TWISTED FROBENIUS BOUNDS IN THE SMOOTH AND PROJECTIVE CASE (ACCORDING TO E. HRUSHOVSKI)

by

Damian RÖSSLER

Abstract. — These are the notes of a talk I gave at the CIRM during the meeting "The geometry of the Frobenius automorphism" (which took place during the last week of March 2013). The prerequisites are algebraic geometry at the level of the three first chapters of Hartshorne's book on algebraic geometry.

Please note that this text is not in final form and that it hasn't been scanned for mistakes or misprints very thoroughly.

1. Introduction

The aim of the following text is to prove the following results, which constitutes a step toward the main result (Th. 1.1) of [4].

Let B be a scheme of finite type over $\mathbb{Z}$.

Let $\pi : V \to B$ be morphism of finite type.

We shall write

$$B^2 := B \times B$$

and

$$V^2 = V \times V.$$

We view V^2 as a B^2-scheme in the natural way.

Let $\beta : B' \to B^2$ be a morphism of finite presentation. Let $p_1 : B' \to B$ (resp. $p_2 : B' \to B$) be the morphism obtained by composing β with the first projection (resp. the second projection) $B^2 \to B$. Note that there is a natural isomorphism

$$V^2 \times_{B^2} B' \simeq p_1^* V \times_{B'} p_2^* V.$$

We shall write $\pi_1 : V^2 \times_{B^2} B' \to p_1^* V$, $\pi_2 : V^2 \times_{B^2} B' \to p_2^* V$ for the natural projections.

Let now $S \hookrightarrow V^2 \times_{B^2} B'$ be a closed immersion.

1.1. The projective and smooth case. — In this subsection, we make the following supplementary hypotheses:

- π is a smooth, projective and of constant relative dimension d;
- there is an open subscheme $U_1 \subseteq p_1^*V$ such that the restriction to U_1 of the natural projection $S \to p_1^*V$ is finite and flat of constant degree δ_1;
- U_1 is dense in every fibre of p_1;
- there is an open subscheme $U_2 \subseteq p_2^*V$ such that the restriction to U_2 of the natural projection $S \to p_2^*V$ is finite and flat of constant degree δ_2;
- U_2 is dense in every fibre of p_2;

Theorem 1.1 **(slight refinement of Th. 11.2 in [4]).** — *There exists a constant $C > 0$ with the following property.*

Let $a : \mathrm{Spec}\,\bar{\mathbb{F}}_p \to B$ be a geometric point with values in $\bar{\mathbb{F}}_p$ and let $n \in \mathbb{N}^*$. Define

$$a_2 := (a, a \circ \mathrm{Frob}^{\circ n}_{\mathrm{Spec}\ \bar{\mathbb{F}}_p}) = (a, \mathrm{Frob}^{\circ n}_{B_{\bar{\mathbb{F}}_p}} \circ a) : \mathrm{Spec}\,\bar{\mathbb{F}}_p \to B^2.$$

Let now $b : \mathrm{Spec}\,\bar{\mathbb{F}}_p \to B'$ and suppose that $\beta(b) = a_2$. Then

$$|\deg(S_b \cdot \Gamma_{\mathrm{Frob}^{(p^n)}_{V_a/\mathrm{Spec}\,\bar{\mathbb{F}}_p}}) - \delta_1 \cdot p^{nd}| \leqslant C \cdot p^{n(d-1/2)}$$

The notation $\Gamma_{\mathrm{Frob}^{(p^n)}_{V_a/\mathrm{Spec}\,\bar{\mathbb{F}}_p}}$ refers to the graph in $V_a \times_{\bar{\mathbb{F}}_p} V_a^{(p^n)}$ of the relative Frobenius morphism $\mathrm{Frob}^{(p^n)}_{V_a/\mathrm{Spec}\,\bar{\mathbb{F}}_p} : V_a \to V_a^{(p^n)}$ (see below for the latter).

Here $S_b \cdot \Gamma_{\mathrm{Frob}^{(p^n)}_{V_a/\mathrm{Spec}\,\bar{\mathbb{F}}_p}}$ is the degree of the intersection product of the cycles S_a and $\Gamma_{\mathrm{Frob}^{(p^n)}_{V_a/\mathrm{Spec}\,\bar{\mathbb{F}}_p}}$ in the Chow theory ring $A^*(V_a \times_{\bar{\mathbb{F}}_p} V_a^{(p^n)}) = \mathrm{CH}^*(V_a \times_{\bar{\mathbb{F}}_p} V_a^{(p^n)})$ of $V_a \times_{\bar{\mathbb{F}}_p} V_a^{(p^n)}$.

Notations. If k is a field, a variety over k is a morphism of schemes $X \to \mathrm{Spec}\ k$, which is separated and of finite type. We say that the variety is geometrically integral if $X \times_k \bar{k}$ is integral. A subvariety of X/k is a closed integral subscheme of X.

If S is an irreducible scheme, we write η_S for the generic point of S.

If X is a scheme of characteristic $p > 0$ we write Frob_X for the absolute Frobenius morphism $X \to X$. If $X \to S$ is an S-scheme and S is of characteristic p (ie there exists a morphism $S \to \mathrm{Spec}\ \mathbb{F}_p$), we write $X^{(p^n)}$ for the base-change of X by the n-th power $\mathrm{Frob}_S^{\circ n}$ of the absolute Frobenius morphism on S. We are then provided with a canonical S-morphism $\mathrm{Frob}^{(p^n)}_{X/S} : X \to X^{(p^n)}$, called the relative Frobenius morphism. See [6, 3.2.4] for details.

If R is a commutative ring, an anticommutative $\mathbb{Z}$-graded R-algebra A is the following set of data:

- a ring A,

- a $\mathbb{Z}$-grading $\bigoplus_{n\in\mathbb{Z}} A_n = A$ of A as a ring,

- together with a ring morphism $R \to A_0$, where the image of R lies in the centre of A_0,

such for any $a_n \in A_n$ and $a_m \in A_m$, we have $a_n \cdot a_m = (-1)^{nm} a_m \cdot a_n$.

1.2. Theorem 1B in [4]. — The following theorem is Theorem 1B in [4], which implies the main theorem 1.1. We include its statement to underline the analogy with Theorem 1.1.

We make the following hypotheses.

- there is an open subscheme $U_1 \subseteq p_1^* V$ such that the restriction to U_1 of the natural projection $S \to p_1^* V$ is finite and flat of constant degree δ_1;
- U_1 is dense in every fibre of p_1;
- there is an open subscheme $U_2 \subseteq p_2^* V$ such that the restriction to U_2 of the natural projection $S \to p_2^* V$ is étale of constant degree δ_2;
- U_2 is dense in every fibre of p_2;
- the natural projection $S \to p_2^* V$ is quasifinite.

Theorem 1.2 **(Th. 1B in [4] with separability assumption (see Lemma 10.19))**
There exists a constant C and an open subset $B'' \subseteq B'$ with the following property.

Let $a : \mathrm{Spec}\,\bar{\mathbb{F}}_p \to B$ be a geometric point with values in $\bar{\mathbb{F}}_p$ and let $n \in \mathbb{N}^*$. Define

$$a_2 := (a, a \circ \mathrm{Frob}^{\circ n}_{\mathrm{Spec}\ \bar{\mathbb{F}}_p}) = (a, \mathrm{Frob}^{\circ n}_{B_{\bar{\mathbb{F}}_p}} \circ a) : \mathrm{Spec}\,\bar{\mathbb{F}}_p \to B^2.$$

Let now $b : \mathrm{Spec}\,\bar{\mathbb{F}}_p \to B''$ and suppose that $\beta(b) = a_2$. Then

$$|\#(S_b \cap \Gamma_{\mathrm{Frob}^{(p^n)}_{V_a/\mathrm{Spec}\,\bar{\mathbb{F}}_p}}) - \delta_1 \cdot p^{nd}| \leqslant C \cdot p^{n(d-1/2)}$$

for all $n > C$.

2. Preliminaries

2.1. The Chow ring. — .

In this subsection, we shall briefly recall the definition of the Chow ring of a smooth variety over a field and describe some of its basic properties. The fundamental reference for this material is Fulton's book [2]. Let X be a variety over a field k.

A *cycle* over X is a formal $\mathbb{Z}$-linear combination of subvarieties of X. So the cycles form an abelian group, which is the free abelian group generated by all the subvarieties of X.

If $n \in \mathbb{N}$, an *n-cycle* is a cycle $\sum_i m_i Z_i$, such that Z_i is of dimension n.

An *effective cycle* is a cycle $\sum_i m_i Z_i$ such that $m_i \geqslant 0$ for all i.

If S is a closed subscheme of X, we define the cycle class $[S]$ of S by the formula

$$[S] := \sum_C \mathrm{length}_{\mathcal{O}_{S,\eta_C}}(\mathcal{O}_{S,\eta_C})C$$

Here C runs through the irreducible components of S, endowed with the reduced structure induced by X.

We now turn to the definition of rational equivalence of cycles.

Let $n \geqslant 0$. Let $W_1, \dots W_r$ be $n+1$-dimensional subvarieties of $X \times_k \mathbb{P}^1_k$. We suppose that $W_1, \dots, W_r$ dominate $\mathbb{P}^1_k$ via the second projection. Write $W_{i,0} := W \cap X \times 0$ (resp. $W_{i,\infty} := W \cap X \times \infty$) for the scheme theoretic intersection in $X \times_k \mathbb{P}^1_k$ between W and $X \times 0$ (resp. the scheme theoretic intersection in $X \times_k \mathbb{P}^1_k$ between W and $X \times \infty$).

In this situation, the n-cycle $\sum_i [W_{i,0}]$ is said to be *rationally equivalent* to the n-cycle $\sum_i [W_{i,\infty}]$. We write

$$\sum_i [W_{i,0}] \sim \sum_i [W_{i,\infty}].$$

Rational equivalence of n-cycles can be shown to be an equivalence relation $\sim$ and we define

$$A_n(X) = n\text{-cycles on } X/\!\sim$$

as the set of classes of rationally equivalent n-cycles on X. We define furthermore

$$A_*(X) := \bigoplus_n A_n(X).$$

Dually, we shall write $A^l(X) := A_{\dim(X)-l}(X)$ and $A^*(X) := \bigoplus_l A^l(X)$.

We shall also need the following notion.

Definition 2.1. — *Let Y and Z be subvarieties of X. We say that Y and Z meet properly if for every irreducible component W of $Y \cap Z$, we have*

$$\mathrm{codim}_X(Y) + \mathrm{codim}_X(Z) = \mathrm{codim}(W).$$

If $C_1 = \sum_i m_i Y_i$ and $C_2 = \sum_j n_j Z_j$ are two cycles on X, we say that C_1 and C_2 *meet properly* if every Y_i meets every Z_j properly.

Let us now suppose until the end of this subsection that X is smooth over k.

Suppose that Y and Z are subvarieties of X, which meet properly. If W is an irreducible component of $Y \cap Z$, we write

$$i(W, Y \cdot Z, X) := \sum_{j \geqslant 0} (-1)^j \cdot \mathrm{length}_{\mathcal{O}_{X,\eta_W}} (\underline{\mathrm{Tor}}_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z))$$

The invariant $i(W, Y \cdot Z, X)$ is called the intersection multiplicity of W with respect to Y and Z in X. It can be shown that $i(W, Y \cdot Z, X) \geqslant 1$ (see [2, 20.4]). We then define the intersection product of Z and Y by the formula

$$[Z] \cdot [Y] := \sum_{W \text{ irred. comp. of } Y \cap Z} i(W, Y \cdot Z, X) \cdot W.$$

We linearly extend this intersection product to any two cycles, which meet properly.

Theorem 2.2. — *(moving lemma) Let C_1 be an cycle and C_2 be a n-cycle on X. Then there exists a n-cycle C_2', which is rationally equivalent to C_2 and which meets C_1 properly. Furthermore if C_2'' is another n-cycle, which is rationally equivalent to C_2 and which meets C_1 properly then the cycles $C_2' \cdot C_1$ is rationally equivalent to $C_2'' \cdot C_1$.*

Proof. See the talk by J.-B. Bost and also [2, 11.4] & [8]. □

Theorem 2.2 can be used to show that the intersection product descends to a bilinear pairing $A^*(X) \otimes A^*(X) \to A^*(X)$, which makes $A^*(X)$ into a commutative $\mathbb{N}$-graded ring.

Let $f : X \to Y$ be a morphism. Suppose until the end of the subsection that X and Y are projective and that k is algebraically closed.

There exists a *push-forward map*

$$f_* : A^*(X) \to A^*(Y) \tag{1}$$

defined as follows. Let Z be a subvariety of X. Consider the extension of function fields $k(Z)|k(f(Z))$. If this extension is infinite, then we define $f_*(Z) = 0$. Otherwise, we define

$$f_*(Z) := [k(Z) : k(f(Z))] \cdot f(Z).$$

As usual, we linearly extend this definition to cycles. This descends to the map (1). The push-forward map is by construction a map of groups but it does not in general respect the ring structure and the grading of the Chow groups.

Furthermore, there exists a canonical *pull-back map* $f^* : A^*(Y) \to A^*(X)$. Suppose first that f is smooth. Let Z be a subvariety of Y. Then we define

$$f^*([Z]) = \sum_{C \text{ irr. comp. of } f^{-1}(Z)} C$$

and we extend this linearly. In general, without the assumption of smoothness on f, the operation f^* is described by the formula

$$f^*(y) = p_{X*}(p_Y^*(y) \cdot \Gamma_f)$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the obvious projections and $\Gamma_f \hookrightarrow X \times Y$ is the graph f. In can be shown that the pull-back map respects the ring structure and the grading of Chow groups. Thus if we restrict $A^*(\cdot)$ to smooth and projective varieties over k, we obtain a contravariant functor from the category of smooth and projective varieties over k to the category of $\mathbb{Z}$-graded anti-commutative $\mathbb{Z}$-algebras.

Finally, the pull-back and push-forward operations are related by the *projection formula*:

$$f_*(x \cdot f^*(y)) = y \cdot f_*(x). \tag{2}$$

See [2, Ex. 8.1.7, chap. 8] for this.

N.B. Pull-back maps can be defined in other situations than the one described above (see [2, 1.7, 6.5]) but we won't need these variants.

2.2. Refinements of Bezout's theorem and of the moving lemma. The norm of a cycle. — Let X be a smooth variety over a field k. Let Y (resp. Z) be a subvariety of X. Let L be a line bundle on X. If X is projective, we write

$$\deg_L(Y) := \deg_X([Y] \cdot \mathrm{c}_1(L)^{\mathrm{codim}_X(Y)})$$

(and similarly for Z). Here $\mathrm{c}_1(L) \in A^1(X)$ is the *first Chern class* of L. It can be represented by the divisor of any rational section of L. See [2, 3.2] for more details.

Theorem 2.3 **(very refined Bezout theorem; Lemma 10.12 in** [4]**)**
Suppose that X is projective and that L is very ample. Suppose that Y and Z meet properly. Then we have

$$\sum_{W \text{ irred. comp. of } Y \cap Z} i(W, Y \cdot Z, X) \cdot \deg_L(W) \leqslant \deg_L(Y) \cdot \deg_L(Z).$$

Proof. See [2, 12.3, esp. Ex. 12.3.1, 11.4.3 and also 6.1, 20.4]. □

N.B. The classical form of Bezout's theorem (which is less difficult to prove) gives the weaker inequality

$$\sum_{W \text{ irred. comp. of } Y \cap Z} \deg_L(W) \leqslant \deg_L(Y) \cdot \deg_L(Z).$$

Proposition 2.4 **(refinement of Theorem 2.2; for a weaker version see Lemma 10.15 in** [4]**)**
Suppose that k is algebraically closed. Suppose that L is very ample and that X is projective. Let C_1 be a

cycle and C_2 be an effective n-cycle on X. Then there exists a n-cycle $C_2' = \sum_i m_i\, W_i$, which is rationally equivalent to C_2, which meets C_1 properly and such that

$$\begin{aligned}\sum_i |m_i| \cdot \deg_L(W_i) &\leqslant [\deg_L(X) \sum_{i=0}^{e-1} (\deg_L(X) - 1)^i + (\deg_L(X) - 1)^e] \deg_L(C_2) \\ &\leqslant (2\dim(X) + 1) \deg_L(X)^{2\dim(X)} \deg_L(C_2)\end{aligned}$$

Proof. (sketch). In [8, par. 3, p. 95], we are provided with
– an integer e such that $0 \leqslant e \leqslant 2\dim(X)$;
– effective cycles $C_2^0, C_2^2, \dots, C_2^{e-1}$;
– effective cycles $X_0, \dots, X_e$, where $X_0 = C_2$;
such that $\sum_{i=0}^{e-1} (-1)^i C_2^i + (-1)^e X_e$ is rationally equivalent to C_2 and meets C_1 properly. We are also provided with the equalities

$$\deg_L(X_i) = \deg_L(C_2^{i-1}) - \deg_L(X_{i-1})$$

for all $i \geqslant 1$. Furthermore, using the refined form of Bezout's theorem, it can be shown that

$$\deg_L(C_2^i) \leqslant \deg_L(X) \cdot \deg_L(X_i).$$

We deduce from all this that

$$\deg_L(X_i) \leqslant (\deg_L(X) - 1) \deg_L(X_{i-1})$$

and thus

$$\deg_L(X_i) \leqslant (\deg_L(X) - 1)^i \deg_L(C_2).$$

Thus we get the inequality

$$\begin{aligned}\deg_L [\sum_{i=0}^{e-1} C_2^i + X_e] &\leqslant \deg_L(X) \deg_L(C_2) \sum_{i=0}^{e-1} (\deg_L(X) - 1)^i + (\deg_L(X) - 1)^e \deg_L(C_2) \\ &= \deg_L(C_2) \cdot [\deg_L(X) \sum_{i=0}^{e-1} (\deg_L(X) - 1)^i + (\deg_L(X) - 1)^e]. \quad \square\end{aligned}$$

The refined form of the moving lemma warrants the

***Definition 2.5* (the norm of a cycle; see notation 10.13 in [4]).** — *Suppose that X is projective and L very ample. Let U be a cycle on X. We define*

$$|U| = |U|_L := \sup_S \inf\{\deg_L(U') + \deg_L(U'')\}_{U',U''} \in \mathbb{N} \cup \{\infty\}$$

here S runs through all the cycles of X and U', U'' runs through all the pairs of effective cycles on X such that U is rationally equivalent to $U' - U''$ and such that U' & U'' cut S properly. The quantity $|U|_L$ is called the norm of U.

Note that by Proposition 2.4, if we write $U = U_1 - U_2$,where U_1 and U_2 are effective, then we have

$$|U|_L \leqslant (2\dim(X) + 1) \deg_L(X)^{2\dim(X)} (\deg_L(U_1) + \deg_L(U_2)), \tag{3}$$

which implies that the norm is finite (which is the content of Lemma 10.15 in [4]).

We also note the important

***Lemma 2.6* (Cor 10.14 in [4]).** — *If U and V are cycles on X, where X is supposed projective and equipped with a very ample line bundle L. Then we have*

$$|U \cdot V|_L \leqslant |U|_L \cdot |V|_L$$

(here $\cdot$ is the intersection product in Chow theory).

Proof. Unwind the definitions and apply Theorem 2.3. □

2.3. Correspondences. — Let X, Y, Z be smooth varieties over a field k. Let $n \geqslant 0$.

A *correspondence* (resp. n-correspondence, resp. effective correspondence) from X to Y is a cycle (resp. n-cycle, resp. effective cycle) on $X \times_k Y$.

We write $\mathrm{Corr}(X,Y)$ (resp. $\mathrm{Corr}_n(X,Y)$, $\mathrm{Corr}^{\mathrm{eff}}(X,Y)$) for the set of correspondences (resp. n-correspondences, resp. effective correspondences) from X to Y. We write $\mathrm{Corr}^{\mathrm{rat}}(X,Y)$ for the set of rational equivalence classes of correspondences from X to Y. By definition, we have $A^*(X \times Y) \simeq \mathrm{Corr}^{\mathrm{rat}}(X,Y)$ so that $\mathrm{Corr}^{\mathrm{rat}}(X,Y)$ has a natural ring structure.

If X, Y, Z are projective over k, we define a bilinear pairing

$$\circ : \mathrm{Corr}^{\mathrm{rat}}(Y,Z) \otimes \mathrm{Corr}^{\mathrm{rat}}(X,Y) \to \mathrm{Corr}^{\mathrm{rat}}(X,Z)$$

by the formula

$$\beta \circ \alpha := \pi_{XZ,*}(\pi^*_{XY}(\alpha) \cdot \pi^*_{YZ}(\beta)).$$

Furthermore, if $\alpha \in \mathrm{Corr}^{\mathrm{rat}}(X,Y)$ and $\tau : X \times Y \to Y \times X$ is the isomorphism, which swaps the factors, then we define the transpose of α by the formula

$${}^t\alpha := \tau_*(\alpha) \in \mathrm{Corr}(Y,X).$$

***Proposition 2.7* (Lemma 10.17 (3) in [4]).** — *Suppose that X, Y, Z are projective varieties of dimension d_0. Let L_X, L_Y, L_Z be a very ample line bundles on X, Y, Z. Let $L_{\mathrm{XY}} := L_X \boxtimes L_Y$, $L_{\mathrm{YZ}} := L_Y \boxtimes L_Z$, $L_{\mathrm{XZ}} := L_X \boxtimes L_Z$. Let $V \in \mathrm{Corr}^{\mathrm{rat}}(Y,Z)$ and $U \in \mathrm{Corr}^{\mathrm{rat}}(X,Y)$. We have the inequality*

$$|V \circ U|_{L_{\mathrm{XZ}}} \leqslant \binom{2d_0}{d_0}^2 |U|_{L_{\mathrm{XY}}} \cdot |V|_{L_{\mathrm{YZ}}}.$$

Proof. Exercise for the reader. Uses Lemma 2.6. □

Proposition 2.7 justifies the following terminology. Keep the assumptions of Proposition 2.7. We define

$$||U|| := ||U||_{L_{XY}} := \binom{2d}{d}^2 |U|_{L_{XY}}$$

so that the inequality in Proposition 2.7 can be rewritten as

$$||V \circ U||_{L_{\mathrm{XZ}}} \leqslant ||U||_{L_{\mathrm{XY}}} \cdot ||V||_{L_{\mathrm{YZ}}}.$$

2.4. l-adic cohomology and the Weil conjectures. —

2.4.1. *Weil cohomologies.* — Suppose in this subsection that k is an algebraically closed field. Let K be a field of characteristic 0. A Weil cohomology theory is a contravariant functor $X \mapsto H^*(X)$ from the category of smooth and projective varieties over k to the category of $\mathbb{Z}$-graded anticommutative K-algebras, satisfying the following six properties:

(1) (finiteness) The group $H^i(X)$ vanishes unless $0 \leqslant i \leqslant 2\dim(X)$ and they are finite-dimensional.

(2) (Poincaré duality) There is a canonical "trace morphism" $\mathrm{tr}_X : H^{2\dim(X)}(X) \to K$. For any $0 \leqslant i \leqslant 2\dim(X)$, the bilinear map

$$H^i(X) \times H^{2\dim(X)-i} \to K$$

given by the formula $\mathrm{tr}_X(x \cdot y)$ is non-degenerate.

Property (2) gives a canonical isomorphism of K-vector spaces between $H^*(X)$ and its dual K-vector space $H^*(X)^\vee$ for any smooth and projective variety X. Let $X \to Y$ be a k-morphism between two smooth and projective varieties over k. We write $f^* : H^*(Y) \to H^*(X)$ for the corresponding map of graded anticommutative K-algebras (*"pull-back"*) and $f_* : H^*(X) \to H^*(Y)$ for the K-linear map (*"push-forward"*) obtained by taking the map dual to f^* and identifying $H^*(X) \to H^*(Y)$ with their duals via (2). Note that f_* does not in general respect the underlying grading or the underlying ring structure.

(3) (Künneth formula) For any X, Y projective and smooth varieties over k, the natural projections induce an isomorphism $H^*(X) \otimes_K H^*(Y) \to H^*(X \times Y)$.

(4) (cycle maps) There exists a natural transformation of contravariant functors $\gamma_{(\cdot)} : A^*(\cdot)_K \to H^{\mathrm{ev}}(\cdot)$, called the cycle class map. Furthermore, $\gamma_{(\cdot)}$ is compatible with push-forwards.

Here $H^{\mathrm{ev}}(\cdot)$ is the K-subalgebra of $H^*(\cdot)$, consisting of elements whose homogenous components are even integers. This subalgebra is by construction commutative and defines a subfunctor.

(5) (weak Lefschetz theorem) Let $h : X \hookrightarrow Y$ be the inclusion of a smooth section of an ample line bundle on Y. Then $h^* : H^i(Y) \to H^i(Y)$ is an isomorphism for $0 \leqslant i \leqslant \dim(X) - 2$ and is injective if $i = \dim(X) - 1$.

(6) (strong Lefschetz theorem) Let $x \in H^2(X)$ be the cycle of the section of a ample line bundle. Let $0 \leqslant i \leqslant \dim(X)$. Then the map $H^i(X) \to H^{2\dim(X)-i}(X)$ given by multiplication by the $(\dim(X) - i)$-th power of x is an isomorphism.

Let $k = \mathbb{C}$ and $K = \mathbb{Q}$ in the next sentence only. An example of a Weil cohomology theory is $X \to H^*(X(\mathbb{C}), \mathbb{Q})$, which associates with X the singular cohomology with coefficients in $\mathbb{Q}$ of the complex points of X.

If $p > 0$, l is a prime number $\neq p$, $k = \bar{\mathbb{F}}_p$ and $K = \mathbb{Q}_l$ then an example of a Weil cohomology theory is Grothendieck's l-adic cohomology theory $X \mapsto H^*(X, \mathbb{Q}_l)$. For an introduction to the construction of this cohomology theory, which involves the introduction of a new notion of topology, we refer to [3].

Let X and Y be smooth and projective varieties over k. Let $\alpha \in \mathrm{Corr}^{\mathrm{rat}}(X, Y)$. Then we obtain a map of K-vector spaces $\alpha^* : H^*(Y) \to H^*(X)$ by the formula

$$y \mapsto p_{X*}(\alpha \cdot p_Y^*(y))$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the obvious projections. We define $\alpha_* : H^*(X) \to H^*(Y)$ by the identity

$$\alpha_* := ({}^t\alpha)^*$$

If Z is a third smooth and projective variety over k and $\gamma \in \mathrm{Corr}^{\mathrm{rat}}(Y, Z)$ then we have

$$\alpha^* \circ \gamma^* = (\gamma \circ \alpha)^*$$

If α is the graph of a morphism $f : X \to Y$, then $\alpha^* = f^*$. See [2].

2.4.2. *The Lefschetz fixed point theorem.* — Let X be a smooth and projective variety of dimension d_0 over an algebraically closed field k. Let $\alpha \in \mathrm{Corr}_d(X, X)$. The Lefschetz fixed point theorem asserts that for any Weil cohomology theory $H^*(\cdot)$, we have

$$\deg(\alpha \cdot \Delta_X) = \sum_{i \geqslant 0} (-1)^i \mathrm{Trace}(\alpha^* : H^i(X) \to H^i(X)).$$

Her Δ_X is the graph if the identity morphism of X (ie the diagonal). This formula is a formal consequence of the axioms for Weil cohomologies described in the last subsubsection. See [5, before Th. 3.1]. Here is a generalization of this formula. Let X and Y be smooth projective varieties of dimension d_0 over k. Let $\alpha, \beta \in \mathrm{Corr}_{d_0}(X, Y)$. Then

$$\deg(\alpha \cdot \beta) = \deg(({}^t\beta \circ \alpha) \cdot \Delta_X) = \sum_{i \geqslant 0} (-1)^i \mathrm{Trace}(\alpha^* \beta_* : H^i(X) \to H^i(X)).$$

This last formula follows from the first one together with [2, Ex. 16.1.3, chap. 16, p. 309].

2.4.3. *Deligne's theorem.* — We keep the notations of the last subsubsection but we let $p > 0$, l a prime number $\neq p$, $k = \bar{\mathbb{F}}_p$ and $K = \mathbb{Q}_l$.

Let X be a smooth and projective variety over k. Suppose that X has a model over a finite field $\mathbb{F}_{p^r}$ (ie "X is defined over $\mathbb{F}_{p^r}$"). Then $X^{(p^r)} \simeq X$ so that the relative Frobenius morphism $\mathrm{Frob}^{(p^r)}_{X/k} : X \to X^{(p^r)}$ gives an endomorphism $X \to X$. A natural question is: what can one say about the eigenvalues of the map $\mathrm{Frob}^{(p^r),*}_{X/k} : H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$ induced on the i-th group of l-adic cohomology ?

Theorem 2.8. — *Let α_i be an eigenvalue of the map $\mathrm{Frob}^{(p^r),*}_{X/k} : H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$. Then for any embedding $\tau : \mathbb{Q}_l \hookrightarrow \mathbb{C}$, we have $|\tau(\alpha_i)| = p^{ri/2}$. Furthermore, α_i is an algebraic integer.*

Proof. For the proof (which requires a lot of machinery, esp. the theory of Lefschetz pencils), see [1]. $\square$

3. Proof of Theorem 1.1

We shall first prove the following two results from [4].

Let X be a smooth and projective variety of dimension d_0 over $\bar{\mathbb{F}}_p$. Let L be a very ample line bundle on X.

***Theorem 3.1* (Prop. 11.11 in [4]).** — *Let $\alpha \in \mathrm{Corr}_{d_0}(X, X)$ and let $M := L \boxtimes L$ on $X \times X$. Let $j \geqslant 0$. Then the characteristic polynomial of the linear map $\alpha^* : H^j(X, \mathbb{Q}_l) \to H^j(X, \mathbb{Q}_l)$ has rational coefficients the absolute values of its roots $\leqslant ||\alpha||_M$.*

***Theorem 3.2* (Cor. 11.12 in [4]).** — *Let $n, j \geqslant 0$. Let $\alpha \in \mathrm{Corr}_{d_0}(X, X^{(p^n)})$. Then the characteristic polynomial of the linear map $({}^t\alpha \circ \Gamma_{\mathrm{Frob}^{(p^n)}_{X/\bar{\mathbb{F}}_p}})^* : H^j(X, \mathbb{Q}_l) \to H^j(X, \mathbb{Q}_l)$ has rational coefficients and the absolute value of its eigenvalues is $\leqslant p^{nj/2} \cdot ||{}^t\alpha||_{L \boxtimes L^{(p^n)}}$.*

Theorem 3.2 contains Theorem 3.1 as a special case but the proof of the former uses the latter.

Proof. (of Theorem 3.1). Suppose that X, as well as L and all the irreducible components of a fixed representative of α have a common model over a finite field $\mathbb{F}_q$, where q is a power of p. Call these models X_0, L_0 and α_0. Let $n \geqslant 1$. We view $\mathrm{Frob}^{(q)}_{X/\bar{\mathbb{F}}_p}$ as a an endomorphism of X.

By Deligne's theorem 2.8 and the Chinese remainder theorem, there is a polynomial $P_j(t) \in \mathbb{Q}(t)$, such that the restriction of $P_j(\mathrm{Frob}^{(q)}_{X/\bar{\mathbb{F}}_p})$ to $H^j(X, \mathbb{Q}_l)$ is the identity and such that the restriction of $P_j(\mathrm{Frob}^{(q)}_{X/\bar{\mathbb{F}}_p})$ to $H^i(X, \mathbb{Q}_l)$ vanishes for all $i \neq j$. By the Lefschetz fixed point theorem (see subsubsection 2.4.2), we have

$$\alpha^{\circ n} \cdot P_j(\mathrm{Frob}_{X_0/\mathbb{F}_q}) = \mathrm{Tr}(\alpha^{\circ n,*} : H^j(X, \mathbb{Q}_l) \to H^j(X, \mathbb{Q}_l)).$$

Now we compute, using Proposition 2.7:

$$\begin{aligned} &|\alpha^{\circ n} \cdot P_j(\mathrm{Frob}_{X_0/\mathbb{F}_q})| \leqslant |\alpha^{\circ n} \cdot P_j(\mathrm{Frob}_{X_0/\mathbb{F}_q})|_M \\ &\leqslant ||\alpha^{\circ n} \cdot P_j(\mathrm{Frob}_{X_0/\mathbb{F}_q})||_M \leqslant ||\alpha^{\circ n}||_M \cdot ||P_j(\mathrm{Frob}_{X_0/\mathbb{F}_q})||_M \leqslant ||\alpha||^n_M \cdot ||P_j(\mathrm{Frob}_{X_0/\mathbb{F}_q})||_M \end{aligned}$$

and thus

$$\frac{|\mathrm{Tr}(\alpha^{\circ n,*} : H^j(X, \mathbb{Q}_l) \to H^j(X, \mathbb{Q}_l))|}{||\alpha||^n_M} = O(1)$$

which implies the result. $\square$

Proof. (of Theorem 3.2). Our first claim is that we may assume that X has a model over $\mathbb{F}_p$. To see this, let $\mathbb{F}_q = \mathbb{F}_{p^r}$ be a finite field such that X and and all the components of a fixed representative of α have models X_0 and α_0 over $\mathbb{F}_q$. Now let $X_1 := X_0 \times_{\mathbb{F}_p} \mathbb{F}_q$. Notice that there is a natural isomorphism of schemes over $\mathbb{F}_q$

$$X_1 \simeq \coprod_{\sigma \in \mathrm{Gal}(\mathbb{F}_q|\mathbb{F}_p)} X_0^\sigma = \coprod_{k \in \mathbb{Z}/r\mathbb{Z}} X_0^{(p^k)}.$$

and the $\mathbb{F}_q$-scheme X_1 is equipped with a natural descent structure to $\mathbb{F}_p$, given by the natural action by permutation of the Galois group $\mathrm{Gal}(\mathbb{F}_q|\mathbb{F}_p)$ on the components of $\coprod_{\sigma \in \mathrm{Gal}(\mathbb{F}_q|\mathbb{F}_p)} X_0^\sigma$. The corresponding scheme over $\mathbb{F}_p$ is of course just X_0 viewed as an $\mathbb{F}_p$-scheme.

Notice that we have natural isomorphisms

$$X_1^{(p^n)} = \coprod_{k \in \mathbb{Z}/r\mathbb{Z}} X_0^{(p^{k+n})}$$

and

$$\coprod_{k \in \mathbb{Z}/r\mathbb{Z}} X_0^{(p^{k+n})} \simeq \coprod_{k \in \mathbb{Z}/r\mathbb{Z}} X_0^{(p^k)} = X_1$$

the latter being given by cyclic permutation of the indices. We leave it to the reader to verify the following facts:

- the resulting isomorphism $X_1^{(p^n)} \simeq X_1$ is the isomorphism arising from the model of X_1 over $\mathbb{F}_p$;

- the morphism $X_1 \to X_1$ arising by base-change from the relative Frobenius morphism $X_0 \to X_0$ (where X_0 is viewed as a scheme over $\mathbb{F}_p$) is the composition of the relative Frobenius morphism $X_1 \to X_1^{(p^n)}$ with the above isomorphism $X_1^{(p^n)} \simeq X_1$.

Finally, under the isomorphism $X_1^{(p^n)} \simeq X_1$, we define the correspondence $\alpha_1 \in \mathrm{Corr}(X_1, X_1)$ by the formula

$$\alpha_1 = \coprod_{k \in \mathbb{Z}/r\mathbb{Z}} \alpha_0^{(p^k)}.$$

The correspondence α_1 is compatible with the descent structure to $\mathbb{F}_p$ and thus arises by base-change from a correspondence on the model of X_1 on $\mathbb{F}_p$. Now we compute

$$\begin{aligned}
& \mathrm{Tr}({}^t\alpha_1 \circ \Gamma_{\mathrm{Frob}^{(p^n)}_{X_1/\mathbb{F}_q}})^* : H^j(X_{1,\bar{\mathbb{F}}_p}, \mathbb{Q}_l) \to H^j(X_{1,\bar{\mathbb{F}}_p}, \mathbb{Q}_l)) \\
= & \sum_{k\in\mathbb{Z}/r\mathbb{Z}} \mathrm{Tr}({}^t\alpha_0^{(p^k)} \circ \Gamma_{\mathrm{Frob}^{(p^n)}_{X_0^{(p^k)}/\mathbb{F}_q}})^* : H^j(X^{(p^k)}_{0,\bar{\mathbb{F}}_p}, \mathbb{Q}_l) \to H^j(X^{(p^k)}_{0,\bar{\mathbb{F}}_p}, \mathbb{Q}_l)) \\
= & \sum_{k\in\mathbb{Z}/r\mathbb{Z}} \mathrm{Tr}({}^t\alpha_0 \circ \Gamma_{\mathrm{Frob}^{(p^n)}_{X_0/\mathbb{F}_q}})^{(p^k),*} : H^j(X^{(p^k)}_{0,\bar{\mathbb{F}}_p}, \mathbb{Q}_l) \to H^j(X^{(p^k)}_{0,\bar{\mathbb{F}}_p}, \mathbb{Q}_l)) \\
= & \sum_{k\in\mathbb{Z}/r\mathbb{Z}} \mathrm{Tr}({}^t\alpha_0 \circ \Gamma_{\mathrm{Frob}^{(p^n)}_{X_0/\mathbb{F}_q}})^* : H^j(X_{0,\bar{\mathbb{F}}_p}, \mathbb{Q}_l) \to H^j(X_{0,\bar{\mathbb{F}}_p}, \mathbb{Q}_l)) \\
= & \; r \cdot \mathrm{Tr}({}^t\alpha \circ \Gamma_{\mathrm{Frob}^{(p^n)}_{X/\bar{\mathbb{F}}_p}})^* : H^j(X, \mathbb{Q}_l) \to H^j(X, \mathbb{Q}_l))
\end{aligned}$$

and hence, assuming the conclusion of Theorem 3.2 holds for X_1, we get that

$$r \cdot |\mathrm{Tr}({}^t\alpha \circ \Gamma_{\mathrm{Frob}^{(p^n)}_{X/\bar{\mathbb{F}}_p}})^* : H^j(X, \mathbb{Q}_l) \to H^j(X, \mathbb{Q}_l))| \leqslant p^{nj/2} \cdot ||{}^t\alpha_1||_{L_1 \boxtimes L_1^{(p^n)}} = p^{nj/2} \cdot r \cdot ||{}^t\alpha||_{L\boxtimes L^{(p^n)}}$$

whence the result for X.

We may thus assume that X, L and α have models X_0, L_0 and α_0 over $\mathbb{F}_p$. In this case $X^{(p^n)} \simeq X$ and we also have that the cycle classes in l-adic cohomology of $\Gamma_{\mathrm{Frob}_{X/\bar{\mathbb{F}}_p}} \circ {}^t\alpha$ and ${}^t\alpha \circ \Gamma_{\mathrm{Frob}_{X/\bar{\mathbb{F}}_p}}$ coincide under this identification. To see this, we apply the formula [2, Prop. 16.1.1 (c), (i) & (ii), chap. 16, p. 306] and we obtain that

$$ {}^t\alpha \circ \Gamma_{\mathrm{Frob}_{X/\bar{\mathbb{F}}_p}} = (\mathrm{Frob}_{X/\bar{\mathbb{F}}_p} \times \mathrm{Id}_X)^*({}^t\alpha) \tag{4}$$

and

$$ \Gamma_{\mathrm{Frob}_{X/\bar{\mathbb{F}}_p}} \circ {}^t\alpha = (\mathrm{Id}_X \times \mathrm{Frob}_{X/\bar{\mathbb{F}}_p})_*({}^t\alpha) \tag{5}$$

in $A^d(X \times X)$. Now to show that the cycle classes of (4) and (5) coincide, it sufficient to show that

$$(\mathrm{Frob}_{X/\bar{\mathbb{F}}_p} \times \mathrm{Id}_X)_*(\mathrm{Frob}_{X/\bar{\mathbb{F}}_p} \times \mathrm{Id}_X)^*({}^t\alpha) \quad = \quad (\mathrm{Frob}_{X/\bar{\mathbb{F}}_p} \times \mathrm{Id}_X)_*(\mathrm{Id}_X \times \mathrm{Frob}_{X/\bar{\mathbb{F}}_p})_*({}^t\alpha) = \mathrm{Frob}_{X\times X/\bar{\mathbb{F}}_p,*}({}^t\alpha)$$

and by the projection formula (see (2)), we have

$$(\mathrm{Frob}_{X/\bar{\mathbb{F}}_p} \times \mathrm{Id}_X)_*(\mathrm{Frob}_{X/\bar{\mathbb{F}}_p} \times \mathrm{Id}_X)^*({}^t\alpha) = \deg(\mathrm{Frob}_{X/\bar{\mathbb{F}}_p} \times \mathrm{Id}_X) \cdot {}^t\alpha = p^{2d_0} \cdot {}^t\alpha$$

Summarizing, to show that the cycle classes of $\Gamma_{\mathrm{Frob}_{X/\bar{\mathbb{F}}_p}} \circ {}^t\alpha$ and ${}^t\alpha \circ \Gamma_{\mathrm{Frob}_{X/\bar{\mathbb{F}}_p}}$ coincide, it is sufficient to show that

$$\mathrm{Frob}_{X\times X/\bar{\mathbb{F}}_p,*}({}^t\alpha) = p^{2d_0} \cdot {}^t\alpha.$$

It is not clear that this holds, but it is sufficient to show that the cycle classes of both side coincide. This is a consequence of the projection formula again and the fact that

$$\mathrm{Frob}^*_{X_0\times X_0/\mathbb{F}_p}({}^t\alpha_0) = p^{d_0} \cdot {}^t\alpha_0.$$

To see this, one has to combine the following facts, which do not fall under the scope of these notes. First, pull-back by the absolute Frobenius morphism coincides with the p-th Adams operation in the Grothendieck group of locally free sheaves (for lack of a better reference, see [7, Intro.]). Then use the fact that the Chern character isomorphism between the Grothendieck group and Chow theory (see [2, 15.1]) respects the Adams operations. Now that we now that the cycle classes of $\Gamma_{\mathrm{Frob}_{X/\bar{\mathbb{F}}_p}} \circ {}^t\alpha$ and ${}^t\alpha \circ \Gamma_{\mathrm{Frob}_{X/\bar{\mathbb{F}}_p}}$ coincide, we may apply Deligne's theorem 2.8 together with Theorem 3.1 to conclude the proof of Theorem 3.2. $\square$

Proof of Theorem 1.1. Using Theorem 3.2 and Deligne's theorem 2.8, we compute that

$$|\deg(S_b\cdot\Gamma_{\mathrm{Frob}^{(p^n)}_{V_a/\mathrm{Spec}\,\mathbb{F}_p}})-\delta_2\cdot p^{nd}|=|\sum_{i\geqslant 0}(-1)^i\mathrm{Trace}(({}^tS_b\circ\Gamma_{\mathrm{Frob}^{(p^n)}_{V_a/\mathrm{Spec}\,\mathbb{F}_p}})^*:H^i(V_a)\to H^i(V_a))-\delta_1\cdot p^{nd}|\leqslant$$

$$\leqslant\quad ||{}^tS_b||_{\mathcal{L}\boxtimes\mathcal{L}^{(p^n)}}[p^{n(d-1/2)}+p^{n(d-1)}+\dots]\leqslant(2d-1)||{}^tS_b||_{\mathcal{L}\boxtimes\mathcal{L}^{(p^n)}}p^{n(d-1/2)}$$

Now the constant $||{}^tS_b||_{\mathcal{L}\boxtimes\mathcal{L}^{(p^n)}}$ is uniformly bounded for all b, because the degree is uniformly bounded and because of inequality (3). □

References

[1] Pierre Deligne, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307 (French).

[2] William Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.

[3] P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$; Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

[4] Ehud Hrushovski, *The elementary theory of the Frobenius automorphism*, Preprint. Version of July 24th 2012.

[5] Steven L. Kleiman, *The standard conjectures*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 3–20.

[6] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné; Oxford Science Publications.

[7] Richard Pink and Damian Rössler, *On the Adams-Riemann-Roch theorem in positive characteristic*, Math. Z. **270** (2012), no. 3-4, 1067–1076, DOI 10.1007/s00209-011-0841-7.

[8] Joel Roberts, *Chow's moving lemma*, Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer School in Math.), Wolters-Noordhoff, Groningen, 1972, pp. 89–96. Appendix 2 to: "Motives" (*Algebraic geometry, Oslo 1970* (Proc. Fifth Nordic Summer School in Math.), pp. 53–82, Wolters-Noordhoff, Groningen, 1972) by Steven L. Kleiman.

Damian RÖSSLER (6)

6. Institut de Mathématiques, Equipe Emile Picard, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse cedex 9, FRANCE, E-mail: rossler@math.univ-toulouse.fr