FOURIER - SATO TRANSFORM

AND LUSZTIG SYMMETRIES

Hyperbolic calculus

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Contents

§1. Quantum group of a hyperplane arrangement

1.1. A quantum group of a hyperplane arrangement. Let $V = \mathbb{C}^n$, $\mathcal{H} = \{H_i\}$ a finite collection of hyperplanes $H_i : f_i(x) = 0$ where f_i are linear functions with *real* coefficients.

This arrangement gives rise to a complex stratification $\mathcal{S}_{\mathbb{C}}$ of V and to a real stratification $\mathcal{S}$ of $V_{\mathbb{R}} = \mathbb{R}^n$.

For $S, S' \in \mathcal{S}$ we write $S \leq S'$ if $S \subset \bar{S}'$.

Let us call two strata S, S' *neighbours* if $\dim S = \dim S'$, $L(S) = L(S')$, and there exists S'' (*a wall* between S and S') $S'' \leq S, S'' \leq S'$, $\dim S'' = \dim S - 1$.

A triple of real strata (S_1, S_2, S_3) is called a *collinear triple* if there exist $x_i \in S_i$ lying on the same line, and such that $x_2 \in [x_1, x_3]$.

Let us define a category $Hyp(\mathcal{S})$ whose objects will be called *hyperbolic sheaves over* $\mathcal{S}$, which are the following linear algebra data:

— a collection of complex vector spaces $E = \{E_S,\ S \in \mathcal{S}\}$;

— for each $S \leq S'$ we have two linear maps: $\gamma_{SS'} : E_S \longrightarrow E_{S'}$ (generalization), and $\delta_{S'S} : E_{S'} \longrightarrow E_S$ (boundary), transitive wrt $S \leq S' \leq S''$.

They should also sarisfy the following properties:

(i) (idempotence) $\gamma_{SS'}\delta_{S'S} = \mathrm{Id}(E_{S'})$.

Let S, S' be arbitrary strata. Choose a stratum $S'' \leq S, S'' \leq S'$, and define a *flopping map*

$$\phi_{SS'} := \gamma_{S''S'}\delta_{SS''}.$$

Due to (i) this definition does not depend on S''.

(ii) (collinearity) If (S, S', S'') is a collinear triple,

$$\phi_{SS''} = \phi_{S'S''}\phi_{SS'}.$$

(iii) (invertibility) If S and S' are neighbours, $\phi_{SS'}$ is an isomorphism.

In other words, the category $Hyp(\mathcal{S})$ is a category $Rep(\mathcal{A}(\mathcal{S}))$ of representations in *Vect* of certain associative algebra $\mathcal{A}(\mathcal{S})$.

1.2. Let $\mathcal{M} \in Perv(V; \mathcal{S}_{\mathbb{C}})$. Let $i : V_{\mathbb{R}} \hookrightarrow V$. One can show that

$$Ri^!(\mathcal{M}) \in \mathcal{D}^b_c(V_{\mathbb{R}}, \mathcal{S})$$

which is a priori a complex of sheaves, is actually a single sheaf. Denote by

$$E(\mathcal{M})_A = \Gamma(A; Ri^!(\mathcal{M})) \in Vect,\ \ A \in \mathcal{S}$$

its fibers, and by

$$\gamma_{AB} : E(\mathcal{M})_A \longrightarrow E(\mathcal{M})_B,\ \ A \leq B$$

the generalization maps.

One can show that

$$E(\mathcal{M}^*)_A \stackrel{\sim}{=} E(\mathcal{M})^*_A$$

where $\mathcal{M}^*$ is the Verdier dual sheaf, whence maps

$$\delta_{BA}(\mathcal{M}) := \gamma_{AB}(\mathcal{M}^*)^*$$

1.3. Theorem, [KS]. *The association $\mathcal{M} \mapsto E(\mathcal{M})$ gives rise to a functor*

$$E: \ Perv(V; \mathcal{S}_S) \longrightarrow Hyp(\mathcal{S})$$

which is an equivalence of categories.

1.4. $R\Gamma(V;\mathcal{M})$ AND $R\Gamma_c(V;\mathcal{M})$ IN TERMS OF $E(\mathcal{M})$.

Suppose our arrangement is central, i.e. $\{0\}$ is one of its faces. Let $\mathcal{S}_i \in \mathcal{S}$ denote the subset of faces of dimension i.

If $E(\mathcal{M}) = (E_A, \gamma, \delta)$ then

$$R\Gamma_c(V;\mathcal{M}) : 0 \longrightarrow E_0 \longrightarrow \oplus_{\ell\in\mathcal{S}_1} E_\ell \longrightarrow \ldots,$$

the differential being γ's with signs. The complex sits in nonnegative degrees.

Dually,

$$R\Gamma(V;\mathcal{M}) : \ldots \longrightarrow \oplus_{\ell\in\mathcal{S}_1} E_\ell \longrightarrow E_0 \longrightarrow 0,$$

the differential being δ's with signs. The complex sits in negative degrees.

1.5. Elementary version: the braid groupoid. Let

$$U := V \setminus \cup_i H_i$$

The fundamental groupoid $\Pi(U)$ admits the following description:

Objects: *chambers*, i.e. strata C of maximal dimension.

Morphisms.

Generators: for each two chambers C, C' we have one generator $\phi_{CC'} : C \longrightarrow C'$.

Relations: for each collinear triple (C, C', C''),

$$\phi_{CC''} = \phi_{C'C''}\phi_{CC'}.$$

§2. Fourier - Sato transform

2.1. Fourier - Sato transformation. Cf. [KaScha]. Let V be a complex finite dimensional vector space, V^* its complex dual,

$$P = \{(x, \ell) \in V \times V^* |\ \Re\ell(x) \geq 0\} \subset V \times V^*.$$

Let $p_1 : P \longrightarrow V,\ \ p_2 : P \longrightarrow V^*$ be the projections.

Let $Perv(V)$ denote the abelian category of *monodromic* perverse sheaves over V.

The Fourier - Sato transformation

$$FS : Perv(V) \xrightarrow{\sim} Perv(V^*)$$

is defined by

$$FS(\mathcal{M}) = p_{2*}p_1^!\mathcal{M} = p_{2!}p_1^*\mathcal{M}$$

see [KaScha], Definition 3.7.8.

2.1.1. Fourier - Sato and vanishing cycles. Let $f \in V^*$, $V_f = \{f^{-1}(0) \subset V\}$,

$$i_{\{f\}} : \{f\} \hookrightarrow V^*.$$

We have the vanishing cycles functor

$$\Phi_f : \ Perv(V) \longrightarrow Perv(V_f).$$

Then the fiber

$$i^*_{\{f\}} FS(\mathcal{M}) = R\Gamma_c(V_f; \mathcal{M}).$$

2.2. Let us return to the framework of §1. Let $\mathcal{H}^*$ denote an arrangement in V^* whose hyperplanes are orthogonals $H^* = \ell^\perp$ where $\ell = \cap H_j \subset V$ is a line. Let $\mathcal{S}^*$ denote the corresponding stratification of $V^*_{\mathbb{R}}$.

Warning: $\mathcal{H} \subset \mathcal{H}^{**}$, but $\mathcal{H}^{**}$ is much bigger if $n = \dim V > 2$.

The Fourier - Sato transformation acts as

$$FS : \mathcal{D}^b_c(V, \mathcal{S}) \longrightarrow \mathcal{D}^b_c(V^*, \mathcal{S}^*).$$

2.2.1. Relation to vanishing cycles. For example, let

$$V^{*o} = V \setminus \cup_{H^* \in \mathcal{H}^*} H^*$$

A point in V^{*o} is nothing else as a linear function $f : V \longrightarrow \mathbb{C}$ in general position to $\mathcal{H}$.

For $\mathcal{M} \in \mathcal{D}^b_c(V, \mathcal{S})$ let

$$\Phi_f(\mathcal{M}) \in D^b(f^{-1}(0))$$

denote the sheaf of vanishing cycles. It is concentrated at $0 \in f^{-1}(0)$, and the fiber

$$\Phi_f(\mathcal{M})_0 = FS(\mathcal{M})_f$$

Thus, over V^{*o} the sheaf $FS(\mathcal{M})$ decribes the variation of the space of vanishing cycles when a function f varies.

2.3. Now let $\mathcal{M} \in Perv(V, \mathcal{S}_{\mathbb{C}})$, $E = E(\mathcal{M}) \in Hyp(\mathcal{S})$, $\mathcal{M}^\vee = FS(\mathcal{M}) \in Perv(V^*; \mathcal{S}^*)$.

Let us describe $E^\vee := E(\mathcal{M}^\vee)$ in terms of E.

First let $A^\vee \in \mathcal{S}^*$ be a chamber. Choose $f \in A^\vee$, and denote

$$V_f^+ = \{x \in V_{\mathbb{R}} |\ f(x) > 0\}.$$

Consider a complex

$$E(A^\vee)^\bullet :$$

$$0 \longrightarrow E_{\{0\}} \longrightarrow \oplus_{B \subset V_f^+, \dim B = 1} E_B \longrightarrow \oplus_{B \subset V_f^+, \dim B = 2} E_B \longrightarrow \dots \tag{2.3.1}$$

concentrated in degrees ≥ 0. The boundary maps are γ's with appropriate signs.

Dually, we can consider a complex

$$E(A^\vee)^\bullet_\delta : 0 \longrightarrow E_{\{0\}} \longleftarrow \oplus_{B \subset V_f^+, \dim B = 1} E_B \longleftarrow \oplus_{B \subset V_f^+, \dim B = 2} E_B \longleftarrow \dots$$

concentrated in degrees ≤ 0, whose boundary maps are δ's with appropriate signs.

2.4. Main Acyclicity Theorem.

(i) *The complexes* $E(A^\vee)^\bullet$ *and* $E(A^\vee)^\bullet_\delta$ *are acyclic except for degree* 0.

(ii) *Its zeroth cohomology computes the vanishing cycles*

$$E(A^\vee) := H^0(E(A^\vee)^\bullet) \overset{\sim}{=} E^\vee_{A^\vee} \overset{\sim}{=} H^0(E(A^\vee)^\bullet_\delta)$$

2.5. Now let $A^\vee \in S^*$ be an arbitrary face, $A^\vee \neq 0$.

As previously, choose $f \in A^\vee$, and consider a complex similar to (2.3.1):

$$E(A^\vee)^\bullet :$$

$$0 \longrightarrow E_{\{0\}} \longrightarrow \oplus_{B\subset V_f^+, \dim B=1} E_B \longrightarrow \oplus_{B\subset V_f^+, \dim B=2} E_B \longrightarrow \dots \tag{2.5.1}$$

concentrated in degrees ≥ 0.

The boundary maps are γ's with signs.

2.6. Theorem. (i) *The complex $E(A^\vee)^\bullet$ is acyclic except for degree* 0*. Its zeroth cohomology computes*

$$E(A^\vee) := H^0(E(A^\vee)^\bullet) \overset{\sim}{=} E^\vee_{A^\vee}$$

(ii)

$$E^\vee(0) = E_0.$$

Part (ii) is a version of Braden's theorem.

2.8. EXAMPLE: THREE LINES ON THE PLANE

2.9. EXAMPLE WITH LIE OPERAD

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§3. Lusztig symmetries and vanishing cycles

3.1. Braid group actions. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $R \subset \mathfrak{h}^*$ the set of roots with respect to $\mathfrak{h}$. Let us fix a Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$; let $\{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ be the corresponding set of simple roots.

Let L be a finite dimensional $\mathfrak{g}$-module. The Weyl group W of $\mathfrak{g}$ acts on the set of weights of L.

This action may be lifted to an action on L of an extended Weyl group ("Tits - Weyl group") defined by Tits, which is an extension

$$0 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^r \longrightarrow \tilde{W} \longrightarrow W \longrightarrow 1$$

where $r = \dim \mathfrak{h}$, cf. [Tits].

This action may be q-deformed.

Let $q \in \mathbb{C}^*$; consider the quantum deformation $U_q\mathfrak{g}$ of $U\mathfrak{g}$. Let us suppose for simplicity that q is generic (not a root of unity).

Let

$$\mathfrak{h}^o = \mathfrak{h} \setminus \cup_{\alpha \in R}\ \alpha^\perp$$

The braid group Br of R (resp. the pure braid group PBr) is defined by

$$Br = \pi_1(\mathfrak{h}^o/W),\ \ PBr = \pi_1(\mathfrak{h}^o)$$

They fit into an extentsion

$$1 \longrightarrow PBr \longrightarrow Br \stackrel{\pi}{\longrightarrow} W \longrightarrow 1.$$

According to Lusztig [L], Prop. 41.2.4, a finite dimensional module L over $U_q\mathfrak{g}$ is acted upon by Br.

The generators $T_i, i \in I$, of Br act as certain combinations of the operators $E_i, F_i \in U_q\mathfrak{g}$.

For $b \in Br$ and a weight subspace $L_\mu \subset L$, $\mu \in \mathfrak{h}^*$,

$$b(L_\mu) \subset L_{\pi(\mu)},$$

whence the pure braid group PBr respects weight subspaces $L_\mu \subset L$.

3.2. Vanishing cycles and weight components. For a dominant integral weight λ, let $L(\lambda)$ be the irreducible $U_q\mathfrak{g}$-module with highest weight λ.

Let $J \subset I$; $\beta_J = \sum_{i\in J} \alpha_i$.

We are going to describe geometrically the weight subspace

$$L(\lambda)_J := L(\lambda)_{\lambda_J},\ \lambda_J = \lambda - \beta_J.$$

Let us consider the space $\mathbb{A}^J = \mathbb{C}^J$ with coordinates $t_j,\ j \in J$. Inside it, let us consider hypersurfaces

$$H_j = \{t_j = 0\},\ H_{jk} = \{t_j = t_k\} \subset \mathbb{A}^J,$$

and the open complement

$$\mathbb{A}^{Jo} = \mathbb{A}^J \setminus (\cup H_j) \setminus (\cup H_{kl}).$$

We have a one-dimensional local system $\mathcal{L}_J$ over $\mathbb{A}^{Jo}$ with monodromies

$q^{-(\lambda,\alpha_j)}$ around H_j ,

and

$q^{(\alpha_j,\alpha_{j'})}$ around $H_{jj'}$.

Let $\mathcal{M}_J$ denote a perverse sheaf over $\mathbb{A}^J$, the intermediate extension of $\mathcal{L}_J$.

Consider a function

$$f: \ \mathbb{A}^J \longrightarrow \mathbb{A}^1 = \mathbb{C}, \ f((t_j)) = \sum_J t_j.$$

The sheaf of vanishing cycles

$$\Phi_f(\mathcal{M}_J) \in Perv(f^{-1}(0))$$

is supported at the origin $0 \in f^{-1}(0)$

One of the main results of [BFS] establishes an isomorphism of vector spaces

$$\Phi_f(\mathcal{M}_J)_0 \stackrel{\sim}{=} L(\lambda)_J.$$

More generally, for any $J' \subset J$, the component $L(\lambda)_{J'}$ is realized as an appropriate space of vanishing cycles living on a subspace $\mathbb{A}^{J \setminus J'} \subset \mathbb{A}^J$.

The operators

$$\mathrm{var} = E_i : \ L(\lambda)_K \overset{\longleftarrow}{\longrightarrow} L(\lambda)_{K \setminus \{i\}} \ : F_i = \mathrm{can}$$

of the quantum group are being identified with the operators var and can acting on vanishing cycles.

A similar description holds true for any weight component (one has to use the spaces of divisors on $\mathbb{A}^1$), and for any finite dimensional $U_g \mathfrak{g}$-module.

3.3. Geometric braid group action. Now let us vary the function f. Let

$$\mathfrak{h}_J = \oplus_{j\in J}\mathbb{C}\alpha_j$$

(recall that we have identified $\mathfrak{h}$ with $\mathfrak{h}^*$).

For each

$$c = \sum_J c_j\alpha_j \in \mathfrak{h}_J,$$

consider a function

$$f_c : \mathbb{A}^\beta \longrightarrow \mathbb{A}^1,\ f(t_j) = \sum_{j\in J} c_j t_j.$$

For generic c again the sheaf $\Phi_{f_c}(\mathcal{M}_\beta)$ will be concentrated at $0 \in f_c^{-1}(0)$, and when c varies, we get a local system of vector spaces over some open part of $\mathfrak{h}_J$, whose fiber at c is $\Phi_{f_c}(\mathcal{M}_\beta)_0$.

One can show that for q sufficiently close to 1, this local system is well defined over $\mathfrak{h}_J^0$ (a priori it has singularities at a bigger set of hyperpanes).

3.3.1. Theorem. *Let q be formal at the infinitesimal neighbourhood of* 1*. The resulting representation of* $\pi_1(\mathfrak{h}_J^\circ) \subset PBr(\mathfrak{g})$ *on* $\Phi_f(\mathcal{M}_\beta)_0 = L(\lambda)_\mu$ *is equivalent to the Lusztig representation.*

3.3.2. Conjecture. *The same holds true for any q.*

3.4. Comments. Relation to the theory from §1.

Operators

$$\delta \longleftrightarrow E_i$$

$$\gamma \longleftrightarrow F_i$$

§4. Combinatorics of Young tableaux

and duality for representations of S_n and $GL_n(\mathbb{F}_q)$

4.1. Representations of symmetric groups. Let $A = \mathbb{C}[S_{n+1}]$.

Denote by:

$[n] = \{1, \dots, n\}$; Sub_n the set of subsets of $[n]$;

$\mathcal{P}_{n+1}$ the set of partitions of $[n+1]$ = the set of Young diagrams with $n+1$ boxes;

$\mathcal{T}_\lambda$ the set of standard Young tableaux of shape λ, for $\lambda \in \mathcal{P}_{n+1}$;

$$\mathcal{T}_{n+1} = \cup_{\lambda \in \mathcal{P}_{n+1}} \text{ for } \mathcal{T}_\lambda.$$

For each $T \in \mathcal{T}_{n+1}$ we have the corresponding Young symmetrizer $y_T \in A$,

$$y_T^2 = y_T, \ \ y_T y_{T'} = 0 \ \ T \neq T'.$$

The left ideal

$$L_T := A \cdot y_T \subset A \tag{4.1.1}$$

is an irreducible representation of S_{n+1}; $L_T \overset{\sim}{=} L_{T'}$ iff T and T' have the same shape.

We have

$$A = \oplus_{T \in \mathfrak{T}_{n+1}} L_T, \tag{4.1.2}$$

cf. [W], Theorem 4.3.J.

4.2. A.Postnikov's descent map and projectors. Let T be a standard Young tableau of shape λ. We say that an index i in $\{1, ..., n\}$ is *a descent of* T if the number $i+1$ is located in T below the number i (that is, the row containing $i+1$ is below the row containing i).

Let $Des(T)$ denote the set of all descents of T.

For example, for $T = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 8 & 9 \\ \hline 3 & 5 & 7 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}$ we have $Des(T) = \{2, 4, 5\}$.

This way we get a map

$$Des : \mathcal{T}_{n+1} \longrightarrow Sub_n. \tag{4.2.1}$$

For each $I \in \mathcal{P}_n$, we denote

$$\mathcal{T}_I := Des^{-1}(I),$$

and we define Postnikov projectors

$$p'_I = \sum_{T \in \mathcal{T}_I} y_T \in A, \tag{4.2.2}$$

and

$$p_I = \sum_{J \subset I} p'_J \tag{4.2.3}$$

4.3. Kostka numbers and multiplicities.

To each $\lambda \in \mathcal{P}_{n+1}$ there corresponds a subgroup $S_\lambda \subset S_{n+1}$ on the one hand, and (an isomorphism class of) an irreducible representation L_λ of S_{n+1} on the other, such that

$$M_\lambda := \mathrm{Ind}_{S_\lambda}^{S_{n+1}}(1_{S_\lambda}) \overset{\sim}{=} \oplus_{\mu \geq \lambda} L_\mu^{K_{\lambda\mu}}, \tag{4.3.1}$$

with $K_{\lambda\lambda} = 1$, cf. [Ko], [F], [FH], Corollary 4.39.

4.4. Numbers $\kappa_{\lambda,I}$.

We define a map

$$\mu : Sub_n \longrightarrow \mathcal{P}_{n+1} \tag{4.4.1}$$

as follows. Given a subset $J = \{j_1 < j_2 < \ldots < j_r\} \subset \{1, 2, \ldots, n\}$, we consider a decomposition $(j_1, j_2 - j_1, \ldots, j_r - j_{r-1}, n + 1 - j_r)$ of $n + 1$, and we denote the corresponding partition by $\mu(J)$.

For example, if $n = 4$, then $\mu(13) = (221)$.

4.4.1. Remark. Let $G = GL(n+1)$. The set Sub_n may be identified with the set of G-conjugacy classes of parabolics $P \subset G$, whereas $\mathcal{P}_{n+1}$ may be identified with the set of G-conjugacy classes of nilpotent elements $x \in Lie(G)$.

The map (4.4.1) assigns to P the class of a generic nilpotent $x \in Lie(U(P))$.

Dually, we could assign to P the class of a generic nilpotent $y \in Lie(L(P))$; this would give the conjugate partition.

4.4.2. Definition. *We define* small Kostka numbers: *for* $\lambda \in \mathcal{P}_{n+1}, I \in Sub_n$,

$$\kappa_{\lambda,I} = \sum_{J \subset I} (-1)^{|J|-|I|} K_{\lambda,\mu(J)}.$$

4.4.3. Proposition. *We have*

$$K_{\lambda,\mu(I)} = \sum_{J\subset I} \kappa_{\lambda,J}. \qquad (4.4.3.1)$$

This formula defines the numbers $\kappa_{\lambda,I}$ *uniquely.*

4.5. Theorem (A.Postnikov) *The number* $\kappa_{\lambda,I}$ *equals the number of SYT's of shape* λ *with descent set* $Des(T) = I$*.*

4.6. A hyperbolic sheaf over $\mathbb{R}^n$.

4.6.1. Consider $V = \mathbb{C}^n \supset V_{\mathbb{R}} = \mathbb{R}^n$ equipped with the coordinate arrangement

$$\mathcal{H} = \{H_i : x_i = 0,\ 1 \le i \le N\}.$$

Let $\mathcal{S}$ be the corresponding stratification of $V_{\mathbb{R}}$. For each $S \in \mathcal{S}$ its linear span

$$L(S) = H_I := \cap_{i \in I}\ H_i$$

for some $I \subset [n]$.

In this manner we get a surjective map

$$\nu :\ \mathcal{S} \longrightarrow Sub_n$$

We have $|\mathcal{S}| = 3^n$, and

$$|\nu^{-1}(I)| = \binom{|I|}{n}.$$

In fact, Sub_n is in bijection with the set of complex strata $\mathcal{S}_{\mathbb{C}}$, and ν is the complexification map.

4.6.2. Recall that for each $T \in \mathcal{T}_{n+1}$ we have an irreducible constituent

$$L_T \subset A = \mathbb{C}[S_{n+1}],$$

cf. (4.1.1), and for any $I \in Sub_n$ the submodules

$$L_I = \oplus_{T \in \mathcal{T}_I} L_T$$

and

$$M_I = \oplus_{J \subset I} L_J$$

We define S_{n+1}-modules

$$E_S := M_{\nu(S)},\ S \in \mathcal{S}.$$

For $S' \geq S$ we have obvious inclusions

$$\delta_{S'S} : E_{S'} \hookrightarrow E_S$$

and projections

$$\gamma_{SS'} :\ E_S \hookrightarrow E_{S'}$$

4.6.3. Theorem - definition. *The collection*

$$E = (E_S, \gamma_{SS'}, \delta_{S'S})$$

is a $Rep(S_{n+1})$*-valued hyperbolic sheaf over* $\mathcal{S}$ *We call it* the Postnikov sheaf.

HYPERBOLIC FIBERS OF E: INDUCED MODULES

4.6.4. Proposition. *Recall the map* $\mu : Sub_n \longrightarrow \mathcal{P}_{n+1}$, (4.4.1). *We have isomorphisms of representations*

$$E_{S_I} \overset{\sim}{=} M_{\mu(I)}.$$

4.6.5. Let $\mathcal{M} \in Perv(V; \mathcal{S}_{\mathbb{C}})$ be the perverse sheaf corresponding to E.

Recall that the poset $\mathcal{S}_{\mathbb{C}}$ may be identified with Sub_n, in such a way that $[n]$ corresponds to $\{0\}$, and $\emptyset$ corresponds to the unique open stratum.

We denote this bijection $I \mapsto S_I$.

For $\lambda \in \mathcal{P}_{n+1}$, $I \in Sub_n$ denote an irreducible perverse sheaf

$$\mathcal{L}_{\lambda,I} := i_{I*}\underline{L}_\lambda$$

where $i_I := \bar{S}_I \hookrightarrow V$, and $\underline{L}_\lambda$ is the (shifted) constant sheaf with fiber L_λ.

Then

$$\mathcal{M} \overset{\sim}{=} \oplus_{I \in Sub_n} \mathcal{L}_{\lambda,I}^{\kappa_{\lambda,I}}$$

is the decomposition of $\mathcal{M}$ into irreducible constituents in $Perv(V, \mathcal{S}_{\mathbb{C}}; Rep(S_{n+1}))$, and the small Kostka numbers $\kappa_{\lambda,I}$ are the multiplicities.

4.7. DUAL SHEAF AND *Alt*.

The arrangement $\mathcal{S}$ is self-dual. The dual hyperbolic sheaf $E^\vee$ has a general fiber isomorphic to the alternating representation *Alt* of S_{n+1}.

More spevifically, its fiber at the main octant is the complex of vanishing cycles

$$E_\emptyset^{\vee\bullet}$$

is a resolution of *Alt* by the induced modules.

4.8. More generally, let $L \in Rep(S_{n+1})$.

For every $\lambda \in \mathcal{P}_{n+1}$ we have

$$M(L)_\lambda := \mathrm{Ind}_{S_\lambda}^{S_{n+1}}(L) \overset{\sim}{=} M_\lambda \otimes L$$

We define a $Rep(S_{n+1})$-valued hyperbolic sheaf over $\mathcal{S}$

$$E(L) := E \otimes L$$

with fibers

$$E(L)_I := E_I \otimes L$$

We have

$$E(L)^\vee = E(L^\vee)$$

where

$$L^\vee = L \otimes Alt$$

("transposition of a Young diagram").

DEFORMATION: $GL_{n+1}(\mathbb{F}_q)$ STORY

4.9. Let $G = GL_{n+1}(\mathbb{F}_q)$; fix a Borel subgroup $B \subset G$. The ordered set Sub_n is in bijection with the set of parabolics $P \supset B$ (*standard parabolics*). For $I \in Sub_n$ we denote P_I the corresponding parabolic, so that $P_\emptyset = B$, and $P_{[n]} = G$.

If $\mu(I) = \mu(I') \in \mathcal{P}_{n+1}$, the parabolics P_I an $P_{I'}$ are called *associated* (Langlands); they are isomorphic.

Let us denote

$$M_\emptyset = M_{\emptyset,q} = \mathrm{Ind}_B^G 1_B = Fun(G/B, \mathbb{C});$$

it is a q-analog of the regular representation of S_{n+1}. Its G-submodules are called *unitary*. Let

$$\mathrm{Unirep}(G) \subset \mathrm{Rep}(G)$$

denote the full subcategory of unitary representations.

Hecke algebra

Let

$$A_q = H_{n+1,q} = Hecke(G, B)$$

be the algebra of B-biinvariant functions $f : G \longrightarrow \mathbb{C}$, with the convolution as a multiplication.

Alternatively,

$$H_{n+1,q} = End_G(\mathrm{Ind}_B^G 1_B).$$

This algebra admits as a $\mathbb{C}$-base, the set $\{T_w,\ w \in S_{n+1}\}$, with multiplication defined by

$$(T_{s_i} + 1)(T_{s_i} - q) = 0,$$

where $s_i, 1 \leq i \leq n$, are the standard generators of S_{n+1}, and

$$T_w \cdot T_{w'} = T_{ww'}$$

if $\ell(ww') = \ell(w) + \ell(w')$, cf.[Iw], [L]; it is a q-deformation of $\mathbb{C}[S_{n+1}]$.

Steinberg - Iwasawa isomorphism

According to Steinberg, [St], one has an algebra isomorphism

$$st : A_q := H_{n+1} \overset{\sim}{=} \mathbb{C}[S_{n+1}] = A, \tag{4.11.1}$$

cf. also [L] and references therein.

Morita equivalence

$M_{\emptyset,q}$ is an $A_q - G$-bimodule, and it defines a Morita equivalence between two categories. Namely, two functors

$$HU : \ \mathrm{Rep}(A_q) \longrightarrow \mathrm{Unirep}(G), \ HU(N) = M_{\emptyset,q} \otimes_{A_q} N$$

and

$$UH : \ \mathrm{Unirep}(G) \longrightarrow \mathrm{Rep}(A_q), \ UH(L) = M_{\emptyset,q} \otimes_G L$$

are mutually inverse equivalences of categories.

We have

$$M_{\emptyset,q} \stackrel{\sim}{=} \oplus_{N\in Irrrep(A_q)} N \otimes HU(N) \stackrel{\sim}{=} \oplus_{L\in Irrrep(G)} UH(L) \otimes L.$$

4.12. *Parabolic induction*

For $I \in Sub_n$ let $U_I \subset P_I$ denote the unipotent radical, $L_I \subset P_I$ a Levi subgroup.

The subspace $L^{U_I} \subset L$ is an L_I-module since U_I normalizes L_I; using the canonical projection $P_I \longrightarrow P_I/U_I \stackrel{\sim}{=} L_I$, we consider it as a P_I-module.

Parabolic induction functors

$$Par_I : \ Rep(G) \longrightarrow Rep(G)$$

are defined by

$$Par_I(M) = \mathrm{Ind}_{P_I}^{G}(M^{U_I}).$$

Under the equivalences UH, HU the parabolic induction goes to the parabolic induction.

4.13. Curtis - Alvis duality. Let
$L \in \mathrm{Unirep}(G), N = HU(L) \in Rep(S_{n+1})$.

The image under the equivalence st

$$M_q(N) := st_*(M(N)) \in Hyp(\mathcal{S}; \mathrm{Rep}(A_q))$$

is a hyperbolic sheaf with values in $\mathrm{Rep}(A_q)$.

Applying the functor UH we get a $\mathrm{Unirep}(G)$-valued hyperbolic sheaf

$$M_q(L) := UH(M_q(N)) \in Hyp(\mathcal{S}; \mathrm{Unirep}(G)),$$

a "hyperbolic localization"of L.

Its (hyperbolic) fibers are induced G-modules, the general fiber being L itself.

Consider the generic fiber of the dual sheaf $M_q(L)^\vee$ in the main octant, aka its complex of vanishing cycles for the function $f(x) = \sum x_i$:

$$M_q(L)_0^{\vee\bullet}.$$

Let us denote by

$$L^\vee := H^0(M_q(L)_0^{\vee\bullet}$$

its only nonzero cohomology.

The operation $L \mapsto L^\vee$ is the known Curtis - Alvis duality on $\mathrm{Rep}(G)$.

For example

$$1_G^\vee = St_G$$

(the Steinberg module).

We have

$$M_q(L^\vee) = M_q(L)^\vee$$

In other words, the hyperbolic localization takes CA duality to Fourier - Sato duality.

References

[BFS] R.Bezrukavnikov, M.Finkelberg, V.Schechtman, Factorizable sheaves and quantum groups.

[FKS] M.Finkelberg, M.Kapranov, V.Schechtman, Fourier transform on hyperplane arrangements, in preparation.

[FS] M.Finkelberg, V.Schechtman, Microlocal approach to Lusztig symmetries.

[KS] M.Kapranov, V.Schechtman, Perverse sheaves over real hyperplane arrangements.

[KaScha] M.Kashiwara, P.Schapira, Sheaves on manifolds.

[L] G.Lusztig, Introduction to quantum groups

[Tits] J.Tits, Normalisateurs de tores I. Groupes de Coxeter étendus

[W] H.Weyl, Classical groups