)$$

such that in the sequence

$$i_{11},\ldots i_{nq_n}$$

there are exactly $m_s$ indices $i_{ab} = s$ for all $1 \leq s \leq r$.

Let us pick a map $u' : [r] \longrightarrow [m]$ such that $uu' = \mathrm{Id}_{[r]}$ and denote for brevity $t_i := t_{u'(i)}$, $1 \leq i \leq r$.

Define the differential forms

$$\omega_I(z;t) = d\log(t_{i_1} - t_{i_2}) \wedge \ldots \wedge d\log(t_{i_{q-1}} - t_{i_q}) \wedge d\log(t_{i_q} - z),$$
$$\omega_{\bar{I}}(z;t) = \omega_{I_1}(z_1;t) \wedge \ldots \wedge \omega_{I_n}(z_n;t)$$

Now comes the main definition:

$$\omega(z;t) = \sum_{\bar{I} \in \mathcal{P}(\lambda,n)} \sum_{\sigma \in \Sigma_{\bar{I}}} (-1)^{\sigma} \sigma(\omega_{\bar{I}}(z;t)) f_{\bar{I}} v \in M_{\Lambda-\lambda}$$

Here $\Sigma_{\bar{I}} \subset \Sigma_m$ denotes certain subgroup associated with $\bar{I}$ acting on variables $t_i$.

**5.5. Theorem**, [FMTV].

$$d\log\phi_\mu \wedge \omega = (\sum_i \mu^{(i)} dz_i + L) \wedge \omega \tag{5.5.1}$$

*where*

$$L = \sum_{i<j} \Omega^{(ij)} d\log(z_i - z_j).$$

This is a purely combinatorial identity proved by using the *Gelfand identity*. For $\mu = 0$ it is proven in [SV], Thm. 7.2.5".

**5.6. Theorem**, [FMTV]. *Let $\gamma(z)$ be a $z$-horizontal family of cycles in the fibres of the projection*

$$\mathbb{C}^m \times \mathbb{C}^n \longrightarrow \mathbb{C}^n,\ (t,z) \mapsto z.$$

*Then*

$$v(\mu,z) = \int_{\gamma(z)} \omega(z,t)\phi(\mu,z;t)^{1/\kappa} dt$$

*is a solution of both systems* (5.3.1) *and* (5.3.2).

For $\mu = 0$ this is the main result of [SV].

**Proof of the first half.** The identity (5.5.1) implies (since $d\log(\phi_\mu^{1/\kappa})$):

$$d\phi_\mu^{1/\kappa}\omega = (\sum_i \mu^{(i)} dz_i + L)\phi_\mu\omega. \tag{5.6.1}$$

Now apply to both sides the operator

$$I(?) = \int_\gamma (i_{\partial/\partial z_i}.?)dt \tag{5.6.2}$$

The result of applying to the RHS will be the RHS of (5.3.1) applied to $v$.

On the other hand, $d\omega = 0$, whence

$$d\phi_\mu^{1/\kappa} \wedge \omega = d(\phi_\mu^{1/\kappa} \wedge \omega);$$

decomposing $d = d_z + d_t$ and using

$$\int_\gamma (d_t?)dt = 0$$

we deduce that $I$ applied to the LHS of (5.6.1) will result in the LHS of (5.3.1). $\square$

**5.7. Example and questions.** Recall the Matsuo equations $(M)_\lambda$, cf. 4.2 and its rational limit.

Take $\mathfrak{g} = \mathfrak{sl}_n$, thus $W = S_n$, and $M = V^{\otimes n}$ where $V$ is the vector representation. Then take the KZ equation with values in the weight space $M_0$; this space is $n! = |W|$-dimensional. This equation may be identified with the rational limit of the Matsuo equation for the root system $A_{n-1}$, cf. [M], 6.3.

Thus we have for them the integral formulas for the solutions. Applying the (rational limit of) Matsuo theorem 4.10 we get the integral formulas for the zonal spherical functions.

Applying the limit near the walls of a Weyl chamber we get integral formulas (in the form of Selberg integrals) for the Harish-Chandra $c$-function in this case. It would be interesting to compare them with the original Harish-Chandra integrals and the GK formula.

What does the Matsuo transformation with the a solution of the full FMTV equation?

**5.7.** The FMTV equations are a particular case of more general KZ equations with irregular singularities, cf. [GL], [NS].

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