The Adams-Riemann-Roch theorem and applications

PIMS, Sept. 11th 2007

The Adams operations

Let X be a noetherian scheme.

Definition

For any $k \geqslant 1$, the k-th Adams operation is the only functorial ring endomorphism

$$\psi^k : \mathrm{K}(X) \to \mathrm{K}(X)$$

such that

$$\psi^k(L) = L^{\otimes k}$$

for any line bundle L on X.

The diagram in question

Let $f : X \to Y$ be a proper morphism of noetherian schemes. A natural question is: does the following diagram commute ?

$$\begin{array}{ccc}
\mathrm{K}(X) & \xrightarrow{\psi^k} & \mathrm{K}(X) \\
\downarrow{\scriptstyle \mathrm{R}^\bullet f_*} & & \downarrow{\scriptstyle \mathrm{R}^\bullet f_*} \\
\mathrm{K}(Y) & \xrightarrow{\psi^k} & \mathrm{K}(Y)
\end{array}$$

Cannibals & Riemann-Roch I

This answer is again NO. Nevertheless, the deviation from commutativity can be measured precisely. To describe this deviation, we need the

Definition (Bott's cannibalistic classes)

The operations $(\theta^k)_{k\in\mathbb{N}^*}$ functorially associate elements of $K(X)$ to vector bundles on X. The following three properties determine them uniquely.

- *For every line bundle L on X, we have*

$$\theta^k(L) = 1 + L + L^{\otimes 2} + \cdots + L^{\otimes(k-1)}.$$

- *For any exact sequence of vector bundles*

$$0 \to E' \to E \to E'' \to 0$$

on X we have $\theta^k(E')\theta^k(E'') = \theta^k(E)$.

Cannibals & Riemann-Roch II

The deviation is now described by the

Theorem (Adams-Riemann-Roch theorem; Grothendieck et al.)

Suppose that X and Y are quasi-projective over an affine noetherian scheme and that f is smooth and projective. Then

- *The element $\theta^k(\Omega_f)$ is invertible in $\mathrm{K}(X)[\frac{1}{k}]$.*
- *The equality*

$$\psi^k(\mathrm{R}^\bullet f_*(x)) = \mathrm{R}^\bullet f_*(\theta^k(\Omega_f)^{-1} \otimes \psi^k(x))$$

  holds in $\mathrm{K}(Y)[\frac{1}{k}]$.

Adams-Riemann-Roch & the de Rham complex

We shall apply the Adams-Riemann-Roch theorem to the element

$$x = \Lambda_{-1}(\Omega_f) = \sum_{k \geqslant 0} (-1)^k \Lambda^k(\Omega_f).$$

We shall need the

Lemma

$$\psi^k(\Lambda_{-1}(\Omega_f)) = \theta^k(\Omega_f)\Lambda_{-1}(\Omega_f)$$

For example, if $\mathrm{rk}(\Omega_f) = 1$, the lemma amounts to the equality

$$1 - \Omega_f^{\otimes k} = (1 + \Omega_f + \Omega_f^{\otimes 2} + \cdots + \Omega_f^{\otimes (k-1)}) \otimes (1 - \Omega_f).$$

Computations I

Using the Lemma, we can compute

$$\begin{aligned}\psi^k(\mathrm{R}^\bullet f_*(\Lambda_{-1}(\Omega_f))) &= \mathrm{R}^\bullet f_*(\theta^k(\Omega_f)^{-1}\Lambda_{-1}(\Omega_f)) \\ &= \mathrm{R}^\bullet f_*(\Lambda_{-1}(\Omega_f))\end{aligned}$$

which implies that

$$\boxed{\sum_{j\geqslant 0}(-1)^j(\psi^k - \mathrm{Id})(H^j_{\mathrm{dR}}(X/Y)) = 0 \quad (*)}$$

in $\mathrm{K}(Y)[\frac{1}{k}]$.

Computations II

We shall consider the image of (*) under certain Chern classes. Consider the symmetric functions

$$x_1^t + x_2^t + \cdots + x_r^t$$

where $t, r \geqslant 1$. The associated Chern classes will be denoted by $\mathrm{ch}_t(\bullet)$. The classes $\mathrm{ch}_t(\bullet)$ give rise to morphisms of abelian groups

$$\mathrm{ch}_t : \mathrm{K}(\bullet) \to \mathrm{CH}^t(\bullet)$$

such that

$$\mathrm{ch}_t(L) = \mathrm{c}_1(L)^t$$

for any line bundle.

Computations III

The effect of $\mathrm{ch}_t(\bullet)$ on the Adams operations is described by the equation

$$\mathrm{ch}_t(\psi^k(\bullet)) = k^t \cdot \mathrm{ch}_t(\bullet).$$

Applying ch_t to (*), we thus get that

$$(k^t - 1) \cdot \Big(\sum_{j \geqslant 0} (-1)^j \mathrm{ch}_t(H^j_{\mathrm{dR}}(X/Y)) \Big) = 0$$

in $\mathrm{CH}^t(Y)[\frac{1}{k}]$.

Computations IV

Since the last equality is true for any $k \geqslant 2$, we even obtain that

$$\mathrm{gcd}\{(k^t-1)k^\infty\}_{k\geqslant 2}\cdot\Big(\sum_{j\geqslant 0}(-1)^j\mathrm{ch}_t(H^j_{\mathrm{dR}}(X/Y))\Big)=0$$

in $\mathrm{K}(Y)$.

In can be shown that

$$N_t := \mathrm{gcd}\{(k^t-1)k^\infty\}_{k\geqslant 2} = \begin{cases} 2\cdot\prod_{p-1|t} p^{\mathrm{ord}_p(t)+1} & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd}\end{cases}$$

Notice that if t is even, then

$$\prod_{p-1|t} p^{\mathrm{ord}_p(t)+1} = \mathrm{Denominator}(B_t/t).$$

A conjecture

This leads to the following conjecture:

Conjecture

Suppose that the base scheme is $\mathbb{C}$ and that Y is smooth. The equality

$$N_t \cdot \mathrm{ch}_t(H^j_{\mathrm{dR}}(X/Y)) = 0$$

holds in $\mathrm{CH}^t(Y)$ for any $j \geqslant 0$ and $t \geqslant 1$.

Notice that the weaker statement that $\mathrm{ch}_t(H^j_{\mathrm{dR}}(X/Y))$ is a torsion element of $\mathrm{CH}^t(Y)$ is also conjectural.

In the following, we shall call this weaker statement *the weak conjecture*.

Evidence for the conjecture

- The weak conjecture is true for $j = 1$ [Esnault, Viehweg; Maillot & R.];
- The natural characteristic p analog of the weak conjecture is true, if the Hodge to de Rham spectral sequence degenerates [Maillot & R.];
- The image of the conjecture under the cycle class map is true [Grothendieck, Thomas; Maillot & R. (different proof)].

When $\dim(X) = \dim(Y)$, (*) shows that the conjecture is true. This was also shown earlier by Fulton and MacPherson, using different techniques.

Characteristic p, I

If Y is defined over $\mathbb{F}_l$ (l a prime number), one can exhibit two spectral sequences.

The *conjugate spectral sequence*

$$E^2_{pq} = F^*_Y R^p f_*(\Omega^q_{X/Y}) \Longrightarrow H^{p+q}_{\mathrm{dR}}(X/Y)$$

and the spectral sequence which is the image of the *Hodge to de Rham spectral sequence* under F^*_Y

$$E^1_{pq} = F^*_Y R^q f_*(\Omega^p_{X/Y}) \Longrightarrow F^*_Y H^{p+q}_{\mathrm{dR}}(X/Y).$$

Characteristic p, II

A comparison of these two spectral sequences leads to the following result.

Lemma

If the Hodge to de Rham spectral sequence degenerates then the conjugate spectral sequence also degenerates.

Characteristic p, III

Thus, if the Hodge to de Rham spectral sequence degenerates then

$$\mathrm{ch}_t\big(H^j_{\mathrm{dR}}(X/Y)\big) = \mathrm{ch}_t\big(F_Y^* H^j_{\mathrm{dR}}(X/Y)\big) \in \mathrm{CH}^t(Y).$$

for any $j \geqslant 0$. In other words

$$(1 - l^t) \cdot \mathrm{ch}_t(H^j_{\mathrm{dR}}(X/Y)) = 0$$

in $\mathrm{CH}^t(Y)$.

This last equality implies the positive characteristic analog of the weak conjecture.

Generalisations

- Similar techniques can be used to obtain vanishing statements for Chern classes in an equivariant situation; it that case, the arithmetic of the field generated by the eigenvalues of the automorphism of the Gauss-Manin bundle will enter the picture;
- The above conjecture is also expected to be true if f is a semi-stable fibration and the de Rahm complex is replaced by the de Rahm complex with logarithmic singularities along the singularities of f.