The rate of attraction of super-critical waves in a Fisher-KPP type model with shear flow

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Abstract

We consider in this paper the thermo-diffusive model for flame propagation, which is a reaction-diffusion equation of the KPP (Kolmogorov, Petrovskii, Piskunov) type, posed on an infinite cylinder. Such a model has a family of travelling waves of constant speed, larger than a critical speed c_*. The family of all supercritical waves attract a large class of initial data, and we try to understand how. We describe in this paper the fate of an initial datum trapped between two supercritical waves of the same velocity: the solution will converge to a whole set of translates of the same wave, and we identify the convergence dynamics as that of an effective drift, around which an effective diffusion process occurs.

1 Introduction

1.1 Statement of the main results

The topic of this paper is to estimate how fast the solutions of:

$$\begin{cases} u_t - \Delta u + \alpha(y) u_x = f(u) \quad ((x,y) \in \mathbb{R} \times \mathbb{T}^{N-1}) \\ \lim\limits_{x \to -\infty} u(t,x,y) = 0, \qquad \lim\limits_{x \to +\infty} u(t,x,y) = 1, \end{cases} \tag{1.1}$$

with suitable initial data, will converge to travelling wave profiles. Here α is a sufficiently smooth function (C^∞ to avoid technical difficulties), and $\mathbb{T}^{N-1}$ denotes the $(N-1)$-dimensional torus. The function f will always be supposed to be smooth enough, and positive on $(0,1)$; moreover it will be assumed to be concave in u, and

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0.$$

This model is sometimes known, in the mathematical theory of flame propagation, as the 'thermo-diffusive model'. It is indeed the simplest model with nontrivial flow may be derived from the reacting fluid dynamics equations; the (shear flow) field $\vec{V}(x,y) = (0, \alpha(y))$ is imposed and only the chemical and heat transfer processses are conserved. The model was the object of numerical studies - see for instance [6], [19] - as a relevant preliminary account of the wrinkling of a flame front. See also [7] for its mathematical justification, and [1] for large shear asymptotics. The model has from then remained an important tool to understand the interplay between a flow field and reaction-diffusion processes, see for instance estimates on the burning rate (see [17]) or existence theorems for systems (see [15]).

Travelling waves propagating at the speed c are solutions of the form $\phi(x+ct, y)$, where the function $\phi(\xi, y)$ solves

$$\begin{cases} -\Delta\phi + (c+\alpha(y))\phi_\xi = f(\phi) \quad ((\xi,y)\in\mathbb{R}\times\mathbb{T}^{N-1}) \\ \lim\limits_{\xi\to-\infty}\phi(\xi,y)=0, \quad \lim\limits_{\xi\to+\infty}\phi(\xi,y)=1. \end{cases} \tag{1.2}$$

Their existence and qualitative properties are given by the following

Theorem 1.1 *(Berestycki-Nirenberg [8]) There is c_* such that* (1.2) *has no solution if $c < c_*$, and a unique - up to translation in ξ - solution if $c \ge c_*$. Moreover we have $\partial_\xi\phi_c > 0$.*

Some stability results have been proved before by Mallordy and the second author [21]:

Theorem 1.2 *(Mallordy-Roquejoffre [21]) Consider $c > c_*$ and ϕ_c a solution of* (1.2)*. Choose an initial datum u_0 satisfying*

$$\frac{u_0(x,y)}{\phi_c(x,y)} = O(e^{rx}) \quad \text{as } x\to+\infty.$$

Consider the solution u of the Cauchy problem

$$\begin{cases} u_t - \Delta u + \alpha(y)u_x = f(u) \quad ((x,y)\in\mathbb{R}\times\mathbb{T}^{N-1}), \\ u(0,x,y) = u_0(x,y). \end{cases} \tag{1.3}$$

Then $u(t,x,y) = \phi_c(x+ct,y) + O(e^{-\omega t})$ as $t\to+\infty$, uniformly in $(x,y)\in\mathbb{R}\times\mathbb{T}^{N-1}$.

The goal of this paper is to study what happens when the initial datum u_0 is trapped between two supercritical waves of the same velocity, a slightly more general assumption than that of Theorem 1.2. We are going to prove the following (rather drastic) change in asymptotic behaviour:

Theorem 1.3 *Let $u_0(x,y)$ be a Cauchy datum for* (1.3)*. Assume the existence of $c > c_*$ and $M > 0$ such that*

$$\forall (x,y)\in\mathbb{R}\times\mathbb{T}^{N-1}, \quad \phi_c(x-M,y) \le u_0(x,y) \le \phi_c(x+M,y).$$

Define the initial shift $m_0(x,y)$ as

$$\forall (x,y)\in\mathbb{R}\times\mathbb{T}^{N-1}, \quad \phi_c(x+m_0(x,y),y) = u_0(x,y).$$

Then there exist $D_*(c)\ge 1$ and $V_*(c) > 0$ such that: if $s^{pp}(t,\xi)$ is the solution of

$$s^{pp}_t - D_*(c)s^{pp}_{\xi\xi} + V_*(c)s^{pp}_\xi = 0, \quad s^{pp}(0,\xi,y) = \int_{y'} e^{r_-(c)m_0(\xi,y')}\psi_{r_c}(y')^2\,dy', \tag{1.4}$$

where $r_-(c) > 0$ and ψ_{r_c} is the (positive) principal eigenfunction associated to some explicit elliptic operator, and if

$$m^{pp}(t,\xi) := \frac{1}{r_-(c)}\ln s^{pp}(t,\xi),$$

then we have

$$\sup_{(x,y)\in\mathbb{R}\times\mathbb{T}^{N-1}} |u(t,x,y) - \phi_c(x+ct+m^{pp}(t,x+ct),y)| = O(\frac{1}{t^{1/4}}).$$

It is known - see [10], [26] - that very simple equations like (1.4) can exhibit complex behaviours. In particular, the ω-limit set (in the sense of uniform convergence on every compact - can be a whole interval.

Our result extends and completes in several directions some of our earlier results. Before explaining this, let us add some comments.

- The advection-diffusion (1.4) can be solved explicitely. Hence we may find reasonably sharp conditions ensuring the convergence of u to some translate of the travelling wave ϕ_c.
- We have $D_*(c) > 1$ as soon as α is nonconstant; this is a manifestation of the well-known 'convection-enhanced'-diffusion - see [12].
- When $\alpha = 0$, $V_*(c) = \sqrt{c^2 - c_*^2}$; and the proof breaks down when $c = c_*$ (the reason being that we have $V_*(c_*) = 0$).

The main step of Theorem 1.3 will be the computation of the effective dynamics (1.4), by a Fourier argument combined with some classical functional analysis.

1.2 Comparison with earlier results

In [4], we studied the more general equation of the same type

$$\begin{cases} u_t - \mathrm{div}(A(x,y)Du) + B(x,y).Du = f(x,y,u), \quad ((x,y) \in \mathbb{R} \times \mathbb{T}^{N-1}), \\ \lim\limits_{x \to -\infty} u(t,x,y) = 0, \quad \lim\limits_{x \to +\infty} u(t,x,y) = 1, \end{cases} \tag{1.5}$$

a model that contains as particular cases the one-dimensional equation $u_t - u_{xx} = f(u)$ and the thermo-diffusive model (1.3). Under suitable periodicity and smoothness assumptions on A, B and f, we proved the existence of pulsating waves $\phi_c(t,x,y)$ having a precise asymptotic behaviour and increasing in time, and then that the family of all these pulsating waves provides an attractor for a large class of initial data to (1.5):

Theorem 1.4 *(Bages-Martinez-Roquejoffre [4]) Let $u_0(x,y)$ be a Cauchy datum for* (1.5)*. Assume the existence of $c > c_*$ and $M > 0$ such that $\phi_c(-M,x,y) \leq u_0(x,y) \leq \phi_c(M,x,y)$. There exists a smooth function $m(t,x,y)$ such that $\lim\limits_{t \to +\infty} \|(m_t, Dm, D^2m)(t,.,.)\|_\infty = 0$, and such that*

$$\sup_{(x,y) \in \mathbb{R} \times \mathbb{T}^{N-1}} |u(t,x,y) - \phi_c(t + m(t,x,y), x, y)| \to 0 \quad \textit{as } t \to +\infty.$$

This shift $m(t,x,y)$ satisfies - up to a Hopf-Cole transform - a linear diffusion equation. Our result completes that of [14], which proves the asymptotic stability of all the waves under fatly decaying perturbations. However, at this level of generality, we were not able to provide an effective one dimensional equation, nor an estimate of the rate of attraction of the family of translates of the pulsating waves. The goal of the present paper is to prove such results in the context of the thermo-diffusive model.

1.3 Additional comments and open questions

We hope that the ideas developped here will not only provide a better understanding of the dynamics of super-critical KPP waves, but will also help to understand how the critical wave is attained from fastly decaying initial data. In a forthcoming work [20] we will see how this works for the 1D homegeneous model - already proved by Bramson [9] but where a deterministic proof is still unknown - and on the multi-D model with special cases of advection. The general case is an important issue that goes far beyond scalar reaction-diffusion equations, see [11].

There are several questions close to this work whose answers would be very interesting:

- We concentrated our study in the case where the initial condition of the Cauchy problem is trapped between two translates of the same pulsating wave. It would be very interesting to investigate the behaviour of the solutions under weakened assumptions on the initial condition. An important first step is taken in Hamel-Nadirashvili [13], where it is proved that (almost) every time-global solution of the N-dimensional homogeneous model

$$u_t - \Delta u = u(1-u), \qquad t > 0,\ x \in \mathbb{R}^N, \quad 0 < u < 1$$

  is a (possibly uncountably infinite) convex combination of one-dimesional waves.

- The case of initial conditions trapped between two critical waves ($c = c_*$) is not treated here. We do not know if a result similar to Theorem 1.3 exists and can be proved.

1.4 Description of the proof of Theorem 1.3,plan of the paper

Since the travelling wave ϕ_c is strictly increasing in its first variable, we are able to define

$$n^*(t,x,y) := (\phi_c(\cdot,y))^{-1}(u(t,x,y)),$$

which satisfies

$$u(t,x,y) = \phi_c(n^*(t,x,y),y),$$

hence informations on n^* give informations on u. And to have informations on n^*, we write the differential problem that it satisfies. Unfortunately, this problem is fully nonlinear, but it admits an "approximate" solution $n(t,x,y)$, obtained linearizing the problem at $-\infty$. The final task is then

- to obtain informations on the approximate solution n,

- to study the difference between u and its "approximation" $u^{app}(t,x,y) := \phi_c(n(t,x,y),y)$.

These two things are in fact closely related.

The plan of the paper is the following:

- in Section 2 we explain how to construct the approximate solution n, and we study the difference between u and u^{app} admitting some properties of n; some parts of section 2 are directly adapted from [4], where a more general case is considered, but we included them in a sake of completeness and clarity for the reader;

- in Section 3 we prove the admitted properties of n, given in Proposition 2.3; it is specific for the thermo-diffusive paper and is the heart of the present paper; the proof of Theorem 1.3 lies on it.

- finally in Section 5 we give some applications of our results.

2 The construction of the approximate solution u^{app}

2.1 The moving frame

First we note that our assumption on the initial condition u_0 and the maximum principle imply that the solution $u(t,x,y)$ of (1.3) satisfies:

$$\phi_c(x+ct-M,y) \leq u(t,x,y) \leq \phi_c(x+ct+M,y),$$

hence it is natural to consider the problem in the moving frame given by the classical change of variables $(t,x) \mapsto (t, \xi = x+ct)$. Now if $u(t,x,y)$ is a solution of the Cauchy Problem for (1.3), the function $\widetilde{u}$ defined by

$$\widetilde{u}(t,\xi,y) := u(t,\xi - ct, y)$$

satisfies $u(t,x,y) = \widetilde{u}(t,x+ct,y)$, and thus is solution of the Cauchy problem

$$\begin{cases} \widetilde{u}_t + c\widetilde{u}_\xi - \Delta\widetilde{u} + \alpha(y)\widetilde{u}_\xi = f(\widetilde{u}), & t>0, \xi \in \mathbb{R}, \\ \widetilde{u}(0,\xi) = u(0,\xi) = u_0(\xi). \end{cases} \tag{2.1}$$

Then consider

$$\beta(y) := c + \alpha(y),$$

and the differential operator

$$NL_c[\widetilde{U}] := \widetilde{U}_t - \Delta\widetilde{U} + \beta(y)\widetilde{U}_\xi - f(\widetilde{U}) :$$

then the function u is solution of the differential equation of (1.3) if and only if

$$NL_c[\widetilde{u}] = 0. \tag{2.2}$$

A travelling wave ϕ_c satisfies

$$NL_c[\phi_c] = 0, \quad \phi_c(-\infty,y) = 0, \quad \phi_c(+\infty,y) = 1.$$

2.2 Specific behaviour of travelling waves

The behaviour of a travelling wave at $-\infty$ is of course by now well-established. Linearizing (2.2) at 0, we obtain

$$-\Delta\widetilde{v} + \beta(y)\widetilde{v}_\xi - f'(0)\widetilde{v} = 0. \tag{2.3}$$

An *exponential solution* to (2.3) is a solution of the form

$$\widetilde{v}(\xi,y) = e^{\lambda\xi}\psi(y), \quad \xi \in \mathbb{R}, y \in \mathbb{T}^{N-1},$$

where $\lambda > 0$ is called a *characteristic exponent*. The function $\psi(y)$ has to solve

$$\mathcal{L}(\lambda)\psi := -\Delta\psi + \lambda\beta(y)\psi = (\lambda^2 + f'(0))\psi, \ y \in \mathbb{T}^{N-1}, \tag{2.4}$$

and because ϕ_c has to be positive, ψ has to be positive. Hence, given $\lambda > 0$, it is natural to consider the operator $\mathcal{L}(\lambda)$ defined by

$$\mathcal{L}(\lambda)\psi := -\Delta_y\psi + \lambda\beta(y)\psi, \tag{2.5}$$

and acting on the space $C^2(\mathbb{T}^{N-1})$. Denote $\nu_1(\lambda)$ the first eigenvalue of $\mathcal{L}(\lambda)$ and ψ_λ an associated eigenfunction, problem (2.4) amounts to solving

$$\nu_1(\lambda) = \lambda^2 + f'(0). \tag{2.6}$$

It is well known that $\nu_1(\lambda)$ is a simple, nondegenerate eigenvalue for $\mathcal{L}(\lambda)$ and Kato-Rellich's Theorem implies the existence of an analytic extension of ν_1 in a complex domain containing the right half-line. In particular, $\lambda \mapsto \nu_1(\lambda)$ is C^1. The more special features of the function ν_1 are summarized in the following

Theorem 2.1 *([8]). The function ν_1 is concave, and equation* (2.6) *has solutions if and only if $c \geq c_*$. For $c > c_*$, there are two solutions* $0 < r_-(c) < r_+(c)$.

This implies, in particular:

$$V_*(c) := \frac{d\nu_1}{d\lambda}(r_-(c)) - 2r_-(c) > 0. \tag{2.7}$$

Moreover, we have an integral formula for $\frac{d\nu_1}{d\lambda}(r_-(c))$: let ψ_λ be the unique principal eigenfunction of $\mathcal{L}(\lambda)$ such that $\psi_\lambda(y) > 0$, $\displaystyle\int_{\mathbb{T}^{N-1}} \psi_\lambda^2 = 1$; then

$$\frac{d\nu_1}{d\lambda}(r_-(c)) = \int_{\mathbb{T}^{N-1}} \beta\psi_{r_-(c)}^2. \tag{2.8}$$

(This follows from an asymptotic analysis similar to the one performed in the proof of Lemma 3.1.)

The additional information that we need for ϕ_c is the

Theorem 2.2 *([8]) If $c > c_*$, there is $\delta > 0$ such that the wave ϕ_c has the following asymptotic behaviour as $\xi \to -\infty$:*

$$\phi_c(\xi, y) = e^{r_-(c)\xi}\psi_{r_c}(y)(1 + O(e^{\delta\xi})). \tag{2.9}$$

2.3 Exact local shift

Motivated by [4], we will look for an approximation of the solution of (2.2) with initial datum u_0 as in Theorem 1.3 under the form

$$\widetilde{u}^{app}(t, \xi, y) = \phi_c(\xi + m(t, \xi, y), y),$$

where the (approximate) shift m will satisfy a suitable parabolic problem.

First consider a general shifted wave: given a sufficiently smooth function $m : (0, +\infty) \times \mathbb{R} \times \mathbb{T}^{N-1} \to \mathbb{R}$, consider

$$T^{(m)}\phi_c(t, \xi, y) := \phi_c(\xi + m(t, \xi, y), y).$$

Of course, when m is identically zero, we have $T^{(0)}\phi_c = \phi_c$, and $NL_c[T^{(0)}\phi_c] = 0$. Let us compute in the general case $NL_c[T^{(m)}\phi_c]$: some computations lead to

$$NL_c[T^{(m)}\phi_c] = \Big(m_t - \Delta m + \beta(y) m_\xi - 2\frac{D\partial_\xi \phi_c}{\partial_\xi \phi_c}.Dm - \frac{\partial_{\xi\xi}\phi_c}{\partial_\xi \phi_c}|Dm|^2\Big)\partial_\xi \phi_c,$$

where, in the right handside, m and its derivatives are evaluated at (t, ξ, y), and $\partial_\xi \phi_c$ and its derivatives are evaluated at $(\xi + m(t, \xi, y), y)$. Hence $NL_c[T^{(m)}\phi_c] = 0$ if and only if

$$m_t - \Delta m + \beta(y) m_\xi - 2\frac{D\partial_\xi \phi_c}{\partial_\xi \phi_c}.Dm - \frac{\partial_{\xi\xi}\phi_c}{\partial_\xi \phi_c}|Dm|^2 = 0. \tag{2.10}$$

This is the problem satisfied by the exact shift

$$m^*(t, \xi, y) := \phi_c(\cdot, y)^{-1}(\widetilde{u}(t, \xi, y)) - \xi,$$

which is well defined since $\partial_x \phi_c > 0$.

To study the properties of the solution of this problem seems difficult, hence our strategy will be to find a parabolic problem that will be: as close as possible of the previous one, but simpler. This will permit us to study the properties of its solution m; of course in this case the functions $\widetilde{u}$ and $T^{(m)}\phi_c$ will have no reason to coincide. Then we will consider the difference $\widetilde{u} - T^{(m)}\phi_c$, and estimate its asymptotic behaviour as $t \to +\infty$. What came as a surprise to us is that this very simple and natural strategy actually enables us to say something about very general models, see also [4].

Even if we cannot say many things on the exact shift $m^*(t, \xi, y)$, we can see that it has the following property: for all $t > 0$, $m^*(t, \cdot, \cdot)$ is of class $C^1(\mathbb{R} \times \mathbb{T}^{N-1})$ and is bounded in the natural C^1-norm: indeed,

- first, it is clear that m^* is bounded, and more precisely, $m^*(t, \xi, y) \in [-M, M]$; this follows from the assumption that $u_0(\xi, y) \in [\phi_c(\xi - M, y), \phi_c(\xi + M, y)]$, using the weak maximum principle;

- next, from parabolic regularity, $m^*(t, \cdot, \cdot)$ is of class $C^1(\mathbb{R} \times \mathbb{T}^{N-1})$;

- at last,

$$m^*_\xi(t, \xi, y) = \frac{\widetilde{u}_\xi(t, \xi, y)}{\phi_c'(\xi + m^*(t, \xi, y))} - 1 = \frac{\widetilde{u}_\xi(t, \xi, y)}{\phi_c'(\xi, y)} \frac{\phi_c'(\xi, y)}{\phi_c'(\xi + m^*(t, \xi, y))} - 1,$$

hence $m^*_\xi(t, \cdot, \cdot)$ is bounded if and only if $\frac{\widetilde{u}_\xi}{\phi_c'}$ is bounded; and it follows from local parabolic estimates that this is true, once again using the fact that the weak maximum principle implies that $\phi_c(\xi - M, y) \leq \widetilde{u}(t, \xi, y) \leq \phi_c(\xi + M, y)$ for all $t > 0$ and all $\xi \in \mathbb{R}$, $y \in \mathbb{T}^{N-1}$. The same property holds for $D_y m^*$.

Hence, in particular, $m^*(1, \cdot, \cdot)$ is of class $C^1(\mathbb{R} \times \mathbb{T}^{N-1})$ and is bounded in the natural C^1-norm. We will use this remark in the following.

2.4 The approximate shift and the associated shifted wave

Now we are going to approximate the nonlinear parabolic problem (2.10) satisfied by m^*. Due to elliptic regularity, the exponential deviation in $O(e^{(r_-(c)+\delta)\xi})$ of ϕ_c from $e^{r_-(c)\xi}\psi_{r_c}(y)$ also holds for the derivatives. Using this, let us study the asymptotic behaviour of the coefficients as $\xi \to -\infty$: neglecting in a first approach the residual term, we obtain

$$\partial_\xi \phi_c(\xi, y) = r_-(c)e^{r_-(c)\xi}\psi_{r_c}(y) + \dots,$$

$$\partial_{\xi\xi}\phi_c(\xi, y) = r_-(c)^2 e^{r_-(c)\xi}\psi_{r_c}(y) + \dots,$$

$$D\partial_\xi \phi_c(\xi, y) = r_-(c)^2 e^{r_-(c)\xi}\psi_{r_c}(y)e_1 + r_-(c)e^{r_-(c)\xi}D_y\psi_{r_c}(y) + \dots.$$

Hence we are driven to consider the approximate problem

$$m_t - \Delta m + (\beta(y) - 2r_-(c))m_\xi - 2\frac{D_y\psi_{r_c}}{\psi_{r_c}}.D_y m - r_-(c)|Dm|^2 = 0, \tag{2.11}$$

and to define the approximate shift m as the solution of this nonlinear parabolic equation, coupled with a suitable initial condition: $m(0) = m^*(0)$ (hence such that $\phi_c(\xi + m^*(0)(\xi, y), y) = u_0(\xi, y))$ if $m^*(0)$ is C^1 and bounded in norm C^1, or $m(1) = m^*(1)$, which is always C^1 and bounded in norm C^1, as a consequence of the assumption that the initial condition u_0 is trapped between two translates of ϕ_c. In the following we assume that $m(0) = m^*(0)$.

We will prove in section 3 the following properties of the approximate shift m:

Proposition 2.3 *The approximate shift m solution of the Cauchy problem* (2.11) *has the following properties:*

(i) m is bounded on $(0, +\infty) \times \mathbb{R} \times \mathbb{T}^{N-1}$, and more precisely

$$\forall t > 0, \forall \xi \in \mathbb{R}, \forall y \in \mathbb{T}^{N-1}, \quad m(t, \xi, y) \in [-\|m^*(0)\|_\infty, \|m^*(0)\|_\infty];$$

(ii) m is C^1 on $(0, +\infty) \times \mathbb{R} \times \mathbb{T}^{N-1}$, and $(t, \xi, y) \mapsto Dm(t, \xi, y)$ is bounded on $(0, +\infty) \times \mathbb{R} \times \mathbb{T}^{N-1}$;

(iii) $\|Dm(t, .)\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{N-1})} = O(t^{-1/4})$.

Admit Proposition 2.3 for a moment. Then considering the associated shifted wave

$$\widetilde{u}^{app}(t, \xi, y) := \phi_c(\xi + m(t, \xi, y), y), \tag{2.12}$$

we are in position to study the difference $\widetilde{u} - \widetilde{u}^{app}$.

2.5 The difference between the solution and the approximate solution, admitting Proposition 2.3

2.5.1 Preliminary result

Given $\delta \in (0, r_+(c) - r_-(c)))$, let Y_δ be the space of all continuous functions $\widetilde{u}$ on $\mathbb{R} \times \mathbb{T}^{N-1}$ such that

$$e^{-(r_-(c)+\delta)\xi}\widetilde{u}(\xi, y) \quad \text{is a bounded uniformly continuous function,}$$

endowed with the natural norm $\|\tilde{u}\|_{Y_\delta} := \sup_{(\xi,y)\in\mathbb{R}\times\mathbb{T}^{N-1}} |\tilde{u}(\xi,y)|\, e^{-(r_-(c)+\delta)\xi}$.

The following (by now standard) result is useful to study the asymptotic behaviour as $t \to +\infty$ of solutions of parabolic equations decaying sufficiently fast in space, and we will use it to study the difference $\widetilde{u} - \widetilde{u}^{app}$:

Lemma 2.4 *Consider the equation*

$$\widetilde{v}_t - \Delta\widetilde{v} + \beta(y)\widetilde{v}_\xi - f'(0)\widetilde{v} = 0. \tag{2.13}$$

Then, given $\delta \in (0, r_+(c) - r_-(c))$, there is $C_\delta \geq 1$ and $\omega_\delta > 0$ such that, if $\tilde{v}(0,.) := \tilde{v}_0$ is in Y_δ, then the solution $\tilde{v}$ of (2.13) emerging from $\tilde{v}_0$ satisfies: for all $t > 0$, $\tilde{v}(t,\cdot,\cdot) \in Y_\delta$, and

$$\|\tilde{v}(t,\cdot,\cdot)\|_{Y_\delta} \leq C_\delta e^{-\omega_\delta t}\|\tilde{v}_0\|_{Y_\delta}. \tag{2.14}$$

Proof of Lemma 2.4. It follows from the construction of a suitable positive super-solution to (2.13), exponentially decaying in time: first, given $\lambda > 0$ and $\omega \geq 0$, let us consider

$$\tilde{V}^{(\lambda,\omega)}(t,\xi,y) := \psi_\lambda(y)e^{\lambda\xi-\omega t}. \tag{2.15}$$

Then the definition of ψ_λ immediately implies that

$$\tilde{V}_t^{(\lambda,\omega)} - \Delta\tilde{V}^{(\lambda,\omega)}) + \beta(y)\tilde{V}_\xi^{(\lambda,\omega)} - f'(0)\tilde{V}^{(\lambda,\omega)} = (\nu_1(\lambda) - \lambda^2 - f'(0) - \omega)\tilde{V}^{(\lambda,\omega)}. \tag{2.16}$$

Since we want $\tilde{V}^{(\lambda,\omega)}$ to be positive and we want it to be a super-solution to (2.13), we need to find λ and ω such that

$$\nu_1(\lambda) - \lambda^2 - f'(0) - \omega \geq 0.$$

This can be done, noting that it follows from Theorem 2.1 that, given $\lambda \in (r_-(c), r_+(c))$, we have

$$\nu_1(\lambda) - \lambda^2 - f'(0) > 0.$$

Hence consider

$$\lambda = \lambda_\delta := r_-(c) + \delta, \text{ and } \quad \omega = \omega_\delta := \nu_1(r_-(c)+\delta) - \lambda_\delta^2 - f'(0).$$

Then we obtain that $\tilde{V}^\delta := \tilde{V}^{(\lambda_\delta,\omega_\delta)}$ is solution of (2.13). Now, since ψ_λ is positive on the compact $\mathbb{T}^{N-1}$, denoting $\tilde{C}_\delta := \frac{1}{\inf_{\mathbb{T}^{N-1}} \psi_{r_-(c)+\delta}}$, we have for all $\tilde{v}_0 \in Y_\delta$

$$|\tilde{v}_0(\xi,y)| \leq \tilde{C}_\delta\psi_{r_-(c)+\delta}(y)e^{(r_-(c)+\delta)\xi}\|\tilde{v}_0\|_{Y_\delta} = \tilde{C}_\delta\|\tilde{v}_0\|_{Y_\delta}\tilde{V}^\delta(0,\xi,y).$$

And then, the weak maximum principle implies, for $(t,\xi,y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}^{N-1}$:

$$|\tilde{v}(t,\xi,y)| \leq \tilde{C}_\delta\|\tilde{v}_0\|_{Y_\delta}\tilde{V}^\delta(t,\xi,y) = \tilde{C}_\delta\|\tilde{v}_0\|_{Y_\delta}\psi_{r_-(c)+\delta}(y)e^{(r_-(c)+\delta)\xi-\omega_\delta t},$$

which implies that $\tilde{v}(t,\cdot,\cdot) \in Y_\delta$ for all $t > 0$, and that (2.14) is satisfied with $C_\delta = \frac{\sup_{\mathbb{T}^N} \psi_{r_-(c)+\delta}}{\inf_{\mathbb{T}^N} \psi_{r_-(c)+\delta}}$. □

2.5.2 Convergence, admitting Proposition 2.3

Now, admitting Proposition 2.3, we have everything to prove the convergence result of Theorem 1.3: due to elliptic regularity, the exponential deviation of ϕ_c from $e^{r_-(c)\xi}\psi_{r_c}(y)$ also holds for the derivatives, and it follows from the choice of the approximate shift that

$$\begin{aligned} NL_c[\widetilde{u}^{app}] &= \Big(m_t - \Delta m + \beta(y) m_\xi - 2\frac{D\partial_\xi\phi_c}{\partial_\xi\phi_c}.Dm - \frac{\partial_{\xi\xi}\phi_c}{\partial_\xi\phi_c}|Dm|^2\Big)\partial_\xi\phi_c \\ &= \Big(2r_-(c)m_\xi + 2\frac{D_y\psi_{r_c}}{\psi_{r_c}}.D_y m + r_-(c)|Dm|^2 - 2\frac{D\partial_\xi\phi_c}{\partial_\xi\phi_c}.Dm - \frac{\partial_{\xi\xi}\phi_c}{\partial_\xi\phi_c}|Dm|^2\Big)\partial_\xi\phi_c \\ &= \Big(2(r_-(c) - \frac{\partial_\xi\partial_\xi\phi_c}{\partial_\xi\phi_c})m_\xi + 2(\frac{D_y\psi_{r_c}}{\psi_{r_c}} - \frac{D_y\partial_\xi\phi_c}{\partial_\xi\phi_c}).D_y m + (r_-(c) - \frac{\partial_{\xi\xi}\phi_c}{\partial_\xi\phi_c})|Dm|^2\Big)\partial_\xi\phi_c \\ &= O(1)(\|Dm(t)\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{N-1})} + \|Dm(t)\|^2_{L^\infty(\mathbb{R}\times\mathbb{T}^{N-1})})e^{\delta\xi}\partial_\xi\phi_c. \end{aligned}$$

Since Dm is bounded, and $\partial_\xi\phi_c$ goes exponentially fast to 0 as $\xi \to +\infty$, there is some $r > 0$ such that

$$NL_c[\widetilde{u}^{app}] = \begin{cases} O(e^{(r_-(c)+\delta)\xi})\|Dm(t)\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{N-1})} \text{ for } \xi \le 0, \\ = O(e^{-r\xi})\|Dm(t)\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{N-1})} \text{ for } \xi \ge 0. \end{cases}$$

Now denote

$$a(t,\xi,y) := -\frac{f(\widetilde{u}(t,\xi,y)) - f(\widetilde{u}^{app}(t,\xi,y))}{\widetilde{u}(t,\xi,y) - \widetilde{u}^{app}(t,\xi,y)}.$$

Since f is concave, we have

$$-f'(0) \le a(t,\xi,y) \le -f'(1). \tag{2.17}$$

Of course, the interest in introducing this function a comes from the following fact:

$$NL_c[\widetilde{u}] - NL_c[\widetilde{u}^{app}] = (\widetilde{u} - \widetilde{u}^{app})_t - \Delta(\widetilde{u} - \widetilde{u}^{app}) + \beta(y)(\widetilde{u} - \widetilde{u}^{app})_\xi + a(t,\xi,y)(\widetilde{u} - \widetilde{u}^{app}),$$

hence

$$(\widetilde{u} - \widetilde{u}^{app})_t - \Delta(\widetilde{u} - \widetilde{u}^{app}) + \beta(y)(\widetilde{u} - \widetilde{u}^{app})_\xi + a(t,\xi,y)(\widetilde{u} - \widetilde{u}^{app}) = -NL_c[\widetilde{u}^{app}]. \tag{2.18}$$

Let us denote by $g(t,\xi,y)$ the right-hand side, and $\widetilde{w}$ the solution of the Cauchy problem

$$\begin{cases} \widetilde{w}_t - \Delta\widetilde{w}_{\xi\xi} + \beta(y)\widetilde{w}_\xi - f'(0)\widetilde{w} = |g(t,\xi)|, \\ \widetilde{w}(0,\xi,y) = |\widetilde{u} - \widetilde{u}^{app}|(0,\xi,y) = |u_0(\xi,y) - \phi_c(\xi + m_0^*(\xi,y),y)| = 0. \end{cases} \tag{2.19}$$

We are going to prove the following facts:

- **Claim 2.5** : *for all $t > 0$ and all $\xi \in \mathbb{R}$, $y \in \mathbb{T}^{N-1}$, we have $|\widetilde{u} - \widetilde{u}^{app}|(t,\xi,y) \le \widetilde{w}(t,\xi,y)$;*
- **Claim 2.6** : *for all $\xi_0 \in \mathbb{R}$, $\sup_{\xi\le\xi_0, y\in\mathbb{T}^{N-1}} \widetilde{w}(t,\xi,y) = O(t^{-1/4})$;*
- **Claim 2.7** : *there exists $\xi_0 \in \mathbb{R}$ such that $\sup_{\xi\ge\xi_0, y\in\mathbb{T}^{N-1}} |\widetilde{u} - \widetilde{u}^{app}|(t,\xi,y) = O(t^{-1/4})$.*

It is clear that Claims 2.5-2.7 imply Theorem 1.3: by Claims 2.5 and 2.6,

$$\sup_{\xi\leq\xi_0, y\in\mathbb{T}^{N-1}} |\widetilde{u}-\widetilde{u}^{app}|(t,\xi,y) \leq \sup_{\xi\leq\xi_0, y\in\mathbb{T}^{N-1}} \widetilde{w}(t,\xi,y) = O(t^{-1/4}),$$

and adding Claim 2.7, we obtain that

$$\sup_{\xi\in\mathbb{R}, y\in\mathbb{T}^{N-1}} |\widetilde{u}-\widetilde{u}^{app}|(t,\xi,y) = O(t^{-1/4}).$$

Hence it remains to prove these claims.

Claim 2.5 is a consequence of the weak maximum principle. Indeed, first $\widetilde{w}$ is nonnegative; next, $\widetilde{u}-\widetilde{u}^{app}-\widetilde{w}$ satisfies

$$(\widetilde{u}-\widetilde{u}^{app}-\widetilde{w})_t-\Delta(\widetilde{u}-\widetilde{u}^{app}-\widetilde{w})+\beta(y)(\widetilde{u}-\widetilde{u}^{app}-\widetilde{w})_{\xi\xi}+a(\widetilde{u}-\widetilde{u}^{app}-\widetilde{w}) = g-|g|-(a+f'(0))\widetilde{w} \leq 0,$$

and

$$(\widetilde{u}-\widetilde{u}^{app}-\widetilde{w})(0,\xi,y) = 0,$$

hence $\widetilde{u}-\widetilde{u}^{app}-\widetilde{w}\leq 0$. In the same way, $\widetilde{u}-\widetilde{u}^{app}+\widetilde{w}\geq 0$, hence Claim 2.5 is proved.

Claim 2.6 is a consequence of Lemma 2.4: by Duhamel's formula:

$$\widetilde{w}(t,\xi,y) = \int_0^t e^{-(t-s)(-\partial_{\xi\xi}+\beta\partial_\xi-f'(0))}|g(s,.)|\,ds;$$

but it follows from the definition of g, the properties of ϕ_c and Proposition 2.3 that there is some $\delta\in(0,r_+(c)-r_-(c))$ and $C_0>0$ such that, for all $t>0$, $\xi\in\mathbb{R}$ and $y\in\mathbb{T}^{N-1}$ there holds

$$|g(t,\xi,y)| \leq \frac{C_0}{1+t^{1/4}}\frac{1}{1+e^{-(r_-(c)+\delta)\xi}};$$

then Lemma 2.4 implies that

$$e^{-(t-s)(-\partial_{\xi\xi}+\beta\partial_\xi-f'(0))}|g(s,.)|(\xi) \leq e^{(r_-(c)+\delta)\xi}e^{-\omega(\delta)(t-s)}\frac{C_0}{1+s^{1/4}}.$$

Then

$$|\widetilde{w}(t,\xi,y)| \leq C_0'\,e^{(r_-(c)+\delta)\xi}\int_0^t e^{-\omega(\delta)(t-s)}\frac{ds}{1+s^{1/4}}.$$

Since

$$\int_0^t \frac{e^{\omega(\delta)s}}{1+s^{1/4}}\,ds \sim_{t\to+\infty} \frac{1}{\omega(\delta)}\frac{e^{\omega(\delta)t}}{1+t^{1/4}},$$

we obtain the following bound, valid for all $(t,\xi,y)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{T}^{N-1}$:

$$|\widetilde{w}(t,\xi,y)| \leq \frac{C_1}{1+t^{1/4}}e^{(r_-(c)+\delta)\xi},$$

hence Claim 2.6 is proved.

Claim 2.7 is a consequence of Claim 2.6, using the following classical argument: first, there is some $q_0>0$ and $\eta>0$ such that $-f'(y)\geq q_0$ for all $y\in(1-\eta,1)$; choose ξ_0 such that $\phi_c(\xi_0-M,y)>1-\eta$ for all $y\in\mathbb{T}^{N-1}$; next, because u_0 is trapped between two waves of the same speed, this order is preserved for all time; hence $\widetilde{u}(t,\xi,y)\geq\phi_c(\xi-M,y)$ for all $t>0$ and all $\xi\in\mathbb{R}, y\in\mathbb{T}^{N-1}$; at last, since we already know that the shift m is bounded between $-M$ and M, the mean value theorem implies that $a(t,\xi,y) \geq q_0$ for all $t \geq 0$ and $\xi \geq \xi_0$, $y \in \mathbb{T}^{N-1}$. And thus the difference $\widetilde{u} - \widetilde{u}^{app}$ satisfies

$$\begin{cases} (\widetilde{u} - \widetilde{u}^{app})_t - \Delta(\widetilde{u} - \widetilde{u}^{app})_\xi + \beta(\widetilde{u} - \widetilde{u}^{app})_\xi + a(t,\xi)(\widetilde{u} - \widetilde{u}^{app}) = O(\frac{1}{1+t^{1/4}}), \quad t > 0, \xi \geq \xi_0, \\ (\widetilde{u} - \widetilde{u}^{app})(t,\xi_0,y) = O(\frac{1}{1+t^{1/4}}), \quad t > 0, \\ (\widetilde{u} - \widetilde{u}^{app})(0,\xi,y) = 0, \quad \xi \geq \xi_0. \end{cases}$$

$\frac{C}{(1+\varepsilon t)^{1/4}}$ is a super-solution if C and ε are well chosen (C sufficiently large, ε sufficiently small), hence the weak maximum principle implies Claim 2.7, and the proof of Theorem 1.3 is complete. □

3 Asymptotic behaviour of the approximate shift: preparation

3.1 Hopf-Cole transform of the approximate shift

The Hopf-Cole transform $s(t,\xi,y) = e^{r_-(c)m(t,\xi,y)}$ allows us to transform the nonlinear problem (2.11) into a linear parabolic equation: s is solution of

$$s_t - \Delta s + (\beta(y) - 2r_-(c))s_\xi - 2\frac{D_y\psi_{r_c}}{\psi_{r_c}}.D_y s = 0, \quad s(0) = e^{r_-(c)m(0)}. \tag{3.20}$$

Then standard theory on linear parabolic equations imply (i) (weak maximum principle) and (ii) (regularity) of Proposition 2.3. Concerning (iii): consider

$$q(t,\xi,y) = \psi_{r_c}(y)s(t,\xi,y);$$

then some computations allow us to check that q satisfies the parabolic problem

$$q_t + L_{r_c}q = 0, \quad q(0) = \psi_{r_c}(y)e^{r_-(c)m(0)}, \tag{3.21}$$

where

$$\begin{aligned} L_{r_c}q &= -\Delta q + (\beta - 2r_-(c))q_\xi + (r_-(c)\beta(y) - r_-(c)^2 - f'(0))q \\ &= (-\Delta_\xi + \mathcal{L}(r_-(c)) - (r_-(c)^2 + f'(0)))q + (\beta - 2r_-(c))q_\xi \\ &= -\Delta_\xi q + (\mathcal{L}(r_-(c)) - \nu_1(r_-(c)))q + (\beta - 2r_-(c))q_\xi. \end{aligned} \tag{3.22}$$

For commodity, define the operator $\mathcal{L}_0$ acting on $C^2(\mathbb{T}^{N-1})$ as

$$\mathcal{L}_0 := \mathcal{L}(r_-(c)) - \nu_1(r_-(c)) : \tag{3.23}$$

$\mathcal{L}_0$ is symmetric, and its first eigenvalue is 0, associated to ψ_{r_c}.

Let us describe our startegy. Because the variable ξ does not appear explicitly in the coefficients of (3.21), we may perform a Fourier transform in ξ, analyze the resulting equation in one less space variable, and transform back. We are going to do it on the heat kernel (or fundamental solution) of (3.23), i.e. the solution $\pi(t,\xi,\xi'y,y')$ of the equation with initial datum $\delta_{\xi=\xi',y=y'}$, in other words $e^{-tL_{r_c}}\delta_{\xi=\xi',y=y'}$. It is obvious to give a precise meaning to this with test functions, because we will work with Fourier transforms we will not do it.

Observe also that $e^{-tL_{r_c}}\delta_{\xi=\xi',y=y'}$ regularizes immediately and that it has unifom H^k bounds at $t=1$. This will also be used without further mention.

We note that, in general, it is not at all obvious to estimate heat kernels in the full range of (t,x,x',y,y'). Quite precise expressions are known for self-adjoint operators [23] or, for general operators, in the homogenisation range - i.e. $|x-x'|$ of the order $\sqrt{t}$; see [22]. None of them are totally useful for our purpose.

3.2 The fundamental solution and its Fourier transform

Given $\xi'\in\mathbb{R}$, $y'\in\mathbb{T}^{N-1}$, consider the solution $\pi(t,\xi,\xi',y,y')$ of

$$\pi_t+L_{r_c}\pi=0,\quad \pi(0,\xi,\xi',y,y')=\delta_{\xi=\xi'}\delta_{y=y'}. \tag{3.24}$$

Then we have a formula for the solution q of (3.21):

$$q(t,\xi,y)=\int_{\xi'\in\mathbb{R}}\int_{y'\in\mathbb{T}^{N-1}}\pi(t,\xi,\xi',y,y')q_0(\xi',y')\,d\xi'\,dy'. \tag{3.25}$$

Moreover, since the equation (3.24) is translation-invariant in ξ, we have

$$\pi(t,\xi,\xi',y,y')=\pi(t,\xi-\xi',0,y,y');$$

hence

$$q(t,\xi,y)=\int_{\xi'\in\mathbb{R}}\int_{y'\in\mathbb{T}^{N-1}}\pi(t,\xi-\xi',0,y,y')q_0(\xi',y')\,d\xi'\,dy'. \tag{3.26}$$

Now consider the Fourier transform in ξ of π:

$$\hat\pi(t,\hat\xi,\xi',y,y')=\int_{\xi\in\mathbb{R}}e^{-i\xi\hat\xi}\pi(t,\xi,\xi',y,y')\,d\xi. \tag{3.27}$$

The function $\hat\pi$ is solution of the following problem:

$$\hat\pi_t+\mathcal{L}_0\hat\pi+i\hat\xi(\beta-2r_-(c))\hat\pi+\hat\xi^2\hat\pi=0,\quad \hat\pi(0,\hat\xi,\xi',y,y')=e^{-i\hat\xi\xi'}\delta_{y=y'}, \tag{3.28}$$

and we will study this problem in the following. The Fourier inversion formula says that

$$\pi(t,\xi,\xi',y,y')=\frac{1}{2\pi}\int_{\hat\xi\in\mathbb{R}}e^{i\xi\hat\xi}\hat\pi(t,\hat\xi,\xi',y,y')\,d\hat\xi,$$

and then it allows us to transform (3.26) into

$$q(t,\xi,y)=\frac{1}{2\pi}\int_{\xi'\in\mathbb{R}}\int_{y'\in\mathbb{T}^{N-1}}\Big(\int_{\hat\xi\in\mathbb{R}}e^{i(\xi-\xi')\hat\xi}\hat\pi(t,\hat\xi,0,y,y')\,d\hat\xi\Big)q_0(\xi',y')\,d\xi'\,dy'. \tag{3.29}$$

Hence, informations on $\hat\pi$ will provide informations on q. We begin by studying the spectrum of the operator that appears in (3.28), and in particular its principal eigenvalue.

Consider

$$\mathcal{L}_{\hat\xi}:=\mathcal{L}_0+i\hat\xi(\beta-2r_-(c))+\hat\xi^2. \tag{3.30}$$

The operator $\mathcal{L}_0$ is symmetric and its first eigenvalue is 0. Denote ω_1 its second eigenvalue (ω_1 is positive). By analyticity, there exists some $\hat\xi_0>0$ such that, for all complex numbers $\hat\xi$ of modulus less than $2\hat\xi_0$, we have:

- $\mathcal{L}_{\hat{\xi}} = \mathcal{L}_0 + i\hat{\xi}(\beta - 2r_-(c)) + \hat{\xi}^2$ has a simple eigenvalue $\mu_{1,\hat{\xi}}$, analytic in $\hat{\xi}$, with eigenfunction $e_{\hat{\xi}}(y)$ such that $\|e_{\hat{\xi}}\|_{L^2(\mathbb{T}^{N-1})} = 1$;
- the rest of the spectrum of $\mathcal{L}_{\hat{\xi}}$ has real part larger than $\frac{\omega_1}{2}$.

We are going to compute an expansion of $\mu_{1,\hat{\xi}}$ and $e_{\hat{\xi}}$ in the neighborhood of $\hat{\xi} = 0$. This will be the key to analyze (3.28).

Lemma 3.1 *Denote $\langle \cdot, \cdot \rangle$ the scalar product on $L^2(\mathbb{T}^{N-1})$. We have the following asymptotics:*

$$e_{\hat{\xi}} = \psi_{r_c} + i\hat{\xi}\psi_{r_c}^{(1)} + \hat{\xi}^2\psi_{r_c}^{(2)} + O(\hat{\xi}^3), \tag{3.31}$$

$$\mu_{1,\hat{\xi}} = i\hat{\xi}V_*(c) + D_*(c)\hat{\xi}^2 + O(\hat{\xi}^3), \tag{3.32}$$

where

- *$V_*(c)$ is given by*
$$V_*(c) = \langle \beta(y) - 2r_-(c))\psi_{r_c}, \psi_{r_c}\rangle, \tag{3.33}$$
- *$\psi_{r_c}^{(1)}$ is solution of*
$$\mathcal{L}_0\psi_{r_c}^{(1)} = V_*(c)\psi_{r_c} - (\beta - 2r_-(c))\psi_{r_c}, \tag{3.34}$$
- *$D_*(c)$ is given by*
$$D_*(c) = 1 + \langle \mathcal{L}_0\psi_{r_c}^{(1)}, \psi_{r_c}^{(1)}\rangle, \tag{3.35}$$
- *and $\psi_{r_c}^{(2)}$ is solution of*

$$\mathcal{L}_0\psi_{r_c}^{(2)} = -V_*(c)\psi_{r_c}^{(1)} + (\beta - 2r_-(c))\psi_{r_c}^{(1)} + (D_*(c) - 1)\psi_{r_c}. \tag{3.36}$$

Proof. The existence of the asymptotic expansions (3.31) and (3.32) follows by analyticity, hence what we have to prove is that the coefficients appearing in (3.31) and (3.32) are given by the formulae (3.33)-(3.36). We already know that $e_0 = \psi_{r_c}$, hence we know the first term of the expansion of $e_{\hat{\xi}}$. Next, writing $\mathcal{L}_{\hat{\xi}}e_{\hat{\xi}} = \mu_{1,\hat{\xi}}e_{\hat{\xi}}$, hence

$$(\mathcal{L}_0 + i\hat{\xi}(\beta - 2r_-(c)) + \hat{\xi}^2)(\psi_{r_c} + i\hat{\xi}\psi_{r_c}^{(1)} + \hat{\xi}^2\psi_{r_c}^{(2)} + O(\hat{\xi}^3)) \\ = (i\hat{\xi}V_*(c) + D_*(c)\hat{\xi}^2 + O(\hat{\xi}^3))(\psi_{r_c} + i\hat{\xi}\psi_{r_c}^{(1)} + \hat{\xi}^2\psi_{r_c}^{(2)} + O(\hat{\xi}^3)),$$

we obtain

$$\begin{cases} \mathcal{L}_0\psi_{r_c}^{(1)} + (\beta - 2r_-(c))\psi_{r_c} = V_*(c)\psi_{r_c}, \\ \mathcal{L}_0\psi_{r_c}^{(2)} - (\beta - 2r_-(c))\psi_{r_c}^{(1)} + \psi_{r_c} = -V_*(c)\psi_{r_c}^{(1)} + D_*(c)\psi_{r_c}. \end{cases} \tag{3.37}$$

Take the scalar product of the first equation of (3.37) with ψ_{r_c}:

$$V_*(c) = \langle V_*(c)\psi_{r_c}, \psi_{r_c}\rangle = \langle \mathcal{L}_0\psi_{r_c}^{(1)} + (\beta - 2r_-(c))\psi_{r_c}, \psi_{r_c}\rangle \\ = \langle \psi_{r_c}^{(1)}, \mathcal{L}_0\psi_{r_c}\rangle + \langle(\beta - 2r_-(c))\psi_{r_c}, \psi_{r_c}\rangle = \langle(\beta - 2r_-(c))\psi_{r_c}, \psi_{r_c}\rangle,$$

hence $V_*(c)$ is given by (3.33). This implies that $V_*(c)\psi_{r_c} - (\beta - 2r_-(c))\psi_{r_c}$ is orthogonal to ψ_{r_c}, hence it belongs to the image of $\mathcal{L}_0$, which implies the existence of $\psi_{r_c}^{(1)}$ solution of (3.34). Now the scalar product of the second equation of (3.37) with ψ_{r_c}:

$$\begin{aligned} D_*(c) = \langle D_*(c)\psi_{r_c}, \psi_{r_c}\rangle &= \langle \mathcal{L}_0\psi_{r_c}^{(2)} - (\beta - 2r_-(c))\psi_{r_c}^{(1)} + \psi_{r_c} + V_*(c)\psi_{r_c}^{(1)}, \psi_{r_c}\rangle \\ &= 1 + \langle \psi_{r_c}^{(1)}, V_*(c)\psi_{r_c} - (\beta - 2r_-(c))\psi_{r_c}\rangle = 1 + \langle \psi_{r_c}^{(1)}, \mathcal{L}_0\psi_{r_c}^{(1)}\rangle, \end{aligned}$$

hence (3.35) is proved. At last, this implies that $(\beta - 2r_-(c))\psi_{r_c}^{(1)} + \psi_{r_c} - V_*(c)\psi_{r_c}^{(1)} + D_*(c)\psi_{r_c}$ is orthogonal to ψ_{r_c}, which implies the existence of $\psi_{r_c}^{(2)}$ satisfying (3.36). □

Note that $\hat{\xi}_0$ can be chosen small enough such that the asymptotic expansion of $\mu_{1,\hat{\xi}}$ satisfies

$$|\mu_{1,\hat{\xi}} - i\hat{\xi}V_*(c)| = |D_*(c)\hat{\xi}^2 + O(\hat{\xi}^3)| \in [\frac{D_*(c)}{2}\hat{\xi}^2, 2D_*(c)\hat{\xi}^2].$$

□

This result brings several remarks:

- first, the expression $V_*(c)$ given by (3.33) is consistent with the one given in (2.7), thanks to the value of $\nu_1'(r_-(c))$ given in (2.8);
- next, although $\psi_{r_c}^{(1)}$ is defined up to a multiple of ψ_{r_c}, there are enough orthogonality relations so that the quantity $D_*(c)$ is uniquely defined;
- at last, notice that we have $D_*(c) > 1$ as soon as α is nonconstant, this is a manifestation of the well-known 'convection-enhanced'-diffusion.

3.3 Decomposition of $\hat{\pi}$ in high and low frequencies, study of the high frequency part

The analysis of $\hat{\pi}$ is broken in two parts: 'high' frequencies $|\hat{\xi}| \geq \hat{\xi}_0$, and 'low' frequencies $|\hat{\xi}| \leq \hat{\xi}_0$. Take a C^∞ function $\gamma(\hat{\xi})$, supported in $[-\hat{\xi}_0, \hat{\xi}_0]$, equal to 1 in $[\frac{-\hat{\xi}_0}{2}, \frac{\hat{\xi}_0}{2}]$. Set

$$\hat{\pi} = \gamma(\hat{\xi})\hat{\pi} + (1 - \gamma(\hat{\xi}))\hat{\pi} =: \hat{\pi}^\ell + \hat{\pi}^h.$$

The function $\hat{\pi}^h$ satisfies

$$\hat{\pi}_t^h + \mathcal{L}_0\hat{\pi}^h + i\hat{\xi}(\beta - 2r_-(c))\hat{\pi}^h + \hat{\xi}^2\hat{\pi}^h = 0, \quad \hat{\pi}^h(0, \hat{\xi}, \xi', y, y') = e^{-i\hat{\xi}\xi'}(1 - \gamma(\hat{\xi}))\delta_{y=y'}. \quad (3.38)$$

Denote $\|\cdot\|_{L_y^2}$ the norm associated to the scalar product $\langle\cdot,\cdot\rangle$.

3.3.1 Main results concerning $\hat{\pi}^h$ and its inverse Fourier transform π^h

Lemma 3.2 *The function $\hat{\pi}^h$ has the following property: given $k, k' \in \mathbb{N}$, there exists $C_{k,k'} > 0$, $\omega_{k,k'} > 0$ such that*

$$\|\partial_{\hat{\xi}}^k D_y^{k'}\hat{\pi}^h\|_{L_y^2}^2 \leq C_{k,k'}e^{-\omega_{k,k'}\hat{\xi}^2 t}. \quad (3.39)$$

This will imply the following

Corollary 3.3 *The inverse Fourier transform of $\hat{\pi}^h$:*

$$\pi^h(t,\xi,\xi',y,y') := \frac{1}{2\pi}\int_{\hat{\xi}\in\mathbb{R}} e^{i(\xi-\xi')\hat{\xi}}\hat{\pi}^h(t,\hat{\xi},0,y,y')\,d\hat{\xi}$$

has the following property: given $k,k'\in\mathbb{N}$, there exists $C_{k,k'}>0$, $\omega_{k,k'}>0$ such that, for all $t\geq 1$, for all $\xi,\xi'\in\mathbb{R}$, for all $y,y'\in\mathbb{T}^{N-1}$ we have

$$(1+(\xi-\xi')^2)|\partial_\xi^k D_y^{k'}\pi^h(t,\xi,\xi',y,y')|\leq C_{k,k'}e^{-\omega_{k,k'}\hat{\xi}_0^2 t}. \tag{3.40}$$

This implies in particular that there exists $C>0$, $\omega>0$ such that, for all $t\geq 1$, for all $\xi\in\mathbb{R}$, for all $y,y'\in\mathbb{T}^{N-1}$ we have

$$\int_{\xi'\in\mathbb{R}}\Big(|\pi^h|+|\partial_\xi\pi^h|+|D_y\pi^h|\Big)\leq Ce^{-\omega\hat{\xi}_0^2 t}. \tag{3.41}$$

3.3.2 Proof of Lemma 3.2.

First we prove (3.39) for $k=0=k'$, hence that the L^2_y-norm of $\hat{\pi}^h$ goes exponentially to 0 as $t\to+\infty$. Multiply the equation satisfied by $\hat{\pi}_h$ by its conjugate $\overline{\hat{\pi}^h}$, integrate on $\mathbb{T}^{N-1}$ and take the real part of this expression:

$$0=\Re e\int_{\mathbb{T}^{N-1}}(\hat{\pi}^h_t+\mathcal{L}_0\hat{\pi}^h+i\hat{\xi}(\beta-2r_-(c))\hat{\pi}^h+\hat{\xi}^2\hat{\pi}^h)\overline{\hat{\pi}^h}.$$

We study all the terms in this expression: first,

$$\Re e\int_{T^{N-1}}\hat{\pi}^h_t\overline{\hat{\pi}^h}=\frac{1}{2}\frac{\partial}{\partial t}\|\hat{\pi}^h\|^2_{L^2_y};$$

next,

$$\Re e\int_{\mathbb{T}^{N-1}}\mathcal{L}_0\hat{\pi}^h\overline{\hat{\pi}^h}=\int_{\mathbb{T}^{N-1}}(\mathcal{L}_0(\Re e\hat{\pi}^h))\Re e\hat{\pi}^h+(\mathcal{L}_0(\Im m\hat{\pi}^h))\Im m\hat{\pi}^h,$$

and all this is nonnegative, since $\mathcal{L}_0$ is symmetric and its principal eigenvalue is zero; next

$$\Re e\int_{\mathbb{T}^{N-1}}i\hat{\xi}(\beta-2r_-(c))\hat{\pi}^h\overline{\hat{\pi}^h}=\Re e\Big(i\int_{\mathbb{T}^{N-1}}\hat{\xi}(\beta-2r_-(c))|\hat{\pi}^h|^2\Big)=0;$$

at last,

$$\Re e\int_{\mathbb{T}^{N-1}}\hat{\xi}^2\hat{\pi}^h\overline{\hat{\pi}^h}=\hat{\xi}^2\|\hat{\pi}^h\|^2_{L^2_y}.$$

Then we obtain that

$$\frac{1}{2}\frac{\partial}{\partial t}\|\hat{\pi}^h\|^2_{L^2_y}+\hat{\xi}^2\|\hat{\pi}^h\|^2_{L^2_y}\leq 0,$$

which implies that $t\mapsto e^{2\hat{\xi}^2 t}\|\hat{\pi}^h\|^2_{L^2_y}$ is nonincreasing, which implies that

$$\|\hat{\pi}^h\|^2_{L^2_y}\leq C_0 e^{-2\hat{\xi}^2 t},$$

hence (3.39) holds true for $k=0=k'$.

Let us prove now that (3.39) holds true for $k = 1$, $k' = 0$. We remark that $\partial_{\hat\xi}\hat\pi^h$ is solution of the nonhomogeneous parabolic equation

$$(\partial_{\hat\xi}\hat\pi^h)_t + \mathcal{L}_0(\partial_{\hat\xi}\hat\pi^h) + i\hat\xi(\beta - 2r_-(c))(\partial_{\hat\xi}\hat\pi^h) + \hat\xi^2(\partial_{\hat\xi}\hat\pi^h) = -i(\beta - 2r_-(c))\hat\pi^h - 2\hat\xi\hat\pi^h;$$

then we proceed in the same way:

$$\Re e \int_{\mathbb{T}^{N-1}} \Big((\partial_{\hat\xi}\hat\pi^h)_t + \mathcal{L}_0(\partial_{\hat\xi}\hat\pi^h) + i\hat\xi(\beta - 2r_-(c))(\partial_{\hat\xi}\hat\pi^h) + \hat\xi^2(\partial_{\hat\xi}\hat\pi^h)\Big)\overline{\partial_{\hat\xi}\hat\pi^h}$$
$$= \Re e \int_{\mathbb{T}^{N-1}} \Big(-i(\beta - 2r_-(c))\hat\pi^h - 2\hat\xi\hat\pi^h\Big)\overline{\partial_{\hat\xi}\hat\pi^h},$$

hence

$$\frac{1}{2}\frac{\partial}{\partial t}\|\partial_{\hat\xi}\hat\pi^h\|^2_{L^2_y} + \hat\xi^2\|\partial_{\hat\xi}\hat\pi^h\|^2_{L^2_y} \le \Re e \int_{\mathbb{T}^{N-1}} \Big(-i(\beta - 2r_-(c))\hat\pi^h - 2\hat\xi\hat\pi^h\Big)\overline{\partial_{\hat\xi}\hat\pi^h}$$
$$\le C\|\hat\pi^h\|^2_{L^2_y} + \frac{\hat\xi^2}{2}\|\partial_{\hat\xi}\hat\pi^h\|^2_{L^2_y},$$

which implies that

$$\frac{\partial}{\partial t}\|\partial_{\hat\xi}\hat\pi^h\|^2_{L^2_y} + \hat\xi^2\|\partial_{\hat\xi}\hat\pi^h\|^2_{L^2_y} \le 2C\|\hat\pi^h\|^2_{L^2_y} \le C'e^{-2\hat\xi^2 t},$$

hence

$$\frac{\partial}{\partial t}(e^{\hat\xi^2 t}\|\partial_{\hat\xi}\hat\pi^h\|^2_{L^2_y}) \le C'e^{-2\hat\xi^2 t},$$

which implies that

$$\|\partial_{\hat\xi}\hat\pi^h\|^2_{L^2_y} \le C_1 e^{-\hat\xi^2 t}.$$

In the same way, the same property holds for $D_y\hat\pi^h$ since it is solution of a similar nonhomogeneous parabolic equation.

Finally, since β is smooth enough, we can repeat the argument, and the same property holds for all the derivatives of $\hat\pi^h$ of the form $\partial^k_{\hat\xi} D^{k'}_y \hat\pi^h$. □

3.3.3 Proof of Corollary 3.3.

First we prove (3.40) for $k = 0 = k'$: we have

$$(1 + (\xi - \xi')^2)\pi^h(t, \xi, \xi', y, y') = (1 + (\xi - \xi')^2)\int_{\hat\xi} e^{i(\xi-\xi')\hat\xi}\hat\pi^h(t, \hat\xi, 0, y, y')\, d\hat\xi$$
$$= \int_{\hat\xi} e^{i(\xi-\xi')\hat\xi}(\hat\pi^h(t, \hat\xi, 0, y, y') - \hat\pi^h_{\hat\xi\hat\xi}(t, \hat\xi, 0, y, y'))\, d\hat\xi.$$

Now, Sobolev's embeddings ensure the existence of C and ω such that

$$\sup_{y,y'}(|\hat\pi^h(t, \hat\xi, 0, y, y')| + |\hat\pi^h_{\hat\xi\hat\xi}(t, \hat\xi, 0, y, y')| \le Ce^{-\omega\hat\xi^2 t},$$

hence

$$(1 + (\xi - \xi')^2)|\pi^h(t, \xi, \xi', y, y')| \le \int_{|\hat\xi|\ge\hat\xi_0/2} Ce^{-\omega\hat\xi^2 t}\, d\hat\xi \le C(\hat\xi_0)e^{-\omega\hat\xi_0^2 t}.$$

The same property holds for all the derivatives of π^h, by the same argument. This proves (3.40), and (3.41) follows from (3.40):

$$\int_{\xi'\in\mathbb{R}} |\pi^h| \leq \int_{\xi'\in\mathbb{R}} C_{0,0} e^{-\omega_{0,0}\hat{\xi}_0^2 t} \frac{d\xi'}{(1+(\xi-\xi')^2)} = \frac{\pi}{2} C_{0,0} e^{-\omega_{0,0}\hat{\xi}_0^2 t},$$

and the same property holds for $\partial_\xi \pi^h$ and $D_y \pi^h$. □

4 Influence of the low frequencies

4.1 Part 1: decomposition of $\hat{\pi}^\ell$

The function $\hat{\pi}^\ell$ satisfies

$$\hat{\pi}_t^\ell + \mathcal{L}_0 \hat{\pi}^\ell + i\hat{\xi}(\beta - 2r_-(c))\hat{\pi}^\ell + \hat{\xi}^2 \hat{\pi}^\ell = 0, \quad \hat{\pi}^\ell(0, \hat{\xi}, \xi', y, y') = e^{-i\hat{\xi}\xi'} \gamma(\hat{\xi}) \delta_{y=y'}. \tag{4.42}$$

Note that both E_1 and E_2 depend on $\hat{\xi}$, we will have to keep this in mind but, in order to reduce the complexity of the different expressions, this dependence will not be mentionned explicitely.
Since $\mu_{1,\hat{\xi}}$ is an isolated eigenvalue of $\mathcal{L}_{\hat{\xi}}$, $\{\mu_{1,\hat{\xi}}\}$ is a spectral bounded set. Then there exist a projection E_1 associated to this spectral set, and a projection E_2 associated to the spectral set given by the complement of $\mu_{1,\hat{\xi}}$ in the spectrum of $\mathcal{L}_{\hat{\xi}}$ (Theorem 1.5.2 of [16]), such that

$$\hat{\pi}^\ell = E_1(\hat{\pi}^\ell) + E_2(\hat{\pi}^\ell).$$

The images of E_1 and E_2 are stable under $\mathcal{L}_{\hat{\xi}}$, hence $E_1(\hat{\pi}^\ell)$ and $E_2(\hat{\pi}^\ell)$ will satisfy similar parabolic equations. In the following we study these projections. We start by studying the projection $E_1(\hat{\pi}^\ell)$, that will give the more important contribution in the behaviour of the Hopf-Cole transform q.

4.2 Influence of the low frequencies, part 2: study of $E_1(\hat{\pi}^\ell)$ and its inverse Fourier transform $\pi^{\ell,1}$

4.2.1 The "principal parts" of $E_1(\hat{\pi}^\ell)$ and $\pi^{\ell,1}$

First, we note that since everything depends analytically on $\hat{\xi}$, the projection E_1 can be written:

$$E_1(g) = \langle g, \Phi_1(\hat{\xi}, y)\rangle e_{\hat{\xi}},$$

where $\xi \mapsto \Phi_1(\hat{\xi}, y)$ is analytic. Moreover, the operator $\mathcal{L}_0$ is symmetric, hence its eigenspaces are orthogonal, hence $\Phi_1(0, y) = \psi_{r_c}(y)$. This implies that, in a neighborhood of $\hat{\xi} = 0$, we have

$$\Phi_1(\hat{\xi}, y) = \psi_{r_c}(y) + O(\hat{\xi}).$$

Next we note that the projection $E_1(\hat{\pi}^\ell)$ satifies the following (simple) equation:

$$\partial_t E_1(\hat{\pi}^\ell) + \mu_{1,\hat{\xi}} E_1(\hat{\pi}^\ell) = 0,$$

thus the formula:

$$E_1(\hat{\pi}^\ell)(t,\hat{\xi},\xi',y,y') = e^{-\mu_{1,\hat{\xi}}t}E_1(\hat{\pi}^\ell)(0,\hat{\xi},\xi',y,y')$$
$$= e^{-\mu_{1,\hat{\xi}}t}\langle\gamma(\hat{\xi})e^{-i\hat{\xi}\xi'}\delta_{y=y'},\Phi_1(\hat{\xi},y)\rangle e_{\hat{\xi}} = e^{-\mu_{1,\hat{\xi}}t}\gamma(\hat{\xi})e^{-i\hat{\xi}\xi'}\overline{\Phi_1(\hat{\xi},y')}e_{\hat{\xi}}(y). \quad (4.43)$$

This implies, taking into account the asymptotic expansions (3.31) and (3.32) of $e_{\hat{\xi}}$ and $\mu_{1,\hat{\xi}}$ in the neighborhood of $\hat{\xi}=0$, that

$$E_1(\hat{\pi}^\ell)(t,\hat{\xi},\xi',y,y') = e^{-t(i\hat{\xi}V_*(c)+D_*(c)\hat{\xi}^2+O(\hat{\xi}^3))}\gamma(\hat{\xi})e^{-i\hat{\xi}\xi'}(\psi_{r_c}(y')+O(\hat{\xi}))(\psi_{r_c}(y)+O(\hat{\xi})).$$

Let us call the "principal part of $E_1(\hat{\pi}^\ell)$" the following:

$$\hat{\pi}^{pp}(t,\hat{\xi},\xi',y,y') := e^{-t(i\hat{\xi}V_*(c)+D_*(c)\hat{\xi}^2)}e^{-i\hat{\xi}\xi'}\psi_{r_c}(y')\psi_{r_c}(y). \quad (4.44)$$

The associated inverse Fourier transform will be

$$\pi^{pp}(t,\xi,\xi',y,y') := \frac{1}{2\pi}\int_{\hat{\xi}\in\mathbb{R}} e^{i(\xi-\xi')\hat{\xi}}\hat{\pi}^{pp}(t,\hat{\xi},0,y,y')\,d\hat{\xi}$$
$$= \frac{1}{2\pi}\psi_{r_c}(y)\psi_{r_c}(y')\int_{\hat{\xi}\in\mathbb{R}} e^{i(\xi-\xi')\hat{\xi}}e^{-t(i\hat{\xi}V_*(c)+D_*(c)\hat{\xi}^2)}\,d\hat{\xi}. \quad (4.45)$$

This is what we call the "principal part of $\pi^{\ell,1}$". In a classical way, we have

$$\frac{\pi^{pp}(t,\xi,\xi',y,y')}{\psi_{r_c}(y)\psi_{r_c}(y')} = \frac{1}{2\pi}\int_{\hat{\xi}\in\mathbb{R}} e^{i(\xi-\xi')\hat{\xi}}e^{-t(i\hat{\xi}V_*(c)+D_*(c)\hat{\xi}^2)}\,d\hat{\xi}$$
$$= \frac{1}{2\pi}\int_{\hat{\xi}\in\mathbb{R}} e^{i(\xi-\xi'-tV_*(c))\hat{\xi}}e^{-D_*(c)t\hat{\xi}^2}\,d\hat{\xi}$$
$$= \frac{1}{\sqrt{4\pi D_*(c)t}}e^{-(\xi-\xi'-tV_*(c))^2/(4D_*(c)t)}. \quad (4.46)$$

This allows us to give a simple interpretation of the so-called principal part π^{pp}: π^{pp} satisfies the transport-diffusion equation

$$\pi_t^{pp} - D_*(c)\pi_{\xi\xi}^{pp} + V_*(c)\pi_\xi^{pp} = 0, \quad \pi^{pp}(0,\xi,\xi',y,y') = \psi_{r_c}(y)\psi_{r_c}(y')\delta_{\xi=\xi'}. \quad (4.47)$$

It remains to prove that we have the right to use that terminology of "principal part": we are going to study the difference between $\pi^{\ell,1}$ and its principal part π^{pp}.

4.2.2 The difference between $\pi^{\ell,1}$ and its principal part π^{pp}: statement of the main results

Lemma 4.1 *Denote $\pi^{\ell,1}$ the inverse Fourier transform of $E_1(\hat{\pi}^\ell)$:*

$$\pi^{\ell,1}(t,\xi,\xi',y,y') := \frac{1}{2\pi}\int_{\hat{\xi}\in\mathbb{R}} e^{i(\xi-\xi')\hat{\xi}}E_1(\hat{\pi}^\ell)(t,\hat{\xi},0,y,y')\,d\hat{\xi}$$

and

$$\pi^R(t,\xi,\xi',y,y') := \pi^{\ell,1}(t,\xi,\xi',y,y') - \pi^{pp}(t,\xi,\xi',y,y').$$

Then there exists M_0 such that we have the following estimate: for all $t\geq 1$, for all $\xi,\xi'\in\mathbb{R}$, for all $y,y'\in\mathbb{T}^{N-1}$,

$$(t+(\xi-\xi'-V_*(c)t)^2)|\pi^R(t,\xi,\xi',y,y')| \leq M_0. \quad (4.48)$$

This implies the following important integral estimate on π^R:

Corollary 4.2

$$\forall t \geq 1, \forall \xi \in \mathbb{R}, \forall y, y' \in \mathbb{T}^{N-1}, \quad \int_{\xi' \in \mathbb{R}} |\pi^R(t, \xi, \xi', y, y')| \leq \frac{4M_0}{\sqrt{t}}. \tag{4.49}$$

Finally, we will need similar estimates on the first order derivatives of π^R: we will prove the following

Lemma 4.3 *(i) Concerning $\partial_\xi \pi^R$: there exists some M_1 such that for all $t \geq 1$, for all $\xi, \xi' \in \mathbb{R}$, for all $y, y' \in \mathbb{T}^{N-1}$, we have*

$$(t + (\xi - \xi' - V_*(c)t)^2)|\sqrt{t}\partial_\xi \pi^R(t, \xi, \xi', y, y')| \leq M_1; \tag{4.50}$$

this implies that

$$\forall t \geq 1, \forall \xi \in \mathbb{R}, \forall y, y' \in \mathbb{T}^{N-1}, \quad \int_{\xi' \in \mathbb{R}} |\partial_\xi \pi^R(t, \xi, \xi', y, y')| \leq \frac{4M_1}{t}. \tag{4.51}$$

(ii) Concerning $D_y \pi^R$: there exists some M_2 such that for all $t \geq 1$, for all $\xi, \xi' \in \mathbb{R}$, for all $y, y' \in \mathbb{T}^{N-1}$, we have

$$(t + \frac{(\xi - \xi' - V_*(c)t)^2}{\sqrt{t}})|D_y\Big(\frac{\pi^R(t, \xi, \xi', y, y')}{\psi_{r_c}(y)}\Big)| \leq M_2; \tag{4.52}$$

this implies that

$$\forall t \geq 1, \forall \xi \in \mathbb{R}, \forall y, y' \in \mathbb{T}^{N-1}, \quad \int_{\xi' \in \mathbb{R}} |D_y\Big(\frac{\pi^R(t, \xi, \xi', y, y')}{\psi_{r_c}(y)}\Big)| \leq \frac{4M_2}{t^{1/4}}. \tag{4.53}$$

4.2.3 Proof of Lemma 4.1.

First we prove that $t\pi^R(t, \xi, \xi', y, y')$ is bounded. Remember that we are dealing with low frequencies, and we have in fact

$$\pi^{\ell,1}(t, \xi, \xi', y, y') = \frac{1}{2\pi} \int_{-\hat{\xi}_0}^{\hat{\xi}_0} e^{i(\xi-\xi')\hat{\xi}} E_1(\hat{\pi}^\ell)(t, \hat{\xi}, 0, y, y')\, d\hat{\xi}.$$

Next, $\hat{\xi}_0$ has been chosen so that the asymptotic expansion of $\mu_{1,\hat{\xi}}$ satisfies

$$|\mu_{1,\hat{\xi}} - i\hat{\xi}V_*(c)| = |D_*(c)\hat{\xi}^2 + O(\hat{\xi}^3)| \in [\frac{D_*(c)}{2}\hat{\xi}^2, 2D_*(c)\hat{\xi}^2].$$

Now consider a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, nondecreasing, and $\rho(t) \to +\infty$ as $t \to +\infty$ and $\frac{\rho(t)}{\sqrt{t}} \to 0$ as $t \to +\infty$. (We will choose $\rho(t) = t^{1/4}$.) Then, first

$$\begin{aligned} \Big|\frac{1}{2\pi} \int_{|\hat{\xi}| \geq \rho(t)/\sqrt{t}} e^{i(\xi-\xi')\hat{\xi}} E_1(\hat{\pi}^\ell)(t, \hat{\xi}, 0, y, y')\, d\hat{\xi}\Big| &\leq C \int_{|\hat{\xi}| \geq \rho(t)/\sqrt{t}} e^{-D_*(c)t\hat{\xi}^2/2}\, d\hat{\xi} \\ &= C \int_{\sigma \geq \rho(t)^2} e^{-D_*(c)\sigma/2} \frac{d\sigma}{2\sqrt{t}\sqrt{\sigma}} = O(1)\frac{e^{-D_*(c)\rho(t)^2/2}}{\sqrt{t}\rho(t)}; \end{aligned} \tag{4.54}$$

next, in the same fashion,

$$\Big|\frac{1}{2\pi}\int_{|\hat\xi|\geq \rho(t)/\sqrt{t}} e^{i(\xi-\xi')\hat\xi}\hat\pi^{pp}(t,\hat\xi,0,y,y')\,d\hat\xi\Big| \leq C\int_{|\hat\xi|\geq\rho(t)/\sqrt{t}} e^{-D_*(c)t\hat\xi^2}\,d\hat\xi = O(1)\frac{e^{-D_*(c)\rho(t)^2}}{\sqrt{t}\rho(t)}. \quad (4.55)$$

Let us finally consider the last term

$$\frac{1}{2\pi}\int_{|\hat\xi|\leq\rho(t)/\sqrt{t}} e^{i(\xi-\xi')\hat\xi}E_1(\hat\pi^\ell)(t,\hat\xi,0,y,y')\,d\hat\xi - \frac{1}{2\pi}\int_{|\hat\xi|\leq\rho(t)/\sqrt{t}} e^{i(\xi-\xi')\hat\xi}\hat\pi^{pp}(t,\hat\xi,0,y,y')\,d\hat\xi :$$

Since $\dfrac{\rho(t)}{\sqrt{t}}\to 0$ as $t\to+\infty$, we have for $|\hat\xi|\leq\rho(t)/\sqrt{t}$:

$$\begin{aligned} &e^{i(\xi-\xi')\hat\xi}E_1(\hat\pi^\ell)(t,\hat\xi,0,y,y') \\ &\quad= e^{i(\xi-\xi')\hat\xi}e^{-t(i\hat\xi V_*(c)+D_*(c)\hat\xi^2+O(\hat\xi^3))}\langle\delta_{y=y'},\psi_{r_c}+O(\hat\xi)\rangle(\psi_{r_c}+O(\hat\xi)) \\ &\quad= e^{i(\xi-\xi')\hat\xi}e^{-t(i\hat\xi V_*(c)+D_*(c)\hat\xi^2)}(1+O(t\hat\xi^3))(\langle\delta_{y=y'},\psi_{r_c}\rangle\psi_{r_c}+O(\hat\xi)) \\ &\quad= e^{i(\xi-\xi')\hat\xi}\hat\pi^{pp}(t,\hat\xi,0,y,y')(1+O(\hat\xi)+O(t\hat\xi^3)), \end{aligned}$$

hence

$$\begin{aligned} &\Big|\frac{1}{2\pi}\int_{|\hat\xi|\leq\rho(t)/\sqrt{t}} e^{i(\xi-\xi')\hat\xi}E_1(\hat\pi^\ell)(t,\hat\xi,0,y,y')\,d\hat\xi - \frac{1}{2\pi}\int_{|\hat\xi|\leq\rho(t)/\sqrt{t}} e^{i(\xi-\xi')\hat\xi}\hat\pi^{pp}(t,\hat\xi,0,y,y')\,d\hat\xi\Big| \\ &\quad\leq \int_{|\hat\xi|\leq\rho(t)/\sqrt{t}}(O(\hat\xi)+O(t\hat\xi^3))e^{-tD_*(c)\hat\xi^2}\,d\hat\xi \\ &\quad= O(1)\int_0^{\rho(t)^2}(\frac{\sqrt{\sigma}}{\sqrt{t}}+t\frac{\sigma^{3/2}}{t^{3/2}})e^{-D_*(c)\sigma/2}\frac{d\sigma}{2\sqrt{t}\sqrt{\sigma}} = O(\frac{1}{t}). \quad (4.56) \end{aligned}$$

Finally, choosing for example $\rho(t)=t^{1/4}$, we see that (4.54)-(4.56) imply that $\pi^R(t,\xi,\xi',y,y') = O(\frac{1}{t})$, uniformly in ξ, ξ', y, y', hence the first part of (4.48).

For the second part of (4.48), first we integrate by parts (twice) to kill the factor $(\xi-\xi'-V_*(c)t)^2$: we have

$$\begin{aligned} &(\xi-\xi'-V_*(c)t)^2\pi^{pp}(t,\xi,\xi',y,y') \\ &\quad= \frac{1}{2\pi}\psi_{r_c}(y)\psi_{r_c}(y')\int_{\hat\xi\in\mathbb{R}}(2tD_*(c)-4D_*(c)^2t^2\hat\xi^2)e^{i(\xi-\xi')\hat\xi}e^{-t(i\hat\xi V_*(c)+D_*(c)\hat\xi^2)}\,d\hat\xi, \end{aligned}$$

next, in the same way:

$$\begin{aligned} &(\xi-\xi'-V_*(c)t)^2\pi^{\ell,1}(t,\xi,\xi',y,y') \\ &\quad= \frac{1}{2\pi}\int_{\hat\xi\in\mathbb{R}} e^{i(\xi-\xi'-V_*(c)t)\hat\xi}\frac{\partial^2}{\partial\hat\xi^2}\Big(-e^{(iV_*(c)\hat\xi-\mu_{1,\hat\xi})t}\gamma(\hat\xi)\overline{\Phi_1(\hat\xi,y')}e_{\hat\xi}(y)\Big)\,d\hat\xi, \end{aligned}$$

and some easy computations show that

$$\frac{\partial^2}{\partial\hat{\xi}^2}\Big(-e^{(iV_*(c)\hat{\xi}-\mu_{1,\hat{\xi}})t}\gamma(\hat{\xi})\overline{\Phi_1(\hat{\xi},y')}e_{\hat{\xi}}(y)\Big)$$
$$= e^{(iV_*(c)\hat{\xi}-\mu_{1,\hat{\xi}})t}\gamma(\hat{\xi})\psi_{r_c}(y)\psi_{r_c}(y')(2tD_*(c)-4D_*(c)^2t^2\hat{\xi}^2+O(t\hat{\xi})+O(t^2\hat{\xi}^3))$$
$$+ e^{(iV_*(c)\hat{\xi}-\mu_{1,\hat{\xi}})t}O(t\hat{\xi})\chi_{|\hat{\xi}|\in(\hat{\xi}_0/2,\hat{\xi}_0)},$$

where $\chi_{|\hat{\xi}|\in(\hat{\xi}_0/2,\hat{\xi}_0)}$ is equal to 1 if $|\hat{\xi}|\in(\hat{\xi}_0/2,\hat{\xi}_0)$ and 0 elsewhere. Now we can proceed in the same way: we estimate separately the integrals on $|\hat{\xi}|\geq\rho(t)/\sqrt{t}$ and $|\hat{\xi}|\leq\rho(t)/\sqrt{t}$ as we did in (4.54)-(4.56), and we obtain

$$(\xi-\xi'-V_*(c)t)^2|\pi^{\ell,1}(t,\xi,\xi',y,y')-\pi^{pp}(t,\xi,\xi',y,y')|$$
$$\leq O(1)t\rho(t)e^{-D_*(c)\rho(t)^2/2}+O(1)t\rho(t)e^{-D_*(c)\rho(t)^2}+O(1), \quad (4.57)$$

these last three terms being respectively the estimates corresponding to (4.54), (4.55) and (4.56). With the same function $\rho(t)=t^{1/4}$, (4.57) allows us to complete the proof of the estimate (4.48), and the proof of Lemma 4.1. □

4.2.4 Proof of Corollary 4.2.

The estimate (4.49) directly comes from (4.48): indeed, (4.48) implies that

$$|\pi^R(t,\xi,\xi',y,y')|\leq\inf\{\frac{M_0}{t},\frac{M_0}{(\xi-\xi'-V_*(c)t)^2}\},$$

hence

$$\int_{\xi'\in\mathbb{R}}|\pi^R(t,\xi,\xi',y,y')|\,d\xi'=\int_{|\xi-\xi'-V_*(c)t|\leq\sqrt{t}}|\pi^R|\,d\xi'+\int_{|\xi-\xi'-V_*(c)t|\geq\sqrt{t}}|\pi^R|\,d\xi'$$
$$\leq\int_{|\xi-\xi'-V_*(c)t|\leq\sqrt{t}}\frac{M_0}{t}\,d\xi'+\int_{|\xi-\xi'-V_*(c)t|\geq\sqrt{t}}\frac{M_0}{(\xi-\xi'-V_*(c)t)^2}\,d\xi'$$
$$\leq\frac{2M_0\sqrt{t}}{t}+\frac{2M_0}{\sqrt{t}}=\frac{4M_0}{\sqrt{t}}.$$

This concludes the proof of Corollary 4.2. □

4.2.5 Proof of Lemma 4.3.

(i) The expression of $\sqrt{t}\partial_\xi\pi^R$ is similar to the expression of π^R, with the additional multiplicative factor $i\sqrt{t}\hat{\xi}$ inside the integrals. Hence, proceeding in the same way, we obtain (4.50), and in the same way as in Corollary 4.49, this implies that (4.51). (ii) Next we see that

$$D_y\Big(\frac{\pi^R(t,\xi,\xi',y,y')}{\psi_{r_c}(y)}\Big)=D_y\Big(\frac{\pi^{\ell,1}(t,\xi,\xi',y,y')-\pi^{pp}(t,\xi,\xi',y,y')}{\psi_{r_c}(y)}\Big)$$
$$=D_y\Big(\frac{\pi^{\ell,1}(t,\xi,\xi',y,y')}{\psi_{r_c}(y)}\Big)=\frac{1}{2\pi}\int_{\hat{\xi}\in\mathbb{R}}D_y\Big(e^{i(\xi-\xi')\hat{\xi}}e^{-\mu_{1,\hat{\xi}}t}\gamma(\hat{\xi})\overline{\Phi_1(\hat{\xi},y')}\frac{e_{\hat{\xi}}(y)}{\psi_{r_c}(y)}\Big)\,d\hat{\xi}$$
$$=\frac{1}{2\pi}\int_{\hat{\xi}\in\mathbb{R}}e^{i(\xi-\xi')\hat{\xi}}e^{-\mu_{1,\hat{\xi}}t}\gamma(\hat{\xi})\overline{\Phi_1(\hat{\xi},y')}D_y\Big(\frac{e_{\hat{\xi}}(y)}{\psi_{r_c}(y)}\Big)\,d\hat{\xi}.$$

Since $e_0(y) = \psi_{r_c}(y)$, we have

$$D_y\Big(\frac{e_{\hat\xi}(y)}{\psi_{r_c}(y)}\Big) = O(\hat\xi),$$

hence

$$D_y\Big(\frac{\pi^R(t,\xi,\xi',y,y')}{\psi_{r_c}(y)}\Big) = O(\frac{1}{t}),$$

which is the first part of (4.52). Next

$$\begin{aligned}
&(\xi-\xi'-V_*(c)t)^2 D_y\Big(\frac{\pi^R(t,\xi,\xi',y,y')}{\psi_{r_c}(y)}\Big)\\
&\quad= \frac{-1}{2\pi}\int_{\hat\xi}\Big(\frac{\partial^2}{\partial\hat\xi^2}e^{i(\xi-\xi'-V_*(c)t)\hat\xi}\Big)\Big(e^{(iV_*(c)\hat\xi-\mu_{1,\hat\xi})t}\gamma(\hat\xi)\overline{\Phi_1(\hat\xi,y')}D_y\Big(\frac{e_{\hat\xi}(y)}{\psi_{r_c}(y)}\Big)\Big)\,d\hat\xi\\
&\quad= \frac{-1}{2\pi}\int_{\hat\xi}e^{i(\xi-\xi'-V_*(c)t)\hat\xi}\frac{\partial^2}{\partial\hat\xi^2}\Big(e^{(iV_*(c)\hat\xi-\mu_{1,\hat\xi})t}\gamma(\hat\xi)\overline{\Phi_1(\hat\xi,y')}D_y\Big(\frac{e_{\hat\xi}(y)}{\psi_{r_c}(y)}\Big)\Big)\,d\hat\xi\\
&\quad= \frac{1}{2\pi}\int_{-\hat\xi_0}^{\hat\xi_0}e^{i(\xi-\xi'-V_*(c)t)\hat\xi}e^{(iV_*(c)\hat\xi-\mu_{1,\hat\xi})t}\Big(O(t)+O(t^2\hat\xi^2)+O(t\hat\xi)\Big)\,d\hat\xi,
\end{aligned}$$

hence, thanks to the same change of variables $\sigma = t\hat\xi^2$, we get

$$(\xi-\xi'-V_*(c)t)^2 D_y\Big(\frac{\pi^R(t,\xi,\xi',y,y')}{\psi_{r_c}(y)}\Big) = O(\sqrt{t}),$$

which is the second part of (4.52). Finally, as in Corollary 4.2, (4.52) implies (4.53), which completes the proof of Lemma 4.3. □

4.3 Influence of the low frequencies, part 3: study of the projection $E_2(\hat\pi^\ell)$ and of its inverse Fourier transform

4.3.1 Statement of the main results concerning $E_2(\hat\pi^\ell)$ and of its inverse Fourier transform $\pi^{\ell,2}$

We are going to prove the following

Lemma 4.4 *First,*

$$\|E_2(\hat\pi^\ell)\|_{L^2_y} \le Ce^{-\omega_1 t}. \tag{4.58}$$

Moreover, given n, there exists $\hat\xi_0 > 0$ small enough, and $C > 0$, such that, given $k, k' \in \{0,\cdots,n\}$ such that $k+k' \ge 1$, we have

$$\|\partial^k_{\hat\xi}D^{k'}_y E_2(\hat\pi^\ell)\|_{L^2_y} \le Ct^{k+k'-1}e^{-D_*(c)\hat\xi^2 t/2}. \tag{4.59}$$

It will imply the following

Corollary 4.5 *Denote $\pi^{\ell,2}$ the inverse Fourier transform of $E_2(\hat\pi^\ell)$:*

$$\pi^{\ell,2}(t,\xi,\xi',y,y') := \frac{1}{2\pi}\int_{\hat\xi\in\mathbb{R}} e^{i(\xi-\xi')\hat\xi}E_2(\hat\pi^\ell)(t,\hat\xi,0,y,y')\,d\hat\xi.$$

Then there exists $\omega > 0$ and $C > 0$ such that, for all $t \ge 1$, all $\xi \in \mathbb{R}$, all $y, y' \in \mathbb{T}^{N-1}$ we have

$$\int_{\xi'\in\mathbb{R}}\Big(|\pi^{\ell,2}(t,\xi,\xi',y,y')| + |\partial_\xi\pi^{\ell,2}(t,\xi,\xi',y,y')| + |D_y\pi^{\ell,2}(t,\xi,\xi',y,y')|\Big)\,d\xi' \le Ce^{-\omega t}. \tag{4.60}$$

4.3.2 Proof of Lemma 4.4.

First we have

$$(\partial_t + \mathcal{L}_{\hat{\xi}}) E_2(\hat{\pi}^\ell) = 0.$$

Since $|\hat{\xi}| \leq \hat{\xi}_0$, the spectrum of $\mathcal{L}_{\hat{\xi}}$, except $\mu_{1,\hat{\xi}}$, lies in the half-plane of complex numbers of real part larger than $\omega_1/2$. Hence, multiplying by $\overline{E_2(\hat{\pi}^\ell)}$, integrating and taking the real part, we obtain that

$$\frac{1}{2}\frac{\partial}{\partial t}\|E_2(\hat{\pi}^\ell)\|^2_{L^2_y} + \frac{\omega_1}{2}\|E_2(\hat{\pi}^\ell)\|^2_{L^2_y} \leq 0,$$

which implies that

$$\|E_2(\hat{\pi}^\ell)\|^2_{L^2_y} \leq C_0 e^{-\omega_1 t},$$

hence (4.58) holds true.

Now, let us study the $\hat{\xi}$-derivatives of $E_2(\hat{\pi}^\ell)$, the main reason being that, as in the proof of Corollary 3.3, we will need integrability bounds of $\pi^{\ell,2}$, which derive from L^∞ bounds of $\partial^2_{\hat{\xi}} E_2(\hat{\pi}^\ell)$. Hence we have to study first $\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell)$: it is solution of the nonhomogeneous parabolic problem

$$(\frac{\partial}{\partial t} + \mathcal{L}_{\hat{\xi}})(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell)) = -i(\beta(y) - 2r_-(c))E_2(\hat{\pi}^\ell) - 2\hat{\xi} E_2(\hat{\pi}^\ell).$$

Now decompose $\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell)$ into

$$\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell) = E_1(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell)) + E_2(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell)).$$

First, it is easy to verify that $\|E_2(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))\|_{L^2_y}$ goes exponentially fast to 0: indeed, $E_2(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))$ is solution of

$$(\frac{\partial}{\partial t} + \mathcal{L}_{\hat{\xi}})(E_2(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))) = E_2(-i(\beta(y) - 2r_-(c))E_2(\hat{\pi}^\ell) - 2\hat{\xi} E_2(\hat{\pi}^\ell)),$$

and we already know that the right hand side goes exponentially fast to 0; integrating and taking the real part, we obtain that

$$\frac{1}{2}\frac{\partial}{\partial t}\|E_2(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))\|^2_{L^2_y} + \frac{\omega_1}{4}\|E_2(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))\|^2_{L^2_y} \leq C\|E_2(\hat{\pi}^\ell)\|^2_{L^2_y},$$

which implies that

$$\|E_2(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))\|^2_{L^2_y} \leq C e^{-\omega_1 t/2}. \tag{4.61}$$

Concerning $E_1(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))$: it is solution of

$$(\frac{\partial}{\partial t} + \mu_{1,\hat{\xi}})(E_1(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))) = E_1(-i(\beta(y) - 2r_-(c))E_2(\hat{\pi}^\ell) - 2\hat{\xi} E_2(\hat{\pi}^\ell)),$$

hence

$$\begin{aligned} E_1(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))(t, \hat{\xi}, \xi', y, y') &= E_1(\partial_{\hat{\xi}} E_2(\hat{\pi}^\ell))(0, \hat{\xi}, \xi', y, y') e^{-\mu_{1,\hat{\xi}} t} \\ &\quad + \int_0^t e^{-\mu_{1,\hat{\xi}}(t-s)} E_1(-i(\beta(y) - 2r_-(c))E_2(\hat{\pi}^\ell) - 2\hat{\xi} E_2(\hat{\pi}^\ell))\, ds. \end{aligned}$$

Using the asymptotic expansion (3.32), we have

$$\| \int_0^t e^{-\mu_{1,\hat\xi}(t-s)} E_1(-i(\beta(y) - 2r_-(c))E_2(\hat\pi^\ell) - 2\hat\xi E_2(\hat\pi^\ell))\, ds \|_{L^2_y}$$

$$\leq C e^{(-D_*(c)\hat\xi^2 + M\hat\xi^3)t} \int_0^t e^{(D_*(c)\hat\xi^2 + M\hat\xi^3)s} e^{-\omega_1 s}\, ds \leq C' e^{(-D_*(c)\hat\xi^2 + M\hat\xi^3)t},$$

if $\hat\xi_0$ is small enough so that $D_*(c)\hat\xi_0^2 + M\hat\xi_0^3 < \omega_1$. Hence

$$\|E_1(\partial_{\hat\xi} E_2(\hat\pi^\ell))\|_{L^2_y} \leq C e^{(-D_*(c)\hat\xi^2 + M\hat\xi^3)t}. \tag{4.62}$$

Hence (4.61) and (4.62) imply that

$$\|\partial_{\hat\xi} E_2(\hat\pi^\ell)\|_{L^2_y} \leq C e^{(-D_*(c)\hat\xi^2 + M\hat\xi^3)t} \leq C e^{-D_*(c)\hat\xi^2 t/2} \tag{4.63}$$

if $\hat\xi_0$ has been chosen small enough. This proves (4.59) when $k = 1$, $k' = 0$.

Next we can repeat the procedure: its derivative $D_y E_2(\hat\pi^\ell)$ satisfies

$$(\partial_t + \mathcal{L}_{\hat\xi})(D_y E_2(\hat\pi^\ell)) = -i\hat\xi D_y \beta E_2(\hat\pi^\ell). \tag{4.64}$$

Multiplying by $\overline{D_y E_2(\hat\pi^\ell)}$, integrating and taking the real part, we obtain that

$$\frac{1}{2}\frac{\partial}{\partial t}\|D_y E_2(\hat\pi^\ell)\|^2_{L^2_y} + \frac{\hat\xi^2}{2}\|D_y E_2(\hat\pi^\ell)\|^2_{L^2_y} \leq C\|E_2(\hat\pi^\ell)\|^2_{L^2_y},$$

hence, thanks to (4.58), there exists C_1 such that

$$\|D_y E_2(\hat\pi^\ell)\|^2_{L^2_y} \leq C_1 e^{-t\hat\xi^2}, \tag{4.65}$$

which is an estimate similar to (4.59) with $k = 0$, $k' = 1$. To obtain exactly (4.59), it is sufficient to decompose

$$D_y E_2(\hat\pi^\ell) = E_1(D_y E_2(\hat\pi^\ell)) + E_2(D_y E_2(\hat\pi^\ell)),$$

and proceed as previously for $\partial_{\hat\xi} E_2(\hat\pi^\ell)$.

Next we turn to $\partial^2_{\hat\xi} E_2(\hat\pi^\ell)$: it is solution of the nonhomogeneous parabolic problem

$$(\frac{\partial}{\partial t} + \mathcal{L}_{\hat\xi})(\partial^2_{\hat\xi} E_2(\hat\pi^\ell)) = -2i(\beta(y) - 2r_-(c))\partial_{\hat\xi} E_2(\hat\pi^\ell) - 2\hat\xi\partial_{\hat\xi} E_2(\hat\pi^\ell) - 2E_2(\hat\pi^\ell),$$

and the L^2_y norm of the right hand side goes to as 0 as $e^{(-D_*(c)\hat\xi^2 + M\hat\xi^3)t}$. Then, we decompose into a sum of projections, and we see that

$$\|E_2(\partial^2_{\hat\xi} E_2(\hat\pi^\ell))\|^2_{L^2_y} \leq C e^{-\omega_1 t/2}. \tag{4.66}$$

Concerning the other projection $E_1(\partial^2_{\hat\xi} E_2(\hat\pi^\ell))$: Duhamel's formula gives that

$$E_1(\partial^2_{\hat\xi} E_2(\hat\pi^\ell))(t, \hat\xi, \xi', y, y') = E_1(\partial^2_{\hat\xi} E_2(\hat\pi^\ell))(0, \hat\xi, \xi', y, y') e^{-\mu_{1,\hat\xi} t}$$

$$+ \int_0^t e^{-\mu_{1,\hat\xi}(t-s)} E_1(-2i(\beta(y) - 2r_-(c))\partial_{\hat\xi} E_2(\hat\pi^\ell) - 2\hat\xi\partial_{\hat\xi} E_2(\hat\pi^\ell) - 2E_2(\hat\pi^\ell))\, ds,$$

which easily implies that

$$\|E_1(\partial^2_{\hat\xi} E_2(\hat\pi^\ell))\|_{L^2_y} \le Ce^{(-D_*(c)\hat\xi^2+M\hat\xi^3)t} + Ce^{(-D_*(c)\hat\xi^2+M\hat\xi^3)t}\int_0^t e^{(D_*(c)\hat\xi^2+M\hat\xi^3)s}e^{(-D_*(c)\hat\xi^2+M\hat\xi^3)s}\,ds \le Ce^{(-D_*(c)\hat\xi^2+M\hat\xi^3)t} + Ce^{(-D_*(c)\hat\xi^2+3M\hat\xi^3)t}t,$$

hence

$$\|\partial^2_{\hat\xi} E_2(\hat\pi^\ell)\|_{L^2_y} \le Ce^{-D_*(c)\hat\xi^2 t/2}t \tag{4.67}$$

if $\hat\xi_0$ is small enough, which proves (4.59) when $k=2$, $k'=0$.

The same strategy and a suitable induction argument leads to the validity of Lemma 4.4. □

4.3.3 Proof of Corollary 4.5.

Sobolev's embeddings imply that $H^N(\mathbb{T}^{N-1}) \subset L^\infty(\mathbb{T}^{N-1})$, and there exists some $\theta > 0$ and $C > 0$ such that, for all $f \in H^{N+1}(\mathbb{T}^{N-1})$, we have

$$\|f\|_{L^\infty(\mathbb{T}^{N-1})} \le C(\|f\|_{L^2(\mathbb{T}^{N-1})} + \|f\|^\theta_{L^2(\mathbb{T}^{N-1})}\|D_y^{N+1}f\|^{1-\theta}_{L^2(\mathbb{T}^{N-1})}).$$

Then (4.58) and (4.59) imply that there exists $\omega > 0$ and $C > 0$ such that, for all $t \ge 1$, all $\hat\xi, \xi' \in \mathbb{R}$, all $y, y' \in \mathbb{T}^{N-1}$ we have

$$|E_2(\hat\pi^\ell)(t, \hat\xi, \xi', y, y')| \le Ce^{-\omega t}. \tag{4.68}$$

In the same way, Sobolev's embeddings imply that in fact all the y-derivatives of $E_2(\hat\pi^\ell)$ go exponentially fast to 0, uniformly in $\hat\xi, \xi', y, y'$. Then, since $E_2(\hat\pi^\ell)$ is compactly supported in $\hat\xi$, this implies an L^∞ bound for $\pi^{\ell,2}$:

$$|\pi^{\ell,2}(t, \xi, \xi', y, y')| \le Ce^{-\omega t}.$$

Now, as in Corollary 3.3, we consider

$$(\xi-\xi')^2\pi^{\ell,2}(t,\xi,\xi',y,y') = -\frac{1}{2\pi}\int_{\hat\xi\in\mathbb{R}} e^{i(\xi-\xi')\hat\xi}\partial^2_{\hat\xi}E_2(\hat\pi^\ell)(t,\hat\xi,0,y,y')\,d\hat\xi.$$

(4.59) and Sobolev's embeddings imply that there

$$|\partial^2_{\hat\xi}E_2(\hat\pi^\ell)(t,\hat\xi,\xi',y,y')| \le Ct^{N+2}e^{-D_*(c)\hat\xi^2 t/2}.$$

Then

$$(\xi-\xi')^2|\pi^{\ell,2}(t,\xi,\xi',y,y')| \le Ct^{N+3/2}.$$

This seems a rough estimate, but as we did before:

$$|\pi^{\ell,2}(t,\xi,\xi',y,y')| \le \inf\{e^{-\omega t}, \frac{t^{N+3/2}}{(\xi-\xi')^2}\} \le \inf\{e^{-\omega t}, \frac{e^{\omega t/2}}{(\xi-\xi')^2}\},$$

which implies that

$$\int_{\xi'\in\mathbb{R}} |\pi^{\ell,2}(t,\xi,\xi',y,y')| \le Ce^{-\omega t/4}.$$

The same strategy yields that the same estimate holds for the first order derivatives $\partial_\xi\pi^{\ell,2}$ and $D_y\pi^{\ell,2}$, hence (4.60) holds true. □

4.4 Application of these estimates to the Hopf-Cole transform q.

We decomposed successively $\hat{\pi}$ into

$$\hat{\pi} = \hat{\pi}^h + \hat{\pi}^\ell = \hat{\pi}^h + \hat{\pi}^{\ell,1} + \hat{\pi}^{\ell,2} = \hat{\pi}^h + \hat{\pi}^{pp} + \hat{\pi}^R + \hat{\pi}^{\ell,2},$$

and this yields

$$\pi = \pi^h + \pi^{pp} + \pi^R + \pi^{\ell,2},$$

and

$$q = q^h + q^{pp} + q^R + q^{\ell,2}$$

with the natural definition:

$$q^j(t,\xi,y) := \int_{\xi'\in\mathbb{R}} \int_{y'\in\mathbb{T}^{N-1}} \pi^j(t,\xi,\xi',y,y') q_0(\xi',y')\, d\xi'\, dy'.$$

We are going to study these four terms, and prove the following

Lemma 4.6 *(i) q^{pp} has the following property: it is the solution of the advection diffusion equation*

$$q_t^{pp} - D_*(c) q_{\xi\xi}^{pp} + V_*(c) q_\xi^{pp} = 0, \quad q^{pp}(0,\xi,y) = \psi_{r_c}(y) \int_{y'} q_0(\xi,y')\psi_{r_c}(y')\, dy'. \tag{4.69}$$

(ii) q^h has the following property: for all $k, k' \in \mathbb{N}$, there exists $C_{k,k'} > 0$, $\omega_{k,k'} > 0$ such that we have:

$$\forall t \geq 1, \forall \xi \in \mathbb{R}, \forall y \in \mathbb{T}^{N-1}, \quad |\partial_\xi^k D_y^{k'} q^h(t,\xi,y)| \leq C_\alpha e^{-\omega_\alpha t} \|q_0\|_\infty. \tag{4.70}$$

(iii) q^R has the following property:

$$\forall t \geq 1, \forall \xi \in \mathbb{R}, \forall y \in \mathbb{T}^{N-1}, \quad |q^R(t,\xi,y)| \leq \frac{4M_0}{\sqrt{t}} \|q_0\|_\infty. \tag{4.71}$$

(iv) $q^{\ell,2}$ has the following property: there exists $C > 0$ and $\omega > 0$ such that

$$\forall t \geq 1, \forall \xi \in \mathbb{R}, \forall y \in \mathbb{T}^{N-1}, \quad |q^{\ell,2}(t,\xi,y)| \leq C e^{-\omega t}. \tag{4.72}$$

Proof. Part (i) is a consequence of (4.47). Part (ii) follows Corollary 3.3. Part (iii) follows from Corollary 4.2, and Part (iv) is a direct consequence of Corollary 4.5. □

4.5 Application of these estimates to the Hopf-Cole transform s.

Now let us consider the decomposition of the Hopf-Cole transform s

$$e^{r_-(c) m(t,\xi,y)} = s = s^h + s^{pp} + s^R + s^{\ell,2}$$

with the definition:

$$s^j(t,\xi,y) := \frac{1}{\psi_{r_c}(y)} q^j(t,\xi,y) = \frac{1}{\psi_{r_c}(y)} \int_{\xi'\in\mathbb{R}} \int_{y'\in\mathbb{T}^{N-1}} \pi^j(t,\xi,\xi',y,y') q_0(\xi',y')\, d\xi'\, dy'.$$

We are going to study these four terms, and prove the following

Corollary 4.7 *(i) The function s^{pp} is solution of the advection diffusion*

$$s^{pp}_t - D_*(c)s^{pp}_{\xi\xi} + V_*(c)s^{pp}_\xi = 0, \quad s^{pp}(0,\xi,y) = \int_{y'} q_0(\xi,y')\psi_{r_c}(y')\,dy', \tag{4.73}$$

hence it does not depend on y and is bounded between two positive constants.

(ii) s^h converges exponentially fast to 0: there is some $\omega > 0$ such that

$$\sup_{\xi,y} |s^h(t,\xi,y)| = O(e^{-\omega t}). \tag{4.74}$$

(iii) s^R converges uniformly to 0 as $t^{-1/2}$:

$$\sup_{\xi,y} |s^R(t,\xi,y)| = O(\frac{1}{\sqrt{t}}). \tag{4.75}$$

(iv) $s^{\ell,2}$ converges exponentially fast to 0: there is some $\omega > 0$ such that

$$\sup_{\xi,y} |s^{\ell,2}(t,\xi,y)| = O(e^{-\omega t}). \tag{4.76}$$

Proof. (i) follows from the parabolic equation (4.69) satisfied by q^{pp} and the weak maximum principle, (ii) follows from (4.70) of Lemma 4.6. (iii) is a consequence of Corollary 4.2, and (iv) comes from Corollary 4.5. □

Finally, we need estimates on the first order derivatives of s.

Corollary 4.8 *(i) $Ds^{pp} = \partial_\xi s^{pp} = O(t^{-1/2})$ uniformly in ξ, y.*

(ii) There is $\omega > 0$ such that $Ds^h = O(e^{-\omega t})$ uniformly in ξ, y.

(iii) $Ds^R = O(t^{-1/4})$ uniformly in ξ, y. More precisely, $\partial_\xi s^R = O(t^{-1})$ and $D_y s^R = O(t^{-1/4})$.

(iv) There is $\omega > 0$ such that $Ds^{\ell,2} = O(e^{-\omega t})$ uniformly in ξ, y.

Proof of Corollary 4.8. Part (i) follows from the advection diffusion equation (4.73) satisfied by s^{pp}. Part (ii) follows from Corollary 3.3. Part (iii): first we derive from (4.51) that

$$\partial_\xi s^R(t,\xi,y) = \frac{1}{\psi_{r_c}(y)} \int_{\xi'\in\mathbb{R}} \int_{y'\in\mathbb{T}^{N-1}} \partial_\xi \pi^R(t,\xi,\xi',y,y') q_0(\xi',y')\,d\xi'\,dy' = O(\frac{1}{t})\|q_0\|_\infty;$$

next we see that

$$D_y s^R(t,\xi,y) = \int_{\xi'} \int_{y'} D_y\Big(\frac{\pi^R(t,\xi,\xi',y,y')}{\psi_{r_c}(y)}\Big) q_0(\xi',y')\,d\xi'\,dy',$$

hence (4.53) implies (iii).

Part (iv) comes directly from Corollary 4.5. □

4.6 Application to these estimates to the approximate shift m

We derive from Corollary 4.8 the following

Corollary 4.9 *Denote*

$$m^{pp}(t,\xi) := \frac{1}{r_-(c)} \ln s^{pp}(t,\xi) \tag{4.77}$$

Then the approximate shift m *has the following properties*

$$\sup_{\xi,y} |m(t,\xi,y) - m^{pp}(t,\xi)| = O(\frac{1}{\sqrt{t}}), \tag{4.78}$$

and

$$\sup_{\xi,y} |m_\xi(t,\xi,y)| = O(\frac{1}{\sqrt{t}}), \quad \sup_{\xi,y} |D_y m(t,\xi,y)| = O(\frac{1}{t^{1/4}}). \tag{4.79}$$

Proof. It derives immediately from Corollaries 4.7 and 4.8: by the mean value theorem,

$$m - m^{pp} = \frac{1}{r_-(c)} \ln s - \frac{1}{r_-(c)} \ln s^{pp} = O(s - s^{pp}) = O(s^h + s^R + s^{\ell,2}) = O(\frac{1}{\sqrt{t}});$$

next

$$m_\xi = \frac{1}{r_-(c)} \frac{s_\xi}{s} = \frac{1}{r_-(c)} \frac{s^h_\xi + s^{pp}_\xi + s^R_\xi + s^{\ell,2}_\xi}{s} = O(\frac{1}{\sqrt{t}}),$$

and in the same way $D_y m = O(t^{-1/4})$. □

5 Conclusion and examples

Corollary 4.9 gives us that $\|Dm\|_\infty = O(t^{-1/4})$, and this completes the proof of Proposition 2.3. Now we have a quite precise description of the asymptotic behaviour of the solution u of the Cauchy problem associated to (1.3): Proposition 2.3 gives that the approximate shift m satisfies $\|Dm\|_\infty = O(t^{-1/4})$, and in subsection 2.5, we have seen that this implies that the difference between u and the approximate solution u^{app} satisfies $\widetilde{u}(t,\xi,y) - \widetilde{u}^{app}(t,\xi,y) = O(t^{-1/4})$ uniformly in ξ and y. But then it is natural to introduce the "principal part" of u:

$$\widetilde{u}^{pp}(t,\xi,y) := \phi_c(\xi + m^{pp}(t,\xi), y), \tag{5.80}$$

and we have immediately

$$\begin{aligned} \widetilde{u}(t,\xi,y) - \widetilde{u}^{pp}(t,\xi,y) &= \widetilde{u}(t,\xi,y) - \widetilde{u}^{app}(t,\xi,y) + \phi_c(\xi + m(t,\xi,y), y) - \phi_c(\xi + m^{pp}(t,\xi), y) \\ &= O(\frac{1}{t^{1/4}}) + O(m - m^{pp}) = O(\frac{1}{t^{1/4}}) + O(\frac{1}{\sqrt{t}}) = O(\frac{1}{t^{1/4}}). \end{aligned}$$

The main gain is that $s^{pp}(t,\xi) = e^{r_-(c) m^{pp}(t,\xi)}$ satisfies the very simple one dimensional advection diffusion equation 4.73, whose solution is explicit, using the classical heat kernel. As a consequence, let us mention these two simple examples, extracted from a more complete study given in [4]:

- assume that $m_0(\xi, y) \to m_0(-\infty, y)$ as $\xi \to -\infty$; then, considering

$$s_0(-\infty) := \int_{y'} e^{r_-(c)m_0(-\infty,y')} \psi_{r_c}(y')^2 \, dy',$$

the explicit formula of s^{pp} allows us to prove that

$$\widetilde{u}^{pp}(t, \xi, y) - \phi_c(\xi + \frac{1}{r_-(c)} \ln s_0(-\infty), y)$$

converges uniformly to 0 as $t \to +\infty$, uniformly in ξ and y (with an explicit decay rate, depending on the convergence of $m_0(\xi, y) - m_0(-\infty, y)$ to 0); hence, the solution u converges to some translate of the travelling wave:

$$\sup_{\xi,y} |\widetilde{u}(t, \xi, y) - \phi_c(\xi + \frac{1}{r_-(c)} \ln s_0(-\infty), y)| \to_{t \to +\infty} 0;$$

- assume now that $s^{pp}(t, 0)$ "slowly oscillates" at $-\infty$ (see [4] for a precise definition); then u^{pp}, and consequently u, does not converge to any translate of the travelling wave ϕ_c.

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