Around the "full" Mordell-Lang conjecture

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The generalized Mordell-Lang conjecture in char. 0

The following theorem is the result of the work of Faltings, Serre, Hindry and McQuillan:

Theorem (generalized Mordell-Lang conjecture)

Let A be an abelian variety over $\mathbb{C}$. Let $X \hookrightarrow A$ be a closed subvariety, which is not the translate of an abelian subvariety of A. Let $\Gamma \subseteq A(\mathbb{C})$ be a group such that $\Gamma \otimes \mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space. Then $X \cap \Gamma$ is not Zariski dense in X.

The special case of the generalized Mordell-Lang conjecture where Γ is a group of torsion points (resp. is a finitely generated group) is referred to as the Manin-Mumford conjecture (resp. the Mordell-Lang conjecture).

Special cases

The following are consequences of GML.

Let C be a curve of genus $\geqslant 2$ defined over a number field K.

Theorem (Mordell conjecture; Faltings [1984])

$C(K)$ is finite.

Theorem (Manin-Mumford conjecture for C; Raynaud [1983])

Let $P \in C(\overline{K})$. Then the set

$$\{Q \in C(\overline{K}) | \exists n \in \mathbb{N}^* \text{ such that } nQ - nP \text{ is principal}\}$$

is finite.

Positive characteristic

It is natural to wonder if the analog of GML is true in positive characteristic. The following is known.

Let K be an algebraically closed field of char. $p > 0$. Let A be an abelian variety over K. Suppose that A has no factors, which are defined over a finite field.

Let $X \hookrightarrow A$ be a closed subvariety of A. Suppose that X is not the translate of an abelian subvariety of A.

Finally, let $\Gamma \subseteq A(K)$ be a group st $\Gamma \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p$-module of finite rank.

Theorem (Hrushovski [1996])

$X \cap \Gamma$ is not Zariski dense in X.

The "full" generalized Mordell-Lang conjecture

Can one replace $\mathbb{Z}_p$ by $\mathbb{Q}$ in Hrushovski's theorem ?

The "full" generalized Mordell-Lang conjecture (FGML) asserts that this is the case.

The conjecture FGML is known to be true if

- X is a curve (Kim, 1997)
- Γ is a torsion group (Pink-R. (2004), Scanlon (2005))
- A is of a certain type (e.g. a product of elliptic curves); see below

One can show show (R., Ziegler, Benoît-Bouscaren-Pillay) that

$$\text{FGML for a torsion group} \Rightarrow \text{GML in char. } p$$

but FGML does not seem reducible (to this date) to FGML for a torsion group.

An equivalent form

Ghioca-Moosa-Scanlon have shown that the following conjecture is equivalent to FGML.

Let L be finitely generated field of char. $p > 0$.

Let A be an abelian variety over L. Suppose that $A_{\bar{L}}$ has no factors, which are defined over a finite field.

Let $X \hookrightarrow A$ be a closed subvariety of A, such that $X_{\bar{L}}$ is not the translate of an abelian subvariety of $A_{\bar{L}}$.

Let $L^{\mathrm{perf}} = L^{p^{-\infty}}$ be the maximal purely inseparable extension of L.

Conjecture

$X(L^{\mathrm{perf}})$ is not Zariski dense in X.

Related questions

The conjecture of Ghioca-Moosa-Scanlon (and thus FGML) is obviously verified if $A(L^{\mathrm{perf}})$ is finitely generated. In this context, P. Ziegler formulated the following conjecture (unpublished) that must have also crossed the minds of Ghioca and Scanlon.

Let A and L be as in the previous frame.

Conjecture

Suppose that A is ordinary. Then $A(L^{\mathrm{perf}})$ is finitely generated.

Notice that this conjecture implies that FGML is verified for ordinary abelian varieties.

When $\dim(A) = 1$, it was shown to be true by Ghioca (2005).

The higher dimensional situation

If $\dim(A) \geqslant 1$ one can show the following.

Suppose that L is the function field of a smooth curve over a finite field.

Theorem (R.)

Suppose that A is an ordinary abelian variety over L.

Suppose that there is a place v of good reduction of A over L such that A_v is an abelian variety of p-rank 0.

Then $A(L^{\mathrm{perf}})$ is finitely generated.

In particular, FGML holds in the situation of Theorem 0.5.

Notice that this result implies the result of Ghioca for elliptic curves.

Semiampleness of Hodge bundles over $\mathbb{C}$

We now turn to a perhaps seemingly unrelated subject.

Let C be a smooth, proper and connected curve over the complex numbers $\mathbb{C}$. Let $\mathcal{G} \to C$ be a smooth group scheme over C and let $\epsilon_{\mathcal{G}} : C \to \mathcal{G}$ be its zero-section.

Suppose that the fibre of $\mathcal{G}$ over the generic point of C is an abelian variety

Theorem (Griffiths)

The inequality $\mu_{\min}(\epsilon_{\mathcal{G}}^* \Omega^1_{\mathcal{G}/C}) \geqslant 0$ *holds.*

Here $\mu_{\min}(\cdot)$ takes the minimal slope (for the Harder-Narasimhan filtration).

In other words, $\epsilon_{\mathcal{G}}^* \Omega^1_{\mathcal{G}/C}$ is semiample.

Semiampleness in positive characteristic

The positive characteristic analog of Griffiths's theorem is false (counterexample by Moret-Bailly).

Nevertheless, the following result can be proven.

Let C be a smooth, proper and connected curve over $\bar{\mathbb{F}}_p$. Let $\mathcal{G} \to C$ be a smooth group scheme over C and let $\epsilon_{\mathcal{G}} : C \to \mathcal{G}$ be its zero-section.

Proposition (R.)

Suppose that the fibre of $\mathcal{G}$ over the generic point of C is an ordinary abelian variety. Then

$$\bar{\mu}_{\min}(\epsilon_{\mathcal{G}}^{*}\Omega^1_{\mathcal{G}/C}) \geqslant 0.$$

Here $\bar{\mu}_{\min}(\cdot)$ takes the Frobenius stabilized minimal slope.

Ampleness

When is $\epsilon_{\mathcal{G}}^{*}\Omega^{1}_{\mathcal{G}/C}$ actually ample ?

It turns out that in positive characteristic, a simple sufficient condition can be given.

Let $C, \mathcal{G}, \epsilon$ be as in the last frame.

Proposition (R.)

Suppose that the fibre of $\mathcal{G}$ over the generic point of C is an ordinary abelian variety.

Suppose furthermore that there is a closed point $v \in C$ such that $\mathcal{G}_v$ is an abelian variety of p-rank 0.

Then

$$\bar{\mu}_{\min}(\epsilon_{\mathcal{G}}^{*}\Omega^{1}_{\mathcal{G}/C}) > 0.$$

In other words, $\epsilon_{\mathcal{G}}^{*}\Omega^{1}_{\mathcal{G}/C}$ is ample.

Back to purely inseparable points

The relevance of the few last propositions to our original concern is the following result.

Let C be a smooth, proper and connected curve over $\mathbb{F}_q$. Let $\mathcal{G} \to C$ be a smooth group scheme over C and let $\epsilon_{\mathcal{G}} : C \to \mathcal{G}$ be its zero-section. Let $L := \kappa(C)$ be the function field of C.

Theorem (R.)

Suppose that $\mathcal{G}$ is semiabelian and that the fibre of $\mathcal{G}$ over the generic point of C is an ordinary, principally polarized abelian variety.

Suppose that $\epsilon_{\mathcal{G}}^* \Omega^1_{\mathcal{G}/C}$ is ample.

Then $\mathcal{G}(L^{\mathrm{perf}})$ is finitely generated.

Structure of the proof of the last theorem

Write $\omega := \epsilon_{\mathcal{G}}^{*}\Omega^{1}_{\mathcal{G}/C}$.

Fix $P \in \mathcal{G}(L^{q^{-n}})\backslash\mathcal{G}(L^{q^{-n+1}})$.

The point P corresponds to a commutative diagram of $\mathbb{F}_q$-schemes

$$\begin{array}{ccc} & & \mathcal{G} \\ & \overset{P}{\nearrow} & \big\downarrow \pi_L \\ \mathrm{Spec}\ L & \xrightarrow[\mathrm{Frob}_L^n]{} & \mathrm{Spec}\ L \end{array}$$

such that the residue field extension $L|\kappa(P(\mathrm{Spec}\ L))$ is of degree 1.

In particular, the map of L-vector spaces $P^{*}\Omega^{1}_{\mathcal{G}/\mathbb{F}_q} \to \Omega^{1}_{L/\mathbb{F}_q}$ arising from the diagram is non-zero.

Now recall that there is a canonical exact sequence

$$0 \to \pi_L^*\Omega^1_{L/\mathbb{F}_q} \to \Omega^1_{\mathcal{G}/\mathbb{F}_q} \to \Omega^1_{\mathcal{G}/L} \to 0.$$

Furthermore the map

$$\mathrm{Frob}_L^{n,*}\Omega^1_{L/\mathbb{F}_q} \stackrel{F_L^{n,*}}{\to} \Omega^1_{L/\mathbb{F}_q}$$

vanishes.

Also, we have a canonical identification $\Omega^1_{\mathcal{G}/L} = \pi_L^*\omega_L$.

Thus the natural surjection $P^*\Omega^1_{\mathcal{G}/\mathbb{F}_q} \to \Omega^1_{L/\mathbb{F}_q}$ gives rise to a non-zero map

$$\boxed{\phi_n : \mathrm{Frob}_L^{n,*}\omega_L \to \Omega^1_{L/\mathbb{F}_q}}$$

What are the poles and zeros of the map ϕ_n ?

The next lemma answers this question.

Let E be the reduced closed subset, which is the union of the points $s \in S$, such that the fibre $\mathcal{A}_s$ is not complete.

Lemma

The morphism ϕ_n extends to a morphism of vector bundles

$$\mathrm{Frob}_C^{n,*}\omega \to \Omega^1_{C/\mathbb{F}_q}(E).$$

The proof of this lemma is easy if E is empty but the proof of the general case requires the existence of relative compactifications of semiabelian schemes, in the form provided in the book by Chai-Faltings, as well as some log-geometry.

To deduce the theorem from the lemma, just notice that if ω is ample then ϕ_n must vanish if $n >> 0$.

Further speculations

Here are some natural conjectures/questions.

The work of Faltings, Griffiths, Viehweg and others should show that the answer to the following question is positive, but I haven't be able to determine this yet.

Let C be a smooth, proper and connected curve over the complex numbers $\mathbb{C}$. Let $\mathcal{G} \to C$ be a smooth group scheme over C and let $\epsilon_{\mathcal{G}} : C \to \mathcal{G}$ be its zero-section.

Question

Suppose that the fibre of $\mathcal{G}$ over the generic point of C is an abelian variety of trace 0 over $\mathbb{C}$.

Is $\epsilon_{\mathcal{G}}^*\Omega^1_{\mathcal{G}/C}$ ample ?

The natural positive characteristic analog of the last question would be following.

Let C be a smooth, proper and connected curve over $\bar{\mathbb{F}}_p$. Let $\mathcal{G} \to C$ be a smooth group scheme over C and let $\epsilon_{\mathcal{G}} : C \to \mathcal{G}$ be its zero-section.

Conjecture

Suppose that the fibre of $\mathcal{G}$ over the generic point of C is an ordinary abelian variety, which has trace 0 over $\bar{\mathbb{F}}_p$.

Then $\epsilon_{\mathcal{G}}^{*}\Omega^1_{\mathcal{G}/C}$ is ample.

Finally, I would like to propose the following conjecture, which is a generalization of a variant of the Bombieri-Lang conjecture in positive characteristic.

Conjecture

Let X be a variety of general type over a finitely generated field L of characteristic $p > 0$.

Suppose that X has no generically finite rational cover by a variety W, such that $W_{\bar{L}}$ has a model over a finite field.

Then $X(L^{\mathrm{perf}})$ is not Zariski dense in X.

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