A new approach to elliptic units

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Preamble: the cyclotomic units

A **unit** is an algebraic integer whose inverse is also an algebraic integer.

A **cyclotomic unit** is an algebraic integer of the form

$$\boxed{1 - \exp(2i\pi\frac{k}{m})}$$

where $(k, n) = 1$ and m is a composite number.

This is the prime example of a unit, which is not a root of unity. They are built from the torsion points of the torus $\mathbb{G}_m$.

Elliptic units

Elliptic units are built from the torsion points of an elliptic curve with potential good reduction over an algebraic number field.

Let $E = \mathbb{C}/[1, \tau]$ be such a curve.

Let z be a point of order m of E, where m is composite.

The complex number

$$\boxed{\left(e^{-z\cdot\mathrm{quasiperiod}(z)/2}\sigma(z)\Delta(\tau)^{\frac{1}{12}}\right)}$$

is the **elliptic unit** attached to z.

The Néron-Tate height

Recall that the **Néron-Tate height** $\mathrm{NT}(\cdot)$ is the only height function associated to the origin of an elliptic curve, such that

(1) $m^2 \cdot \mathrm{NT}(P) = \mathrm{NT}(m \cdot P)$ for all $m \geqslant 2$;
(2) $\mathrm{NT}(0) = 0$.

Formula for the Néron-Tate height

Let $E = \mathbb{C}/[1, \tau]$ be an elliptic curve with a model $\mathcal{E}$

$$y^2 = x^3 + Ax + B$$

over $\mathcal{O}_K$ (K a number field).
Let $P := (x, y) \in K^2$ be a point on E, which reduces into the smooth locus of $\mathcal{E}$.

Theorem (Tate)

$$\begin{aligned} \mathrm{NT}(P) &= \frac{1}{[K:\mathbb{Q}]}\Big[\log|N_{K/\mathbb{Q}}(16(4A^3 + 27B^2))| \\ &- \frac{1}{2}\log|N_{K/\mathbb{Q}}(\mathrm{Denominator}(x))| \\ &- \sum_{v \text{ ar.}} n_v \cdot \log|e^{-z(P_v)\mathrm{quasiperiod}(z(P_v))/2}\sigma(z(P_v))\Delta(\tau_v)^{\frac{1}{12}}| \Big] \end{aligned}$$

Mazur and Tate's refinement of the Néron-Tate height I

Mazur and Tate constructed a refinement $\mathrm{MT}(\cdot)$ of the Néron-Tate height $\mathrm{NT}(\cdot)$.

The refined height $\mathrm{MT}(\cdot)$ has values in the group

$$\widehat{\mathrm{Cl}}(\mathcal{O}_K) := \Big(\bigoplus_{v \text{ n.}-\text{ar.}} \mathbb{Z} \bigoplus_{v \text{ ar.}} \mathbb{R} \Big) \,/\, \{ \oplus_{v \text{ n.}-\text{ar.}} v(k) \oplus_{v \text{ ar.}} \log |k|_v^{-2}, \;\; k \in K^* \}$$

which is a quotient of the idele class-group of K.

Mazur and Tate's refinement of the Néron-Tate height II

- The group $\widehat{\mathrm{Cl}}(\mathcal{O}_K)$ fits in a diagram

$$\begin{array}{ccccccccc}
\mathcal{O}_K^* & \xrightarrow{reg} & \mathbb{R}^{\#(\text{ar. v.})} & \longrightarrow & \widehat{\mathrm{Cl}}(\mathcal{O}_K) & \longrightarrow & \mathrm{Cl}(\mathcal{O}_K) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \widehat{\mathrm{Cl}}(\mathbb{Z}) & & & &
\end{array}$$

where the first row is exact and the map *reg* is $(-2) \times$ the Dirichlet regulator.

- $\mathrm{MT}(P)$ is mapped to $[K : \mathbb{Q}] \cdot \mathrm{NT}(P)$ by the map

$$\widehat{\mathrm{Cl}}(\mathcal{O}_K) \to \widehat{\mathrm{Cl}}(\mathbb{Z}) = \mathbb{R}.$$

Formula for the refined height

Theorem (Mazur-Tate)

The refined height $\mathrm{MT}(P)$ *of* P *is given by the formula*

$$\bigoplus_{v\ \mathrm{n.-ar.}} \Big(v(16(4A^3 + 27B^2)) - \frac{1}{2} v(\mathrm{Denominator}(x(P))) \Big)$$

$$\bigoplus_{v\ \mathrm{ar.}} -\log |e^{-z(P_v)\mathrm{quasiperiod}(z(P_v)/2)} \sigma(z(P_v)) \Delta(\tau_v)^{\frac{1}{12}}|^2.$$

Interpretation of the formula of Mazur and Tate

- The term $v(16(4A^3+27B^2))$ comes from the bad reduction of E.
- The term $-\frac{1}{2}v(\mathrm{Denominator}(x(P)))$ is the intersection multiplicity of the section of $\mathcal{E}$ defined by P with the unit section of $\mathcal{E}$.
- The term $-\log|e^{-z(P_v)\mathrm{quasiperiod}(z(P_v)/2)}\sigma(z(P_v))\Delta(\tau_v)^{\frac{1}{12}}|^2$ is an archimedean intersection multiplicity and is best understood via Arakelov theory.

Properties of the refined height

Just as the Néron-Tate height, the refined height $\mathrm{MT}(\cdot)$ has the following properties:

(1) $\mathrm{MT}(m \cdot P) = m^2 \cdot \mathrm{MT}(P)$;

(2) $\mathrm{MT}(0) = 0$.

The refined height and elliptic units

Proposition

Let P be an m-torsion point on an elliptic curve over K with good reduction everywhere. Suppose that P is defined over K. If m is composite, then

$$m^2 \cdot \bigoplus_{v \text{ ar.}} -\log|e^{-z(P_v)\mathrm{quasiperiod}(z(P_v))/2}\sigma(z(P_v))\Delta(\tau_v)^{\frac{1}{12}}|^2$$

is in the image of the Dirichlet regulator map.

Hence

$$|e^{-z(P_v)\mathrm{quasiperiod}(z(P_v))/2}\sigma(z(P_v))\Delta(\tau_v)^{\frac{1}{12}}|^{-2m^2}$$

is a unit for each archimedean v and satisfies an obvious reciprocity law.

Question: can one generalise to abelian varieties the theorem of Mazur and Tate and apply the above method to associate units to torsion points on higher-dimensional abelian varieties ?

We shall give a partial answer to this question using Arakelov theory. The next few slides recall the relevant results from that theory.

Hermitian line bundles

The first notion we shall need is the following.

Definition

A hermitian line bundle $\overline{L}$ on an arithmetic variety X over $\mathcal{O}_K$ is a line bundle L on X together with a hermitian metric h_σ on $L_\sigma(\mathbb{C})$ for each $\sigma \in \mathrm{Hom}(K, \mathbb{C})$. The h_σ are required to be conjugation invariant.

Example: the Hodge bundle

Let $p : \mathcal{A} \to U$ be an abelian scheme of relative dimension g over an open subscheme U of $\mathrm{Spec}\ \mathcal{O}_{\mathrm{K}}$. Let $\omega := p_*\big(\Lambda^g(\Omega_{\mathcal{A}/U})\big)$ be the direct image of the Hodge bundle of $\mathcal{A}$. It carries a natural metric given by the formula

$$\langle \eta, \lambda \rangle := \frac{i^g(-1)^{g(g+1)/2}}{(2\pi)^g} \int_{\mathcal{A}_\sigma(\mathbb{C})} \eta \wedge \overline{\lambda}.$$

We shall write $\overline{\omega}$ for the resulting hermitian line bundle.

Cubist structures on hermitian line bundles over abelian schemes

Let $p : \mathcal{A} \to U$ be an abelian scheme as before. A hermitian line bundle $\overline{L}$ on $\mathcal{A}$ will be called **cubist** if L satisfies the theorem of the cube on $\mathcal{A}/U$ and if the isomorphism which is part of the latter theorem is an isometry.

Lemma (Moret-Bailly; Breen)

$\overline{L}$ is cubist if and only if $\overline{L}|_0 \simeq \overline{\mathcal{O}}_U$ and the first Chern form of $\overline{L}(\mathbb{C})$ is translation invariant.

The first arithmetic Chern class of hermitian line bundles

Let $\overline{M}$ be a hermitian line bundle on U. Let s be a section of M. We define the **first arithmetic Chern class** $\widehat{c}_1(\overline{M})$ of $\overline{M}$ by the formula

$$\widehat{c}_1(\overline{M}) := \bigoplus_{v \text{ n.-ar.}} v(Z(s)) \bigoplus_{v \text{ ar.}} -\log\ h_{M,v}(s_v, s_v) \in \widehat{\mathrm{Cl}}(U)$$

This definition does not depend on the choice of s.

The height of Mazur and Tate via cubist hermitian line bundles

We consider again the abelian scheme $p : \mathcal{A} \to U$. Let $\overline{L}$ be a cubist, hermitian line bundle on $\mathcal{A}$ such that L_K is ample and symmetric.

Let $P \in \mathcal{A}_K(K)$. Let $s_P : U \to \mathcal{A}$ be the resulting section.

Theorem (Moret-Bailly)

The refined height $\mathrm{MT}(P, L_K)$ of the point P with respect to the line bundle L_K is $\widehat{c}_1(s_P^*(\overline{L}))$.

A result of Moret-Bailly

Let A be an abelian variety of dim. g and let $\Theta \subseteq A$ be an effective symmetric divisor. Suppose that $\mathcal{O}(\Theta)$ defines a principal polarisation. Endow $\mathcal{O}(\Theta)$ with the unique hermitian metric h whose first Chern form is translation invariant and such that

$$\int_A h(s_\Theta, s_\Theta)\ dh = 2^{-g/2}.$$

Proposition (Moret-Bailly)

Let $\theta(z)$ be the theta function associated to A and Θ. Then

$$h(s_\Theta(z), s_\Theta(z)) = \det(\Im(\Omega))^{\frac{1}{2}} e^{-2\pi y(\Im(\Omega))^{-1}y} |\theta(z)|^2$$

where $z = x + i \cdot y \in \mathbb{C}^g$ and $A(\mathbb{C}) \simeq \mathbb{C}^g/\Omega$.

The "formule clé"

We consider again the abelian scheme $p : \mathcal{A} \to U$. Let e be its unit section. Suppose that Θ is an effective symmetric divisor defining a principal polarisation on each fibre of p. Make $\mathcal{O}(\Theta)$ into a hermitian line bundle as in the previous slide.

Theorem (Moret-Bailly; Chai-Faltings)

There is a canonical isometric isomorphism

$$e^*\overline{\mathcal{O}}(\Theta)^{\otimes 8} \simeq \overline{\omega}^{\otimes 4}.$$

A generalisation of the formula of Mazur and Tate

We still consider $p : \mathcal{A} \to U$. Let $A := A_K$ be its generic fiber and let $P \in A(K)$. Let $s_P : U \to \mathcal{A}$ be the resulting section.

We get:

$$\begin{aligned}
8 \cdot \mathrm{MT}(P, \mathcal{O}(\Theta)_K) &= 8 \cdot \widehat{c}_1(s_P^*(\overline{\mathcal{O}}(\Theta))) - 4 \cdot \widehat{c}_1(\overline{\omega}) = \\
&= \bigoplus_{v \text{ n.-ar.}} 8 \cdot (\text{Inter. Mult. of } \mathrm{s_{P,v}} \text{ and } \Theta_\mathrm{v}) \cdot e_v^{-1} \\
&\quad \bigoplus_{v \text{ ar.}} -8 \cdot \log \Big[\det(\Im(\Omega_v))^{\frac{1}{2}} \cdot \\
&\quad e^{-2\pi y(P_v)(\Im(\Omega_v))^{-1} y(P_v)} |\theta(z(P_v))|^2 \Big] \\
&\quad -4 \cdot \widehat{c}_1(\omega)
\end{aligned}$$

Application to the construction of units of higher genus I

We shall now specialise the previous formula to the case where P is a torsion point and $\mathcal{A}$ is the Néron model of the Jacobian of a curve of genus $g = 2$.

The following result will replace the condition on m to be composite.

Theorem (Boxall, Grant)

Let C be a curve of genus 2, embedded into its Jacobian via a Weierstrass point. Then C does not meet any non-trivial 3-torsion point.

Application to the construction of units of higher genus II

Proposition

Let C be a curve of genus 2 and suppose that C has potential good reduction over a number field outside $S \subseteq \operatorname{Spec} \mathbb{Z}$. Let $P \neq Q$ be non-trivial points of order 3 on $\operatorname{Jac}(C)$. Then the real number

$$\frac{|\theta(z(P))|e^{-\pi^t y(P)(\Im(\Omega_C)^{-1})y(P)}}{|\theta(z(Q))|e^{-\pi^t y(Q)(\Im(\Omega_C)^{-1})y(Q))}}$$

is a unit outside $S \cup 3$.

Application to the construction of units of higher genus III

In special cases the invariant $\widehat{c}_1(\overline{\omega})$ can be computed, as in the following example.

Proposition

Let C be the curve $y^2 = x^5 - 1$. Let P be a non-trivial 3-torsion point on $\mathrm{Jac}(C)$. Then the real number

$$|\theta(z(P))|^2 e^{-2\pi {}^t y(P)(\Im(\Omega_C)^{-1})y(P)}\Gamma(\frac{1}{5})^5\Gamma(\frac{2}{5})^3\Gamma(\frac{3}{5})^{-1}$$

is a unit outside $3 \cup 5$.