Some PDE's with fractional diffusion.

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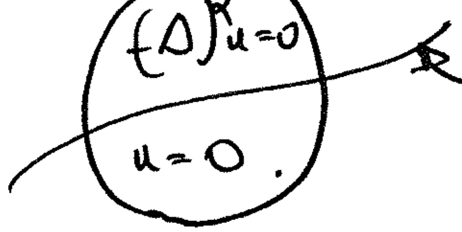
Remerciements.

Introduction.

The initial plan was to give a survey of the PDE's with integral diffusion that I am studying with various colleagues. I will however adopt a different strategy: go, in a reasonable depth, into the study of a free boundary problem for the fractional laplacian. In doing so, this will allow me to review some various analytic methods involving operators with integral diffusion.

So, the problem under study is the following: given are $\alpha \in (0,1)$ and $A > 0$. We study the properties of an unknown function $u(x)$ $(x \in \mathbb{R}^N)$ solving,

$$\begin{cases} (-\Delta)^\alpha u(x) = 0 & (x>0,\ x \in B_1). \\ [u] = 0 & \text{on } \Gamma = \partial\{u=0\} \\ \displaystyle\lim_{\substack{x \to x_0 \\ u(x)>0}} \frac{u(x) - u(x_0)}{|(x-x_0)\cdot \nu|^\alpha} = A. \end{cases}$$



Γ : $[u] = 0$
Hölder quotient imposed

<u>def</u>. $(-\Delta)^\alpha u(x) = c_\alpha \displaystyle\int \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}}\, dy$.

- Convergent integral if $\alpha < \frac{1}{2}$.
- Principal value if $\alpha \geq \frac{1}{2}$.

⚠ u should be known everywhere in $\mathbb{R}^N$.

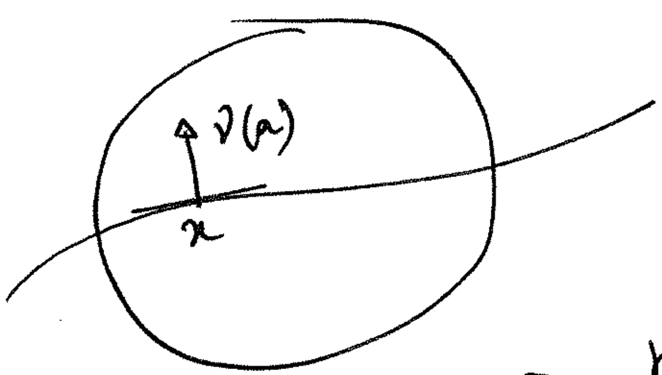
The questions that one might ask are numerous:

- Qualitative properties of u & the free boundary?
- Regularity?

- Possible singularities?

This class of problems have not arisen alone. They have been precedented by the analogue for the usual laplacian:

$$\begin{cases} -\Delta u = 0. & (x \in B_1,\ u(x) > 0). \\ [u] = 0 & \text{on } \Gamma = \partial\{u = 0\}. \\ \dfrac{\partial u}{\partial \nu} = A & \text{on } \Gamma. \end{cases}$$



We ask the same questions. There is a huge collection of works on that class of problems, where the Caffarelli name appears everywhere.

- <u>Regularity issues.</u>

- Assume $\Gamma \cap B_1$ is a "flat" piece of free boundary. Then it is a $C^{1,\gamma}$ graph in $B_{1/2}$ (hence analytic in $B_{1/2}$ thanks to a remark of Kinderlehrer, Nirenberg, Spruck)
- Same conclusion if $\Gamma \cap B_1$ is a Lipschitz piece

of surface.

These results are due to Caffarelli.

<u>Existence of singularities.</u>

. There exist solutions of the FBP for the laplacian for which Γ is not a flat piece of surface, or a lipschitz piece of surface:

- $N=8$: Jerison - de Silva.
- $N \geqslant 9$: Kamburov.

When one sees such dimensions, one irresistibly thinks of minimal surface theory. And one is right: the dS-J example is built on the Simons cone example, the K example is built on the Bombieri - de Giorgi - Giusti cone.

There is a complete theory about the obstacle problem (less singular than this one, but the free boundary properties are easier to grasp: Brézis - Stampacchia - Kinderlehrer - Caffarelli - ...). ~~If we come bac~~
The problem reads $-\Delta u + \mathbb{1}_{\{u>0\}} = 0$.

Coming back to the fractional Laplacian, there is a complete theory by Caffarelli-Salsa-Silvestre (2008).

So, let us come back to the fractional Laplacian with Hölder quotient imposed. We will discuss here weak formulations (and we know that we are not introducing them for the pleasure of doing it: Murphy's law, or the theorem of maximal bother, tells us that there should be singularities around - although we cannot prove their existence at the moment.), then we will give some properties. And so, the plan of this minicourse that I am proposing is:

I). General properties, explicit solutions.

II). Motivations.
(Generalising what is known for the usual Laplacian is nice, but the pb has motivations on its own).

III). Variational aspects of (1).

Variational formulation, existence of the FB, rough properties of the FB. (work w. Caffarelli and Sire).

IV). Regularity results for the FB.

$\alpha = \frac{1}{2}$, a flat piece of surface is $C^{1,\gamma}$ (work w. de Silva. Analyticity requires a bit more work, it is a bit more than a remark).

So, I will not have deviated too much from what was ~~un~~ initially announced.

I). General properties, explicit solutions.

1°). The fractional laplacian.

$$(-\Delta)^{\alpha} u(x) = c_{\alpha} \int \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} \, dy .$$

c_{α} is adjusted so that its symbol is $|\xi|^{2\alpha}$:

$$\mathcal{F}\left(\left((-\Delta)^{\alpha} u\right)\right)(\xi) = |\xi|^{2\alpha} \hat{u}(\xi) .$$

In particular, $(-\Delta)^{\alpha} u(x) \xrightarrow[\alpha \to 1]{} -\Delta u(x)$.

This, in particular, forces some behaviour of c_α as $\alpha \to 1$: $c_\alpha \underset{\alpha \to 1}{\sim} 1-\alpha$.

<u>Fundamental solution</u>, $E_\alpha(x) = \dfrac{d_\alpha}{|x|^{N-2\alpha}}$.

– <u>Dirichlet problem</u> –

$$-\Delta u = f \quad \Omega, \qquad u = 0.$$

Everyone knows that has a unique solution, say, for $f \in L^2(\Omega)$ and Ω ~~regular~~ reasonably smooth.

For the fractional Laplacian the well-posed problem is:

$$(-\Delta)^\alpha u = f \qquad u = 0$$

One reason is that the energy ~~at~~ involved is
$$a(u,v) = \int_{\mathbb{R}^N} (-\Delta)^\alpha u(x)\, v(x)\, dx$$
$$= \frac{1}{2}\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2\alpha}}\, dx\, dy.$$

(Well, do not worry about the $\frac{1}{2}$ ... all the constants will be, unless quite necessary, assumed to be equal to 1).

There is a regularity theory — similar to that of the Laplacian — that we will not touch. See Caffarelli – Silvestre [Caf S2].

These are the basic elements we will work with. Let us now come to what is going to save us from treating $(-\Delta)^\alpha$ as a true integral operator.

2°). The Caffarelli – Silvestre formula and applications.

The following algorithm has been known since at least Poisson.

Let $u(x)$ be bounded, reasonably smooth. Solve the following:

$$\begin{cases} -\Delta v = 0 & (x \in \mathbb{R}^N,\ y > 0). \\ v(x,0) = u(x). \end{cases}$$

In particular, $v(x,y)$ is given by the Poisson kernel:

$$v(x,y) = \int \frac{y\, u(x')}{\left[(x-x')^2 + y^2\right]^{\frac{N+1}{2}}}\, dx'.$$

Then $(-\Delta)^{1/2} u(x) = \frac{-\partial v}{\partial y}(x, 0)$.

The contribution of C-S is to have generalised this formula.

<u>Th.</u> (CafS1). $u(x)$ bounded, or at most sub-linear. $\alpha \in (0,1)$ and $\beta = 1-2\alpha$. Solve

$$\begin{cases} -\operatorname{div} y^\beta \nabla v = 0 & (x \in \mathbb{R}^N,\ y > 0). \\ v(x,0) = u(x). \end{cases}$$

Then $(-\Delta)^\alpha u(x) = -\lim_{y \to 0^+} y^\beta v_y(x,y)$.

<u>Element of proof.</u> Assume everything is smooth. We have:

$$-\operatorname{div}(y^\beta \nabla v) = y^\beta\left(-\Delta v - \frac{\beta}{y} v_y\right).$$

So, we need to solve

$$\begin{cases} -\Delta v - \frac{\beta}{y} v_y = 0. & (y > 0,\ x \in \mathbb{R}^N) \\ v(x,0) = u(x). \end{cases}$$

Fourier in x:

$$\begin{cases} -\hat{v}_{yy} - \frac{\beta}{y}\hat{v}_y + |\xi|^2 \hat{v} = 0. \\ \hat{v}(\xi, 0) = \hat{u}(\xi). \end{cases}$$

All we need to to is solve an ODE in y:

$$\begin{cases} -w'' - \frac{\beta}{y} w' + |\xi|^2 w = 0. \\ w(y=0) = 1;\ w(+\infty) = 0 \\ \text{(except } \xi = 0). \end{cases}$$

$$w(y) = h_\alpha(|\xi| y).$$

$$\begin{cases} -h_\alpha'' - \frac{\beta}{z} h_\alpha' + h_\alpha = 0. \\ h_\alpha(0) = 1, \ h_\alpha(+\infty) = 0. \end{cases}$$

Elementary considerations show the existence of a unique solution $h_\alpha(z) > 0$. Moreover the equation gives (integrate between 0 and $+\infty$):

$$-\frac{d}{dz}(z^\beta h_\alpha') + z^\beta h_\alpha = 0.$$

$$\lim_{z \to 0+} z^\beta h_\alpha'(z) = -\int_0^{+\infty} z^\beta h_\alpha(z)\, dz := -d_\alpha$$

So $\lim_{y \to 0^+} y^\beta w_y(\xi, y) = \lim_{y \to 0^+} |\xi|^{1-\beta} (|\xi| y)^\beta h_\alpha'()$

$$= -|\xi|^{1-\beta} d_\alpha = -|\xi|^{2\alpha} d_\alpha. \ \boxtimes$$

This adds one more dimension, but reduces the study of a nonlocal operator to a local one:

$$-\operatorname{div}(y^\beta \nabla).$$

<u>Some properties.</u>

- $\alpha \in (0, \frac{1}{2})$: $\beta = 1 - 2\alpha > 0$. Weakly degenerate.
- $\alpha \in (\frac{1}{2}, 1)$: $\beta < 0$. Weakly singular diffusion.

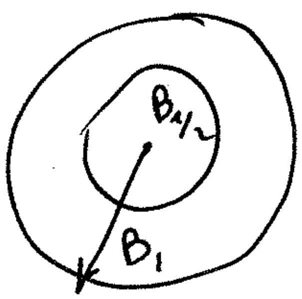
Properties.

$X \in \mathbb{R}^{N+1}$, $X = (x, y)$. $w(X) = y^{\beta}$. The weight w belongs to the class answering the sweet appellation of 2^{nd} Muckenhoupt class: for all B ball of $\mathbb{R}^{N+1}$,

$$\int_B w \, dX \int_B w^{-1} \, dX < +\infty .$$

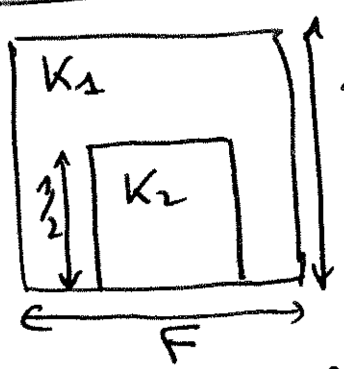
As such it satisfies:

- Harnack inequalities:



$$\operatorname{div}(w(X) \nabla u) = 0 . \quad (B_1)$$
$$u \geq 0 .$$
$$\inf_{B_{1/2}} u(X) \geq K \sup_{B_{1/2}} u(X) .$$

- Harnack inequalities up to the boundary



$u \geq 0$, $v \geq 0$

$0 = -\operatorname{div}(w(X) \nabla u) = -\operatorname{div}(w(X) \nabla v)$.
in K_1.

$u = v = 0$ on F.

Then, in K_2: $\exists C > 0$ s.t.

$$C^{-1} \leq \frac{u(X)}{v(X)} \leq C .$$

All this is due to Fabes, Kenig, Serapioni [FKS].

Some elementary properties.

* $\operatorname{div} y^\beta \nabla u = y^\beta \left(\Delta u + \frac{\beta}{y}\right) u_y$.

Assume (a stupid assumption) that β is an integer. Now, view y as a radial variable in $\beta+1$ dimensions. Now, $u_{yy} + \frac{\beta}{y} u_y$ is the laplacian of a radial function in $\beta+1$ dimensions, and $\Delta u + \frac{\beta}{y} u_y$ is thus the laplacian of u in $N+\beta+1$ dimensions.

* Application: exercise.

(i). Fundamental solution of $-\operatorname{div}(y^\beta \nabla)$:

$$\frac{c_\alpha}{|x|^{N+1+\beta-2}} = \frac{c_\alpha}{|x|^{N-2\alpha}}.$$

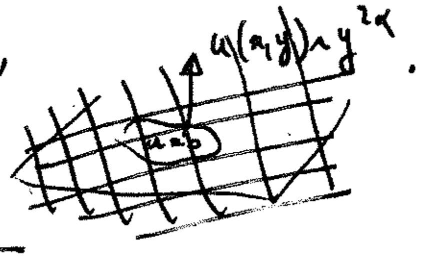
Application. Retrieve the fundamental sol. of $(-\Delta)^\alpha$.

(ii). Poisson kernel.

$$\begin{cases} -\operatorname{div} y^\beta \nabla v = 0. & (\mathbb{R}^{N+1}_+) \\ v(x,0) = u(x). & (y=0). \end{cases}$$

$$v(x,y) = \int P_N(x-x', y)\, u(x')\, dx',$$

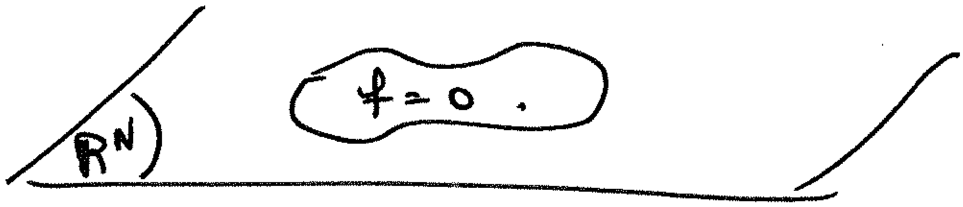
$$P_N(x,y) = \frac{y^{2\alpha}}{(|x|^2+y^2)^{\frac{N+2\alpha}{2}}} \rightsquigarrow$$



this

<u>Consequence</u> : boundary behaviour of a β-harmonic function in the extension at a point where it vanishes.

<u>Situation</u>.



$$u(x,y) = \int P_N(x - x', y)\, dx' f(x').$$

f : vanishes in a compact of $\mathbb{R}^N$ of nonempty interior.

<u>Then</u> : $u(x) = y^{2\alpha} \int_{x' \notin K} \dfrac{f(x')}{|x - x'|^{N+2\alpha}}$, so :

$u(x)$ departs from 0 like $y^{2\alpha}$.

<u>Rq</u>. Compatible w. the case $\alpha = \frac{1}{2}$.

<u>Rq</u>. This is a general fact : assume

$-\operatorname{div} |y|^\beta \nabla v = 0$ in B_1, v vanishes on a compact subset of $\mathbb{R}^N$ of nonempty interior, $v \geq 0$ in B_1. Then, if $x \in \mathring{K}$:

$$C^{-1} |y|^{2\alpha} \leq v(x,y) \leq C\, |y|^{2\alpha} \text{ for } y \sim 0.$$

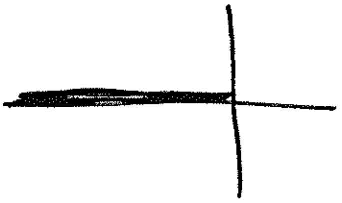
<u>Proof</u>. Harnack up to the boundary.

<u>Application</u>. $(-\partial_{xx})^{\alpha}(x_+)^{\alpha} = 0$ on $\mathbb{R}_+$.

<u>Proof</u>. • $\alpha = \frac{1}{2}$. In the complex plane, define $\operatorname{Arg} z \in (-\pi, +\pi)$,

$$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}, \quad \theta \in (-\pi, +\pi).$$

This is an analytic function in the plane cut by $\mathbb{R}_-$:



On $\mathbb{R}_-$: $\operatorname{Re}\sqrt{z} = 0$.

$\mathbb{R}_+$: $\operatorname{Re}\sqrt{z} = \sqrt{x}$.

$\operatorname{Re}\sqrt{z}$ is harmonic and even $\left(\sqrt{r}\cos\frac{\theta}{2}\right)$. So,

$$\left.\frac{\partial(\operatorname{Re}\sqrt{z})}{\partial y}\right|_{y=0} = 0.$$

• $\alpha \neq \frac{1}{2}$. Inspired by this, set

$$u(X) = r^{\alpha}\cos^{2\alpha}\frac{\theta}{2}; \quad X = (x, y), \quad r = \sqrt{x^2+y^2}.$$

<u>Exercise</u>: $\operatorname{div}(y^{\beta}\nabla u) = 0$.

$$\left(\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \; \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right).$$

$$\frac{1}{y^{\beta}}\operatorname{div}(y^{\beta}\nabla u) = -\Delta u - \frac{\beta}{y}u_y = -\Delta u - \frac{\beta}{r\sin\theta}\left(\sin\theta\, u_r + \frac{\cos\theta}{r}u_\theta\right)$$

$$= 0.$$

After this great achievement(s) (ou pb is non vid), let us move to 3°). <u>Boundary behaviour of α-harmonic func</u>-

tions.

let us start from the following statement:

$$\begin{cases} (-\Delta)^\alpha u = 0 & (\Omega) \\ u = f \geq 0 & (\mathbb{R}^N \setminus \Omega). \end{cases}$$

Then, either $f \equiv 0$ or $u > 0$ in Ω. This is trivial: if u takes a negative min at $x_0 \in \Omega$ we have:

$$(-\Delta)^\alpha u(x_0) = \int_\Omega \frac{u(x_0) - u(y)}{|x-y|^{N+2\alpha}}\, dy$$

< 0 ~~because~~ as soon as f is nonzero and continuous in $\mathbb{R}^N \setminus (\Omega)$.

(In fact continuity is too strong). This very simple proof is in sharp contrast w. the max. pple for the usual laplacian. Let us see what happens at the boundary of Ω for u.

Th. Assume $x_0 \in \partial\Omega$ is regular. such that $u(x_0) = 0$.
Then if $\nu(x_0)$ is the outer normal vector:
$$u(x) \geq C|(x - x_0).\nu(x_0)|^\alpha \quad \text{for } x \in \Omega.$$

For this we need a bit more information on explicit solutions.
Let us already notice that this is not stupid because $(-\partial_{xx})^\alpha (x_+)^\alpha = 0$. Now, for every $f \in C(\mathbb{R}^N \setminus \Omega)$, the problem

$$\begin{cases} (-\Delta)^\alpha u = 0 & (\Omega) \\ u = f & (\mathbb{R}^N \setminus \partial\Omega) \end{cases}$$

has a unique solution u_f. For every $x \in \Omega$, the ~~function~~ map $f \mapsto u_f(x)$ is a linear ~~pos~~ nonnegative form on $C(\mathbb{R}^N \setminus \Omega)$. Hence it is a measure supported by $\mathbb{R}^N \setminus \Omega$ and we denote it by $\omega_x(dy)$. And so, we have:

$$u_f(x) = \int_{\mathbb{R}^N \setminus \Omega} f \, \omega_x(dy),$$

ω_x is the α-harmonic measure of $\mathbb{R}^N \setminus \Omega$ at x.

<u>Remark</u>: if $\alpha = 1$, ω_x is supported by $\partial\Omega$.

<u>Theorem</u>. (M. Riesz, 1937, [Rie]). If $\Omega = B_1$,

$$\omega_x(dy) = k_{\alpha,N} \left(\frac{1-|x|^2}{|y|^2-1} \right)^\alpha \frac{dy}{|x-y|^N}.$$

If $\omega = \mathbb{R}^N \setminus B_1$,

$$\omega_x(dy) = k_{\alpha,N} \left(\frac{|x|^2-1}{1-|y|^2} \right)^\alpha \frac{dy}{|x-y|^N}.$$

- The identity for $\omega_x(dy)$, $x \in \mathbb{R}^N \setminus B_1$, is dedu from that for $x \in B_1$ by inversion: if $(-\Delta)^\alpha u = 0$ in Ω, then $(-\Delta)^\alpha \tilde{u} = 0$ in $\tilde{\Omega}$, $\tilde{\Omega}$ = image of Ω by inversion $x \mapsto \frac{x}{|x|^2}$
$$\tilde{u}(x) = \frac{1}{|x|^{N-2\alpha}} u\left(\frac{x}{|x|^2}\right).$$

- The computation of the formula for $x \in B_1$ is nontrivial.

Application to the boundary behaviour of u.

If Ω is regular at x_0, there is a ball B_{2r} such that $x_0 \in \partial B_{2r}$ and $B_{2r} \subset \Omega$. Consider B_r such that $x_0 \in \partial B_{r/2}$, then $B_{r/2} \subset B_r$.

Let $\hat{B}$ be a very small ball within B_{2r}, $\hat{B} \cap B_r = \emptyset$. Let $f \in C^\infty(\mathbb{R}^N)$ positive be supported in $\hat{B}$. Then solve:
$$\begin{cases} (-\Delta)^\alpha \underline{u} = 0 & (B_r) \\ \underline{u} = f & (\mathbb{R}^N \setminus B_r). \end{cases}$$

Assume O to be at the centre of B_r:

$$\underline{u}(x) = \int \left(\frac{r^2-|x|^2}{|y|^2-r^2}\right)^\alpha \frac{f(y)}{|x-y|^N}\,dy.$$

If $\delta>0$ is small enough, $\delta f \leq u$ in $\hat{B}$, hence $\delta f \leq u$ in $\mathbb{R}^N \setminus B_r$. The maximum principle ($(-\Delta)^\alpha(u-\delta\underline{u}) = 0$ in B_r, $u-\delta\underline{u}\geq 0$ outside B_r) implies $u \geq \delta\underline{u}$ in B_r.

In a neighbourhood of x_0 inside B_r we have

$$\underline{u}(x) \propto |\nu(x_0).(x-x_0)|^\alpha.$$

Hence $u(x) \geq \underline{u}(x) \geq k\,|\nu(x_0).(x-x_0)|^\alpha$ in a neighbourhood of x_0. $\boxtimes$

II). Some motivations.

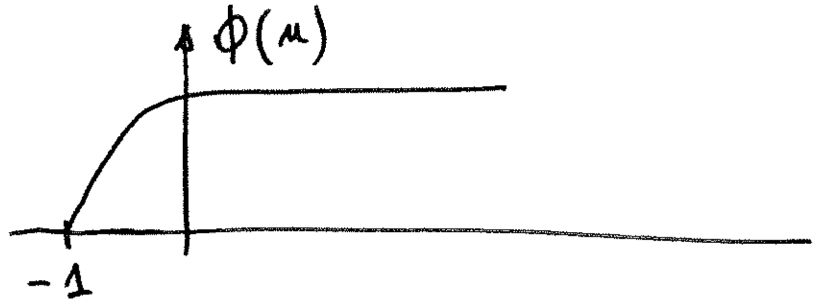
1°). A model for boundary reactions.

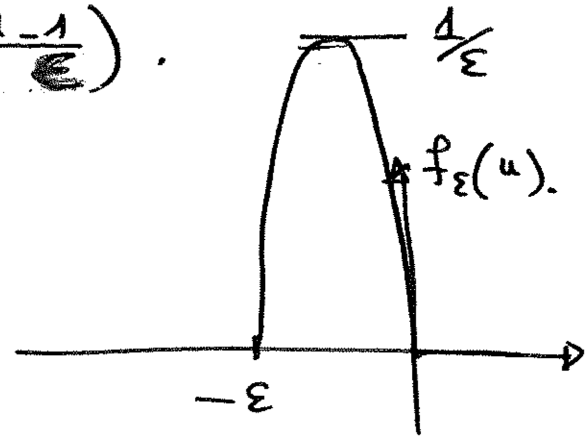
Very close to what Xavier is lecturing about is the following model:

$$\begin{cases} -\Delta u = 0 & (\mathbb{R}^{N+1}_+). \\ \partial_y u = -f(u). & (y=0). \end{cases}$$

This is exactly $(-\Delta)^{1/2} u = f(u)$. What does it have to do with our FBP? Well, here is a sequence of problems to which the FBP is a limiting problem. Consider the sequence of source terms:

$$f_\varepsilon(u) = \frac{1}{\varepsilon}(1-u)\,\phi\left(\frac{u-1}{\varepsilon}\right).$$



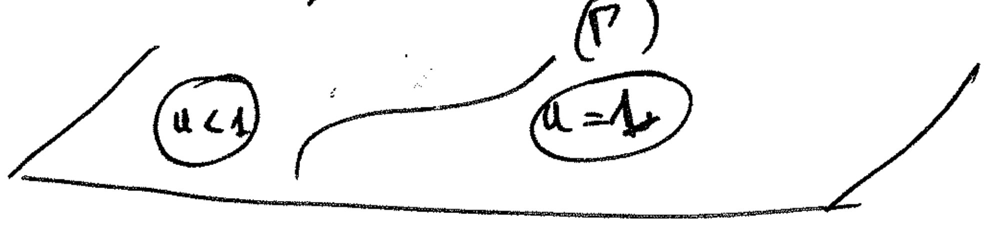


Let us analyse that in a completely formal fashion.

- In the upper half-plane: we still have $-\Delta u = 0$.
- <u>In $\mathbb{R}^N$ $(y=0)$:</u>

 * if $u < 1$: $f_\varepsilon(u) = 0$ for $\varepsilon > 0$ small enough.

 * Assume $u \leq 1$ (which is not impossible):



 * Let us see what happens for values

close to 1. Set $(\xi, \eta) = \dfrac{(x, y)}{\varepsilon^2}$, $v(\xi,\eta) = \dfrac{u(\varepsilon^2\xi, \varepsilon^2\eta)}{\varepsilon}$

we have: $$\frac{\partial v}{\partial \eta}(\xi, 0) = \varepsilon \frac{\partial u}{\partial y}(x, 0)$$
$$= \frac{1}{\varepsilon}(1-u)\,\phi\left(\frac{u-1}{\varepsilon}\right)$$
$$= -v(\xi,0)\,\phi\big(v(\xi,0)\big).$$

Set $w(\xi) = v(\xi, 0)$. Assume:

(i). 0 is a regular free boundary point, i.e. the F.B is a plane in the ξ variable; assume: $(\Gamma) = \{\xi_N = 0\}$.

(ii). ~~It~~ is smooth in a neighbourhood of 0, so that w only depends on ξ_N.

Thus an equation for w is:
$$\begin{cases} \left(-\partial_{\xi_N\xi_N}\right)^{1/2} w + w\,\phi(w) = 0. & (\xi_N \in \mathbb{R}). \\ w(0) = -1, \quad w(+\infty) = 0. \end{cases}$$

<u>Exercise</u>. There is $A > 0$ such that
$$w(\xi_N) = -A\sqrt{|\xi_N|} \quad \text{for } |\xi_N| \ll 0.$$
||

<u>Back to u</u>: for ξ_N large but not huge:
$$v(\xi, 0) \sim -A\sqrt{|\xi_N|}.$$
$$\frac{u(x, 0) - 1}{\varepsilon} \sim -A\sqrt{\frac{|x_N|}{\varepsilon^2}} \sim -\frac{A\sqrt{|x_N|}}{\varepsilon}.$$

This says exactly that the limit of the Hölder quotient of u at the F.B. is $-A$.

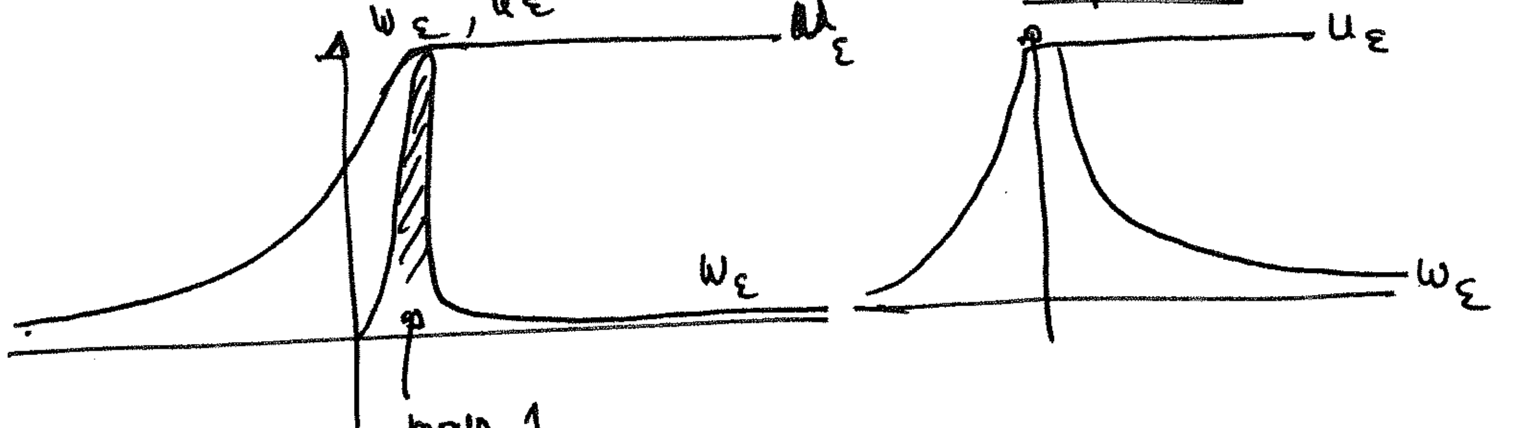
<u>Remark 1.</u> The problem has similarities with that of the usual Laplacian:

- usual Laplacian: $-\Delta u = \frac{1}{\varepsilon^2}(1-u)\,\phi\left(\frac{u-1}{\varepsilon}\right)$.
- Scaling: $(\xi,\eta) = \frac{(x,y)}{\varepsilon}$, $v(\xi,\eta) = \frac{u(\varepsilon\xi,\varepsilon\eta)-1}{\varepsilon}$
- limiting equation: $\begin{cases} -\partial_{\xi_N\xi_N} w + w\,\phi(w) = 0. \\ w(0) = -1,\ w(+\infty) = 0. \end{cases}$

For $\xi_N < 0$ we have $w(\xi_N) = A\,\xi_N$ (A easily computed).

<u>Remark 2.</u> There are important physical differences. $w_\varepsilon = \frac{1}{\varepsilon^2}(1-u)\,\phi\left(\frac{u-1}{\varepsilon}\right)$

<u>Usual Laplacian.</u> <u>Fractional Laplacian.</u>



$w_\varepsilon \longrightarrow \delta_0$.

$w_\varepsilon \longrightarrow \frac{C}{(x_+)^{1/2}}$.

And so, the limiting equation is not the same:

- $-\Delta u = \delta_{\{u=0\}}$.

- $(-\Delta)^{1/2} u = \dfrac{c}{d(x, \{u=0\})^{1/2}}$.

If we look at the fractional laplacian in the extension this is quite predictable: we have
$$\begin{cases} -\Delta u = 0 & (B_1) \\ u = 0 & \text{in } \{u=0\} \cap \mathbb{R}^N. \end{cases}$$

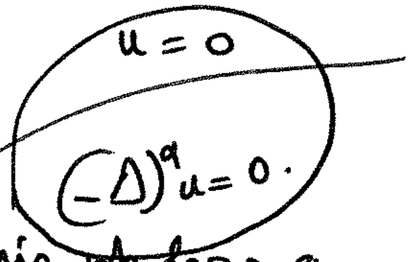
In particular, $u \geq 0$ in $B_1 \setminus \mathbb{R}^N$ and, if $u(x_0) = 0$ we have: $\dfrac{\partial u}{\partial y}(x, 0) > 0$ (Hopf).

<u>Remarks</u>. It would be very nice to do a rigorous analysis of that (work in progress by Caffarelli, Mellet and Sire).

III). <u>Variational aspects of the free boundary problem</u>.

Recall that we are solving

$$[u] = 0, \quad \lim_{x \to x_0} \frac{u(x) - u(x_0)}{|(x - x_0) \cdot \nu(x_0)|^a} = A$$



We are not even sure that this pb has a

a meaning.

It occurs that the classical FB with imposed gradient jump has a variational formulation:

$$\begin{cases} -\Delta u = 0 & (B_1 \cap \{u>0\}). \\ [u]=0,\ \frac{\partial u}{\partial \nu} = A & (\partial\{u=0\}). \end{cases}$$

can be expressed, in weak form, as: minimise the functional

$$\int_{B_1} |\nabla u|^2 \, dx + \mathcal{L}_N(\{u>0\} \cap B_1).$$

I do not want to make anything precise, since I will do it on the fractional model. Let us just notice that:

– the Dirichlet functional is not the worst way of obtaining harmonic functions,

– the area of $\{u>0\}$ is not the worst way of obtaining a normal derivative: perturbations of a set usually occur through the normal of the boundary.

And so, we take our inspiration from that.

Notations.

- Ω : smooth open subset of $\mathbb{R}^{N+1} = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}\}$
- $H^1_\beta(\Omega) =$ the closure of $C^\infty(\bar{\Omega})$ for the norm

$$\int_\Omega |y|^\beta |\nabla u|^2 \, dx\,dy, \qquad \beta = 1 - 2\alpha.$$

- For all $u \in H^1_\beta(\Omega)$ with $u \geq 0$, set

$$J[u,\Omega] = \int_\Omega |y|^\beta |\nabla u|^2 \, dx\,dy + A\mathcal{H}_N\big(\underbrace{\{u>0\} \cap \mathbb{R}^N}_{\cap\,\Omega}\big).$$

We are observing u here on the hyperplane $\mathbb{R}^N$.

Minimisation problem. We wish to study the properties of a function $u(x,y) \geq 0$, satisfying: (i) $u(x,y) = u(x,-y)$.

(ii) $\forall B$ open subset of B_1,

$\forall v \in H^1_\beta(B)$ such that $v|_{\partial B} = u|_{\partial B}$,

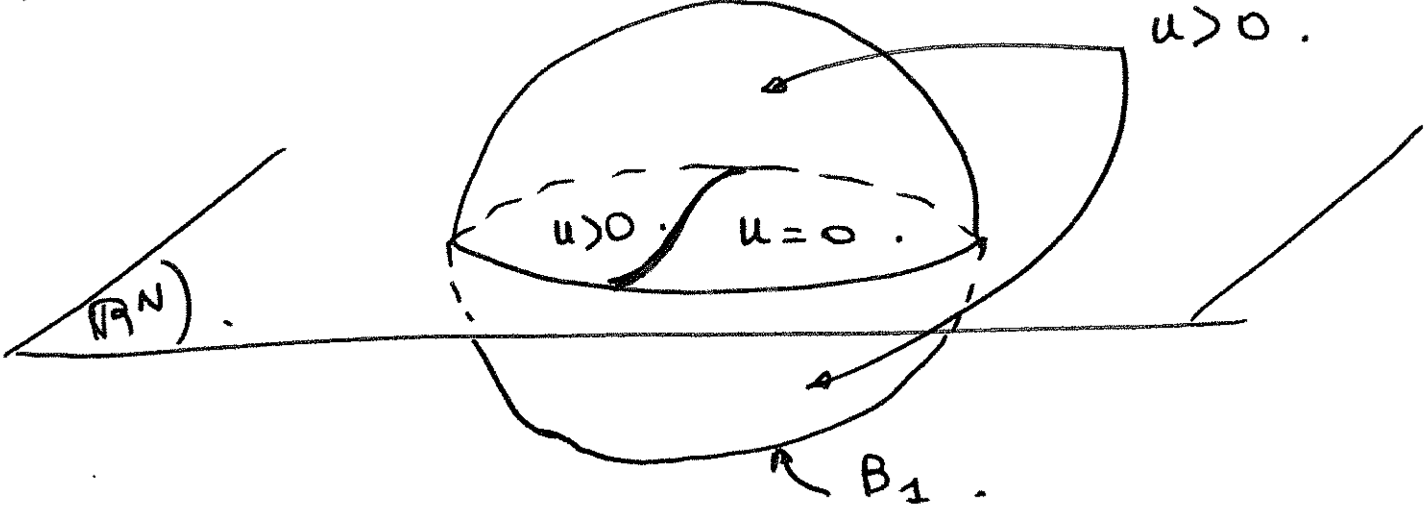
$$J[u,B] \leq J(v,B).$$

Such a function u is called a minimiser of J in B_1.

If we replace B_1 by $\mathbb{R}^{N+1}$, u is called a global minimiser of J in $\mathbb{R}^{N+1}$.

<u>Remark</u>. Same definition as that given by Xavier. This is a standard definition in the calculus of variations.

The goal of this paragraph is to study the "rough" local properties of the minimisers and their free boundary. Here is a picture of the situation.



We expect a minimiser to satisfy $-\mathrm{div}(y^\beta \nabla u)=0$ in $B_1 \setminus \{u=0\}$, hence $u>0$ except on a part of the plane $\mathbb{R}^N$.

1°). <u>Basic properties</u>.

We do not know yet if minimisers are continuous. We do not even know if they exist!

prop. (Existence of minimisers with nontrivial free boundaries). Ω bdd open subset of $\mathbb{R}^N$. There exists $u_0 \geqslant 0$:
- small enough.
- zero on a sufficiently large part of $\partial\Omega$

such that the minimisation pb

$$\begin{cases} \forall v \in H^1_\beta(\Omega) \text{ w. } v|_{\partial\Omega} = u_0, \\ J[u,\Omega] \leq J[v,\Omega] \end{cases}$$

has a solution $u \in H^1_\beta(\Omega)$ such that

$\{u=0\}$ has nonempty interior in $\mathbb{R}^N$

Proof. Exercise based on what will follow. (cf optimal regularity + nondegeneracy).

prop. Let u be a minimiser in B_1. Then

- $-\operatorname{div}(|y|^\beta \nabla u) = 0$ in $\{u>0\}$.

(provided u is continuous).

- Still under the assumption that u is continuous, then $u \in C^2(\{u>0\})$ and

$$\lim_{y\to 0} y^\beta \nabla u(x,y) = 0.$$

- If u is a ~~minimum~~ global minimiser, and

$u(x,0) \leq C|x|^\alpha$, then:

$$(-\Delta)^\alpha u = 0 \quad \text{in } \mathbb{R}^N \cap \{u>0\}.$$

This is the link with the fractional laplacian. All this can certainly be proved with weaker assumptions.

The main result of this part is the

Th. (Optimal regularity). $u \in C^\alpha(B_{\alpha})$.

Remark: we ~~do~~ call this "optimal regularity" because we cannot do better: see $(x_N^+)^\alpha$.

To prove this result we need the following (well-known in the laplacian case) lemma:

Lemma $u \in H^1_\beta(B_1)$ such that

$$-\operatorname{div} |y|^\beta \nabla u = 0 \quad \text{in } B_1.$$

Then, for all $0 < r < R < 1$,

$$\int_{B_r} |y|^\beta |\nabla u|^2 \leq \left(\frac{r}{R}\right)^{N+1+\beta} \int_{B_R} |y|^\beta |\nabla u|^2.$$

Element of proof.

$E_r[u]$ = the Dirichlet integral $= \int_{B_r} |y|^\beta |\nabla u|^2$.

Define : $u_\varepsilon(x,y) = \begin{cases} \frac{1}{1+\varepsilon} u((1+\varepsilon)(x,y)) & |(x,y)| \le \frac{r}{1+\varepsilon} \\ \text{linear radial extension from } \frac{r}{1+\varepsilon} \text{ to } r. \end{cases}$

Fact : $\lim_{\varepsilon \to 0} \frac{E_r[u_\varepsilon] - E_r[u]}{\varepsilon}$

$$= r \frac{dE_r}{dr} - (N+\beta+1) E_r .$$

Indeed : $E_r[u_\varepsilon] = \underbrace{\int_{|x,y| \le \frac{r}{1+\varepsilon}}} + \underbrace{\int_{|x,y| \ge \frac{r}{1+\varepsilon}}}$

$E_r[u] + \cancel{E_r dE}$ $\qquad\qquad$ $\varepsilon \frac{r\, dE_r}{dr}$.

$- (N+1+\beta)\varepsilon E_r + O(\varepsilon^2)$.

Integrate from R down to r.

<u>Exercise</u> - Set $v = |y|^\beta |\nabla u|^2$

(i). $\alpha \le \frac{1}{2}$. Prove that $-\operatorname{div}(|y|^\beta \nabla v) \le 0$.

Retrieve monotonicity formula.

(ii). $\alpha > \frac{1}{2}$. Compute $-\operatorname{div} |y|^\beta \nabla v$ in terms

of u_x, u_y, possibly higher derivatives.
Retrieve a formula of the form

$$\int_{B_r} |y|^\beta |\nabla u|^2 \leq C\left(\frac{r}{R}\right)^{N+1+\beta} \int_{B_R} |y|^\beta |\nabla u|^2.$$

Proof of optimal regularity.

We use a classical result of Morrey: a function $u(x)$ is in C^α if, for some p: (we are in $\mathbb{R}^{N+1}$)

$\forall r>0, \forall x: \int_{B_r} |\nabla u|^p \, dx \leq C r^{1+N-p+\alpha p}$. In other words: $\forall r>0, \forall x: \fint_{B_r} |\nabla u|^p \leq C r^{(\alpha-1)p}$.

Let $X_0 = (x_0, y_0) \in B_1$, for $r>0$ let $h_r^{X_0}$ the harmonic replacement of u in $B_r(X_0)$:

$$\begin{cases} -\operatorname{div}(|y|^\beta \nabla h_r^{X_0}) = 0 & (B_r(X_0)). \\ h = u & (\partial B_r(X_0)). \end{cases}$$

Let us write the minimality of u.

$$\int_{B_r(X_0)} |y|^\beta |\nabla u|^2 + \mathcal{L}_N(\{u>0\} \cap \mathbb{R}^N) \leq \int |y|^\beta |\nabla h_r|^2 + C r^N.$$

In other words:

$$\int_{B_r(X_0)} |y|^\beta (|\nabla u|^2 - |\nabla h_r|^2) \leq C r^N.$$ But

h_r is a harmonic replacement: hence

$$\int_{B_r(X_0)} |y|^\beta \nabla h_r \cdot \nabla (u - h_r) = 0 .$$

So: $$\int_{B_r(X_0)} |y|^\beta |\nabla(u - h_r)|^2 \leq C r^N .$$

Consider $0 < r < \rho$:

$$\int_{B_r(X_0)} |\nabla u|^2 |y|^\beta = \int_{B_r} |\nabla(u - h_\rho) + \nabla h_\rho|^2 |y|^\beta$$

$$\leq 2 \int_{B_r} (|\nabla(u - h_\rho)|^2 + h_\rho^2) |y|^\beta .$$

$$\leq 2 \int_{B_r} h_\rho^2 + 2 \int_{B_\rho} |\nabla(u - h_\rho)|^2 |y|^\beta$$

$$\leq 2 \left(\frac{r}{\rho}\right)^{N+1+\beta} \int_{B_\rho(X_0)} |\nabla u|^2 |y|^\beta + \cancel{2\int_{B_\rho}}$$

$$+ C\rho^N .$$

Set: $r = \delta^{n+1}$, $\rho = \delta^n$, $\mu = \delta^N$. This reads

$$\int_{B_{\delta^{n+1}}(X_0)} |y|^\beta |\nabla u|^2 \leq C\mu^n + C\mu\, \delta^{2(1-\alpha)} \int_{B_{\delta^n}(X_0)} |y|^\beta |\nabla u|^2 .$$

$q = C\delta^{2(1-\alpha)}$ such that $q < 1$:

$$\int_{B_{\delta^n}(X_0)} |y|^\beta |\nabla u|^2 \le C(\delta)\, \mu^n$$

In particular:

$$\int_{B_r(X_0)} |y|^\beta |\nabla u|^2 \le C(\delta)\, r^N .$$

<u>Case 1.</u> $\beta \ge 0$, i.e. $\alpha \le \frac{1}{2}$.

$$\int_{B_r(X_0)} |\nabla u| = \int \frac{1}{|y|^{\beta/2}} |y|^{\beta/2} |\nabla u|$$

$$\le \underbrace{\left(\int \frac{1}{|y|^\beta}\right)^{1/2}}_{r^{\frac{N+1-\beta}{2}}} \underbrace{\left(\int |y|^\beta |\nabla u|^2\right)^{1/2}}_{r^{N/2}} .$$

$$\le C r^{N+\alpha} = C r^{N+1+\alpha-1} .$$

<u>Case 2.</u> $\beta \le 0$, i.e. $\alpha \ge \frac{1}{2}$.

$$\int_{B_r(X_0)} |\nabla u|^2 = \int \frac{1}{|y|^\beta} |y|^\beta |\nabla u|^2$$

$$\le r^{-\beta} \int |y|^\beta |\nabla u|^2 \le C r^{N-\beta} = C r^{N-1+2\alpha}$$

$$= C r^{N+2\alpha-2} ,$$

This proves optimal regularity. ☒

The consequence is that we can rescale around a free boundary point.

Indeed, if u is a minimiser in B_1, $X_0 = (x_0, 0)$ a free boundary point, then

$$w_r(X) = \frac{1}{r^\alpha} u(X_0 + rX)$$

is a minimiser in $B_{1/r}$ (well, we have to cut the radius a little but, say, $B_{a/r}$, $a>0$).

Moreover, the scaling preserves the Hölder norm: if u is α-Hölder w. constant C, then w_r is also α-Hölder w. norm C. Hence rescaling of a minimiser u are a relatively compact family in B_1.

The second thing to make sure about is that a sequence of rescalings does not converge to some thing trivial. This is the role of the

<u>Theorem</u>. u minimiser in B_1, assume that X_0 is a free boundary point. Then:

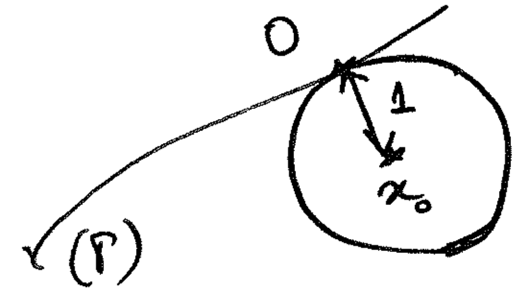
$$u(X) > 0 \Rightarrow u(X) \geq C|X - X_0|^\alpha.$$

The result is proved in 2 steps. The step I am going to prove is the

<u>prop</u>. We have: $u(X) > 0 \Rightarrow u(X) \geq C \, d_{RN}(x, \Gamma)$.

We are going to prove it to use the notions a bit. So,

<u>Proof</u>. We may rescale to arrive at the following situation: we have $d(x_0, \Gamma) = |x_0| = 1$.



Set $u(x_0, 0) = \epsilon$ (with the idea that ϵ is small).

$X_0 := (x_0, 0)$.

By the Harnack inequality, we have

$$u(x, y) \leq C\varepsilon \quad \text{in } B_{1/2}(x_0).$$

$$\text{Define } \bar{u}(x, y) = \begin{cases} (C+1)\varepsilon & \text{outside } B_{1/2}(x_0). \\ 0 & \text{inside } B_{1/4}(x_0). \\ \text{linear radial} & \text{in } B_{1/2}(x_0) \setminus B_{1/4}(x_0) \end{cases}$$

Set $u_*(x, y) = $ • $\inf(u(x, y), \bar{u}(x, y))$ in $B_{1/2}(x_0, 0)$

• $u(x, y)$ outside.

In particular: • $u_*(x,y) = \bar{u}(x,y)$ in $B_{1/2}(x_0, 0) \setminus B_{1/4}(x_0, 0)$

• $u_*(x, y) = u(x, y)$ on $\partial B_1(X_0)$.

And: $\displaystyle\int_{B_1(X_0)} |y|^\beta |\nabla u_*|^2 + \mathcal{L}_N(\{u_* > 0\} \cap \mathbb{R}^N \cap B_1(X_0))$

$$\leq \int_{B_1(X_0)} |y|^\beta |\nabla u_*|^2 + \underbrace{\mathcal{L}_N(\{u > 0\} \cap \mathbb{R}^N \cap B_1(X_0))}_{O} - |B_{1/2} \setminus B_{1/4}|$$

$$= \int_{u = u_*} |y|^\beta |\nabla u_*|^2 + \underbrace{\int_{u_* = \bar{u}} |y|^\beta |\nabla \bar{u}|^2}_{\leq C'\varepsilon^2} - |B_{1/2} \setminus B_{1/4}|$$

If ϵ is too small, this contradicts the minimality of u. ▨

To get the full property, we start from a free boundary point and construct a sequence of points $(x_n)_n$, the mutual distance being controlled, and such that $u(x_{n+1}, 0) \geq (1+\lambda)\, u(x_n, 0)$.

2°). Rough regularity properties of the free boundary - Free boundary relation.

Let us first mention the

<u>Th.</u> u minimiser in B_1, with free boundary Γ. $X_0 \in \Gamma$, $X_0 = (x_0, 0)$. Then:

$$\mathcal{L}_N\left(B_r^N(x_0) \cap \{u > 0\}\right) \geq C r^N.$$

$$\mathcal{L}_N\left(B_r^N(x_0) \cap \{u = 0\}\right) \geq C r^N.$$

In other words,

is forbidden. Of course the free boundary may have much more imagination than that. The main result is the following:

Th. Let $X_0 = (x_0, 0)$ be a regular free boundary point. The sequence of rescalings $w_r(X) = \frac{1}{r^\alpha} u(X_0 + rX)$ converges in $C(B_1)$ to the function $A_\alpha U(x_N, y)$, up to translations and rotations. The constant A_α is given by $A_\alpha = c_{1,\alpha}\left(\int_{-1}^{0} \frac{(1+x)^\alpha}{(-x)^\alpha}dx\left(\int_1^{+\infty}\frac{(1+x)^\alpha}{x^{1+2\alpha}}dx\right)\right)^{-1}$.
The function $U_\alpha(x_N, y)$ is given by
$$U_\alpha(x_N, y) = r^\alpha \cos^{2-2\alpha}\frac{\theta}{2}, \quad r^2 = x_N^2 + y_N^2,$$
$$x_N = r\cos\theta .$$

In particular, in the plane $\mathbb{R}^N$, we have ($w_\infty(x,y) = \lim_{r\to 0} w_r(x,y)$):
$$w_\infty(x,0) = A_\alpha\,(x_N^+)^\alpha .$$
We have to define what a regular point is. At such a regular point x_0, there is a unit vector $\nu(x_0)$ such that
$$B_r^N(x_0)\cap\Gamma \subset \{|(x-x_0)\cdot\nu(x_0)| \leq o(r)\}.$$
__Consequence__. At a regular point, the blow-up sequence converges to a function with straight free boundary.

The reason why this theorem holds is threefold.

① $\bullet$ $w_\infty(X)$ is a global minimiser. It is indeed defined in larger and larger balls.

- By optimal regularity and positive density $\left(\mathcal{L}_N(\{u>0\}\cap B_r) \geq Cr^N,\ \mathcal{L}_N(\{u=0\}\cap B_r) \geq Cr^N\right)$ sequences of minimisers converge to minimisers

<u>Exercise</u> : prove it.

So, in particular :

$$\begin{cases} -\operatorname{div} |y|^\beta \nabla w_\infty = 0 & \text{in } \mathbb{R}^{N+1} \\ w_\infty(X) > 0 \text{ if } X \notin \{x', x_N < 0, y = 0\}. \\ C^{-1}(x_N^+)^\alpha \leq w_\infty(X) \leq C(x_N^+)^\alpha . \end{cases}$$

In particular, $(-\Delta)^\alpha w_\infty(x) = 0$ in $\mathbb{R}^N$.

②. We have the

<u>prop.</u> Let $w(x)$ $(x \in \mathbb{R}^N)$ be such that

- $(-\Delta)^\alpha w(x) = 0$. $(x_N > 0)$.
- $w(x) = 0$ $(x_N < 0)$.
- $C^{-1}(x_N^+)^\alpha \leq w(x) \leq C(x_N^+)^\alpha$.

Then $\exists k > 0 : w(x) = k(x_N^+)^\alpha$.

Proof. Very classical: those who are used to seeing this have already the conclusion. It is like the de Giorgi oscillation lemma but this time at large scales.

Want to prove: for every half-ball B in $\{x_N > 0\}$ with centre at $x_N = 0$ we have: $w(x) = k\,(x_N^+)^\alpha$. So, set

$$h(x) = \frac{w(x)}{(x_N^+)^\alpha}.$$

Take $R > 0$ humongous. For every ρ:

$$M_\rho = \sup_{B_\rho^+} h(x)$$

$$m_\rho = \inf_{B_\rho^+} h(x) \qquad B_\rho^+ = B_\rho \cap \{x_N > 0\}.$$

The Harnack inequality in B_R^+ ~~does not~~ up to the boundary does not depend on R because of the scale invariance.

Try to estimate $M_{R/2} - m_{R/2}$ in terms of $M_R - \cancel{M}\, m_R$.

Case 1. $\exists\, x \in B_{R/2}$: $h(x) = \frac{M_R + m_R}{2}$. (hard case)

Hence we have (careful, one can get stuck quite easily):

$$h(x) - m_R \geq \lambda \sup_{B_{R/2}} (h(x) - m_R)$$

$$\geq \lambda \left(\frac{M_R + m_R}{2} - m_R \right) = \frac{\lambda}{2} (M_R - m_R).$$

Hence $m_{R/2} \geq m_R + \frac{\lambda}{2}(M_R - m_R)$.

And $M_{R/2} - m_{R/2} \leq M_R - m_R - \frac{\lambda}{2}(M_R - m_R)$

$$= \underbrace{\left(1 - \frac{\lambda}{2}\right)}_{<1} (M_R - m_R).$$

<u>Case 2</u>. $\forall x \in B_{R/2}$, $h(x) \leq \frac{M_R + m_R}{2}$.

Then $\cancel{M_R - M}$ $M_{R/2} - m_{R/2} \leq \frac{M_R - m_R}{2}$.

In both cases, $M_{R/2} - m_{R/2} \leq q(M_R - m_R)$.

Geometric decay, R arbitrary $\Rightarrow$ h = constant. ☒

③. So, a global minimiser with straight free boundary has the form $k(x_N^+)^\alpha$.

<u>Identification of k</u>. use the fact that

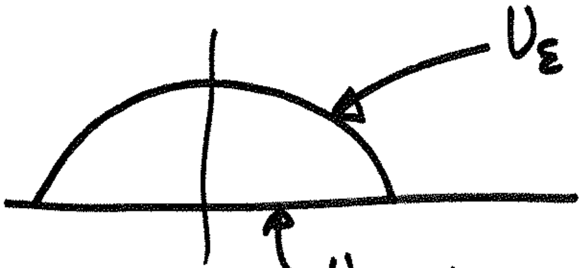
$k(x_N^+)^\alpha$ is a minimiser of

$$J[u, B] = \frac{1}{2}\int_B |y|^\beta |\nabla u|^2 + \mathcal{L}_N(\{u>0\} \cap B \cap \mathbb{R}) .$$

inhomogeneity between the 2 terms .

- Reduce the computation to a computation in $\mathbb{R}^2_+$.
- <u>Perturbation</u> : $u_\varepsilon(x) = \dfrac{(x+\varepsilon)_+^\alpha}{(1+\varepsilon)^\alpha}$.

$$U_\varepsilon(x) : \begin{cases} -\operatorname{div} |y|^\beta \nabla U_\varepsilon = 0 & (\{|x,y|<1,\ y>0\}) \\ U_\varepsilon(x,y) = U_0(x,y) & (|(x,y)|=1) \\ U_\varepsilon(x,0) = u_\varepsilon(x) . \end{cases}$$



$U_0(x,y) = r^\alpha \cos^{2-2\alpha}\frac{\theta}{2}$.

$$E[U] = \frac{1}{2}\int_{B_1 \cap \{y>0\}} y^\beta |\nabla U|^2 .$$

<u>Write</u> : $E[AU_0] + \mathcal{L}_N(\{U_0>0\} \cap \mathbb{R})$

$\leq E[AU_\varepsilon] + \mathcal{L}_N(\{U_\varepsilon > 0\} \cap \mathbb{R}$.

Let $\varepsilon \to 0^+$ and 0^-, retrieve A .

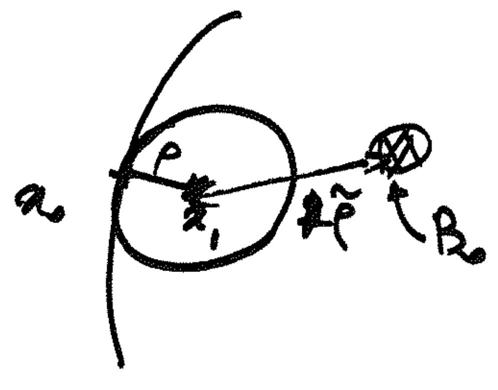
The last general thing one can say about minimisers is the tangent ball property.

<u>def</u>. u: minimiser, $x_0 \in \Gamma$. B: ball of $\mathbb{R}^N$,

- Γ has a tangent ball B from ~~below~~ outside at x_0 if $x_0 \in \partial B$ and $B \subset \{u = 0\}$.
- Γ has a tangent ball B from inside at x_0 if $x_0 \in \partial B$ and $B \subset \{u > 0\}$.

<u>prop</u>. u: minimiser, Γ has a tangent ball at x_0 from inside or outside. Then x_0 is a regular point.

<u>Proof</u>. The picture is the following:



Assume $x_1 = 0$ for simplicity.

f: supported in the small ball, for instance its indicator.

$$u_*(x, 0) = \int_{B_0} \left(\frac{\rho^2 - |x|^2}{|y|^2 - \rho^2} \right)^{\alpha/2} \frac{dy}{|x - y|^N}$$

Set: $\nu(x_0) = \dfrac{x_1 - x_0}{|x_1 - x_0|}$

$u_\alpha(x,0) \sim \lambda |(x-x_0).\nu(x_0)|^\alpha$ for x close to x_0, inside B.

<u>Nondegeneracy</u> $\Rightarrow$ $u(x,0) \geq q_0\, u_\alpha(x,0)$ inside B. Harnack $\Rightarrow$ $u(x,y) \geq q_0'\, u_\alpha(x,y)$ inside ~~B (the~~ the extension of $B = \{(x,y) : |x|^2+y^2 <$

$$u_\alpha(x,y) = \int \frac{y^{2\alpha}\, dx'}{|(x-x')^2+y^2|^{\frac{N+2\alpha}{2}}} .$$

<u>Exercise</u> (if you do not find the solution, go to Caffarelli - Salsa):

$q_r = \sup\{q>0 : u \geq q\, u_\alpha \text{ in } B_r(X_0)\}$.

$(X_0 = (x_0, 0))$.

- $(q_r)_r$ is uniformly bdd as $r \to 0$. (use optimal regularity).

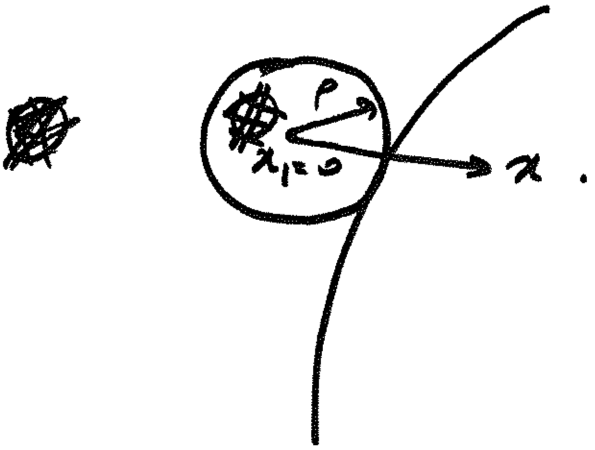
- $q_r \xrightarrow[r\to 0]{\nearrow} q_\infty > 0$.

- As $x \to x_0$, inside B_r, we have ~~$u(x) = q_\infty$~~

$$u(x,0) = q_\infty |(x-x_0).\nu(x_0)|^\alpha + o(|x-x_0|^\alpha).$$

Thus the hyperplane $\perp$ to $\nabla(x_0)$ is a tangent plane to $\{u=0\}$ at x_0.

Tangent ball from outside:

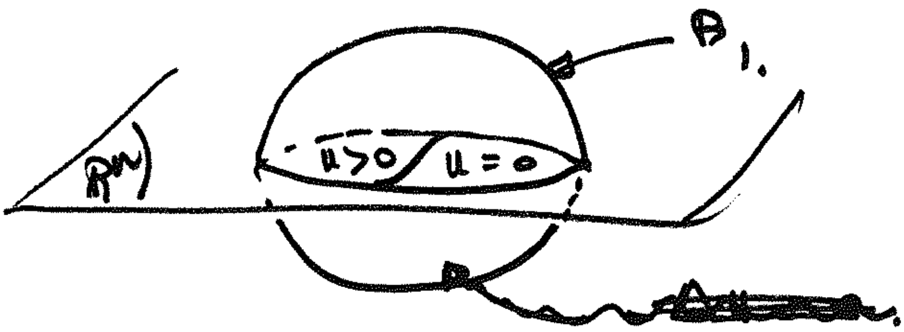


$$u^*(x,0) = \int_{B^\#} \left(\frac{|x|^2-\rho^2}{\rho^2-|y|^2}\right)^{\alpha/2} \frac{dy}{|x-y|^N}.$$

u^* serves as a lower barrier. $\boxtimes$

IV]. Regularity of the free boundary in the case $\alpha = \frac{1}{2}$.

Recall the situation we are studying:



u : minimiser of the energy :

$$J[u,B] = \frac{1}{2}\int |\nabla u|^2 + \mathcal{L}_N\left(\{u>0\}\cap \mathbb{R}^N \cap B\right).$$

<u>Redefine</u> : $U(s,y) = \sqrt{r}\cos\frac{\theta}{2}$

with $r = \sqrt{s^2+y^2}$ and $s = r\cos\theta$.

We have $U(s,0) = \sqrt{s_+}$. Abuse notations by putting $U(x_N, y) = U(X)$.

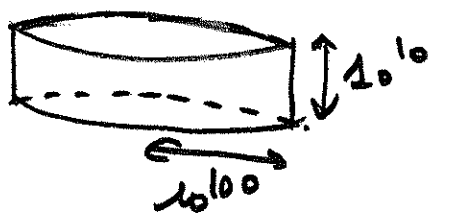
The preceding session had led us to the following result: let $X_0 = (x_0, 0)$ be a F.B point that is regular in the geometric measure theory sense. Then we proved:

$$u(X) \sim B\,U(X) \qquad X \to X_0$$

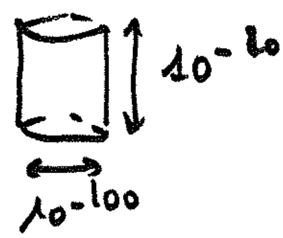
for a well-defined constant B. This amounts to imposing the Hölder quotient at the point X. The question is now: how many such regular points do we have, so we can connect to the FB problem of the beginning of this course? We are going to see that a sufficiently flat piece of free boundary is $C^{1,\gamma}$.

<u>def</u>. let C the cylinder of $\mathbb{R}^N$ (⚠ we are working in $\mathbb{R}^N$ here) $\{(x', x_N) \in B \times [-\delta, \delta]\}$; B is a ball.
The flatness of C is the ratio $\dfrac{\delta}{\mathrm{diam}(B)}$.

It is a dimensionless notion. <u>Ex</u>:



is flat.



is not flat.

<u>def</u>. A set Γ of $\mathbb{R}^N$ is δ-flat if it is included in a δ-flat cylinder.

<u>Th</u>. (de Silva – R.). u: a minimiser of J. (w. $\alpha = \frac{1}{2}$). $X_0 = (x_0, 0)$ a free boundary point. Assume $\Gamma \cap B_1 \cap \mathbb{R}^N$ to be δ-flat.
Then it is a $C^{1,\gamma}$ graph.

<u>Rq</u>. To get to analyticity requires a bit more work. The known way at the moment is to prove $C^{2,\gamma}$ by a refinement of the method. This has been done by de Silva – Savin.

Method of proof: a compactness argument.

The plan of the section is

1°). Reduction of pb + strategy.

2°). Description of the method on minimal surfaces.

3°). Adaptation to our situation.

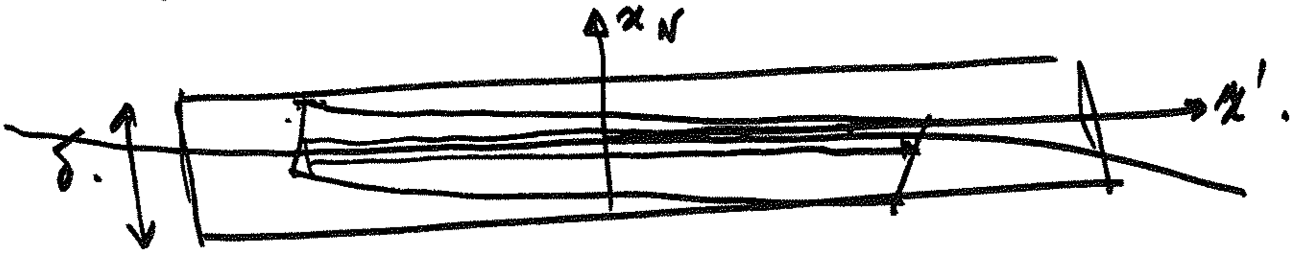
1°). Reduction of pb.

When you wish to reach $C^{1,\gamma}$ from flat, the thing to reach is the following.

Holy Grail. Try to prove: assume $\Gamma \cap B_1$ to be included in a δ-flat cylinder. Assume the existence of $\eta \in (0,1)$ (possibly very small) ~~so that~~ ind. of δ such that:

$\Gamma \cap B_{1-\eta}$ is in a $\delta(1-\eta)$ flat cylinder.

I claim that: if we manage to do it for the FBP, we are done.



Indeed: iterate the situation.

In B_{η^n}, Γ is included in a $((1-\eta))^n\delta$ flat cylinder. Rescaling: $(x,y) \to \frac{x,y}{\eta^n}$,

Γ is still (flatness is dimensionless) included in a $\delta(1-\eta)^n$ flat cylinder of base $\underline{\underline{1}}$.

The iteration occurs at around one point, this means, assuming the point is $(0,0)$:

- $\Gamma \cap \{x' = 0\} = \{(0,0)\}$.
- Γ has a tangent plane at $0,0$. and a normal vector $\nu(0)$
- One may do the process around another FB point (x', x_N). This shows that the FB is a graph $x_N = \varphi(x')$.
- To control the deviation between the normal vectors $\nu(0,0)$ and $\nu(x', \varphi(x'))$: look at the 1^{st} n when $(0,0)$ and $(x', \varphi(x'))$ are in two different cylinders.

$|x - x'| \sim \eta^n$.

$$|\nu(0) - \nu(x', \varphi(x'))| \leq \delta(1-\eta)^n.$$
$$\sim |x - x_0|^{\frac{|\log 1-\eta|}{|\log \eta|}}.$$

As said before, the method used here is a compactness method. We are going to assume the existence of a sequence of flatter and flatter free boundaries in which we cannot do the improvement of flatness, which will give the contradiction.

1^{st} person I know to have used it:

L. Wang Compactness methods for certain degenerate eq$^{\text{ns}}$, JDE, 1994.

2^{nd} person:

O. Savin. Perturbation solutions of fully nonlinear elliptic equations, CPDE, 2007.

I ~~##~~ personally think that this is a very important paper. Proves $C^{2,\sigma}$ smoothness ~~for~~ for elliptic equations only assuming ellipticity of the linearized equation.

<u>Application</u>: the de Giorgi theorem for minimal surfaces. This is what I am going to explain now. And this is one reason why this paper is very important.

2°). <u>Smoothness of minimal surfaces</u>.

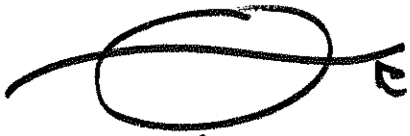
Let us go quickly through the definitions:

<u>def</u>. E admissible in B_1 if $\cancel{\int |D\mathbb{1}_E|}$ $|D\mathbb{1}_E|$ is a measure w. finite mass.

Ω open subset of $\mathbb{R}^N$, E admissible,

$$\mathrm{Per}(E,\Omega) = \int_\Omega |D \mathbb{1}_E| .$$

Exercise.



E nice surface.

check that this is the area in Ω.

Minimal set: E minimal in B_1

$\Longrightarrow \mathrm{Per}(E, B_1) \leq \mathrm{Per}(F, B_1)$

F: admissible set s.t. $E = F$ on ∂B_1.

Th. (de Giorgi). ∂E (to be given a precise def...) δ-flat in B_1. Then it is an analytic graph in $B_{1/2}$.

Ingredients.

(i). A viscosity relation for minimal surfaces.

Fact. If ∂E is a nice minimal hypersurface (ex. a graph in $\mathbb{R}^N$: $x_N = u(x')$.) then it has zero mean curvature.

$$-\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0 .$$

The additional information is that this relation holds in the viscosity sense

__Th.__ (Caffarelli - Cordoba).

- Let ~~[illegible]~~ $(x', \varphi(x'))$ touch ∂E from below.

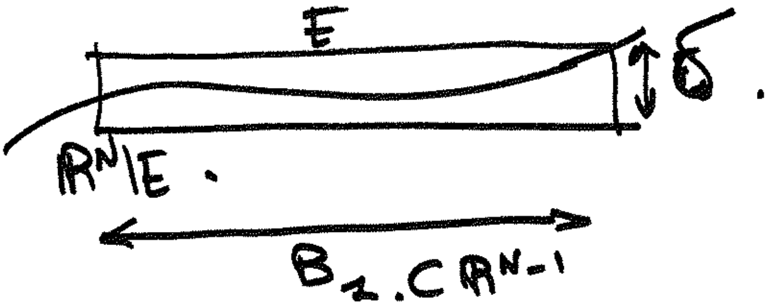
∂E. Then $-\operatorname{div} \dfrac{D\varphi}{\sqrt{1+|D\varphi|^2}} \leq 0$ at contact point.

- Let $(x', \varphi(x'))$ touch ∂E from above.

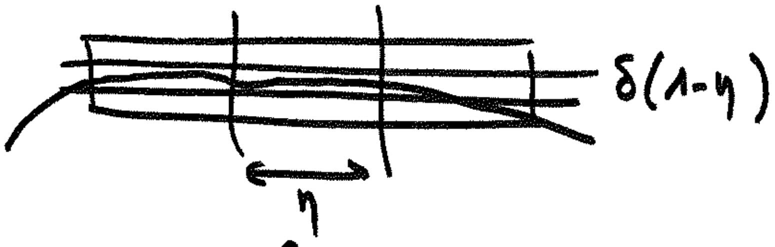
  Then $-\operatorname{div} \dfrac{D\varphi}{\sqrt{1+|D\varphi|^2}} \geq 0$ at contact point.

Proof of this fact not at all easy. Paper not easy to read either!

(ii). A localisation property.



__Th.__ ~~If δ is small enough,~~ There is $\eta > 0$, ~~indep~~ and $\delta_0 > 0$ such that: if $\delta \leq \delta_0$, ~~$\partial E \cap B_\eta$ is in a~~ E we have the following picture:



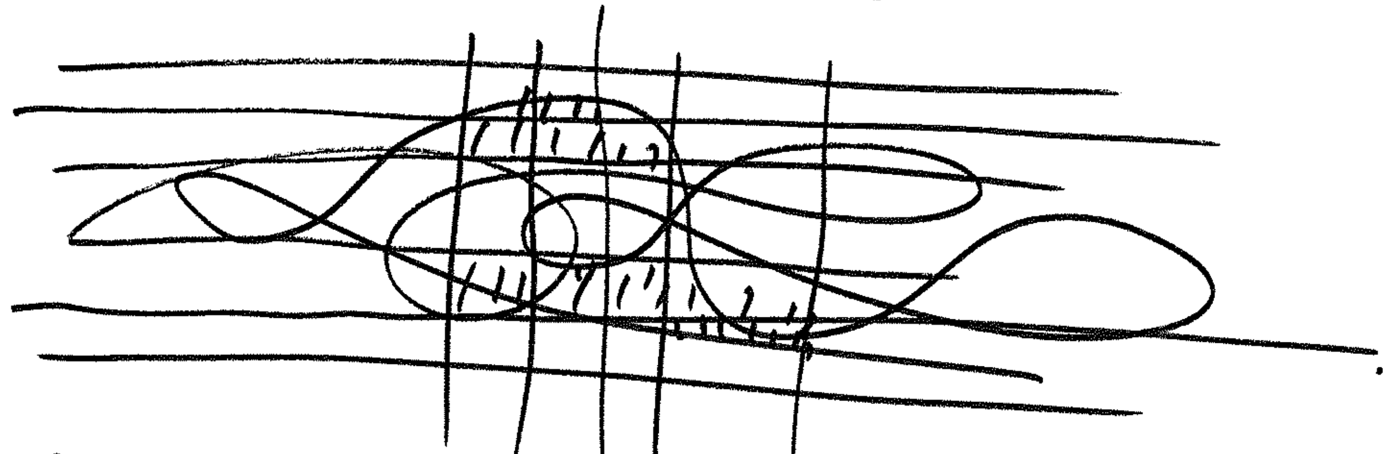
Contradiction argument.

Assume the existence of a sequence (E_{ε_k}) with $\delta = \varepsilon_k$ for which we cannot do the improvement of flatness.

$$\tilde{E}_\varepsilon = \left\{ \left(x', \frac{x_N}{\varepsilon}\right) : (x', x_N) \in E \right\}.$$

(we have magnified E_ε).

Apply localisation property around a point.



Can apply it only a finite number of times but the thickness goes to 0. $\left(\frac{\varepsilon}{\eta^{\lambda}} \leq \delta_0\right)$.

$$\varepsilon \to 0 : \quad \tilde{E}_\varepsilon \longrightarrow \{ (x', x_N) : x_N = \varphi(x') \}.$$

Moreover: φ is Hölder.

<u>YES BUT</u> : $\partial \tilde{E}_\varepsilon$ ~~is a~~ satisfies the curvature equation in the viscosity sense.

$$\text{Eq}^{n} : \quad -\cancel{\varepsilon}\, \operatorname{div} \frac{D\varphi}{\sqrt{1+\varepsilon^2 |D\varphi|^2}} = 0.$$

Viscosity solutions pass to the limit!

Limit: $-\Delta\varphi = 0$.

Are we happy? Not so quick! ~~A~~ Need to prove that a viscosity harmonic fct is smooth.

Th. (Caffarelli). This is true.

But then, what were we complaining about? We can indeed do the improvement of flatness.

3°). ~~B~~ The free boundary problem.

In the remaining part of this lecture, I will discuss, ~~what is no~~ in a very impressionistic way, the extra ingredients that are needed. Pb. We have 2 things: the FB and the unknown.

(0). Reformulation of the problem.

Nondegeneracy + optimal regularity ~~imply~~ allow to reduce to the

Th u: local minimiser. Assume that, in B_1: (this time $\mathbb{R}^{N+1}$):

$$U(X-\varepsilon e_N) \le u(X) \le U(X+\varepsilon e_N).$$

There is $\varepsilon_0 > 0$: if $\varepsilon \le \varepsilon_0$, Γ is a $C^{1,\gamma}$ graph in the e_N direction.

(i). <u>The ingredients</u>.

— Viscosity relation.

<u>Th.</u> (tangent balls).

— localisation property.

<u>Th.</u> $U(X + \varepsilon a_0 e_N) \le u(X) \le U(X + \varepsilon b_0 e_N)$ in B_1. Then there is $\eta > 0$ and $\exists\, \varepsilon_0 > 0$ such that:

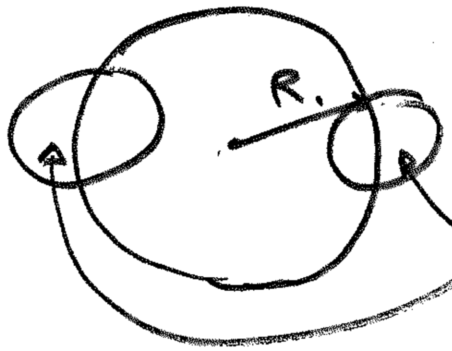
$U(x + \varepsilon a_1 e_N) \le u(X) \le U(X + \varepsilon b_1 e_N)$

in B_η, with $b_1 - a_1 \le (1-\eta)(b_0 - a_0)$.

<u>Proof relies on a subsolution / supersolution</u>.

$R > 0$ large, $V_{R,\delta}(t,y) = \left(1 + \delta \frac{t}{R}\right) U(t,y)$.

$v_{R,\delta}(X) = V_{R,\delta}\left(R^2 - \sqrt{|x'|^2 + (x_N - R)^2},\ y\right)$.

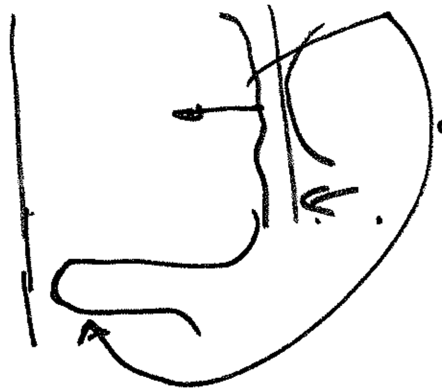


<u>prop.</u>

$v_{R,\delta}$ is a subsol for $\delta = +1$.

v_δ is a supersol for $\delta = -1$.

With these ingredients, one can push the FB inside.



(ii). <u>The contradiction argument.</u>

We have to deal not only with the FB, but with u. So, to we propose to straighten the FB by setting

$$u(X) = U(X - \varepsilon v(X) e_N).$$

This leads to the linearised equation

$$\begin{cases} \Delta(\partial_{x_N} U\, v) = 0 & (X \in \mathbb{R}^{N+1} \setminus \{x_N \leq 0, y = 0\}) \\ \partial_r (U_N v) = 0 & (X \to \{x_N = 0, y = 0\}). \end{cases}$$

The last eq^{n} means $\dfrac{U_N v(X)}{\sqrt{x_N^2 + y^2}} \xrightarrow[\substack{x_N \to 0 \\ y \to 0}]{} 0.$

<u>Th</u>. This equation has an improvement of flatness.

The pf relies on a brute force analysis.