A conjecture on the equivariant analytic torsion forms

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Let G be a finite group and let M be a complex manifold on which G acts by holomorphic automorphisms. Let $f : M \to B$ be a proper holomorphic map of complex manifolds and suppose that G preserves the fibers of f (i.e. $f \circ g = f$ for all $g \in G$). Let η be a G-equivariant holomorphic vector bundle on M and suppose that the direct images $\mathrm{R}^\bullet f_*\eta$ are locally free on B. Let h^η be a G-invariant hermitian metric on η and let ω^M be a G-invariant Kähler metric on M. For each $g \in G$, we shall write M_g for the set of fixed points of g on M. This set is endowed with a natural structure of complex Kähler manifold. We suppose that there is an open dense set U of B such that the restricted map $f^{-1}(U) \to U$ is a submersion. We shall write V for $f^{-1}(U)$ and f_V for the map $f_V : f^{-1}(U) \to U$. Denote by $T_g(\omega^V, h^\eta) \in P^U$ the equivariant analytic torsion form of $\eta|_V$ relatively to f_V and $\omega^V := \omega^M|_V$, in the sense of [4, Par. d)]. Here the space P^U (resp. $P^{U,0}$) is the direct sum of the space of *complex* differential forms of type p, p (resp. the direct sum of the space of complex differential forms of type p, p of the form $\partial\alpha + \overline{\partial}\beta$) on U. Notice that we consider $T_g(\omega^V, h^\eta)$ as a differential form and *not* as an element of $P^U/P^{U,0}$. Let $h'^{\mathrm{R}^\bullet f_*\eta}$ be any C^∞-hermitian metric on $\mathrm{R}^\bullet f_*\eta$ and write $h^{\mathrm{R}^\bullet f_{V,*}\eta|_V}$ for the metric obtained on $\mathrm{R}^\bullet f_{V,*}\eta|_V$ (which is a graded vector bundle on U) by integration along the fibers of the map f_V. We write $\widetilde{\mathrm{ch}}(\mathrm{R}^\bullet f_{V,*}\eta|_V, h^{\mathrm{R}^\bullet f_{V,*}\eta|_V}, h'^{\mathrm{R}^\bullet f_*\eta}) \in P^U$ for the Bott-Chern secondary class comparing the metrics $h^{\mathrm{R}^\bullet f_{V,*}\eta|_V}$ and the restriction of the metric $h'^{\mathrm{R}^\bullet f_*\eta}$ to U. Again, we do not consider the Bott-Chern secondary classes as elements of $P^U/P^{U,0}$. We consider $\widetilde{\mathrm{ch}}_g(\cdot)$ as a special case of the equivariant Bott-Chern singular current $T_g(\cdot)$ defined in [2, Par. 6] (which is defined as a current - not an equivalence class of currents).

Our first conjecture is the following

Conjecture 0.1. *Suppose that f is a submersion in a neighborhood of M_g. Then the differential form $T_g(\omega^V, h^\eta) + \widetilde{\mathrm{ch}}(\mathrm{R}^\bullet f_{V,*}\eta|_V, h^{\mathrm{R}^\bullet f_{V,*}\eta|_V}, h'^{\mathrm{R}^\bullet f_*\eta}) \in P^U$ is locally integrable on B.*

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Our second conjecture concerns the immersion formula. Suppose given a G-equivariant closed immersion of complex manifolds $j : M \hookrightarrow M'$ and a proper holomorphic submersion $k : M' \to B$ such that $k \circ j = f$. Suppose that M' is endowed with a G-invariant Kähler metric $\omega^{M'}$ which extends ω^M. Let

$$\Xi: \ 0 \to \xi_m \to \xi_{m-1} \to \dots \xi_0 \to j_*\eta \to 0$$

be a resolution of the coherent sheaf $j_*\eta$ by G-equivariant vector bundles on M'. Let $h^{\xi_\bullet}$ be G-invariant hermitian metrics on $\xi_\bullet$. Suppose that Bismut's assumption (A) is fulfilled in this setting. Let h^N be the quotient metric on the normal bundle N of the immersion j and $h^{\mathrm{T}k}$ be the metric on the relative tangent bundle $\mathrm{T}k$ of k induced by $\omega^{M'}$. Suppose now as well that $\mathrm{R}k^l_*(\xi_\bullet) = 0$ for all $l > 0$ (resp. $\mathrm{R}f^l_*(\eta_\bullet) = 0$ for all $l > 0$). We write $\widetilde{\mathrm{ch}}_g(h'^{k_*(\Xi)}) \in P^B$ for the equivariant Bott-Chern secondary class of the complex of G-equivariant vector bundles

$$k_*\Xi: \ 0 \to k_*\xi_m \to k_*\xi_{m-1} \to \dots k_*\xi_0 \to f_*\eta \to 0$$

where $k_*\xi_\bullet$ carries the metric obtained by integrating along the fibers of k and $f_*\eta$ carries the metric $h'^{\mathrm{R}^0 f_*\eta}$ chosen above.

Conjecture 0.2. *Suppose that f is a submersion in a neighborhood of M_g. Suppose that conjecture 0.1 holds. Then the equality*

$$\sum_{i=0}^{m}(-1)^i T_g(\omega^{M'}, h^{\xi_i}) - [T_g(\omega^V, h^\eta) + \widetilde{\mathrm{ch}}_g(\mathrm{R}^\bullet f_{V,*}\eta|_V, h^{\mathrm{R}^\bullet f_{V,*}\eta|_V}, h'^{\mathrm{R}^\bullet f_*\eta})] + \widetilde{\mathrm{ch}}_g(h'^{k_*(\Xi)}) =$$
$$= \int_{M_g/B} \mathrm{ch}_g(\eta)\mathrm{R}_g(N)\mathrm{Td}_g(Tf) + \int_{M'_g/B} T_g(\xi_\bullet, h^{\xi_\bullet})\mathrm{Td}_g(\mathrm{T}k, h^{\mathrm{T}k}) -$$
$$- \int_{M_g/B} \mathrm{ch}_g(\eta, h^\eta)\widetilde{\mathrm{Td}}_g(f/k)\mathrm{Td}^{-1}(N, h^N)$$

holds in $P^B_B/P^{B,0}_B$.

The space P^B_B (resp. $P^{B,0}_B$) is the direct sum of the space of *complex* currents of type p,p (resp. the direct sum of the space of complex currents of type p,p of the form $\partial\alpha + \overline{\partial}\beta$) on B. The expression

$$[T_g(\omega^M, h^\eta) + \widetilde{\mathrm{ch}}(\mathrm{R}^\bullet f_{V,*}\eta|_V, h^{\mathrm{R}^\bullet f_{V,*}\eta|_V}, h'^{\mathrm{R}^\bullet f_*\eta})]$$

refers to the element of P^B_B obtained from $T_g(\omega^M, h^\eta) + \widetilde{\mathrm{ch}}(\mathrm{R}^\bullet f_{V,*}\eta|_V, h^{\mathrm{R}^\bullet f_{V,*}\eta|_V}, h'^{\mathrm{R}^\bullet f_*\eta})$ via conjecture 0.1. The terminology is otherwise the same as in Th. 0.1 of [3].

Remarks.

(1) The conjectures 0.1 and 0.2 are the strongest conjecture we would venture to make. For many of the applications that we have in mind, it would be sufficient

to prove them with the supplementary assumption that $f : M \to B$ is *semi-stable*, i.e. that f satisfies the assumptions described in the introduction of [1]. In the terminology of the introduction of [1], we would set $U := S \backslash \Delta$.

(2) One could generalize the conjectures above by assuming only that G is a compact Lie group rather than a finite group.

(3) It seems important to choose the ζ-function normalisation in the definition of *both* the equivariant analytic torsion form and the Bott-Chern secondary class (which is what we have done above). One could probably choose a different type of normalisation (e.g. a normalisation obtained via the deformation to the normal cone) without making conjecture 0.1 false but it seems important to choose the same type of normalisation for the equivariant analytic torsion form and the Bott-Chern secondary class.

(4) Conjectures 0.1 and 0.2 have very interesting applications in arithmetic geometry. They should for instance make conceptual proofs of results of Kudla-Rapoport, Bost-Kühn and Bruinier-Kühn possible, which would be based on an extension of the arithmetic fixed point formula of Köhler-Rössler.

References

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