

On a canonical class of Green currents associated with the unit sections of abelian schemes

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Preamble: the exponential function and the Siegel functions

A **unit** is an algebraic integer whose inverse is also an algebraic integer.

A **cyclotomic unit** is an algebraic integer of the form

$$1 - \exp(2i\pi \frac{k}{m})$$

where $(k, m) = 1$ and m is a composite number.

This is the prime example of a unit, which is not a root of unity. Such units are built from the torsion points of the torus \mathbb{G}_m .

Elliptic units

Elliptic units are built from the torsion points of an elliptic curve with potential good reduction over an algebraic number field.

Let $E = \mathbb{C}/[1, \tau]$ be such a curve.

Let z be a point of order m of E , where m is composite.

The complex number

$$\boxed{e^{-z \cdot \text{quasiperiod}(z)/2} \sigma(z) \Delta(\tau)^{\frac{1}{12}}} \quad (*)$$

is a unit, called the **elliptic unit** attached to z .

The function $(*)$ on $E(\mathbb{C})$ is usually called a **Siegel function**.

The Néron-Tate height

Why do Siegel functions give rise to units ?

To explain why they do so, the best way is to relate them to height functions.

Recall that the **Néron-Tate height** $\text{NT}(\cdot)$ is the only height function associated with the origin of an elliptic curve, such that

$$m^2 \cdot \text{NT}(P) = \text{NT}(m \cdot P)$$

for all $m \geq 2$.

Formula for the Néron-Tate height

Let $E = \mathbb{C}/[1, \tau]$ be an elliptic curve with a model \mathcal{E}

$$y^2 = x^3 + Ax + B$$

over \mathcal{O}_K (K a number field).

Let $P := (x, y) \in K^2$ be a point on E , which reduces into the smooth locus of \mathcal{E} .

Theorem (Tate)

$$\begin{aligned} \text{NT}(P) &= \frac{1}{[K : \mathbb{Q}]} \left[\log |N_{K/\mathbb{Q}}(16(4A^3 + 27B^2))| \right. \\ &\quad - \frac{1}{2} \log |N_{K/\mathbb{Q}}(\text{Denominator}(x))| \\ &\quad \left. - \sum_{v \text{ ar.}} n_v \cdot \log |e^{-z(P_v)\text{quasiperiod}(z(P_v))/2} \sigma(z(P_v)) \Delta(\tau_v)^{\frac{1}{12}}| \right] \end{aligned}$$

Mazur and Tate's refinement of the Néron-Tate height I

One cannot prove that Siegel functions give rise to units using the Néron-Tate height alone, because the latter involves an averaging.

Mazur and Tate constructed a refinement $MT(\cdot)$ of the Néron-Tate height $NT(\cdot)$.

The refined height $MT(\cdot)$ has values in the group

$$\widehat{Cl}(\mathcal{O}_K) := \left(\bigoplus_{v \text{ n.-ar.}} \mathbb{Z} \bigoplus_{v \text{ ar.}} \mathbb{R} \right) / \left\{ \bigoplus_{v \text{ n.-ar.}} v(k) \bigoplus_{v \text{ ar.}} \log |k|_v^{-2}, k \in K^* \right\}$$

which is a quotient of the idele class-group of K .

Mazur and Tate's refinement of the Néron-Tate height II

- The group $\widehat{\text{Cl}}(\mathcal{O}_K)$ fits in a diagram

$$\begin{array}{ccccccc} \mathcal{O}_K^* & \xrightarrow{\text{reg}} & \mathbb{R}^{\#(\text{ar. v.})} & \longrightarrow & \widehat{\text{Cl}}(\mathcal{O}_K) & \longrightarrow & \text{Cl}(\mathcal{O}_K) \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & \widehat{\text{Cl}}(\mathbb{Z}) & & \end{array}$$

where the first row is exact and the map *reg* is $(-2) \times$ the Dirichlet regulator.

- $\text{MT}(P)$ is mapped to $[K : \mathbb{Q}] \cdot \text{NT}(P)$ by the map

$$\widehat{\text{Cl}}(\mathcal{O}_K) \rightarrow \widehat{\text{Cl}}(\mathbb{Z}) = \mathbb{R}.$$

Formula for the refined height

Just as the Néron-Tate height, the refined height $\text{MT}(\cdot)$ has the property that

$$\text{MT}(m \cdot P) = m^2 \cdot \text{MT}(P).$$

Theorem (Mazur-Tate)

The refined height $\text{MT}(P)$ of P is given by the formula

$$\bigoplus_{\substack{v \text{ n.-ar.}}} \left(v(16(4A^3 + 27B^2)) - \frac{1}{2}v(\text{Denominator}(x(P))) \right) \\ \bigoplus_{v \text{ ar.}} -\log \left| e^{-z(P_v)\text{quasiperiod}(z(P_v)/2)} \sigma(z(P_v)) \Delta(\tau_v)^{\frac{1}{12}} \right|^2.$$

Interpretation of the formula of Mazur and Tate

- The term $v(16(4A^3 + 27B^2))$ comes from the bad reduction of E .
- The term $-\frac{1}{2}v(\text{Denominator}(x(P)))$ is the intersection multiplicity of the section of \mathcal{E} defined by P with the unit section of \mathcal{E} .
- The term $-\log |e^{-z(P_v)\text{quasiperiod}(z(P_v)/2)}\sigma(z(P_v))\Delta(\tau_v)^{\frac{1}{12}}|^2$ is an archimedean intersection multiplicity and is best understood via Arakelov theory.

The refined height and elliptic units

Corollary (of the formula of Mazur and Tate)

Let P be an m -torsion point on an elliptic curve over K with good reduction everywhere. Suppose that P is defined over K . If m is composite, then

$$m^2 \cdot \bigoplus_{v \text{ ar.}} -\log \left| e^{-z(P_v)\text{quasiperiod}(z(P_v))/2} \sigma(z(P_v)) \Delta(\tau_v)^{\frac{1}{12}} \right|^2$$

lies in the image of the Dirichlet regulator map.

Hence

$$\left| e^{-z(P_v)\text{quasiperiod}(z(P_v))/2} \sigma(z(P_v)) \Delta(\tau_v)^{\frac{1}{12}} \right|$$

is a unit for each archimedean v .

Questions: do Siegel functions have analogs on any abelian scheme ?

How should the group $\widehat{Cl}(\cdot)$ be defined on a higher-dimensional scheme ?

Is there a natural analog of $MT(\cdot)$ on any abelian scheme ?

Can one generalize the formula of Mazur and Tate to any abelian scheme ?

We shall propose answers to these questions, which are based on Arakelov theory.

Generalization of the group $\widehat{\text{Cl}}(\mathcal{O}_K)$, I

Let X be a regular scheme, which is of finite type over \mathcal{O}_K (K a number field).

Gillet and Soulé defined the **arithmetic Chow group** $\widehat{\text{CH}}^*(X)$.

If $X = \text{Spec } \mathcal{O}_K$, then $\widehat{\text{CH}}^1(X) = \widehat{\text{Cl}}(\mathcal{O}_K)$.

There is for any $g \geq 0$ an exact sequence

$$\underbrace{\text{CH}^{g,g-1}(X)}_{\text{motivic coh. group}} \xrightarrow{\text{cyc}_{\text{an}}} \underbrace{\widetilde{A}^{g-1,g-1}(X_{\mathbb{R}})}_{\text{space of diff. forms}} \xrightarrow{a} \widehat{\text{CH}}^g(X) \rightarrow \underbrace{\text{CH}^g(X)}_{\text{Chow group}} \rightarrow 0$$

analogous to the sequence of Mazur and Tate for $\widehat{\text{Cl}}(\mathcal{O}_K)$.

Generalization of the group $\widehat{\text{Cl}}(\mathcal{O}_K)$, II

The arithmetic Chow group $\widehat{\text{CH}}^*(X)$ is an extension of the ordinary Chow group, which includes differential geometric data on $X(\mathbb{C})$.

It is generated by pairs (Z, g_Z) , where Z is a cycle on X and g_Z is a **Green current** for Z . By definition, such a g_Z has the property that

$$\frac{i}{2\pi} \partial \bar{\partial} g_Z + \underbrace{\delta_{Z(\mathbb{C})}}_{\text{Dirac current}} = \text{smooth current.}$$

For any hermitian vector bundle $\bar{E} := (E, h_{E(\mathbb{C})})$ on X , there are Chern classes $\hat{c}^i(\bar{E}) \in \widehat{\text{CH}}^i(X)$.

The arithmetic Chow groups are covariant for projective and generically smooth morphisms and contravariant for quasi-projective morphisms.

Higher analogs of the Siegel functions I

Let S be a regular scheme, which is of finite type over \mathcal{O}_K .

Let $\pi : \mathcal{A} \rightarrow S$ be an abelian scheme over S of relative dimension g .

We shall write \mathcal{A}^\vee for the dual abelian scheme and $\mathcal{P}/\mathcal{A} \times_S \mathcal{A}^\vee$ for the Poincaré bundle.

This bundle carries a natural hermitian metric and we write $\bar{\mathcal{P}}$ for the corresponding hermitian bundle.

We write S_0 (resp. S_0^\vee) for the unit section of \mathcal{A} (resp. \mathcal{A}^\vee).

Higher analogs of the Siegel functions II

Theorem (existence and unicity of the current $\mathfrak{g}_{\mathcal{A}}$)

There is a real and conjugation invariant current $\mathfrak{g}_{\mathcal{A}}$ of type $(g-1, g-1)$ on $\mathcal{A}^{\vee}(\mathbb{C})$ with the following properties.

- (a) The current $\mathfrak{g}_{\mathcal{A}}$ is a Green current for the unit section S_0^{\vee} of \mathcal{A}^{\vee} .
- (b) We have $(S_0^{\vee}, \mathfrak{g}_{\mathcal{A}}) = (-1)^g p_{\mathcal{A}^{\vee},*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)}$ in $\widehat{\text{CH}}^g(\mathcal{A}^{\vee})_{\mathbb{Q}}$.
- (c) The identity $\mathfrak{g}_{\mathcal{A}} = [n]_* \mathfrak{g}_{\mathcal{A}}$ holds for all $n \geq 2$.

The current $\mathfrak{g}_{\mathcal{A}^{\vee}}$ is uniquely determined by these properties, up to currents of type $\partial(\cdot) + \bar{\partial}(\cdot)$.

Back to $g = 1$

Let $\sigma : S \rightarrow \mathcal{A}$ be a section. We have a pull-back map

$$\sigma^* : \widehat{\text{CH}}^*(\mathcal{A}) \rightarrow \widehat{\text{CH}}^*(S).$$

It can be shown that if $g = 1$, then

$$\text{MT}(\sigma) = (-1)^g \sigma^*(p_{\mathcal{A}^\vee, *}) (\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)}.$$

Furthermore, if $g = 1$ then

$$\mathfrak{g}_{\mathcal{A}} = -\log |e^{-z \cdot \text{quasiperiod}(z)/2} \sigma(z) \Delta(\tau)^{\frac{1}{12}}|^2.$$

Algebraic properties of $\mathfrak{g}_{\mathcal{A}}$

By property (b) in the theorem, for all $n, g \geq 1$, we have

$$[n]^*(p_{\mathcal{A}^\vee, *})(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)} = p_{\mathcal{A}^\vee, *})(\widehat{\text{ch}}(\overline{\mathcal{P}}^{\otimes n})) = n^{2g} \cdot p_{\mathcal{A}^\vee, *})(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)}$$

and thus for any section $\sigma : S \rightarrow \mathcal{A}^\vee$ we have

$$([n] \cdot \sigma)^*(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)} = n^{2g} \cdot \sigma^*(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)}.$$

In particular, if σ is a torsion section and $\sigma(S) \cap S_0^\vee = \emptyset$ we have

$$\sigma^*(\mathfrak{g}_{\mathcal{A}}) \in \text{image}(\text{cyc}_{\text{an}}(\text{CH}^{g, g-1}(S)))_{\mathbb{Q}}.$$

This generalizes the theorem of Mazur and Tate and its Corollary.

Further properties of the current $g_{\mathcal{A}}, I$

Let \mathcal{L} be a rigidified symmetric relatively ample line bundle on \mathcal{A} .

Endow \mathcal{L} with the unique hermitian metric $h_{\mathcal{L}}$, which is compatible with the rigidification and whose curvature form is translation invariant on the fibres of $\mathcal{A}(\mathbb{C}) \rightarrow S(\mathbb{C})$.

Let $\bar{\mathcal{L}} := (\mathcal{L}, h_{\mathcal{L}})$ be the resulting hermitian line bundle.

Let $\phi_{\mathcal{L}} : \mathcal{A} \rightarrow \mathcal{A}^{\vee}$ be the polarisation morphism induced by \mathcal{L} .

Further properties of the current $\mathfrak{g}_{\mathcal{A}}$, II

- (distributivity) Let $\iota : \mathcal{A} \rightarrow \mathcal{B}$ be an isogeny of abelian schemes over S . Then the identity $\iota_*^{\vee}(\mathfrak{g}_{\mathcal{B}}) = \mathfrak{g}_{\mathcal{A}}$ holds.
- ("formula of Beauville and Bloch") The equalities

$$(S_0^{\vee}, \mathfrak{g}_{\mathcal{A}}) = (-1)^g \rho_{\mathcal{A}^{\vee},*}(\widehat{\text{ch}}(\overline{\mathcal{P}})) = \frac{1}{g! \sqrt{\deg(\phi_{\mathcal{L}})}} \phi_{\mathcal{L},*}(\widehat{c}_1(\overline{\mathcal{L}})^g)$$

are verified in $\widehat{\text{CH}}^g(\mathcal{A}^{\vee})_{\mathbb{Q}}$.

- If S is projective over \mathcal{O}_K , then the condition

$$(S_0^{\vee}, \mathfrak{g}_{\mathcal{A}}) = (-1)^g \rho_{\mathcal{A}^{\vee},*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)}$$

(ie (b) in the last theorem) can be replaced by the weaker

$$\frac{i}{2\pi} \partial \bar{\partial} \mathfrak{g}_{\mathcal{A}} + \delta_{S_0^{\vee}(\mathbb{C})} = (-1)^g \rho_{\mathcal{A}^{\vee},*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)}.$$

Spectral interpretation of the current $g_{\mathcal{A}}$, I

Part of the current $g_{\mathcal{A}^\vee}$ can be interpreted as the Bismut-Köhler higher analytic torsion form of the Poincaré bundle.

Let λ be a $(1, 1)$ -form on $\mathcal{A}(\mathbb{C})$ defining a Kähler fibration structure on the fibration $\mathcal{A}(\mathbb{C}) \rightarrow S(\mathbb{C})$.

We suppose that λ is translation invariant on the fibres of the map $\mathcal{A}(\mathbb{C}) \rightarrow S(\mathbb{C})$ as well as conjugation invariant.

We shall write

$$T(\lambda, \overline{\mathcal{P}}^0) \in \tilde{A}(\mathcal{A}^\vee \setminus S_0^\vee) := \bigoplus_{p \geq 0} \tilde{A}^{p,p}(\mathcal{A}^\vee \setminus S_0^\vee)$$

for the Bismut-Köhler higher analytic torsion form of $\overline{\mathcal{P}}$ restricted to the fibration

$$\mathcal{A}(\mathbb{C}) \times_{S(\mathbb{C})} (\mathcal{A}^\vee(\mathbb{C}) \setminus S_0^\vee(\mathbb{C})) \longrightarrow \mathcal{A}^\vee(\mathbb{C}) \setminus S_0^\vee(\mathbb{C}).$$

Spectral interpretation of the current $\mathfrak{g}_{\mathcal{A}}$, II

Let $\bar{\Omega}$ be the sheaf of differentials of \mathcal{A} , endowed with the metric induced by the Kähler fibration. Let $\epsilon : S \rightarrow \mathcal{A}$ be the unit section.

Theorem

The equality

$$\mathfrak{g}_{\mathcal{A}}|_{\mathcal{A}^{\vee}(\mathbb{C}) \setminus S_0^{\vee}(\mathbb{C})} = \mathrm{Td}(\epsilon^* \bar{\Omega}) \cdot T(\lambda, \bar{\mathcal{P}}^0)$$

holds on $\mathcal{A}^{\vee} \setminus S_0^{\vee}$.

In particular $T(\lambda, \bar{\mathcal{P}}^0)^{(g-1)}$ does not depend on λ .

This theorem specialises to the **second Kronecker limit formula**, when $g = 1$.

Fields of definition

Let

$$N_{2g} := 2 \cdot \text{denominator} [(-1)^{g+1} B_{2g} / (2g)],$$

where B_{2g} is the $2g$ -th Bernoulli number. Recall that

$$\sum_{t \geq 1} B_t \frac{u^t}{t!} := \frac{u}{\exp(u) - 1}.$$

Theorem

Suppose that $\sigma : S \rightarrow \mathcal{A}^\vee$ is an n -torsion section, such that $\sigma(S) \cap S_0^\vee = \emptyset$. Then

$$2g \cdot n \cdot N_{2g} \cdot \sigma^* T(\lambda, \overline{\mathcal{P}}^0) \in \text{image}(\text{reg}_{\text{an}}(K_1(S))).$$

In particular, $48 \cdot \sigma^* T(\lambda, \overline{\mathcal{P}}^0) \in \text{image}(\text{reg}_{\text{Dirichlet}}(\mathcal{O}_S^*))$ if $g = 1$.

Further topics

- There should be a connection between $\mathfrak{g}_{\mathcal{A}^\vee}$ and the Hodge realisation of the abelian polylogarithm (work in progress by G. Kings and D. R.).
- If $\dim(S) \leq 1$ then $\mathfrak{g}_{\mathcal{A}}$ is the canonical harmonic Green current associated with S_0^\vee and the above results (aside from the spectral interpretation) are contained in the work of K. Künnemann.