ON THE DESCENT OF DELIGNE GROUPOIDS

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A "standard" statement of the Formal Deformation Theory asserts that the ring of functions on the base of the universal deformation is canonically isomorphic (in characteristic zero) to the 0-th cohomology of a certain dg (or homotopy) Lie algebra associated with a deformation problem. The main point is that this Lie algebra has a "local" nature.

In arbitrary characteristics one should replace dg Lie algebras by cosimplicial formal schemes.

Some statements of this sort has been proven (cf. for example [GM1], [HS2]); but a big part of them one should rather consider as conjectures. The aim of this note is to formulate certain conjectures in homological algebra which imply "Standard statements". The most important points are in sections 3 and 5.

1. Sugawara Lie algebras

For details, see [HS1].

Throughout this note we fix a ground field k. Untill Section 4 we will suppose that char $k = 0$.

1.1. Let A be a commutative k-algebra. Let L be a $\mathbb{Z}$-graded A-module such that all graded components L^i are flat over A.

Let us consider the symmetric algebra $S^\bullet_A(L[1]) = \oplus_{n=1}^\infty S^n(L[1])$. It has a canonical structure of a cocommutative filtered dg A-coalgebra. The increasing filtration is defined by the formula

$$F_i S^\bullet_A(L[1]) = \oplus_{n=1}^i S^n(L[1]). \tag{1}$$

Suppose that an A-linear map

$$d : S^\bullet_A(L[1]) \longrightarrow S^\bullet_A(L[1])$$

of degree 1 is given, such that $d^2 = 0$ making $S^\bullet_A(L[1])$ a *dg coalgebra.*

The map d is a sum of components

$$d_{pq} : S^p_A(L[1]) \longrightarrow S^q_A(L[1])$$

of degree 1. We will suppose that

$$d_{pq} = 0 \ for \ q > p.$$

Given the above data, we will say, following Drinfeld, that a structure of *a Sugawara Lie algebra* over A on L is defined.

The symmetric algebra $S^{\bullet}_A(L[1])$ with the differential d will be called the *standard complex* of L and denoted $C(L)$. Thus, $C(L)$ is a cocommutative filtered dg A-coalgebra.

We will use the following notation for cohomology:

$$H_i^{Lie}(L) := H^{-i}(C(L)), \tag{2}$$

and

$$H^i_{Lie}(L) := \mathrm{Hom}_A(H_i^{Lie}(L), A), \tag{3}$$

$i \in \mathbb{Z}$.

1.2. Let us draw some consequences from this definition. Let us set

$$d_r = \sum_{q-p=r} d_{pq}.$$

We have $d = \sum_{r=0}^{\infty} d_r$. The equation $d^2 = 0$ becomes an infinite set of equations

$$d_0 d_0 = 0;\ d_0 d_1 + d + d_1 d_0 = 0,\ d_0 d_2 + d_1 d_1 + d_2 d_0 = 0,$$

etc.

First, each $S^p(L[1])$ becomes a complex with differential d_{pp}. We have

$$d_{pp} = S^p(d_{11})$$

for all p.

Next, one can see that d is defined uniquely by its components d_{i1}. More precisely, let $T^{\bullet}$ denote a functor of tensor algebra (over A). By definition, a functor of p-th symmetric power S^p is a quotient of T^p over the natural action of the symmetric group Σ_p. Let $x_1 \cdot \ldots \cdot x_p \in S^p$ denote the image of $x_1 \otimes \ldots \otimes x_p \in T^p$ under the natural projection.

Define maps

$$S^p \xrightarrow{\epsilon_p} T^p \xrightarrow{\pi_p} S^p$$

by the formulas

$$\epsilon_p(x_1 \cdot \ldots \cdot x_p) = \sum_{\sigma \in \Sigma_p} \sigma(x_1 \otimes \ldots \otimes x_p),$$

$$\pi_p(x_1 \otimes \ldots \otimes x_p) = \frac{1}{p!} x_1 \cdot \ldots \cdot x_p.$$

We have

$$d_{pq} = q \cdot \pi_q \circ \left(d_{p-q+1,1} \circ \pi_{p-q+1} \otimes 1^{\otimes q-1}\right) \circ \epsilon_p \tag{4}$$

for all $p \geq q$.

All these facts follow from the compatibility of d with the comultiplication.

1.3. Language of brackets. For $n \geq 2$, let us consider a map of graded A-modules of degree 1 given by the composition

$$L^{\otimes n}[n] \xrightarrow{\sim} L[1]^{\otimes n} \xrightarrow{\pi_n} S^n(L[1]) \xrightarrow{d_{n1}} L[1];$$

this is the same as a map of degree $2-n$

$$\delta_n : L^{\otimes n} \longrightarrow L \tag{5}$$

of degree n. We also introduce a differential d on L, i.e. a map of degree 1 $L \longrightarrow L$ equal to $-d_{11}$ (the change of the sign is due to the fact that d_{11} is a differential on $L[1]$.

These maps enjoy the following properties.

(i) *Skew symmetry.* $\delta_n \circ \sigma = (-1)^{|\sigma|} \cdot \delta_n$ for all $\sigma \in \Sigma_n$.

(ii) *Compatibility.*

$$d(\delta_n) = \sum_{i=2}^{n-1} (-1)^{n-i+1} i \cdot \delta_i \circ (\delta_{n-i+1} \otimes 1^{\otimes i-1}) \circ \mathrm{alt}_n,$$

where

$$d(\delta_n) := d_L \circ \delta_n - (-1)^{2-n} \delta_n \circ d_{L^{\otimes n}},$$

$$\mathrm{alt}_n := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{|\sigma|} \cdot \sigma : L^{\otimes n} \longrightarrow L^{\otimes n}.$$

We will use the notation $[x_1, \ldots, x_n]_n$ or simply $[x_1, \ldots, x_n]$ for $\delta_n(x_1 \otimes \ldots \otimes x_n)$.

Conversely, a complex L equipped with "brackets" δ_n, $n \geq 2$ satisfying properties (i) and (ii) above defines a structure of Sugawara Lie algebra on L. More precisely, δ_n define maps d_{n1}, and all other components are reconstructed from them by (4).

1.3.1. *Examples.* (i) $[\ ,\ ] : L \otimes L \longrightarrow L$ is a skew symmetric map commuting with differentials.

(ii) We have

$$\begin{aligned} d([x,y,z]) + [dx,y,z] + (-1)^{|x|}[x,dy,z] + (-1)^{|x|+|y|}[x,y,dz] = \\ = [[x,y],z] + (-1)^{|z|(|x|+|y|)}[[z,x],y] + (-1)^{|x|(|y|+|z|)}[[y,z],x] \end{aligned} \tag{6}$$

for $x \in L^{|x|}$, $y \in L^{|y|}$, $z \in L^{|z|}$.

This justifies

1.3.2. **Definition.** A Sugawara Lie A-algebra L such that all "higher brackets" δ_n, $n \geq 3$, are identically zero, is called *a dg Lie algebra* over A.

Of course, this coincides with the usual definition.

2. Deligne functor and its generalization

2.1. Let A be as in the previous Section, and L a dg Lie algebra over A.

Suppose that

(N) *the adjoint action of L^0 on all L^i, $i \geq 0$, is nilpotent.*

In particlular, the Lie algebra L^0 itself is nilpotent. Let $G(L^0)$ denote the corresponding nilpotent Lie group. Its elements will be denoted $\gamma = \exp(\lambda)$, $\lambda \in L^0$, the multiplication being defined by the Campbell-Hausdorff formula.

Following Deligne (see [GM2]), define a groupoid $\mathcal{G}(L)$ as follows. We define $Ob\ \mathcal{G}(L)$ as the set of all $x \in L^1$ satisfying Maurer-Cartan equation

$$dx + \frac{1}{2}[x, x] = 0 \tag{7}$$

The group $G(L^0)$ acts on this set by the rule

$$\exp(\lambda) \circ x = \exp(\text{ad } \lambda)(x) + \frac{1 - \exp(\text{ad } \lambda)}{\text{ad } \lambda}(d\lambda) \tag{8}$$

By definition, a map $x \longrightarrow y$ in $\mathcal{G}(L)$ is an element $\gamma \in G(L^0)$ such that $\gamma \circ x = y$. Composition is defined in an obvious way.

2.2. We want to generalize this definition. Let L be a Sugawara Lie algebra over A. Suppose that

(N1) *all higher brackets $[x_1, \ldots, x_i]$, $i \geq 3$, are identically zero provided all x_j except possibly one, belong to L^0.*

Then the bracket $[\ ,\ ]$ makes L^0 a Lie algebra and all L^i — L^0-modules. Let us suppose that

(N2) *the action of L^0 on all L^i, $i \geq 0$ is nilpotent.*

Finally, suppose that

(N3) *brackets $[\ , \ldots ,\ ]_i : (L^1)^{\otimes i} \longrightarrow L^2$ are zero for sufficiently large i.*

As before, denote by $G(L^0)$ the nilpotent Lie group corresponding to L^0.

Following Drinfeld, (cf. [Dr]), let us introduce the set $M(L)$ consisting of all $x \in L^1$ satisfying *generalized Maurer-Cartan equation*

$$dx + \sum_{i=1}^{\infty} [x, \ldots, x]_i = 0 \tag{9}$$

As before, consider an action of $G(L^0)$ on L^1 defined by the formula (8). We conjecture that

2.2.1. *this action preserves the subset $M(L)$.*

In the sequel, we suppose that this holds true. Then we can define a groupoid $\mathcal{G}(L)$ with the set of objects being $M(L)$, and morphisms $x \longrightarrow y$ being elements $\gamma \in G(L^0)$ such that $\gamma \circ x = y$.

Let us call a map of Sugawara Lie algebras $L \longrightarrow L'$ a *quasi-isomorphism* if the induced map $C(L) \longrightarrow C(L')$ is a quasi-isomorphism. We expect that

2.2.2. *functor $\mathcal{G}$ takes a quasi-isomorphism of Sugawara Lie algebras to an equivalence of groupoids.*

2.3. Deligne functor. Let $\mathcal{A}rtin_k$ denote the category of artinian k-algebras with the residue field k. Let L be a non-negatively graded Sugawara Lie algebra over k satisfying the assumption (N1) from the previous subsection (for $A = k$).

Given $A \in \mathcal{A}rtin_k$, set $\mathfrak{m} = \mathrm{rad}(A)$. Consider a complex $L_{\mathfrak{m}} := L \otimes_k \mathfrak{m}$. One introduces brackets on $L_{\mathfrak{m}}$ by a formula

$$[x_1 \otimes a_1, \dots, x_i \otimes a_i] = [x_1, \dots, x_i] \otimes a_1 \cdot \ldots \cdot a_i, \ x_j \in L, a_j \in \mathfrak{m}.$$

This way we get a structure of a Sugawara Lie algebra on $L_{\mathfrak{m}}$. Obviously, it satisfies conditions (N1)-(N3); consequently, a groupoid $\mathcal{G}(L_{\mathfrak{m}})$ is defined. Let us denote it $\mathcal{G}(L; A)$.

If L is fixed, we get a functor

$$\mathcal{G}(L) : \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids \tag{10}$$

to the category of groupoids. We can compose it with the functor of the set of connected components $\pi_0 : \mathcal{G}roupoids \longrightarrow \mathcal{S}ets$ and get a functor

$$\pi_0(\mathcal{G}(L)) : \mathcal{A}rtin_k \longrightarrow \mathcal{S}ets \tag{11}$$

Let us denote $\mathcal{A}rtin^{\leq i}$ a full subcategory of $\mathcal{A}rtin_k$ whose objects are algebras A with $\mathrm{rad}(A)^{i+1} = 0$.

Let us consider the standard complex $C(L)$. Recall that it is a filtered cocommutative dg coalgebra over k.

Before we go on, let us make a small digression.

2.3.1. *Unital coalgebras.* Let B be a dg k-coalgebra with a counit $\epsilon : B \longrightarrow k$. Let us call an element $1_B \in B$ *a unit* if $d1_B = 0$, $\Delta(1_B) = 1_B \otimes 1_B$, and $\epsilon(1_B) = 1_k$ (Δ denotes the comultiplication in B). We call a (dg) coalgebra with a unit *a unital (dg) coalgebra.*

A unit defines an evident decomposition $B = B^+ \oplus k \cdot 1_B$, $B^+ := \mathrm{Ker}\ \epsilon$. Following [HS2], 2.1.2, we remark that it defines also a canonical increasing filtration $\{F_iB\}$ by the rule

$$F_iB = \mathrm{Ker}(B \overset{\Delta^{(i+1)}}{\longrightarrow} B^{\otimes i+1} \longrightarrow (B^+)^{\otimes i+1}), \tag{12}$$

the last map being the projection. We call B *cocomplete* if this filtration is exhaustive.

We denote by $\mathcal{C}oalg_k$ a category of cocomplete unital dg coalgebras over k. (We consider a coalgebra as a dg coalgebra concentrated in degree 0.)

2.3.2. *Example.* If A is a complete local k-algebra with the residue field k, the dual k-vector space A^* is a cocomplete unital k-coalgebra. Canonical filtration (12) is dual to the filtration by the powers of the maximal ideal.

Let us return to our Sugawara Lie algebra L. The standard complex $C(L)$ is a cocomplete unital cocommutative dg coalgebra over k; the canonical filtration (12) coincides with (1).

2.4. Lemma. *For every $A \in \mathcal{A}rtin_k$ one has a natural isomorphism*

$$M(L_{\mathrm{m}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}oalg_k}(A^*, C(L)) \tag{13}$$

where $A^* = \mathrm{Hom}_k(A, k)$.

This lemma is due essentially to Drinfeld, see [Dr], Remark 1, p. 2, and was (we believe) the source of the definition of generalized Maurer-Cartan elements. For dg Lie algebras it is a standard fact going back to Quillen.

Let us describe (13) explicitely. For a k-algebra A and a k-coalgebra B, let us denote by $M(A \otimes B)$ the set of elements $z \in A \otimes B$ such that

$$(1_A \otimes \Delta_B)(z) = (\mu_A \otimes 1_{B\otimes B})(z \otimes z) \in A \otimes B \otimes B$$

where $\mu_A : A \otimes A \longrightarrow A$ is the multiplication in A, Δ_B — comultiplication in B.

One sees immediately that $M(A \otimes B)$ coincides with the set of all coalgebra maps $A^* \longrightarrow B$.

In view of this remark, to define a map (13), we have to define a map

$$M(L_{\mathrm{m}}) \longrightarrow M(A \otimes C(L)).$$

It is given by the formula

$$\alpha \mapsto \exp(-\alpha) \in A \otimes S^\bullet(L^1) \subset A \otimes C(L).$$

2.5. Now let us suppose that $H^i(L) = 0$ for $i \leq 0$. In this case $H_i^{Lie}(L) = 0$ for $i < 0$.

Taking the composition of (13) with a natural projection, we get a map

$$\begin{aligned} q : M(L_{\mathrm{m}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}oalg_k}(A^*, C(L)) \longrightarrow \\ \longrightarrow \mathrm{Hom}_{\mathcal{C}oalg_k}(A^*, \tau_{\geq 0}\, C(L)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}oalg_k}(A^*, H_0^{Lie}(L)) \end{aligned} \tag{14}$$

2.5.1. *Conjecture. The map q factors through the quotient $M(L_{\mathrm{m}})/G(L^0_{\mathrm{m}}) = \pi_0(\mathcal{G}(L; A))$.*

We have checked this for dg Lie algebras. Suppose this is true. Then we get a map (denoted by the same letter)

$$q : \pi_0(\mathcal{G}(L; A)) \longrightarrow \mathrm{Hom}_{\mathcal{C}oalg_k}(A^*, H_0^{Lie}(L)) \tag{15}$$

Let us introduce the notation $H^{0,Lie}(L)$ for the k-dual space to $H_0^{Lie}(L)$. It is naturally a complete local k-algebra with the residue field k.

Let us denote by $\mathcal{A}lg_k$ a category of complete local k-algebras with residue field k. We have naturally

$$\mathrm{Hom}_{\mathcal{C}oalg_k}(A^*, H_0^{Lie}(L)) \cong \mathrm{Hom}_{\mathcal{A}lg_k}(H^0_{Lie}(L), A).$$

Thus, we have a map

$$r : \pi_0(\mathcal{G}(L;A)) \longrightarrow \mathrm{Hom}_{\mathcal{A}lg_k}(H^0_{Lie}(L), A) \tag{16}$$

2.6. Conjecture. *The map r is an isomorphism. In other words, a functor $\pi_0(\mathcal{G}(L))$ is prorepresentable by a complete local k-algebra $H^0_{Lie}(L)$.*

(Recall that our running assumption is $H^i(L) = 0$ for $i \leq 0$).

Sketch of the proof for dg Lie algebras. First, check the statement for dg Lie algebras L such that $L^0 = 0$. This can be done directly.

For an arbitrary L, choose a k-subspace $'L^1 \subset L^1$ which is a direct complement to dL^0. Let

$$'L \hookrightarrow L \tag{17}$$

be a dg Lie subalgebra with $'L^i = 0$, for $i \leq 0$, $'L^1 =' L^1$ and $'L^i = L^i$ for $i > 2$. The embedding (17) is a quasi-isomorphism. Consequently, by [GM2], Thm 2.4, the induced map $\mathcal{G}('L;A) \longrightarrow \mathcal{G}(L;A)$ is an equivalence. On the other hand, it is clear that the induced map $C('L) \longrightarrow C(L)$ is a quasi-isomorphism. □

2.7. Remark. It is clear that if $A \in \mathcal{A}rtin_k^{\leq i}$ then the maps

$$\mathrm{Hom}_{\mathcal{C}oalg_k}(A^*, H^0(F_jC(L))) \longrightarrow \mathrm{Hom}_{\mathcal{C}oalg_k}(A^*, H_0^{Lie}(L))$$

are isomorphisms for $j \geq i$.

This means that the above conjecture implies that the restriction of $\pi_0\mathcal{G}(L)$ to $\mathcal{A}rtin_k^{\leq i}$ is pro-representable by $H^0(F_jC(L))^*$ for every $j \geq i$.

3. Descent

3.1. Let A be a commutative k-algebra. Let $L = \{L^{pq}\}$ be a cosimplicial dg Lie algebra over A. We agree that $p \geq 0$ will be a cosimplicial index, and q - "dg" index.

Consider the corresponding non-normalized double complex (with components L^{pq}); let $\Gamma(L)$ denote the associated simple complex.

According to [HS1], there exists certain "almost canonical" structure of a Sugawara Lie algebra on $\Gamma(L)$. "Almost canonical" means that it is unique up to a homotopy.

One can show that one can choose a Sugawara Lie structure on $\Gamma(L)$ in such a way that the condition (N1) from 2.2 is fulfilled. Let choose such a structure.

3.2. Let $\mathcal{G}^\bullet = \{\mathcal{G}^i\}$ be a cosimplicial groupoid. Let us define a groupoid $\Gamma(\mathcal{G}^\bullet)$ as follows. By definition, an object of $\Gamma(\mathcal{G}^\bullet)$ is an object $x \in \mathrm{Ob}\ \mathcal{G}^0$ together with an isomorphism

$$\gamma : d^0(x) \xrightarrow{\sim} d^1(x) \tag{18}$$

satisfying the cocycle condition

$$d^2(\gamma) \circ d^0(\gamma) = d^1(\gamma) \tag{19}$$

(both maps from $d^0d^0(x)$ to $d^2d^1(x)$).

A map from (x,γ_x) to (y,γ_y) is a map $\phi : x \longrightarrow y$ in $\mathcal{G}^0$ such that

(20) $$\gamma_y \circ d^0(\phi) = d^1(\phi) \circ \gamma_x$$

This groupoid is nothing but Grothendieck's "category of descent data" (see [FGA]).

3.3. Descent conjecture. Let us return to the situation 3.1. Let us suppose that all dg Lie algebras $L^{p\bullet}$, $p \geq 0$, satisfy the nilpotency condition (N) from 2.1. Then a cosimplicial groupoid $\mathcal{G}(L) = \{\mathcal{G}(L^{p\bullet}\}$ is defined.

On the other hand, consider "the" Sugawara Lie algebra $\Gamma(L)$. We have already chosen a Sugawara Lie structure on $\Gamma(L)$ in such a way that (N1) from 2.2 is satisfied. It follows from our assumptions that (N2) is also satisfied. Suppose that (N3) is also satisfied (it is probable that one can always choose such a Sugawara Lie structure). Then a groupoid $\mathcal{G}(\Gamma(L))$ is defined.

We conjecture that

there exists a natural equivalence

(21) $$\Gamma\mathcal{G}(L) \xrightarrow{\sim} \mathcal{G}\Gamma(L)$$

3.4. Let us verify the above conjecture "up to the second order". In the situtation 3.1, suppose that $L^{pq} = 0$ for $q \neq 0$; let us denote L^{p0} simply L^p.

By assumption, each L^p is a nilpotent Lie algebra. The associated groupoid $\mathcal{G}(L^p)$ is essentially the corresponding Lie group $G(L^p)$. More precisely, Ob $\mathcal{G}(L^p)$ consists of one element, and Mor $\mathcal{G}(L^p) = G(L^p)$.

Objects of $\Gamma\mathcal{G}(L)$ are elements $\exp(\lambda)$, $\lambda \in L^1$ such that

(22) $$\exp(d^2\lambda) \cdot \exp(d^0\lambda) = \exp(d^1\lambda)$$

A morphism in $\Gamma\mathcal{G}(L)$ has a form

$$\exp(\phi) : \exp(\lambda) \longrightarrow \exp(\mu)$$

where $\phi \in L^0$ is such that

$$\exp(\mu) \cdot \exp(d^0\phi) = \exp(d^1\phi) \cdot \exp(\lambda),$$

i.e.

(23) $$\exp(\mu) = \exp(d^1\phi) \cdot \exp(\lambda) \cdot \exp(-d^0\phi)$$

In the sequel, let us drop in all formulas all brackets of order ≥ 3. We have Campbell-Hausdorff formula:

(24) $$\exp(a) \cdot \exp(b) = \exp(a + b + \frac{1}{2}[a,b])$$

Consequently, (23) becomes

(25) $$\exp(\mu) = \exp(d^1\phi + \lambda - d^0\phi + \frac{1}{2}[d^1\phi, \lambda] - \frac{1}{2}[d^1\phi, d^0\phi] - \frac{1}{2}[\lambda, d^0\phi]) =$$
$$= \exp(d^1\phi - d^0\phi + \lambda + \frac{1}{2}[d^0\phi + d^1\phi, \lambda] + \frac{1}{2}[d^0\phi, d^1\phi])$$

A part of a Sugawara Lie structure on $\Gamma(L)$ we are interested in is the second bracket

$$\delta_2 : \Gamma(L) \otimes \Gamma(L) \longrightarrow \Gamma(L).$$

It may be defined by a skew-symmetrization of the Alexander-Whitney multiplication: for $x^i \in L^i$ we may define

(26)
$$\delta_2(x^p, x^q) = \frac{1}{2}([d^{p+q} \circ \dots d^{p+1}x^p, d^0 \circ \dots \circ d^0 x^q] - (-1)^{pq}[d^{q+p} \circ \dots d^{q+1}x^q, d^0 \circ \dots \circ d^0 x^p])$$

In particular,

(27) $$\delta_2(x^0, x^1) = \frac{1}{2}[d^0x^0 + d^1x^0, x^1]$$

and

(28) $$\delta_2(x^1, y^1) = \frac{1}{2}([d^2x^1, d^0y^1] + [d^2y^1, d^0x^1])$$

Now we can compare groupoids $\mathcal{G}\Gamma(L)$ and $\Gamma\mathcal{G}(L)$. By definition, Ob $\mathcal{G}\Gamma(L)$ is the set of $\lambda \in L^1$ such that $d\lambda + \frac{1}{2}\delta_2(\lambda, \lambda) = 0$, i.e.

$$d^0\lambda - d^1\lambda + d^2\lambda + \frac{1}{2}[d^2\lambda, d^0\lambda] = 0.$$

On the other hand, Ob $\Gamma\mathcal{G}(L)$ is the set of all $\exp(\lambda)$, $\lambda \in L^1$ such that

$$d^2\lambda + d^0\lambda - \frac{1}{2}[d^2\lambda, d^0\lambda] = d^1\lambda$$

(see (22)). Consequently, the rule $\lambda \mapsto \exp(\lambda)$ establishes *isomorphism*

(29) $$\text{Ob } \mathcal{G}\Gamma(L) \xrightarrow{\sim} \text{Ob } \Gamma\mathcal{G}(L)$$

Similarly, a morphism in $\Gamma\mathcal{G}(L)$ is an element $\exp(\phi)$, $\phi \in L^0$ such that

$$\mu = \lambda + d^1\phi - d^0\phi + \frac{1}{2}[d^0\phi + d^1\phi, \lambda] + \frac{1}{2}[d^0\phi, d^1\phi]$$

(cf. (25)), and a map

$$\exp(\phi) : \lambda \longrightarrow \mu$$

in $\mathcal{G}\Gamma(L)$ is defined by an element $\phi \in L^0$ such that

$$\mu = \lambda + \delta_2(\phi, \lambda) - d\phi - \frac{1}{2}\delta_2(\phi, d\phi)$$

— here we have used the formula (8) up to the second order. In other words, we rewrite this as

$$\mu = \lambda + \frac{1}{2}[d^0\phi + d^1\phi, \lambda] - d^0\phi + d^1\phi - \frac{1}{4}[d^0\phi + d^1\phi, d^0\phi - d^1\phi] =$$
$$= \lambda - d^0\phi + d^1\phi + \frac{1}{2}[d^0\phi + d^1\phi, \lambda] + \frac{1}{2}[d^0\phi, d^1\phi]$$

Consequently, the rule $\phi \mapsto \exp(\phi)$ establishes an isomorphism

$$\text{Mor } \mathcal{G}\Gamma(L) \xrightarrow{\sim} \text{Mor } \Gamma\mathcal{G}(L)$$

compatible with (29).

Thus, we have checked the Descent conjecture up to the order 2. Note that the groupoids turned out to be *isomorphic*, not merely equivalent.

3.4.1. **Question.** *Is it possible to introduce a Sugawara Lie structure on $\Gamma(L)$ in such a way that the groupoids $\mathcal{G}\Gamma(L)$ and $\Gamma\mathcal{G}(L)$ are isomorphic?*

As we can guess from the above, an isomorphism could be essentally identity map!

4. Applications

In this Section we indicate how to deduce from the Descent conjecture the description of the universal formal deformation spaces for two natural deformation problems.

(a) **Deformations of schemes.**

4.1. Local problem. Let B be a commutative k-algebra, smooth over k. It defines a deformation functor

$$\mathcal{D}ef_B : \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids \tag{30}$$

as follows. For $A \in \mathcal{A}rtin_k$, Ob $\mathcal{D}ef_B(A)$ is the set of all liftings of B to a flat A-algebra; morphisms are isomorphisms of liftings inducing indentity on B.

Now let us consider a free dg Lie coalgebra spanned by $B[1]$, and let us denote by L_B the graded Lie algebra of its derivations. The multiplication on B, $S^2(B) \longrightarrow B$ defines a derivation of degree 1, i.e. an element of $\mu \in L_B^1$ such that the bracket $[\mu, \cdot]$ has square 0 and defines a structure of a dg Lie algebra on L_B.

4.1.1. **Claim.** *The deformation functor $\mathcal{D}ef_B$ is naturally isomorphic to the Deligne functor $\mathcal{G}(L_B)$ associated to L_B.*

This is more or less tautology.

We can expect that

4.1.2. *$H^0(L_B) = Der_k(B, B)$ and $H^i(L_B) = 0$ for $i \neq 0$.*

We suppose that this is true in the sequel.

4.2. Glueing. Now suppose we have a smooth proper scheme X over k. Let $\mathcal{T}_X$ denote the tangent sheaf. Suppose that $H^0(X;\mathcal{T}_X)=0$. Similarly to the affine case, X defines a deformation functor

$$\mathcal{D}ef_X : \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids.$$

Let us denote $\mathrm{Def}_X = \pi_0\mathcal{D}ef_X$. In our assumptions Def_X is prorepresentable by a complete local k-algebra R_X.

We can sheafify the construction of the previous subsection, and get a sheaf of dg Lie algebras $\mathcal{L}_X$ on X such that $H^0(\mathcal{L}_X)=\mathcal{T}_X$ and $H^i(\mathcal{L}_X)=0$ for $i\neq 0$.

Choose an affine open covering $\mathcal{U}=\{U_i\}$ of X. Let $\mathcal{L}_{\mathcal{U}}$ denote the Čech cosimplicial dg Lie algebra:

$$\mathcal{L}^p_{\mathcal{U}} = \prod_{i_0,\dots,i_p} \Gamma(U_{i_0}\cap\ldots\cap U_{i_p};\mathcal{L}_X).$$

Consider the corresponding Sugawara Lie algebra $\Gamma(\mathcal{L}_{\mathcal{U}})$. Its quasi-isomorphism class does not depend on a choice of $\mathcal{U}$ (cf. [HS1], §6). We will denote it $R\Gamma(X;\mathcal{T}_X)$.

This Sugawara Lie algebra should be isomorphic to "the" dg Lie algebra denoted by the same symbol in [HS2].

We will use the notation

$$H^i_{Lie}(X;\mathcal{T}_X) := H^i_{Lie}(R\Gamma(X;\mathcal{T}_X)).$$

From the previous remarks follows that the functor $\mathcal{D}ef_X$ is isomorphic to $\Gamma\mathcal{G}(\mathcal{L}_{\mathcal{U}})$. Consequently, from the Descent conjecture and 2.6 follows

4.3. Corollary. *One has a canonical isomorphism of local k-algebras*

$$R_X \xrightarrow{\sim} H^0_{Lie}(X;\mathcal{T}_X)$$

(b) **Deformations of group representations.**

4.4. Let G be a group, V a finite dimensional k-vector space,

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

a representation.

Let

$$\mathcal{D}ef_\rho : \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids$$

denote a functor which assigns to an algebra A a groupoid $\mathcal{D}ef_\rho(A)$ whose objects are liftings of ρ to a representation

$$\rho_A : G \longrightarrow \mathrm{GL}(V_A)$$

where $V_A = V\otimes_k A$; for technical reasons, let us suppose that the determinant $\det\ \rho$ is not deformed. Morphisms are isomorphisms of liftings inducing identity on ρ.

We denote by

$$\mathrm{Def}_\rho : \mathcal{A}rtin_k \longrightarrow \mathcal{S}ets$$

the composition of $\mathcal{D}ef_\rho$ with the functor π_0.

Under mild assumptions on ρ, Def_ρ is prorepresentable by a complete local k-algebra R_ρ.

Let

$$\mathrm{Ad}_\rho : G \longrightarrow \mathrm{GL}(\mathfrak{sl}(V))$$

denote the adjoint representation $\mathrm{Ad}_\rho(g)(f)(x) = gf(g^{-1}x)$. So $\mathfrak{gl}(V)$ is a G-module. Suppose that $H^0(G; \mathfrak{sl}(V)) = 0$.

Consider a cosimplicial Lie algebra $C(G; \mathfrak{sl}(V))$ whose associated complex is a complex of cochains of G with coefficients on $\mathfrak{sl}(V)$ with the action of G defined by Ad_ρ. Let $R\Gamma(G; \mathrm{Ad}_\rho)$ denotes "the" Sugawara Lie algebra $\Gamma(C(G; \mathfrak{sl}(V)))$ and denote the cohomology

$$H^i_{Lie}(G; \mathrm{Ad}_\rho) := H^i_{Lie}(R\Gamma(G; \mathrm{Ad}_\rho))$$

Now, from Descent conjecture and 2.6 follows

4.5. Corollary. *One has a canonical isomorphism of local k-algebras*

(31) $$R_\rho \xrightarrow{\sim} H^0_{Lie}(G; \mathrm{Ad}_\rho)$$

5. Multiplicative picture

5.1. Multiplicative version of Deligne construction. Let $G = \{G^i\}$ be a cosimplicial group. Let us assign to it a groupoid $\mathcal{G}(G)$ as follows. We define $\mathrm{Ob}\ \mathcal{G}(G)$ as the set of all $\gamma \in G^1$ such that

$$d^2\gamma \cdot d^0\gamma = d^1\gamma.$$

The group G^0 acts on this set by the rule

$$\beta \circ \gamma = d^1\beta \cdot \gamma \cdot d^0\beta^{-1}.$$

By definition, a map $\gamma \longrightarrow \gamma'$ is an element $\beta \in G^0$ such that $\gamma' = \beta \circ \gamma$. The composition is defined in the evident way. (It is a particular case of the descent construction used in 3.2).

5.2. Compatibility of the above construction with the "Lie algebraic" construction in characteristic 0 is equivalent to Descent conjecture.

5.3. Let us fix a ground field k of *arbitrary* characteristic. Let $G = \{G^i\}$ be a cosimplicial formal group. Applying the previous construction, we get a functor

(32) $$\mathcal{G}(G) : \mathit{Artin}_k \longrightarrow \mathcal{G}\mathit{roupoids}$$

which sends A to $\mathcal{G}(G(A))$.

Similarly to 2.6, we expect that $\pi_0(\mathcal{G}(G))$ is prorepresentable. The following answer was proposed (in different terms) by Beilinson and Mazur.

5.4. Let us consider a simplicial cosimplicial formal scheme $BG = \{BG^p_q\}$ whose component at a fixed cosimplicial index p is the simplicial formal scheme BG^p — the standard model of the classifying space of G^p.

Let $G^p = \mathrm{Spf}(W_p)$ where W_p is a complete k-algebra. Then $BG^p_q = \mathrm{Spf}(W_p^{\otimes q})$. The k-algebras $W_p^{\otimes q}$ form a cosimplicial simplicial k-algebra — "ring of functions on BG".

Let $\mathcal{W}_G$ denote the simple complex associated with the corresponding bicomplex. The cohomology $H^\bullet(\mathcal{W}_G)$ form a commutative dg algebra.

5.5. Conjecture. *Suppose that*

$$\mathrm{Ker}(d^0, d^1) : G^0 \rightrightarrows G^1$$

is trivial. Then $\pi_0(\mathcal{G}(G))$ is prorepresentable by the complete local k-algebra $H^0(\mathcal{W}_G)$.

5.6. Suppose that k has caracteristic 0, and $L = \{L^i\}$ a cosimplicial Lie algebra over k. For each i consider the standard complex $C(L^i)$. Taken together, these complexes form a cosimplicial complex; let us denote $C(L)$ the corresponding simple complex.

On the other hand, consider the standard complex $C(\Gamma(L))$ of "the" Sugawara Lie algebra $\Gamma(L)$ associated to L.

It is very reasonable to expect that

5.6.1. *There exists a unique up to a homotopy natural quasiisomorphism $C(L)$ with $C(\Gamma(L))$.*

If this is true, the "multiplicative" picture of this section is consistent with the "additive" one of previuos sections.

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