Existence and qualitative properties of multidimensional conical bistable fronts

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Abstract. Travelling fronts with conical-shaped level sets are constructed for reaction-diffusion equations with bistable nonlinearities of positive mass. The construction is valid in space dimension 2, where two proofs are given, and in arbitrary space dimensions under the assumption of cylindrical symmetry. General qualitative properties are presented under various assumptions: conical conditions at infinity, existence of a sub-level set with globally Lipschitz boundary, monotonicity in a given direction.

1 Introduction and main results

The purpose of this paper is the construction and study of classical bounded solutions of the following elliptic equation:

$$(1.1)\qquad \Delta u - c\partial_y u + f(u) = 0 \ \text{ in } \mathbb{R}^N = \{z=(x,y),\ x=(x_1,\cdots,x_{N-1})\in\mathbb{R}^{N-1},\ y\in\mathbb{R}\}.$$

Notice that if u solves (1.1), then the function $(t,x,y)\mapsto v(t,x,y)=u(x+ct,y)$ is a travelling wave solution of the reaction-diffusion equation

$$(1.2)\qquad v_t-\Delta v=f(v),\quad t\in\mathbb{R},\quad (x,y)\in\mathbb{R}^N.$$

In the whole paper, we shall use the following
Notation: if $x\in\mathbb{R}^N$, then $|\cdot|$ is the Euclidean norm, and

$$(1.3)\qquad \hat{x}=\frac{x}{|x|}\ \text{for } x\neq 0.$$

The function f is C^2, and, throughout the paper, f is assumed to be of the 'bistable' type: there exists θ in $(0,1)$ such that

$$\left\{\begin{array}{l} f(0)=f(\theta)=f(1)=0,\ f<0 \text{ on } (0,\theta)\cup(1,+\infty),\ f>0 \text{ on } (-\infty,0)\cup(\theta,1),\\ f'(0)<0,\ f'(1)<1,\ f'(\theta)>0.\end{array}\right.$$

Assume $\int_0^1 f > 0$. It is a well-known fact that there is a unique $c_0 > 0$ such that the 1D differential equation

$$U'' - c_0 U' + f(U) = 0 \ \text{ in } \mathbb{R}, \quad U(-\infty) = 0,\ U(+\infty) = 1. \tag{1.4}$$

has a solution. Moreover the profile U is unique (namely, U is unique up to translation). This solution will be of constant use in the paper.

We look for solutions of (1.1) satisfying a conical asymptotic condition of angle α with respect to the direction $-e_N := (0, \cdots, 0, -1)$. A natural condition is the following:

$$\left\{ \begin{array}{lcl} \limsup\limits_{A \to +\infty,\ y \geq A - |x| \cot\alpha} |u(x,y) - 1| & = & 0, \\ \limsup\limits_{A \to -\infty,\ y \leq A - |x| \cot\alpha} |u(x,y)| & = & 0. \end{array} \right. \tag{1.5}$$

For combustion type nonlinearities f satisfying $f \equiv 0$ on $[0,\theta] \cup \{1\}$ and $f > 0$ on $(\theta, 1)$, this condition (1.5) was used in [17], where the existence of solutions to (1.1), (1.5) is proved in 2 space dimensions; furthermore, it was already noted in [15] that this condition (1.5) - which in particular implies that the lines with slopes $\pm \cot\alpha$ are asymptotic to the level lines of the solution - cannot hold in space dimension $N \geq 3$ for $\alpha \neq \pi/2$. This last property holds for combustion type nonlinearities as well as for bistable nonlinearities with positive mass.

The case of a bistable nonlinearity f is a natural equivalent of the papers [15], [17], and presents additional difficulties due to the lack of positivity of the nonlinearity. Equation (1.1) can be interpreted in terms of geometrical motions [2], [11]: the solutions of (1.1) indeed converge, under a suitable scaling, to solutions of eikonal equations. Also, our problem has close connections to some questions around a conjecture of De Giorgi [1], [3], [14], [24].

Since condition (1.5) is then too strong, for $N > 2$, the following less restrictive condition will be used:

$$\left\{ \begin{array}{lcl} \limsup\limits_{A \to +\infty,\ y \geq A + \phi(|x|)} |u(x,y) - 1| & = & 0, \\ \limsup\limits_{A \to -\infty,\ y \leq A + \phi(|x|)} |u(x,y)| & = & 0 \end{array} \right. \tag{1.6}$$

for some globally Lipschitz function ϕ defined in $[0, +\infty)$. We will see that it automatically implies a weak conical condition with some given angle α.

Let us note that, for $N \geq 2$, the planar front $U(y)$ is a solution of (1.1) and (1.5) with $\alpha = \pi/2$. Let us also mention that for $N = 2$, solutions of (1.1), (1.5) are known to exist for α less and close to $\pi/2$ (see [12], [20]).

The main existence results of this paper are the following:

Theorem 1.1 (Existence result in dimension $N = 2$) *In dimension $N = 2$, for each $\alpha \in (0, \pi/2]$, there exists a unique - up to shift in the (x,y) variables - solution (c,u) of (1.1) and (1.5). Furthermore, $0 < u < 1$ in $\mathbb{R}^2$, c is given by $c = c_0/\sin\alpha$, and, up to shift, u is even in x and increasing in $|x|$. The function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^2$ such that $\tau_y < -\cos\alpha$. For each $\lambda \in (0,1)$, the level set $\{u = \lambda\}$ is a globally Lipschitz graph $\{y = \phi_\lambda(x)\}$ whose Lipschitz norm is equal to $\cot\alpha$. Lastly, $u(x + x_n, y - |x_n| \cot\alpha) \to U(\pm x \cos\alpha + y \sin\alpha)$ in $C^2_{loc}(\mathbb{R}^2)$, for any sequence $x_n \to \pm\infty$.*

A part of this result is proved in [23], by the construction of a super-solution coming from the study of travelling waves for the 2D mean curvature motion with drift.

Theorem 1.2 (Existence result in dimension $N \geq 3$) *In dimension $N \geq 3$, for each $\alpha \in (0, \pi/2]$, there exists a solution $(c, u) = (c_0/\sin\alpha, u)$ of (1.1) such that*

- $0 < u < 1$ *in* $\mathbb{R}^N$,
- $u(x, y) = \tilde{u}(|x|, y)$ *and the following monotonicity properties hold:* $\partial_{|x|}\tilde{u} \geq 0$, $\partial_y u > 0$,
- *the function u satisfies (1.6) with $\phi = \phi_\lambda$, for all $\lambda \in (0, 1)$, where $\{u(x, y) = \lambda\} = \{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$,*
- *there holds $\hat{x} \cdot \nabla\phi_\lambda(x) \to -\cot\alpha$ as $|x| \to +\infty$.*

Moreover the function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y < -\cos\alpha$. Lastly, for any unit direction $e \in \mathbb{R}^{N-1}$, for any sequence $r_n \to +\infty$ and for any $\lambda \in (0, 1)$, $u(x + r_n e, y + \phi_\lambda(r_n e)) \to U((x \cdot e)\cos\alpha + y \sin\alpha + U^{-1}(\lambda))$ in $C^2_{loc}(\mathbb{R}^N)$.

In the proofs of the above theorems, we actually construct some solutions of (1.1) satisfying some conditions which are a priori weaker than (1.5) or (1.6). In the following results, we want to give some -as weak as possible- conditions under which a solution of (1.1) satisfies (1.5) or (1.6). For that we introduce a sequence of conditions, which are weaker than (1.5) or (1.6). The minimal one would be the following:

Hypothesis 1.3 We have $\inf u < \theta < \sup u$. Moreover $\partial_y u \geq 0$.

We note that Assumption 1.3 cannot be weakened in our context: indeed, if we drop the condition $\inf u < \theta < \sup u$ we increase a lot the range of solutions that we have to consider. An instance is given by all the solutions that are larger than θ; they satisfy a KPP equation, for which the global attractor is bigger than the space of all probability measures in $\mathbb{R}^N$ (see [19]).

Assumption 1.3 is, we believe, still too weak. Indeed, the one-dimensional problem

$$U'' + f(U) = 0, \quad U(\pm\infty) = 0 \tag{1.7}$$

has a one-dimensional family of solutions; call for instance $U_-(x)$ the unique solution of (1.7) that is even. This solution is unstable with respect to the one-dimensional parabolic problem

$$u_t - u_{xx} = f(u),\ 0 < u < 1.$$

Problem (1.1) with the conditions at infinity

$$u(x, -\infty) = U_-(x), \quad u(x, +\infty) = 1 \tag{1.8}$$

is of the KPP type. It is not unthinkable that (1.1), (1.8) has solutions for a nontrivial range of velocities. Such solutions would have asymptotically vertical level lines and are studied in the forthcoming paper [8].

This motivates the introduction of the following assumptions which are stronger than Hypothesis 1.3 but still a priori weaker than (1.5) or (1.6):

Hypothesis 1.4 There exists a globally Lipschitz function ϕ defined in $\mathbb{R}^+$ such that

$$(1.9)\qquad \begin{cases} \liminf\limits_{A\to+\infty,\ y\geq A+\phi(|x|)} u(x,y) > \theta \\ \limsup\limits_{A\to-\infty,\ y\leq A+\phi(|x|)} u(x,y) < \theta \end{cases}$$

Hypothesis 1.5 The speed c is nonnegative, $\inf_{\mathbb{R}^N} u < \theta$, $\partial_y u \geq 0$ in $\mathbb{R}^N$, $u(x,y) = \tilde{u}(|x|,y)$ and $\partial_{|x|}\tilde{u}(|x|,y) \geq 0$ for all $(x,y) \in \mathbb{R}^{N-1}\times\mathbb{R}$.

Theorem 1.6 (General qualitative properties) *Let $N \geq 2$ and let u be a bounded nonconstant solution of (1.1), with some speed $c \in \mathbb{R}$.*

1) If Hypothesis 1.5 is satisfied, then so is Hypothesis 1.4.

2) If Hypothesis 1.4 is satisfied, then $0 < u < 1$ in $\mathbb{R}^N$, the level set $\{u = \lambda\}$ (for any $\lambda \in (0,1)$) is a globally Lipschitz graph $\{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$. For every $\lambda \in (0,1)$ the function u satisfies

$$(1.10)\qquad \begin{cases} \limsup\limits_{A\to+\infty,\ y\geq A+\phi_\lambda(x)} |u(x,y)-1| = 0, \\ \limsup\limits_{A\to-\infty,\ y\leq A+\phi_\lambda(x)} |u(x,y)| = 0 \end{cases}$$

Furthermore, all functions ϕ_λ have the same Lipschitz norm, say $\cot\alpha$ with $\alpha \in (0,\pi/2]$, $\hat{x}\cdot\nabla\phi_\lambda(x) \to -\cot\alpha$ as $|x| \to +\infty$, and $c = c_0/\sin\alpha$. The function u is decreasing in any unit direction $\tau = (\tau_x,\tau_y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ such that $\tau_y < -\cos\alpha$. Lastly, for any unit direction $e \in \mathbb{R}^{N-1}$, for any sequence $r_n \to +\infty$ and for any $\lambda \in (0,1)$, $u(x+r_ne, y+\phi_\lambda(r_ne)) \to U((x\cdot e)\cos\alpha + y\sin\alpha + U^{-1}(\lambda))$ in $C^2_{loc}(\mathbb{R}^N)$.

Remark 1.7 In any dimension $N \geq 2$, there is no solution (c,u) of (1.1), (1.9) such that $\phi'(r) \to -\cot\alpha$ as $r \to +\infty$ and $\alpha \in (\pi/2,\pi)$ (see Section 2 for the proof, and [15] for a discussion).

In the following paper [18], we prove some uniqueness results for the solutions of (1.1) under Hypotheses 1.4 or 1.5, under the additional assumption of cylindrical symmetry in the case of dimensions $N \geq 3$. In dimension $N = 2$, we also prove a classification result for the solutions of (1.1) such that there exists a globally Lipschitz function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$ with

$$(1.11)\qquad \begin{cases} \liminf\limits_{A\to+\infty,\ y\geq A+\phi(x)} u(x,y) > \theta, \\ \limsup\limits_{A\to-\infty,\ y\leq A+\phi(x)} u(x,y) < \theta. \end{cases}$$

Lastly, in [18], we make more precise the asymptotic behaviour of the level sets of the solutions u which are constructed in Theorems 1.1 and 1.2: they converge exponentially to some straight lines in dimension 2, which allows us for another proof of Theorem 1.1 by a continuation argument. As opposed to that, they move away logarithmically from their asymptotic directions in dimensions $N \geq 3$.

Remark 1.8 One may wonder what happens if the integral of $\displaystyle\int_0^1 f(u)\,du$ is not positive anymore. If it is negative, the only change that has to be done is to change α into $-\alpha$. The case when it is exactly 0 will be the object of a forthcoming paper [8].

Remark 1.9 The waves constructed here are globally stable with respect to (1.2), for large classes of initial data. This is discussed in detail in [17] for the combustion-type nonlinearity. The proofs can be carried over word by word in the bistable case, which is actually simpler. See also [23].

The plan of the paper is the following. In Section 2, we start with general qualitative properties, like the cone of monotonicity or the uniqueness of the speed; this is done under the rather strong Hypothesis 1.4. The main arguments that are used in this section are the sliding and moving plane methods, the difficulty being - as usual in these problems in the whole space - their initialization. This is done *via* (some adaptations of) general comparison lemmas given in [16]. One of the results of Section 2 is that, in all cases under investigation in this paper, the solutions of (1.1), once a globally Lipschitz level set has beeen identified, will converge, along their level lines, to a 1D wave profiles.

In Section 3 we complete the proof of the existence Theorems 1.1 and 1.2. We give two existence proofs, each of them having its own interests. We start with a sub/super-solution method in dimension 2, that had been used in [17] (nonlinearity f of the 'ignition temperature' type). The difficult part is here the construction of a super-solution; surprisingly, the argument of [17] works almost word by word. Finally, because this method was restricted to the space dimension 2, we give in any dimension $N \geq 2$ a second proof using an approximation argument in a finite cylinder: we impose the velocity, and the boundary conditions are taken to be the planar wave. This forces the level sets of the constructed solution to bend, and the main part of the proof consists in proving that this property is kept throughout the approximation process.

2 Qualitative properties

2.1 Monotonicity and uniqueness results under condition (1.5)

In this section, it is enough to assume that the function f is only C^1, and not C^2 as stated in the introduction. Most probably all the main results (including the existence results) still hold under this sole assumption.

The section is devoted to the proof of some uniqueness and monotonicity results. The only relevant space dimension here is $N = 2$; what we wish to prove is summarized in the following

Proposition 2.1 *Let $\alpha \in (0, \pi/2]$ and let u be a bounded solution of (1.1), (1.5) in $\mathbb{R}^2$, with a speed $c \in \mathbb{R}$. Then $0 < u < 1$ in $\mathbb{R}^2$, $c = c_0/\sin\alpha$ and u is decreasing in any unit direction $\tau = (\tau_x, \tau_y)$ such that $\tau_y < -\cos\alpha$. Furthermore, up to shift in the (x, y) variables, u is unique, even in x, increasing in x for positive x if $\alpha < \pi/2$, and $u(x + x_n, y - |x_n|\cot\alpha) \to U(\pm x\cos\alpha + y\sin\alpha)$ in $C^2_{loc}(\mathbb{R}^2)$ for any sequence $x_n \to \pm\infty$.*

For the sake of clarity, the proof is divided into several auxiliary lemmas.

Lemma 2.2 *Under the assumptions of Proposition 2.1, the function u satisfies $0 < u < 1$ in $\mathbb{R}^2$.*

Proof. Let $M = \sup_{\mathbb{R}^2} u$. Observe that $M \geq 1$ from (1.5). Let (x_n, y_n) be a sequence in $\mathbb{R}^2$ such that $u(x_n, y_n) \to M$ as $n \to +\infty$. Call $u_n(x, y) = u(x + x_n, y + y_n)$. Up to extraction of some subsequence, the functions u_n converge in $C^2_{loc}(\mathbb{R}^2)$ to a classical solution u_∞ of (1.1), namely

$$\Delta u_\infty - c\partial_y u_\infty + f(u_\infty) = 0 \text{ in } \mathbb{R}^2,$$

and $u_\infty(0, 0) = M = \max_{\mathbb{R}^2} u_\infty$. Therefore, $f(M) \geq 0$. Since f is negative in $(1, +\infty)$, one gets that $M \leq 1$.

Similarly, one can easily prove that $m := \inf_{\mathbb{R}^2} u \geq 0$. Hence, $0 \leq u \leq 1$ in $\mathbb{R}^2$. The strong maximum principle then yields $0 < u < 1$ in $\mathbb{R}^2$, since u is not constant from (1.5). □

Let us now notice that, if $\alpha = \pi/2$, then u only depends on y (this result follows for instance from Theorem 2 in [4]). Hence, from the uniqueness for problem (1.4), $c = c_0$ and there exists $\tau \in \mathbb{R}$ such that $u(x, y) = U(y + \tau)$ for all $(x, y) \in \mathbb{R}^2$. Proposition 2.1 follows.

In the sequel, one can then consider only the case $\alpha < \pi/2$.

Lemma 2.3 *Under the assumptions of Proposition 2.1, the function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y)$ such that $\tau_y < -\cos\alpha$.*

Proof. The fact that u is decreasing in the direction $(0, -1)$ (namely, $\partial_y u > 0$ in $\mathbb{R}^2$) follows from (1.5) and from the general comparison result stated in Theorem 1.5 in [15] (this result especially uses the fact that the nonlinearity f is nonincreasing in $[0, \delta]$ and $[1 - \delta, 1]$ for some $\delta > 0$).

If now τ satisfies the conditions of Lemma 2.3, we introduce the rotated coordinates:

$$X = -\tau_y x + \tau_x y; \quad Y = -\tau_x x - \tau_y y. \tag{2.1}$$

In the variables (X, Y) the function $v(X, Y) = u(x, y)$ satisfies $\Delta v + c\tau_x \partial_X v + c\tau_y \partial_Y v + f(v) = 0$ in $\mathbb{R}^2$. Furthermore, it follows from (1.5) that there exists a globally Lipschitz function ψ such that

$$\liminf_{A \to +\infty,\ Y \geq A + \psi(X)} v(X, Y) = 1, \qquad \limsup_{A \to -\infty,\ Y \leq A + \psi(X)} v(X, Y) = 0$$

(actually, the curve $\{y = -|x| \cot\alpha\}$ is a globally Lipschitz graph in the frame (X, Y)). One again gets that $\partial_Y v > 0$ in $\mathbb{R}^2$. In other words, $\tau \cdot \nabla u < 0$ in $\mathbb{R}^2$ and the proof of Lemma 2.3 is complete. □

Lemma 2.4 *Under the assumptions of Proposition 2.1, one has $c = c_0/\sin\alpha$ and there are two real numbers $t_\pm$ such that $u(x + x_n, y - |x_n| \cot\alpha) \to U(\pm x \cos\alpha + y \sin\alpha + t_\pm)$ in $C^2_{loc}(\mathbb{R}^2)$ for any sequence $x_n \to \pm\infty$.*

Proof. First of all, it follows from Lemma 2.3 that, by continuity, u is nonincreasing in the two directions having an angle α with respect to the vertical direction $(0, -1)$, namely the directions $(\pm\sin\alpha, -\cos\alpha)$. Therefore, there exist two functions $U_\pm : \mathbb{R} \to [0, 1]$ such that

$$u(x + x_n, y - |x_n| \cot\alpha) \to v_\pm(x, y) = U_\pm(\pm x \cos\alpha + y \sin\alpha)$$

for any sequence $x_n \to \pm\infty$. From standard elliptic estimates, the convergence holds in $C^2_{loc}(\mathbb{R}^2)$ and the functions $v_\pm$ satisfy (1.1). Furthermore, it follows from (1.5) that

$U_{\pm}(+\infty) = 1$ and $U_{\pm}(-\infty) = 0$. In other words, the functions $U_{\pm}$ solve (1.4) with the speed $c\sin\alpha$. From the uniqueness result for problem (1.4), one gets that $c\sin\alpha = c_0$ and there exist two real numbers $t_{\pm}$ such that $U_{\pm}(s) = U(s + t_{\pm})$ for all $s \in \mathbb{R}$. That completes the proof of Lemma 2.4. □

Lemma 2.5 *Under the assumptions of Proposition 2.1, u is unique up to a shift in the (x, y) variables.*

Proof. The proof can be done by using exactly the same method as in Theorem 1.7 in [15] and is not repeated here.[1] □

The evenness and monotonicity properties with respect to x for $x \geq 0$ (up to shift in x) are based on the following two lemmas:

Lemma 2.6 *Let $\underline{u}$ and $\overline{u}$ be two bounded $C^{2,\beta}$ functions (with $\beta > 0$) in the set $\overline{\Omega}$, where $\Omega = \{x < a,\ y > b - |x|\cot\alpha\}$ and $(a, b) \in \mathbb{R}^2$, satisfying*

$$\Delta\underline{u} - c\partial_y\underline{u} + g(\underline{u}) \geq \Delta\overline{u} - c\partial_y\overline{u} + g(\overline{u}) \quad \textit{in } \Omega,$$

$\underline{u} \leq \overline{u}$ on $\partial\Omega$ and $\liminf_{A\to+\infty,\ y\geq A-|x|\cot\alpha}(\overline{u}(x, y) - \underline{u}(x, y)) \geq 0$. Furthermore, assume that there exists $\rho \in \mathbb{R}$ such that g is Lipschitz continuous on $\mathbb{R}$, nonincreasing on $[\rho, +\infty)$, and $\overline{u} \geq \rho$ in Ω. Then $\underline{u} \leq \overline{u}$ in $\overline{\Omega}$.

Proof. Observe first that $\underline{u} \leq \overline{u} + \varepsilon$ in $\overline{\Omega}$ for $\varepsilon > 0$ large enough. Set

$$\varepsilon^* = \inf\,\{\varepsilon > 0,\ \underline{u} \leq \overline{u} + \varepsilon \ \text{ in } \overline{\Omega}\}.$$

One has $\varepsilon^* \geq 0$ and $\underline{u} \leq \overline{u} + \varepsilon^*$ in $\overline{\Omega}$. One shall prove that $\varepsilon^* = 0$ in order to get the desired result. Assume that $\varepsilon^* > 0$, and let (ε_n) and (x_n, y_n) be some sequences such that $\varepsilon_n \overset{<}{\to} \varepsilon^*$ and $\underline{u}(x_n, y_n) > \overline{u}(x_n, y_n) + \varepsilon_n$. From the limiting condition as $y + |x|\cot\alpha \to +\infty$, it follows that the sequence $(y_n + |x_n|\cot\alpha)$ is bounded. Up to extraction of some subsequence, two cases may occur:

Case 1: $(x_n, y_n) \to (x_\infty, y_\infty) \in \overline{\Omega}$ as $n \to +\infty$. One then has $\underline{u}(x_\infty, y_\infty) = \overline{u}(x_\infty, y_\infty) + \varepsilon^*$ and, since $\varepsilon^* > 0$ and $\underline{u} \leq \overline{u}$ on $\partial\Omega$, it follows that $(x_\infty, y_\infty) \in \Omega$.

On the other hand, the function g is nonincreasing on $[\rho, +\infty)$ and $\overline{u} \geq \rho$ in Ω, whence $g(\overline{u}) \geq g(\overline{u} + \varepsilon^*)$ in Ω. From the Lipschitz continuity of g and from the assumptions on $\underline{u}$ and $\overline{u}$, the function $z = \overline{u} + \varepsilon^* - \underline{u}$ satisfies

$$\Delta z - c\partial_y z + \gamma(x, y) z \leq 0 \ \text{ in } \Omega$$

for some bounded function γ. Remember now that z is nonnegative and vanishes at the interior point $(x_\infty, y_\infty) \in \Omega$. The strong maximum principle implies that $z \equiv 0$ in $\overline{\Omega}$, which contradicts the boundary conditions on $\partial\Omega$.

[1] Theorem 1.7 in [15] dealt with problem (1.1) with conical conditions (1.5), for a nonlinearity f of the "combustion" type, namely $f = 0$ in $[0, \theta] \cup \{1\}$, $f > 0$ in $(\theta, 1)$ and $f'(1) < 0$, for $\theta \in (0, 1)$. But the method used in [15] only relies on the fact that the function f is nonincreasing in $[0, \delta]$ and $[1 - \delta, 1]$ for some $\delta > 0$, and can then be applied here with a bistable nonlinearity f.

Case 2: $x_n \to -\infty$ and $y_n + |x_n| \cot\alpha \to A \in [b, +\infty)$. Call $\underline{u}_n(x,y) = \underline{u}(x + x_n, y - |x_n|\cot\alpha)$ and $\overline{u}_n(x,y) = \overline{u}(x + x_n, y - |x_n|\cot\alpha)$. The functions $\underline{u}_n$ and $\overline{u}_n$ converge, up to extraction of some subsequence, to two $C^{2,\beta}$ functions $\underline{u}_\infty$ and $\overline{u}_\infty$ such that

$$\Delta\underline{u}_\infty - c\partial_y\underline{u}_\infty + g(\underline{u}_\infty) \geq \Delta\overline{u}_\infty - c\partial_y\overline{u}_\infty + g(\overline{u}_\infty) \text{ in } H = \{y > b + x\cot\alpha\},$$

$\underline{u}_\infty \leq \overline{u}_\infty$ on ∂H, $\underline{u}_\infty \leq \overline{u} + \varepsilon^*$ in $\overline{H}$, $\underline{u}_\infty(0,A) = \overline{u}_\infty(0,A) + \varepsilon^*$. Therefore, $(0,A) \in H$ and the function $z = \overline{u}_\infty + \varepsilon^* - \underline{u}_\infty$ is nonnegative in $\overline{H}$, vanishes at $(0,A) \in H$ and satisfies

$$\Delta z - c\partial_y z + \gamma(x,y)z \leq 0 \text{ in } H$$

for some bounded function γ. One concludes from the strong maximum principle that $z \equiv 0$ in $\overline{H}$, which is in contradiction with the boundary conditions on ∂H. In other words, case 2 is ruled out too.

As a consequence, $\varepsilon^* = 0$ and the proof of Lemma 2.6 is complete. □

Lemma 2.7 *Let $\underline{u}$ and $\overline{u}$ be two bounded $C^{2,\beta}$ functions (with $\beta > 0$) in the set $\overline{\Omega}$, where $\Omega = \{x < a,\ y < b - |x|\cot\alpha\}$ and $(a,b) \in \mathbb{R}^2$, satisfying*

$$\Delta\underline{u} - c\partial_y\underline{u} + g(\underline{u}) \geq \Delta\overline{u} - c\partial_y\overline{u} + g(\overline{u}) \quad \text{in } \Omega,$$

$\underline{u} \leq \overline{u}$ on $\partial\Omega$ and $\limsup_{A\to-\infty,\ y\leq A-|x|\cot\alpha}(\underline{u}(x,y) - \overline{u}(x,y)) \leq 0$. Furthermore, one assumes that there exists $\rho \in \mathbb{R}$ such that g is Lipschitz continuous on $\mathbb{R}$, nonincreasing on $(-\infty,\rho]$, and $\underline{u} \leq \rho$ in Ω. Then $\underline{u} \leq \overline{u}$ in $\overline{\Omega}$.

Proof. With the same type of arguments as in the proof of Lemma 2.6, one can show that $\underline{u} - \varepsilon \leq \overline{u}$ in $\overline{\Omega}$ for all $\varepsilon > 0$, which gives the desired conclusion. □

Let us now turn to the evenness and monotonicity (in x) properties stated in Proposition 2.1.

Lemma 2.8 *Under the assumptions of Proposition 2.1, there exists $x_0 \in \mathbb{R}$ such that $u(x_0 + x, y) = u(x_0 - x, y)$ for all $(x,y) \in \mathbb{R}^2$ and, if $\alpha < \pi/2$, $\partial_x u(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$ such that $x > x_0$.*

Proof. As already underlined, the function u only depends on y if $\alpha = \pi/2$. One then considers only the case $\alpha < \pi/2$.

Under the notations of Lemma 2.4, call

$$x_0 = \frac{t_- - t_+}{2\cos\alpha}.$$

Let $a < x_0$ be fixed and define $H = \{(x,y),\ x < a\}$ and $v(x,y) = u(2a - x, y)$. One shall now prove that $u > v$ in H.

From the limiting conditions (1.5), there exists $A > 0$ such that $u \geq 1 - \delta$ in $H \cap \{y > A - |x|\cot\alpha\}$ and $v \leq \delta$ in $H \cap \{y < -A - |x|\cot\alpha\}$, where $\delta > 0$ was chosen so that f is nonincreasing in $(-\infty,\delta]$ and $[1-\delta,+\infty)$.

Call now $u^\tau(x,y) = u(x, y+\tau)$ and choose any $\tau \geq 2A$. Notice that $u^\tau(a,y) > v(a,y)$ for all $y \in \mathbb{R}$ since $\partial_y u > 0$ in $\mathbb{R}^2$ and $\tau > 0$. Since both u and v satisfy (1.1) and (1.5), it is easy to check that Lemma 2.6 can be applied to $(\underline{u},\overline{u}) = (v,u^\tau)$ in the set

$\Omega_- = \{x < a,\ y < -A - |x| \cot\alpha\}$. Therefore, $v \le u^\tau$ in $\overline{\Omega_-}$. Similarly, Lemma 2.7 can be applied to the same pair of functions in the set $\Omega_+ = \{x < a,\ y > -A - |x| \cot\alpha\}$, whence $v \le u^\tau$ in $\overline{\Omega_+}$. As a consequence, $v \le u^\tau$ in $\overline{H}$ for all $\tau \ge 2A$.

Call now

$$\tau^* = \inf\ \{\tau > 0,\ v \le u^\tau \text{ in } \overline{H}\}$$

and assume that $\tau^* > 0$. By continuity, the function $z = u^{\tau^*} - v$ is nonnegative in H. Furthermore, it is positive on ∂H (since $\partial_y u > 0$ in $\mathbb{R}^2$ and $\tau^* > 0$) and it satisfies an equation of the type

$$\Delta z - c\partial_y z + b(x,y) z = 0 \ \text{ in } H$$

for some bounded function b. The strong maximum principle yields $z > 0$ in $\overline{H}$.

On the other hand, it follows from Lemma 2.4 that

$$z(x+x_n, y-|x_n|\cot\alpha) \to U(-x\cos\alpha+y\sin\alpha+t_-+\tau^*)-U(-x\cos\alpha+y\sin\alpha+2a\cos\alpha+t_+)$$

in $C^2_{loc}(\mathbb{R}^2)$ for any sequence $x_n \to -\infty$. The assumption made on a means that $2a\cos\alpha + t_+ < t_- < t_-+\tau^*$. Since U is increasing, one especially gets that $\lim_{x\to-\infty} z(x, y_0-|x|\cot\alpha) > 0$ for all $y_0 \in \mathbb{R}$.

It then follows that there exists $\tau_* \in (0,\tau^*)$ such that, for all $\tau \in [\tau_*, \tau^*]$, $u^\tau > v$ in $\{x \le a,\ -A-|x|\cot\alpha \le y \le A-|x|\cot\alpha\}$ and on ∂H. Let τ be any real number in $[\tau_*, \tau^*]$. Since u is increasing in y, $u^\tau \ge 1-\delta$ in $\{x \le a,\ y \ge A - |x|\cot\alpha\}$ and Lemma 2.6 can be applied to the pair $(\underline{u}, \overline{u}) = (v, u^\tau)$, whence $v \le u^\tau$ in this set. Similarly, one can check that Lemma 2.7 can be applied to the pair $(\underline{u}, \overline{u}) = (v, u^\tau)$ in the set $\{x \le a,\ y \le -A-|x|\cot\alpha\}$, whence $v \le u^\tau$ in this set. One concludes that $v \le u^\tau$ in $\overline{H}$ for all $\tau \in [\tau_*, \tau^*]$, which contradicts the minimality of τ^*.

As a consequence, $\tau^* = 0$ and $v \le u$ in $\overline{H}$. Call $w = u - v$. The function w is nonnegative in $\overline{H}$ and it vanishes on ∂H. Furthermore,

$$w(x+x_n, y-|x_n|\cot\alpha) \to U(-x\cos\alpha+y\sin\alpha+t_-)-U(-x\cos\alpha+y\sin\alpha+2a\cos\alpha+t_+) > 0$$

in $C^2_{loc}(\mathbb{R}^2)$ for any sequence $x_n \to -\infty$ (the positivity of the limit holds since U is increasing and $2a\cos\alpha + t_+ < t_-$). The strong maximum principle and Hopf lemma imply that $w > 0$ in H and $\partial_x w < 0$ on ∂H, whence

$$u(x,y) > u(2a-x,y) \ \text{ for all } x < a \text{ and } y \in \mathbb{R}, \quad \partial_x u(a,y) < 0 \ \text{ for all } y \in \mathbb{R},$$

for all $a < x_0$.

Similarly, by using the same sliding method in y, one can prove that

$$u(x,y) < u(2a'-x,y) \ \text{ for all } x < a' \text{ and } y \in \mathbb{R}, \quad \partial_x u(a',y) > 0 \ \text{ for all } y \in \mathbb{R},$$

for all $a' > x_0$. Passing to the limits $a \to x_0$ and $a' \to x_0$ yields that $u(x_0+x, y) = u(x_0-x, y)$ for all $(x,y) \in \mathbb{R}^2$ and the proof of Lemma 2.8 is complete. □

The above Lemmas 2.3, 2.4, 2.5 and 2.8 complete the proof of Proposition 2.1.

2.2 Proof of Theorem 1.6 under Hypothesis 1.4

The first lemmas (Lemma 2.9 to Lemma 2.17) give some properties which are satisfied under some weaker assumptions than Hypothesis 1.4, namely under condition (1.11) only. The next result (Lemma 2.18) assumes condition (1.9), and the last results hold when Hypothesis 1.4 is satisfied.

Lemma 2.9 *Let $N \geq 2$ and let u be a bounded solution of (1.1) satisfying (1.11) for some globally Lipschitz function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$. Then $0 < u < 1$ and*

$$\liminf_{A\to+\infty,\ y\geq A+\phi(x)} u(x,y) = 1, \qquad \limsup_{A\to-\infty,\ y\leq A+\phi(x)} u(x,y) = 0. \tag{2.2}$$

In other words, the function u converges to 0 and 1 uniformly as $y - \phi(x) \to -\infty$ and $+\infty$ respectively.

Proof. Let $M = \sup_{\mathbb{R}^N} u$ and $m = \inf_{\mathbb{R}^N} u$ and notice that $M > \theta > m$ from (1.11). The same arguments as in the proof of Lemma 2.2 can be applied here in any dimension $N \geq 2$ and they imply that $0 \leq m \leq M \leq 1$. The strong maximum principle then yields $0 < u < 1$ in $\mathbb{R}^N$.

Assume now that there exists $\varepsilon > 0$ and a sequence $(x_n, y_n) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $y_n - \phi(x_n) \to +\infty$ and $u(x_n, y_n) \leq 1 - \varepsilon$. Up to extraction of some subsequence, the functions $u_n(x, y) = u(x+x_n, y+y_n)$ converge in $C^2_{loc}(\mathbb{R}^N)$ to a solution u_∞ of (1.1) such that $u(0, 0) \leq 1-\varepsilon$. On the other hand, the assumption (1.11) and the fact that $y_n-\phi(x_n) \to +\infty$ imply that

$$m_\infty := \inf_{\mathbb{R}^N} u_\infty > \theta,$$

whence $\theta < m_\infty < 1$. Let (x'_n, y'_n) be a sequence such that $u_\infty(x'_n, y'_n) \to m_\infty$. Up to extraction of some subsequence, the functions $v_n(x, y) = u_\infty(x + x'_n, y + y'_n)$ converge in $C^2_{loc}(\mathbb{R}^N)$ to a solution v_∞ of (1.1) such that $m_\infty = v_\infty(0, 0) = \min_{\mathbb{R}^N} v_\infty$. Therefore, $f(m_\infty) \leq 0$, which contradicts the positivity of f on $(\theta, 1)$.

Therefore,

$$\liminf_{A\to+\infty,\ y\geq A+\phi(x)} u(x,y) = 1.$$

The uniform limit of u to 0 as $y-\phi(x) \to -\infty$ can be proved the same way. That completes the proof of Lemma 2.9. □

Remark 2.10 The same arguments lead to more general Liouville type results. Namely, let $(a_{ij})_{1\leq i,j\leq N}$ and $(b_i)_{1\leq i\leq N}$ be bounded in $C^{0,\beta}(\mathbb{R}^N)$ (with $\beta > 0$) and assume that $a_{ij}(z)\xi_i\xi_j \geq \nu|\xi|^2$ for all $z \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$ and for some $\nu > 0$ (under the usual summation convention of repeated indices). Let $\varepsilon > 0$ and $f : \mathbb{R}^N \times [-\varepsilon, \varepsilon]$, $(z, s) \mapsto f(z, s)$ be globally Lipschitz in s and Hölder continuous with exponent β in z. Assume that $\inf_{z\in\mathbb{R}^N} f(z, s) > 0$ for all $s \in [-\varepsilon, 0)$ and $\sup_{z\in\mathbb{R}^N} f(z, s) < 0$ for all $s \in (0, \varepsilon]$. Let u be a $C^{2,\beta}(\mathbb{R}^N)$ solution of $a_{ij}(z)\partial^2_{z_iz_j}u + b_i(z)\partial_{z_i}u + f(z, u) = 0$ in $\mathbb{R}^N$ such that $|u| \leq \varepsilon$. Then $u(z) = 0$ for all $z \in \mathbb{R}^N$.

Lemma 2.11 *Under the assumptions of Lemma 2.9, the function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y < -\cos\alpha_0$, where $\alpha_0 \in (0, \pi/2]$ and $\cot\alpha_0$ denotes the Lipschitz norm of ϕ.*

Proof. The proof is similar to that of Lemma 2.3. Namely, if τ satisfies the conditions of Lemma 2.11, then the function $v(X,Y) = u(x,y)$, where $(X,Y) = (\tau_y x - \tau_x y, -\tau_x \cdot x - \tau_y y) \in \mathbb{R}^{N-1} \times \mathbb{R}$, is such that

$$\liminf_{A \to +\infty,\ Y \geq A+\psi(X)} v(X,Y) = 1, \qquad \limsup_{A \to -\infty,\ Y \leq A+\psi(X)} v(X,Y) = 0$$

for some globally Lipschitz function ψ (because $\cot\alpha_0$ is the Lipschitz norm of ϕ, and $\tau_y < -\cos\alpha_0$). Theorem 1.5 in [15] yields $\partial_Y v > 0$ in $\mathbb{R}^N$. In other words, $\tau \cdot \nabla u < 0$ in $\mathbb{R}^N$. □

The following result is actually a more precise version of Lemma 2.11 in dimension $N = 2$.

Lemma 2.12 *Assume $N = 2$. Under the assumptions of Lemma 2.9, if the function ϕ is of class C^1 and $\tan\beta \leq \phi'(x) \leq \tan\gamma$ for all $x \in \mathbb{R}$, with $-\pi/2 < \beta \leq \gamma < \pi/2$, then u is decreasing in any direction $(\cos\varphi, \sin\varphi)$ such that $\gamma - \pi < \varphi < \beta$.*

Proof. It is similar to that of Lemma 2.11. Notice that, apart from the smoothness assumption of ϕ, Lemma 2.11, in dimension $N = 2$, is a special case of Lemma 2.12 with $\beta = \alpha_0 - \pi/2$ and $\gamma = \pi/2 - \alpha_0$. □

Let us now turn back to the general case of dimension $N \geq 2$.

Lemma 2.13 *Under the assumptions of Lemma 2.9, for each $\lambda \in (0,1)$, the level set $\{u(x,y) = \lambda\}$ is a graph $\{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$. All the functions ϕ_λ have the same Lipschitz norm, say $\cot\alpha$ with $\alpha \in (0,\pi/2]$. Lastly, the function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y < -\cos\alpha$, and u satisfies*

$$\liminf_{A \to +\infty,\ y \geq A+\phi_\lambda(x)} u(x,y) = 1, \qquad \limsup_{A \to -\infty,\ y \leq A+\phi_\lambda(x)} u(x,y) = 0 \tag{2.3}$$

for all $\lambda \in (0,1)$.

Proof. It follows from Lemmas 2.9 and 2.11 that, for each $\lambda \in (0,1)$, the level set $\{u(x,y) = \lambda\}$ is the graph $\{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$ of a globally Lipschitz function ϕ_λ, whose Lipschitz norm is denoted by $\cot\alpha_\lambda$ with $\alpha_\lambda \in (0,\pi/2]$.

Let $\lambda_0 \in (0,1)$ be fixed. Because of Lemma 2.9, the quantity $\sup_{x \in \mathbb{R}^{N-1}} |\phi(x) - \phi_{\lambda_0}(x)|$ is finite, and the function u then satisfies the limits (2.3) with ϕ_{λ_0}. The same arguments as in Lemma 2.11 imply that the function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y < -\cos\alpha_{\lambda_0}$. It especially follows that the Lipschitz norm $\cot\alpha_\lambda$ of the graph $\{y = \phi_\lambda(x)\}$ of any level set $\{u(x,y) = \lambda\}$ is such that $\cot\alpha_\lambda \leq \cot\alpha_{\lambda_0}$.

Since λ_0 was arbitrary in $(0,1)$, one concludes that $\cot\alpha_\lambda$ does not depend on λ. In other words, $\alpha = \alpha_\lambda$ does not depend on λ. □

Remark 2.14 By continuity, it especially follows that u is nonincreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y \leq -\cos\alpha$.

Lemma 2.15 *Under the assumptions of Lemma 2.9 and the notations of Lemma 2.13, then, for each $\lambda \in (0,1)$, the function ϕ_λ is of class $C^2(\mathbb{R}^{N-1})$ and its first- and second-order partial derivatives are globally bounded. Furthermore,*

$$\inf_{x \in \mathbb{R}^{N-1}} \partial_y u(x, \phi_\lambda(x)) > 0. \tag{2.4}$$

Proof. Let $\lambda \in (0,1)$ be fixed. Remember first that $\partial_y u > 0$ in $\mathbb{R}^N$. Assume now that there exists a sequence $(x_n) \in \mathbb{R}^{N-1}$ such that $\partial_y u(x_n, \phi_\lambda(x_n)) \to 0$ as $n \to +\infty$. Let

$$u_n(x,y) = u_n(x + x_n, y + \phi_\lambda(x_n)) \ \text{ and } \ \phi_n(x) = \phi_\lambda(x + x_n) - \phi_\lambda(x_n)$$

(notice that the functions ϕ_n are uniformly Lipschitz continuous). Up to extraction of some subsequence, the functions u_n (resp. ϕ_n) converge in $C^2_{loc}(\mathbb{R}^N)$ (resp. locally uniformly in $\mathbb{R}^{N-1}$) to a function u_∞ (resp. ϕ_∞) such that u_∞ solves (1.1) and (2.2) with ϕ_∞ instead of ϕ (because of the limits (2.3) for u). Furthermore, $\partial_y u_\infty(0,0) = 0$. The latter contradicts Lemma 2.11 applied to u_∞. Therefore, (2.4) follows.

Let us now observe that, since u is (at least) of class $C^2(\mathbb{R}^N)$ and $\partial_y u > 0$, it follows from the implicit function theorem that ϕ_λ is of class C^2 as well. A straightforward calculation leads to

$$\partial_{x_i x_j} \phi_\lambda(x) = -\frac{\partial_{x_i x_j} u + \partial_{x_i}\phi_\lambda(x)\partial_{x_j y} u + \partial_{x_j}\phi_\lambda(x)\partial_{x_i y} u + \partial_{x_i}\phi_\lambda(x)\partial_{x_j}\phi_\lambda(x)\partial_y^2 u}{\partial_y u}$$

for all $x \in \mathbb{R}^{N-1}$ and $1 \le i,j \le N-1$, where the function u and its derivatives are taken at $(x, \phi_\lambda(x))$. On the other hand, the function u is globally bounded in C^2 from standard elliptic estimates. Therefore, since $\nabla\phi_\lambda$ is bounded (by $\cot\alpha$ from Lemma 2.13) and $\inf_{x \in \mathbb{R}^{N-1}} \partial_y u(x, \phi_\lambda(x)) > 0$, it follows that $D^2\phi_\lambda$ is bounded as well. □

Lemma 2.16 *Under the assumptions of Lemma 2.9, for each $\lambda \in (0,1)$, there exist $0 < \beta \le \gamma$ such that*

$$\lambda\, e^{\gamma(y - \phi_\lambda(x))} \le u(x,y) \le \lambda\, e^{\beta(y-\phi_\lambda(x))}$$

for all $(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $y \le \phi_\lambda(x)$. Furthermore, the number γ can be chosen independently of λ.

Proof. First of all, it follows from standard elliptic estimates and Harnack inequality that $|\nabla u|/u$ is globally bounded in $\mathbb{R}^N$. Call

$$\gamma = \sup_{(x,y)\in\mathbb{R}^N} \frac{\partial_y u(x,y)}{u(x,y)}.$$

The real number γ is positive since $\partial_y u > 0$ in $\mathbb{R}^N$. It immediately follows that

$$\forall \lambda \in (0,1), \quad \forall y \le \phi_\lambda(x), \quad u(x,y) \ge \lambda\, e^{\gamma(y-\phi_\lambda(x))}.$$

Let $\eta \in (0,1)$ be fixed small enough so that $f(s) \le f'(0)s/2$ for all $s \in [0,\eta]$. One can assume that $\eta \le \delta$, so that f is nonincreasing in $(-\infty, \eta)$. The function $\overline{u}(x,y) = \eta\, e^{\beta(y-\phi_\eta(x))}$ satisfies

$$\Delta\overline{u} - c\partial_y\overline{u} + f(\overline{u}) \le \left(\beta^2 + \beta^2|\nabla\phi_\eta(x)|^2 - \beta\Delta\phi_\eta(x) - c\beta + \frac{f'(0)}{2}\right)\overline{u} \le 0 \ \text{ in } \{y \le \phi_\eta(x)\}$$

for $\beta > 0$ small enough (remember that $\nabla\phi_\eta$ and $\Delta\phi_\eta$ are bounded from Lemma 2.15). Let $\beta > 0$ be as such. It then follows from Lemma 5.1 in [15] that

$$u(x,y) \le \overline{u}(x,y) = \eta\, e^{\beta(y-\phi_\eta(x))} \ \text{ for all } (x,y) \in \mathbb{R}^{N-1}\times\mathbb{R} \text{ such that } y \le \phi_\eta(x). \tag{2.5}$$

Let now λ be any number in $(0,1)$. One claims that there exists $\beta_\lambda > 0$ such that

$$u(x,y) \le \lambda\ e^{\beta_\lambda (y-\phi_\lambda(x))} \tag{2.6}$$

for all (x,y) such that $y \le \phi_\lambda(x)$. Otherwise, there is a sequence of points (x_n,y_n) such that $y_n \le \phi_\lambda(x_n)$ and

$$u(x_n,y_n) > \lambda\ e^{(y_n-\phi_\lambda(x_n))/n}.$$

Up to extraction of some subsequence, three cases may occur:

Case 1: $y_n - \phi_\lambda(x_n) \to -\infty$. As already underlined, $\sup_{x\in\mathbb{R}^{N-1}} |\phi_\lambda(x) - \phi_\eta(x)| < +\infty$ and one then gets a contradiction with (2.5).

Case 2: $y_n - \phi_\lambda(x_n) \to h < 0$. Then $\liminf_{n\to+\infty} u(x_n,y_n) \ge \lambda$. On the other hand, since $\inf_{x\in\mathbb{R}^{N-1}} \partial_y u(x,\phi_\lambda(x)) > 0$, $\partial_y u > 0$ and $\partial^2_{yy} u$ is bounded in $\mathbb{R}^N$, it follows that $\limsup_{n\to+\infty} u(x_n,y_n) < \lambda$. Therefore, Case 2 is ruled out too.

Case 3: $y_n - \phi_\lambda(x_n) \to 0$ as $n \to +\infty$. One gets a contradiction by using the same arguments as in Case 2.

As a consequence, the claim (2.6) is proved and the proof of Lemma 2.16 is complete. □

Similarly, the following exponential bounds hold in the region where u is close to 1:

Lemma 2.17 *Under the assumptions of Lemma 2.9, for each $\lambda \in (0,1)$, there exist $0 < \beta \le \gamma$ such that*

$$(1-\lambda)e^{-\gamma(y-\phi_\lambda(x))} \le 1 - u(x,y) \le (1-\lambda)e^{-\beta(y-\phi_\lambda(x))}$$

for all $(x,y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ such that $y \ge \phi_\lambda(x)$. Furthermore, the number γ can be chosen independently of λ.

Let us now turn to the proof of the precise formula for the speed c under the assumption (1.9), stronger than (1.11):

Lemma 2.18 *Under the assumption (1.9) and the notations of Lemma 2.13, then $c = c_0/\sin\alpha$, where one recalls that c_0 is the unique speed of planar fronts for problem (1.4) with the nonlinearity f.*

Proof. It is based on an idea used in [16]. Call $\psi = \phi_{1/2}$. Let e be a given unit direction of $\mathbb{R}^{N-1}$ and let $\tilde\varphi : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by

$$\tilde\varphi(k+t) = \psi(ke) + t(\psi((k+1)e) - \psi(ke))$$

for all $k \in \mathbb{N}$ and $t \in [0,1)$. Lastly, let $\varphi(x) = \tilde\varphi(|x|)$ for all $x \in \mathbb{R}^{N-1}$. This function φ is clearly Lipschitz continuous and its Lipschitz norm $Lip(\varphi)$, denoted by $\cot\beta$ with $\beta \in (0,\pi/2]$, is such that $Lip(\varphi) = \cot\beta \le Lip(\psi) = \cot\alpha$ (in other words, $\beta \ge \alpha$).

Furthermore, calling ϕ the function given in (1.9), one has

$$\sup_{x\in\mathbb{R}^{N-1}} |\psi(x) - \phi(|x|)| < +\infty$$

because of (2.2) and (2.3). Therefore, $\sup_{t\ge0} |\tilde\varphi(t) - \phi(t)| < +\infty$ and $\sup_{x\in\mathbb{R}^{N-1}} |\varphi(x) - \phi(|x|)| < +\infty$ by radial symmetry. As a consequence,

$$\sup_{x\in\mathbb{R}^{N-1}} |\psi(x) - \varphi(x)| < +\infty$$

and the function u satisfies (2.3) with φ instead of $\phi_{1/2}$.

Therefore, the arguments in Lemma 2.11 imply that the function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y < -\cos\beta$. If β were (strictly) larger than α, each level curve of u would then have a Lipschitz norm less than or equal to $\cot\beta$, and then (strictly) less than $\cot\alpha$. The latter is in contradiction with Lemma 2.13. Therefore, $\beta = \alpha$ and the functions φ and ψ have the same Lipschitz norm, namely $\cot\alpha$.

By construction of φ, there exists then a sequence of integers $(k(n))_{n\in\mathbb{N}}$ such that

$$|\varphi((k(n)+1)e) - \varphi(k(n)e)| = |\psi((k(n)+1)e) - \psi(k(n)e)| \to \cot\alpha$$

as $n \to +\infty$. Up to extraction of some subsequence, two cases may occur:

Case 1: $\varphi((k(n)+1)e) - \varphi(k(n)e) = \psi((k(n)+1)e) - \psi(k(n)e) \to \cot\alpha$ as $n \to +\infty$. Up to extraction of another subsequence, the functions $u_n(x,y) = u(x+k(n)e, y+\psi(k(n)e))$ converge in $C^2_{loc}(\mathbb{R}^N)$ to a solution v of (1.1) such that $v(0,0) = v(e,\cot\alpha) = 1/2$.

By passage to the limit, the function v is nonincreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ such that $\tau_y \leq -\cos\alpha$. It especially follows that $v(te, t\cot\alpha) = 1/2$ for all $t \in [0,1]$.

For any $t \in (0,1]$, the function $w(x,y) = v(x,y) - v(x+te, y+t\cot\alpha)$ is nonpositive, it vanishes at $(0,0)$, and it satisfies an equation of the type $\Delta w - c\partial_y w + b(x,y)w = 0$ in $\mathbb{R}^N$, for some bounded function b (because f is Lipschitz continuous). The strong maximum principle implies that $w(x,y) = 0$ for all $(x,y) \in \mathbb{R}^N$.

Since $t \in (0,1]$ was arbitrary, one concludes that v is constant in the direction $(e,\cot\alpha)$. Furthermore, it follows that $k(n) \to +\infty$: if not, one could have assumed that the sequence $(k(n))$ was bounded (up to extraction of some subsequence), and then the function u itself would have been constant along the direction $(e,\cot\alpha)$, which is in contradiction with the fact that ϕ is radial in (1.9) and then in (2.2).

Since $k(n) \to +\infty$ as $n \to +\infty$ and ϕ is radial in (1.9) and (2.2), there exists then a globally Lipschitz function $\phi_\infty : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \liminf\limits_{A\to+\infty,\ y\geq A+\phi_\infty(x\cdot e)} v(x,y) = 1 \\ \limsup\limits_{A\to-\infty,\ y\leq A+\phi_\infty(x\cdot e)} v(x,y) = 0. \end{cases}$$

Since $v(te, t\cot\alpha) = 1/2$ for all $t \in \mathbb{R}$, one gets that $\sup_{x\in\mathbb{R}^{N-1}} |\phi_\infty(x\cdot e) - x\cdot e\ \cot\alpha| < +\infty$, whence

$$\begin{cases} \liminf\limits_{A\to+\infty,\ y\geq A+x\cdot e\ \cot\alpha} v(x,y) = 1 \\ \limsup\limits_{A\to-\infty,\ y\leq A+x\cdot e\ \cot\alpha} v(x,y) = 0. \end{cases} \tag{2.7}$$

The arguments used in Lemmas 2.3 and 2.11 then imply that the function v is increasing in any direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y > 0$ and $\tau_x \cdot e = 0$. Fix any such $\tau_x \in \mathbb{R}^{N-1}$ such that $\tau_x \cdot e = 0$ and consider the directions $\tau_\pm = (\tau_x, \pm\tau_y)$ with $\tau_y > 0$. The function v is increasing in both directions τ_+ and $-\tau_-$. Letting $\tau_y \to 0^+$ implies that v is constant in the direction $(\tau_x, 0)$. Therefore, v does not depend in the directions of $\mathbb{R}^{N-1}$ which are orthogonal to e. On the other hand, one has already got that v was constant in the direction $(e,\cot\alpha)$. In other words, there exists a function $v_0 : \mathbb{R} \to [0,1]$ such that

$$v(x,y) = v_0(-x\cdot e\ \cos\alpha + y\sin\alpha)$$

for all $(x, y) \in \mathbb{R}^N$. As a consequence, the function $v_0 = v_0(\xi)$ satisfies

$$v_0'' - c \sin\alpha \; v_0' + f(v_0) = 0, \quad 0 \le v_0 \le 1 \quad \text{ in } \mathbb{R}$$

together with $v_0(-\infty) = 0$, $v_0(+\infty) = 1$ (because of (2.7)). The classical uniqueness result for the above equation yields $c \sin\alpha = c_0$.

Case 2: $\varphi((k(n)+1)e) - \varphi(k(n)e) = \psi((k(n)+1)e) - \psi(k(n)e) \to -\cot\alpha$ as $n \to +\infty$. This case can be treated similarly and leads to the conclusion that $c \sin\alpha = c_0$. □

Remark 2.19 Under the assumptions of Lemma 2.18, it follows from Lemmas 2.13 and 2.18 that, if $\alpha = \pi/2$ (namely, the level sets of u are hyperplanes orthogonal to the y direction), then $u(x, y) \equiv U(y + \tau)$ for some $\tau \in \mathbb{R}$ (from the uniqueness result for problem (1.4). One can then assume that $\alpha < \pi/2$ in the sequel.

The above arguments lead to the

Proof of Remark 1.7. Assume that there exists a solution (c, u) of (1.1), (1.9) such that $\phi'(r) \to -\cot\alpha$ as $r \to +\infty$, with $\alpha \in (\pi/2, \pi)$. Let e be a given unit direction of $\mathbb{R}^{N-1}$ and call $u_n(x, y) = u(x + r_n e, y + \phi(r_n))$ for some sequence $r_n \to +\infty$. With the same arguments as in Lemma 2.18 above, the functions u_n converge, up to extraction of some subsequence, to a function v satisfying (2.7), whence $c = c_0/\sin\alpha > c_0$.

Therefore, the function u satisfies

$$\Delta u - c_0 \partial_y u + f(u) = (c - c_0)\partial_y u > 0 \;\text{ in } \mathbb{R}^N,$$

while $U(x, y) = U(y)$ satisfies $\Delta U - c_0\partial_y U + f(U) = 0$. From (1.9) and (2.2), there exists $A \ge 0$ such that $u(x, y) \le \delta$ for all $x \in \mathbb{R}^{N-1}$ and $y \le -A$, and $U(x, y) = U(y) \ge 1 - \delta$ for all $y \ge A$. Using the same type of weak maximum principle as in Lemmas 2.6 and 2.7 (or, more precisely, Lemmas 5.1 and 5.2 in [15]), one gets that $U(y + \tau) \ge u(x, y)$ for all $x \in \mathbb{R}^{N-1}$, for all $y \le -A$ and $y \ge -A$, and for all $\tau \ge 2A$. Let now τ^* be the real number defined by

$$\tau^* = \inf \{\tau \in \mathbb{R},\; U(y + \tau) \ge u(x, y) \text{ for all } (x, y) \in \mathbb{R}^N\}.$$

If $\inf_{x\in\mathbb{R}^{N-1},\; -A\le y\le A}(U(y + \tau^*) - u(x, y)) > 0$, then there exists $\eta > 0$ such that $U(y + \tau) \ge u(x, y)$ for all $(x, y) \in \mathbb{R}^N$ and for all $\tau \in [\tau^* - \eta, \tau^*]$,[2] which contradicts the minimality of τ^*. Therefore, $\inf_{x\in\mathbb{R}^{N-1},\; -A\le y\le A}(U(y + \tau^*) - u(x, y)) = 0$ and the asymptotic conditions (2.2) (with $\phi = \phi(|x|)$, $\phi'(r) \to \cot\alpha$ as $r \to +\infty$ and $\alpha \in (\pi/2, \pi)$) yield the existence of a point $(x_0, y_0) \in \mathbb{R}^N$ such that $U(y_0 + \tau^*) = u(x_0, y_0)$. The nonnegative function $z(x, y) = U(y + \tau^*) - u(x, y)$ vanishes at (x_0, y_0) and satisfies an inequation of the type $\Delta z - c_0\partial_y z + b(x, y)z \le 0$ in $\mathbb{R}^N$. The strong maximum principle implies that $z \equiv 0$, which is in contradiction with the assumption that $\phi'(r) \to \cot\alpha > 0$ as $r \to +\infty$ in (1.9). □

The next results in this section assume that Hypothesis 1.4 is satisfied.

Lemma 2.20 *Under the Hypothesis 1.4 of Theorem 1.6 and the notations of Lemmas 2.13 and 2.18, then* $\hat{x} \cdot \nabla\phi_\lambda(x) \to -\cot\alpha$ *as* $|x| \to +\infty$, *for all* $\lambda \in (0, 1)$, *where* $\hat{x} = x/|x|$.

[2]One applies Lemmas 5.1 and 5.2 of [15] in $\{y \le -A\}$ and in $\{y \ge A\}$ (even if it means assuming that, say, $u(x, y) \ge 1 - \delta/2$ for all $y \ge A$ to make sure that $U(y + \tau) \ge 1 - \delta$ if $y \ge A$ for τ close to τ^*).

Proof. Let $\lambda \in (0,1)$ be fixed. We first recall that $|\nabla\phi_\lambda(x)| \le \cot\alpha$ for all $x \in \mathbb{R}^{N-1}$. As noticed in Remark 2.19, one can assume that $\alpha < \pi/2$ (the conclusion of Lemma 2.20 clearly holds if $\alpha = \pi/2$). Call

$$m = \limsup_{|x|\to+\infty} \ \hat{x}\cdot\nabla\phi_\lambda(x), \quad M = \liminf_{|x|\to+\infty} \ \hat{x}\cdot\nabla\phi_\lambda(x).$$

Since $-\cot\alpha \le m \le \cot\alpha$, three cases may occur:

1. Let us prove that we cannot have $-\cot\alpha < m < \cot\alpha$. Let (x_n) be a sequence such that $|x_n| \to +\infty$ and $\hat{x_n}\cdot\nabla\phi_\lambda(x_n) \to m$ as $n \to +\infty$, and call

$$u_n(x,y) = u(x+x_n, y+\phi_\lambda(x_n)) \ \text{ and } \ \phi_n(x) = \phi_\lambda(x+x_n) - \phi_\lambda(x_n).$$

From standard elliptic estimates, the functions u_n converge in $C^2_{loc}(\mathbb{R}^N)$, up to extraction of some subsequence, to a solution u_∞ of (1.1). One can also assume that $\hat{x_n} \to e \in \mathbb{S}^{N-2}$ as $n \to +\infty$. From Lemma 2.15, the functions ϕ_n converge in $C^1_{loc}(\mathbb{R}^{N-1})$, up to extraction of a subsequence, to a C^1 function ϕ_∞ such that $|\nabla\phi_\infty| \le \cot\alpha$,

$$\forall x \in \mathbb{R}^{N-1}, \quad -\cot\alpha \le e\cdot\nabla\phi_\infty(x) \le m$$

and $e\cdot\nabla\phi_\infty(0) = m$. Furthermore, the function u_∞ solves (2.2) with ϕ_∞ and $u_\infty(x,\phi_\infty(x)) = \lambda$ for all $x \in \mathbb{R}^{N-1}$.

Call $\gamma = \arctan(m)$. The same arguments as in the proof of Lemma 2.18 imply that the function u_∞ can be written as a function of $x\cdot e$ and y only, namely $u_\infty(x,y) = v(x\cdot e, y)$. Therefore, $\phi_\infty(x) = \tilde{\phi}(x\cdot e)$. On the other hand, since $-\cot\alpha = \tan(\alpha-\pi/2) \le \tilde{\phi}' \le m = \tan\gamma$, it then follows from Lemma 2.12 that the function v is decreasing in any direction $(\cos\varphi, \sin\varphi)$ such that $\gamma - \pi < \varphi < \alpha - \pi/2$. By continuity, $\partial_\rho v \le 0$ in $\mathbb{R}^2$, where $\rho = (\cos(\gamma-\pi), \sin(\gamma-\pi)) = -(\cos\gamma, \sin\gamma)$. But since $\tilde{\phi}'(0) = m = \tan\gamma$ and $\{y = \tilde{\phi}(x\cdot e)\}$ is a level curve of v, one concludes that $\partial_\rho v(0,0) = 0$. The nonpositive function $z := \partial_\rho v$ satisfies an elliptic equation with continuous coefficients, and $z(0,0) = 0$. It follows from the strong maximum principle that $z \equiv 0$ in $\mathbb{R}^2$. With the same arguments as in Lemma 2.18, one concludes, in both cases $\gamma \ge 0$ or $\gamma \le 0$, that

$$c \ \sin(\pi/2 - |\gamma|) = c_0 \ \ (= c \ \sin\alpha). \tag{2.8}$$

But $-\cot\alpha < \tan\gamma < \cot\alpha$ from our assumption in Case 2, whence $\pi/2 \ge \pi/2 - |\gamma| > \alpha$ (> 0). Since $c_0 \ne 0$, one gets a contradiction with (2.8).

Similarly, we cannot have $-\cot\alpha < M < \cot\alpha$.

2. Let us rule out the case $M = m = \cot\alpha$. To do so we notice that, should it be the case, a translate of $U(y)$ would be above u by Lemma 2.6 - with $\alpha = 0$. Translating back until we cannot do so anymore, Lemma 2.6 once again implies the existence of a contact point between u and the ultimate translate of U. THis is a contradiction.

3. Let us finaly rule out the case $M = -m = \cot\alpha$. Should it be true, we claim that we could find two sequences $(x_n)_n$ and $(R_n)_n$ such that $\lim_{n\to+\infty} |x_n| = \lim_{n\to+\infty} R_n = +\infty$, and

$$\phi(|x_n|) = \inf_{r\in(|x_n|-R_n, |x_n|+R_n)} \phi(r).$$

Indeed, choose any $\lambda \in (0,1)$. Arguing as in **1**, if $(x_n)_n$ is a sequence such that $\lim_{n\to+\infty} |x_n| = +\infty$, and $\phi'_\lambda(|x_n|) \to m$, then there is a sequence $(\rho_n)_n$ going to $+\infty$ such that $\phi'_\lambda > \dfrac{3m+M}{4}$

on $[|x_n|-\rho_n, |x_n|+\rho_n]$. Similarly, if $(\tilde{x}_n)_n$ is a sequence such that $\lim_{n\to+\infty} |\tilde{x}_n| = +\infty$, and $\phi'_\lambda(|\tilde{x}_n|) \to M$ there is $\tilde{\rho}_n$ going to $+\infty$ such that $\phi'_\lambda < \dfrac{3M+m}{4}$ on $[|\tilde{x}_n|-\tilde{\rho}_n, |\tilde{x}_n|+\tilde{\rho}_n]$. Consequently, the claim is true for the function ϕ_λ, hence for ϕ since $\sup_x |\phi_\lambda(x)-\phi(|x|)| < +\infty$.

Now, the sequence $(u(x_n+x, \phi(|x_n|)+y))_n$ converges, up to a subsequence, to a function u_∞ which satisfies, uniformly in $x \in \mathbb{R}^{N-1}$: $\lim_{y\to-\infty} u(x,y) = 0$. Arguing as in **2** leads to a contradiction. □

Lemma 2.21 *Under the Hypothesis 1.4 of Theorem 1.6, for any unit vector e of $\mathbb{R}^{N-1}$, the functions $u_n(x,y) = u(x+r_ne, y+\phi_\lambda(r_ne))$ converge in $C^2_{loc}(\mathbb{R}^N)$ to $U(x\cdot e\ \cos\alpha + y\sin\alpha + U^{-1}(\lambda))$ for all sequence $(r_n) \to +\infty$ and for all $\lambda \in (0,1)$.*

Proof. The proof uses the same arguments as in the second part of Case 3 of Lemma 2.20 above and is not repeated here. □

The above results actually complete the proof of part 2) of Theorem 1.6.

2.3 Proof of Theorem 1.6 under Hypothesis 1.5

This section gives the proof of part 1) of Theorem 1.6, namely that Hypothesis 1.5 implies Hypothesis 1.4 for a nonconstant bounded solution u of (1.1) in $\mathbb{R}^N$.

Let us begin this section with a Liouville type result about the subsolutions $0 \le u \le \theta$ of $\Delta u + f(u) \ge 0$ in $\mathbb{R}^N$.

Proposition 2.22 *Let $0 \le u \le \theta$ be a C^2 function satisfying $\Delta u + f(u) \ge 0$ in $\mathbb{R}^N$ (in any dimension $N \ge 1$). Then, either $u \equiv 0$ or $u \equiv \theta$ in $\mathbb{R}^N$.*

Proof. Assume that $\inf_{\mathbb{R}^N} u < \theta$. The strong maximum principle yields $u < \theta$ in $\mathbb{R}^N$. Let φ_R and λ_R be the first eigenfunction and first eigenvalue of

$$\begin{cases} -\Delta\varphi_R &= \lambda_R\varphi_R & \text{in } B_R \\ \varphi_R &> 0 & \text{in } B_R \\ \varphi_R &= 0 & \text{on } \partial B_R, \end{cases}$$

where $B_R \subset \mathbb{R}^N$ is the open euclidean ball with center 0 and radius $R > 0$. Let $R > 0$ be chosen large enough so that $\lambda_R \le f'(\theta)/2$ (this is possible since $f'(\theta) > 0$).

Choose now $\eta > 0$ small enough so that $u < \theta - \eta\varphi_R$ in $\overline{B_R}$ and $f(\theta-\eta\varphi_R) \le -\eta\varphi_R f'(\theta)/2$ in B_R. The function $v := \theta - \eta\varphi_R$ then satisfies

$$\Delta v + f(v) \le \eta\lambda_R\varphi_R - \eta\varphi_R f'(\theta)/2 \le 0 \ \text{ in } B_R$$

and $v = \theta$ on ∂B_R.

Let z_0 be any vector in $\mathbb{R}^N$. From the local uniform continuity of u, there exists $\kappa > 0$ such that $u(\cdot + tz_0) < v$ in $\overline{B_R}$ for all $t \in [0,\kappa]$. Call

$$t^* = \sup\{t \in [0,+\infty),\ u(\cdot + t'z_0) < v \text{ in } \overline{B_R} \text{ for all } t' \in [0,t]\}.$$

One has $0 < \kappa \le t^* \le +\infty$. Assume $t^* < +\infty$. Then, $u(\cdot + t^* z_0) \le v$ in $\overline{B_R}$ and there exists $z^* \in \overline{B_R}$ such that $u(z^* + t^* z_0) = v(z^*)$. Since $v = \theta$ on ∂B_R and $u < \theta$ in $\mathbb{R}^N$, it follows that $z^* \in B_R$. On the other hand,

$$\Delta u(\cdot + t^* z_0) + f(u(\cdot + t^* z_0)) \ge 0 \ge \Delta v + f(v) \ \text{ in } B_R.$$

Hence, there exists a bounded function b such that the function $w := v - u(\cdot + t^* z_0)$ satisfies $\Delta w + bw \le 0$ in B_R. Since w is nonnegative and vanishes at the point $z^* \in B_R$, the strong maximum principle yields $w \equiv 0$ in $\overline{B_R}$. This is impossible because $v = \theta$ on ∂B_R and $u < \theta$ in $\mathbb{R}^N$.

Therefore, $t^* = +\infty$. Since $z_0 \in \mathbb{R}^N$ was arbitrary, one gets that $u(z) < v(0) < \theta$ for all $z \in \mathbb{R}^N$. As a consequence, $\sup_{\mathbb{R}^N} u < \theta$. Since $f < 0$ in $(0, \theta)$, one can prove as in Lemma 2.2 that $\sup_{\mathbb{R}^N} u \le 0$, whence $u \equiv 0$ in $\mathbb{R}^N$. That completes the proof of Proposition 2.22. □

Remark 2.23 Similarly, the following result holds. Let $\theta \le u \le 1$ be a C^2 function satisfying $\Delta u + f(u) \le 0$ in $\mathbb{R}^N$ (in any dimension $N \ge 1$). Then, either $u \equiv \theta$ or $u \equiv 1$ in $\mathbb{R}^N$.

Let us now turn to the proof of Theorem 1.6 under Hypothesis 1.5. The proof is divided into several lemmas.

Lemma 2.24 *Let $N \ge 2$ and let u be a bounded nonconstant solution of (1.1) with some speed $c \in \mathbb{R}$. Assume that Hypothesis 1.5 is satisfied. Then $0 < u < 1$ in $\mathbb{R}^N$ and $u(x, y) \to 1$ (resp. $u(x, y) \to 0$) as $y \to +\infty$ (resp. $y \to -\infty$) locally uniformly in $x \in \mathbb{R}^{N-1}$. Furthermore, $\partial_y u > 0$ in $\mathbb{R}^N$.*

Proof. First, the arguments used in Lemma 2.2, which can be used in any dimension N, imply that $0 < u < 1$ in $\mathbb{R}^N$. Observe now that $\Delta u + f(u) = c\partial_y u \ge 0$ in $\mathbb{R}^N$ from Hypothesis 1.5. It then follows from Proposition 2.22 that $\sup_{\mathbb{R}^N} u > \theta$.

Let us now prove that $u(x, +\infty) = 1$ and $u(x, -\infty) = 0$ for all $x \in \mathbb{R}$. Since $0 < u < 1$ and u is nondecreasing with respect to the variable y, there exist two functions $0 \le u_\pm(x) \le 1$ such that $u(x, y) \to u_\pm(x)$ as $y \to \pm\infty$, for all $x \in \mathbb{R}^{N-1}$. From standard elliptic estimates, the functions $u(x, y + y_0)$ converge to $u_\pm(x)$ as $y_0 \to \pm\infty$ in $C^2_{loc}(\mathbb{R}^N)$ and the functions $u_\pm$ satisfy

$$\Delta_x u_\pm + f(u_\pm) = 0 \ \text{ in } \mathbb{R}^{N-1}.$$

Let us now prove that $u_- \equiv 0$. One first notices that $0 \le u_- < 1$ in $\mathbb{R}^{N-1}$ and that u_- can be written as $u_-(x) = \tilde{u}_-(|x|)$ by Hypothesis 1.5. Furthermore, $\tilde{u}_-'(r) \ge 0$ for $r \ge 0$. Call $l = \tilde{u}_-(+\infty) \in [0, 1]$. From standard elliptic estimates, $\tilde{u}_-'(r) \to 0$ as $r \to +\infty$ and l is a zero of f, namely $l = 0$, $l = \theta$ or $l = 1$. If $l = 0$, then $u_- \equiv 0$, which is the desired result.

If $l = 1$, multiply the equation

$$\tilde{u}_-''(r) + \frac{N-2}{r}\tilde{u}_-'(r) + f(\tilde{u}_-(r)) = 0, \ r > 0$$

by $\tilde{u}_-'(r)$ and integrate on $(0, +\infty)$. It follows that

$$\int_{u_-(0)}^{1} f(s)ds = -\int_0^{+\infty} \frac{(N-2)(\tilde{u}_-'(r))^2}{r} dr \le 0.$$

But $0 \leq u_-(0) < 1$ and the assumptions on the profile of f ($f < 0$ on $(0, \theta)$, $f > 0$ on $(\theta, 1)$ and $\int_0^1 f > 0$) lead to a contradiction.

If $l = \theta$, then $0 \leq u_- \leq \theta$ in $\mathbb{R}^{N-1}$ and Proposition 2.22 yields $u_- \equiv 0$ or $u_- \equiv \theta$. The latter is impossible because $\inf_{\mathbb{R}^N} u < \theta$ whence $\inf_{\mathbb{R}^{N-1}} u_- < \theta$.

One concludes that $u_- \equiv 0$ in $\mathbb{R}^{N-1}$. With similar arguments, one infers that $u_+ \equiv 1$ in $\mathbb{R}^{N-1}$.

The nonnegative function $\partial_y u$ satisfies an elliptic equation with continuous coefficients, and it is not identically 0. Therefore, $\partial_y u > 0$ in $\mathbb{R}^N$ from the strong maximum principle. □

Lemma 2.25 *Under the assumptions of Lemma 2.24, each level set $\{(x, y),\ u(x, y) = \lambda\}$, with $\lambda \in (0, 1)$, is a graph $\{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$. Furthermore,*

$$\inf_{x \in \mathbb{R}^{N-1}} \partial_y u(x, \phi_\lambda(x)) > 0$$

for all $\lambda \in (0, \theta) \cup (\theta, 1)$.

Proof. First, it follows from Lemma 2.25 that each level set of u, $\{(x, y) \in \mathbb{R}^N,\ u(x, y) = \lambda\}$, for $\lambda \in (0, 1)$, is a graph $\{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$.

Let now $\lambda \in (0, \theta)$. Assume that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{N-1}$ such that $\partial_y u(x_n, \phi_\lambda(x_n) \to 0$ as $n \to +\infty$. Let e be any fixed unit vector in $\mathbb{R}^{N-1}$. Since u only depends on $|x|$ and y, one can assume that $x_n = r_n e$ with $r_n \geq 0$. Furthermore, $r_n \to +\infty$ because $\partial_y u$ is continuous and positive in $\mathbb{R}^N$. From standard elliptic estimates, the functions

$$u_n(x, y) = u(x + x_n, y + \phi_\lambda(x_n))$$

converge in $C^2_{loc}(\mathbb{R}^N)$ to a solution u_∞ of (1.1) such that $0 \leq u_\infty \leq 1$, $u_\infty(0, 0) = \lambda$, $\partial_y u_\infty \geq 0$ and $\partial_y u_\infty(0, 0) = 0$. Therefore, $\partial_y u_\infty \equiv 0$ from the strong maximum principle. On the other hand, since $r_n \to +\infty$ and u depends on $|x|$ and y only, the function u_∞ eventually depends on $x \cdot e$ only. Namely, $u_\infty(x, y) = v(x \cdot e)$ and v satisfies

$$v''(\xi) + f(v(\xi)) = 0,\ \xi \in \mathbb{R}. \tag{2.9}$$

Furthermore, $v' \geq 0$ in $\mathbb{R}$ because $\partial_{|x|} \tilde{u}(|x|, y) \geq 0$ in $\mathbb{R}^N$ and $r_n \to +\infty$. Call $l_\pm = v(\pm\infty) \in [0, 1]$. Standard elliptic estimates yield $f(l_\pm) = 0$ and $v'(\xi) \to 0$ as $|\xi| \to +\infty$. Moreover, $0 \leq l_- \leq \lambda < \theta$ and $0 < \lambda \leq l_+ \leq 1$. Therefore, $l_- = 0$ and $l_+ = \theta$ or 1. In both cases, multiply (2.9) by v' and integrate over $\mathbb{R}$. It follows that $\int_0^{l_+} f = 0$, which is impossible due to the profile of f.

That shows that $\inf_{x \in \mathbb{R}^{N-1}} \partial_y u(x, \phi_\lambda(x)) > 0$ for all $\lambda \in (0, \theta)$. Lastly, the same result holds similarly with $\lambda \in (\theta, 1)$. □

Lemma 2.26 *Under the assumptions of Lemma 2.24 and the notations of Lemma 2.25, one has $\inf_{x \in \mathbb{R}^{N-1}} \partial_y u(x, \phi_\theta(x)) > 0$.*

Proof. Assume that the conclusion does not hold and let e be a given unit vector of $\mathbb{R}^{N-1}$. As in the proof of Lemma 2.25, there exists then a sequence $r_n \to +\infty$ such that the functions $u_n(x, y) = u(x + r_n e, y + \phi_\theta(r_n e))$ converge in $C^2_{loc}(\mathbb{R}^N)$, up to extraction of some subsequence, to a function $u_\infty(x, y) = v(x \cdot e)$. The function v satisfies (2.9) in $\mathbb{R}$ and $0 \leq v \leq 1$, $v' \geq 0$, $v(0) = \theta$. Since $f(v(\pm\infty)) = 0$ and $\int_{v(-\infty)}^{v(+\infty)} f = 0$, it follows that $v(\pm\infty) = \theta$, namely $v \equiv \theta$.

In other words, the functions u_n converge locally uniformly to the constant θ. Fix now any $\lambda \in (0,\theta)$. It then follows that $\phi_\theta(r_n e) - \phi_\lambda(r_n e) \to +\infty$ as $n \to +\infty$ and, for any compact set $K \subset \mathbb{R}^N$,

$$\limsup_{n\to+\infty} \max_{(x,y)\in K} u(x + r_n e, y + \phi_\lambda(r_n e)) \le \theta.$$

As a consequence, the functions $w_n(x,y) = u(x + r_n e, y + \phi_\lambda(r_n e))$ converge in $C^2_{loc}(\mathbb{R}^N)$, up to extraction of some subsequence, to a function w_∞ satisfying (1.1) and $0 \le w_\infty \le \theta$ in $\mathbb{R}^N$. Furthermore, $w_\infty(0,0) = \lambda$ and $\partial_y w_\infty \ge 0$. Since $c \ge 0$, one has $\Delta w_\infty + f(w_\infty) \ge 0$ and one gets a contradiction with Proposition 2.22.

That completes the proof of Lemma 2.26. □

The same arguments as in the previous lemma lead to the following

Corollary 2.27 *Under the assumptions of Lemma 2.24 and the notations of Lemma 2.25, one has* $\inf_{\lambda\in[\lambda_1,\lambda_2],\ x\in\mathbb{R}^{N-1}} \partial_y u(x,\phi_\lambda(x)) > 0$ *for all* $0 < \lambda_1 \le \lambda_2 < 1$.

The Lipschitz-continuity of the functions ϕ_λ also follow from the previous results:

Lemma 2.28 *Under the assumptions of Lemma 2.24 and the notations of Lemma 2.25, for each* $\lambda \in (0,1)$*, the level curve* ϕ_λ *is globally Lipschitz-continuous.*

Proof. Fix $\lambda \in (0,1)$. It first follows from the implicit function theorem that ϕ_λ is of class C^2. Furthermore, $|\nabla u|$ is globally bounded in $\mathbb{R}^N$ from standard elliptic estimates. Since

$$\nabla\phi_\lambda(x) = -\frac{\nabla_x u(x,\phi_\lambda(x))}{\partial_y u(x,\phi_\lambda(x))}$$

for all $x \in \mathbb{R}^{N-1}$, it follows from Lemmas 2.25 and 2.26 that $\nabla\phi_\lambda$ is globally bounded in $\mathbb{R}^{N-1}$. □

Lemma 2.29 *Under the assumptions of Lemma 2.24 and the notations of Lemma 2.25, let* ϕ *be the function defined in* $\mathbb{R}_+$ *by* $\phi(|x|) = \phi_\theta(x)$ *for all* $x \in \mathbb{R}^{N-1}$*. Then (1.9) holds.*

Proof. From Corollary 2.27, let m be the positive number defined by

$$m := \inf_{\lambda\in[\theta/2,(1+\theta)/2],\ x\in\mathbb{R}^{N-1}} \partial_y u(x,\phi_\lambda(x)) > 0.$$

The mean value theorem yields

$$\forall x \in \mathbb{R}^{N-1}, \quad \frac{1+\theta}{2} - \theta = u(x,\phi_{(1+\theta)/2}(x)) - u(x,\phi_\theta(x)) \ge m(\phi_{(1+\theta)/2}(x) - \phi_\theta(x)).$$

Hence, $0 \le \phi_{(1+\theta)/2}(x) - \phi_\theta(x) \le (1-\theta)/(2m)$ for all $x \in \mathbb{R}^{N-1}$. Therefore,

$$u(x,y) \ge \frac{1+\theta}{2}$$

for all $x \in \mathbb{R}^{N-1}$ and $y \ge \phi_\theta(x) + (1-\theta)/(2m)$ (because $\partial_y u > 0$ in $\mathbb{R}^N$).

Similarly, one can prove that

$$u(x,y) \le \frac{\theta}{2}$$

for all $x \in \mathbb{R}^{N-1}$ and $y \le \phi_\theta(x) - \theta/(2m)$.

As a consequence, the function u satisfies (1.9) with ϕ as defined in Lemma 2.29. $\square$

End of the proof of Theorem 1.6 under Hypothesis 1.5. Let u be a bounded nonconstant solution of (1.1), with $c \in \mathbb{R}$. Assume that Hypothesis 1.5 is satisfied. It follows from the previous Lemmas that u satisfies (1.9) with ϕ as defined in Lemma 2.29. Furthermore, ϕ is globally Lipschitz-continuous because of Lemma 2.28. As a consequence, Hypothesis 1.4 is satisfied and u satisfies all the properties listed in Section 2.2. $\square$

Remark 2.30 *It can be checked that, with the above arguments and with some adaptations of the comparison results in [15], Theorem 1.6 still holds if, in Hypothesis 1.4, the Lipschitz continuity of ϕ is replaced with the weaker assumption that* $\sup_{|x-x'|\le d} |\varphi(x) - \varphi(x')| < \infty$ *for all $d \ge 0$ (globally bounded local oscillation).*

3 Existence results

The first subsection gives a proof of Theorem 1.1 in dimension $N = 2$. The second subsection gives a proof of Theorem 1.2 in dimensions $N \ge 3$, but the proof also works for $N = 2$.

3.1 Existence for $N = 2$ via the sub/super-solutions method

What follows is an adaptation to the bistable case of the proof of existence of a solution in the case with ignition temperature given in [17].

Let $\alpha \in (0, \pi/2]$ be given. We are looking for a solution ϕ of (1.1), i.e.

$$\Delta\phi - c\partial_y\phi + f(\phi) = 0, \quad 0 < \phi < 1 \quad \text{in} \quad \mathbb{R}^2$$

with $c = c_0/\sin\alpha$, satisfying the conditions (1.5) at infinity, i.e.

$$\begin{cases} \lim\limits_{y_0\to+\infty} \inf\limits_{y\ge y_0-|x|\cot\alpha} \phi(x,y) = 1 \\ \lim\limits_{y_0\to-\infty} \sup\limits_{y\le y_0-|x|\cot\alpha} \phi(x,y) = 0 \end{cases}$$

The strategy to prove Theorem 1.1 is to build a solution ϕ between a sub- and a supersolution in the whole plane $\mathbb{R}^2$.

We perform the proof in three steps.

Step 1: *Construction of a subsolution.* A natural candidate for a subsolution is the following function:

$$\underline{\phi}(x,y) = \sup(\phi_0(y\sin\alpha + x\cos\alpha), \phi_0(y\sin\alpha - x\cos\alpha)).$$

Moreover, by construction we have

$$\lim_{y_0\to-\infty} \sup_{\{y\le y_0-|x|\cot\alpha\}} \underline{\phi}(x,y) = 0 \tag{3.1}$$

and

$$\lim_{y_0\to+\infty}\ \inf_{\{y\geq y_0-|x|\cot\alpha\}} \underline{\phi}(x,y)=1. \tag{3.2}$$

Step 2: *Construction of a supersolution.* On the contrary, the construction of a supersolution which is above the subsolution is a nontrivial fact, and requires the use of the solution ψ to an associated free boundary problem.

We define the candidate for the supersolution as (choosing ϕ_0 such that $\phi_0(0)=\theta$, and $\phi_{00}(s)=\theta e^{c_0 s}$):

$$\overline{\phi}(x,y)=\left\{\begin{array}{l}\phi_0(\phi_{00}^{-1}(\theta\ \psi(x,y)))\ \ \text{in } \Omega:=\{\psi<1\}\\ \phi_0(\text{dist}((x,y),\Omega))\ \ \text{in } \mathbb{R}^2\backslash\Omega\end{array}\right.$$

where dist denotes the euclidean distance function and ψ is the unique (up to shift) solution to the following free boundary problem (*see* [16]):

Theorem 3.1 (A free boundary problem,[3] [16]) *For $\alpha\in(0,\dfrac{\pi}{2}]$, $c_0>0$ and $c=c_0/\sin\alpha$, there exists a function ψ satisfying*

$$\left\{\begin{array}{l}\Delta\psi-c\partial_y\psi=0\quad \textit{in } \Omega:=\{\psi<1\},\\ 0<\psi\leq 1\quad \textit{in } \mathbb{R}^2,\\ \dfrac{\partial\psi}{\partial n}=c_0\quad \textit{on } \Gamma:=\partial\Omega,\\ \displaystyle\lim_{y\to-\infty}\ \sup_{\mathcal{C}^-(y,\alpha)}\ \psi=0,\\ \psi=1\quad \textit{in } \mathcal{C}^+(y_0,\pi-\alpha)\quad \textit{for some } y_0\in\mathbb{R},\end{array}\right. \tag{3.3}$$

where $\frac{\partial\psi}{\partial n}$ stands for the normal derivative on Γ of the restriction of ψ to $\overline{\Omega}$. Furthermore, ψ is continuous in $\mathbb{R}^2$, the set $\Gamma=\partial\Omega$ is a C^∞ graph $\Gamma=\{y=\varphi(x),\ x\in\mathbb{R}\}$ such that

$$\sup_{x\in\mathbb{R}}|\varphi(x)+|x|\cot\alpha|<+\infty,$$

Ω is the subgraph $\Omega=\{y<\varphi(x)\}$, the restriction of ψ is C^∞ in $\overline{\Omega}$, and $|\varphi'(x)|\leq\cot\alpha$ in $\mathbb{R}$. Lastly, ψ is nondecreasing in y, even in x and satisfies

$$\partial_x\psi(x,y)\geq 0\quad \textit{for } x\geq 0,\ y<\varphi(x).$$

From Theorem 3.1 and denoting $\gamma_0(x)=-|x|\cot\alpha$, it is easy to see that there exist two positive constants r_0 and C such that

$$\forall r\geq r_0,\quad \underline{\phi}^r(x,y):=\underline{\phi}(x,y-r)\leq\theta\ \ \text{in } \overline{\Omega}$$

and

$$\text{dist}((x,y),\Omega))\geq -C+(y-\gamma_0(x))\sin\alpha\ \ \text{in } \mathbb{R}^2\backslash\Omega=\{\psi=1\}.$$

Because of (3.1), and from the comparison principles proved in [15], it follows that $\underline{\phi}^r\leq\overline{\phi}$ in Ω for all $r\geq r_0$ and then, by construction of $\overline{\phi}$, we get that

$$\underline{\phi}^r\leq\overline{\phi}\ \ \text{in } \mathbb{R}^2$$

[3] This problem arises in models of equidiffusional premixed Bunsen flames in the limit of high activation energy. The existence of a solution ψ of problem (3.3) can be obtained by regularizing approximations, starting from solutions of problems of the type (1.1) with nonlinearities f_ε approximating a Dirac mass at 1.

as soon as $r \geq \max(r_0, C/\sin\alpha)$.

Moreover, notice that the construction of $\overline{\phi}$ implies that

$$\lim_{y_0 \to -\infty} \sup_{\{y \leq y_0 - |x| \cot\alpha\}} \overline{\phi}(x, y) = 0. \tag{3.4}$$

Proposition 3.2 *The function $\overline{\phi}$ is a supersolution of (1.1) in the vicosity sense.*

The proof is postponed and let us first complete the proof of Theorem 1.1.

Step 3: *Existence of a solution.* Choose a real number r such that $r \geq \max(r_0, C/\sin\alpha)$. By using the Perron method for viscosity solutions (*see* [10] and H. Ishii [22], Theorem 7.2 page 41), we get the existence of a vicosity solution ϕ of $\Delta\phi - c\partial_y\phi + f(\phi) = 0$, which satisfies:

$$0 \leq \underline{\phi}^r \leq \phi \leq \overline{\phi} \leq 1 \quad \text{in } \mathbb{R}^2.$$

Now by the regularity theory for viscosity solutions (see [9]), it follows that ϕ is $C^{2+\beta}$ (with $\beta > 0$), and then ϕ is a classical solution of (1.1). Finally ϕ satisfies the conditions at infinity (1.5) because of (3.2) and (3.4). This completes the proof of Theorem 1.1. □

The proof of Proposition 3.2 is based on the following result:

Lemma 3.3 *Let ξ_0 be the function defined by*

$$\xi_0(x, y) = \phi_{00}^{-1}(\theta\ \psi(x, y)) \quad \textit{in } \overline{\Omega} = \{y \leq \varphi(x)\},$$

where ψ is the solution to the free boundary problem given by Theorem 3.1, and $\phi_{00}(s) = \theta e^{c_0 s}$. Then

$$|\nabla\xi_0| \leq 1 \quad \textit{in } \overline{\Omega}.$$

Proof. We have

$$\begin{cases} \Delta\xi_0 + c_0\left(|\nabla\xi_0|^2 - \dfrac{\partial_y\xi_0}{\sin\alpha}\right) = 0 \ \text{ in } \Omega = \{\xi_0 < 0\}, \\ \xi_0 = 0 \ \text{ and } \dfrac{\partial\xi_0}{\partial n} = 1 \ \text{ on } \Gamma = \partial\{\xi_0 < 0\} \end{cases}$$

since $\phi_{00}(s) = \theta e^{c_0 s}$ for all $s \leq 0$. A straightforward computation gives, for $v = |\nabla\xi_0|^2$:

$$\Delta v + b \cdot \nabla v = 2|D^2\xi_0|^2,$$

where $b = 2c_0\nabla\xi_0 - c_0/\sin\alpha\ e_y$ and $e_y = (0, 1)$.

Let us define $M = \sup v$. We want to prove that $M \leq 1$. Let us assume that $M > 1$. We know that $v = 1$ on Γ and $v(x, y) \to 1$ as $|x| \to +\infty$ and $d((x, y), \Gamma)$ stays bounded. From the maximum principle we conclude that there exists a sequence of points (x_n, y_n) such that $v(x_n, y_n) \to M$, $d((x_n, y_n), \Gamma) \to +\infty$, and the sequence of functions $v_n(x, y) = v(x_n + x, y_n + y)$ converges to the function $v_\infty(x, y)$ which from the strong maximum principle satisfies $v_\infty(x, y) \equiv M$. Moreover $\xi_{0,n}(x, y) = \xi_0(x_n + x, y_n + y) - \xi_0(x_n, y_n)$ converges to a function $\xi_{0,\infty}(x, y)$ such that $v_\infty = |\nabla\xi_{0,\infty}|^2$, and $D^2\xi_{0,\infty} \equiv 0$.

On the other hand, the following function

$$w(x, y) = e^{c_0(y\sin\alpha + x\cos\alpha)} + e^{c_0(y\sin\alpha - x\cos\alpha)} = e^{c_0\sin\alpha(y - g(x)))}$$

is a solution of the equation $\Delta w - \frac{c_0}{\sin\alpha}\partial_y w = 0$ on the whole space. As a consequence of the comparison principle on the globally Lipschitz subgraph $\Omega = \{y < \varphi(x)\}$, we deduce that there exists two constants $y_1 > y_2$ such that

$$e^{c_0 \sin\alpha(y-g(x)-y_1)} \leq \psi \leq e^{c_0 \sin\alpha(y-g(x)-y_2)} \quad \text{in} \quad \Omega$$

We deduce that the function $\xi_0 = \frac{1}{c_0}\ln(\psi) + \frac{1}{c_0}\ln\theta$ satisfies:

$$\begin{aligned}
&\sin\alpha[y - (g(x_n+x) - g(x_n)) + y_2 - y_1] \\
&\leq \xi_0(x_n+x, y_n+y) - \xi_0(x_n, y_n) \\
&\leq \sin\alpha[y - (g(x_n+x) - g(x_n)) + y_1 - y_2]
\end{aligned}$$

and then

$$|\xi_0(x_n+x, y_n+y) - \xi_0(x_n, y_n)| \leq |y_1 - y_2|\sin\alpha + \sqrt{x^2+y^2}$$

because by construction we have $|g'| \leq \cot\alpha$. Therefore at the limit we get

$$|\xi_{0,\infty}(x,y)| \leq |y_1 - y_2|\sin\alpha + \sqrt{x^2+y^2}$$

with $\xi_{0,\infty}(0,0) = 0$. Because $M > 1$, this is in contradiction with $\nabla\xi_{0,\infty} \equiv \nu\sqrt{M}$ for a constant vector ν satisfying $||\nu|| = 1$. This ends the proof of the lemma. □

Let us now turn to the

Proof of Proposition 3.2. Let us define

$$I[u] := \Delta u - c\partial_y u + f(u).$$

A straightforward calculation shows that $\overline{\phi}$ is a classical super-solution of $I[\overline{\phi}] \leq 0$ in the open set $\Omega = \{\overline{\phi} < \theta\}$. Moreover the gradient of $\overline{\phi}$ is continuous across $\Gamma = \partial\{\overline{\phi} < \theta\}$, which is smooth.

Let us now consider the function $\xi(x,y) = \phi_0^{-1}(\overline{\phi}(x,y))$, now defined in the whole plane $\mathbb{R}^2$. We have

$$J[\xi] := \frac{I[\overline{\phi}]}{\phi_0'(\xi)} = \Delta\xi + c_0\left(|\nabla\xi|^2 - \frac{\partial_y\xi}{\sin\alpha}\right) + G(\xi)\left(1 - |\nabla\xi|^2\right) \tag{3.5}$$

in the viscosity sense in $\mathbb{R}^2$, where $G(\xi) = f(\phi_0(\xi))/\phi_0'(\xi)$. Putting together Lemma 3.3 and the fact that $G(\xi) \leq 0$ for $\xi \leq 0$, the end of the proof follows exactly the same lines as in the proof of Proposition 2.2 in [17] and is not repeated here. □

3.2 Existence of a cylindrical solution in $\mathbb{R}^N$ with $N \geq 2$

This subsection is devoted to the proof of Theorem 1.2. The proof is based on the existence of solutions of some approximated problems in bounded cylinders. Then, a passage to the limit in the whole space $\mathbb{R}^N$ gives the existence of a solution in the sense of Theorem 1.2.

Let us first begin with an auxiliary comparison principle in bounded cylinders.

Lemma 3.4 *Let ω be a bounded connected open subset of $\mathbb{R}^{N-1}$ and let $a<b$ be two real numbers. Call $\Omega=\omega\times(a,b)$ (with the generic notations (x,y) for the points in Ω). Let $c\in\mathbb{R}$ and let $f:\mathbb{R}\to\mathbb{R}$ be a Lipschitz-continuous function. Let $\underline{u}$ and $\overline{u}$ be two functions of class C^2 in Ω and continuous in $\overline{\Omega}$. Assume that*

$$\Delta\overline{u}-c\partial_y\overline{u}+f(\overline{u})\le\Delta\underline{u}-c\partial_y\underline{u}+f(\underline{u})\quad in\ \Omega$$

and $\underline{u}\le\overline{u}$ on $\partial\Omega$. Assume furthermore that $\underline{u}(x,y)$ and $\overline{u}(x,y)$ are increasing in the variable $y\in(a,b)$ for all $x\in\partial\omega$. Lastly, assume that $\underline{u}(x,a)<\overline{u}(x,y)$ for all $(x,y)\in\overline{\omega}\times(a,b]$ and that $\underline{u}(x,y)<\overline{u}(x,b)$ for all $(x,y)\in\overline{\omega}\times[a,b)$.

Then $\underline{u}\le\overline{u}$ in $\overline{\Omega}$.

Proof. This result is similar to some classical results in [6] (see also [15]) and it uses a sliding method in the direction y. For the sake of completeness, the proof is given here.

For $\lambda\in(0,b-a)$, call $\Omega_\lambda=\omega\times(a,a+\lambda)$ and

$$u^\lambda(x,y)=\overline{u}(x,y+b-a-\lambda).$$

Both functions $\underline{u}$ and u^λ are then defined and continuous (at least) in $\overline{\Omega_\lambda}$ and they are of class C^2 in Ω_λ. It follows from the assumptions of Lemma 3.4 that $\underline{u}<u^\lambda$ in $\overline{\Omega_\lambda}$ for $\lambda>0$ small enough.

Set

$$\lambda^*=\sup\{\lambda\in(0,b-a),\ \underline{u}<u^\mu\text{ in }\overline{\Omega_\mu}\text{ for all }\mu\in(0,\lambda)\}>0$$

and assume that $\lambda^*<b-a$. One has $\underline{u}\le u^{\lambda^*}$ in $\overline{\Omega_{\lambda^*}}$ and there exists $(x,y)\in\overline{\Omega_{\lambda^*}}$ such that $\underline{u}(x,y)=u^{\lambda^*}(x,y)=\overline{u}(x,y+b-a-\lambda^*)$. If $(x,y)\in\partial\omega\times(a,a+\lambda^*)$, then $a<y<y+b-a-\lambda^*<b$ and, from the assumptions of Lemma 3.4,

$$\underline{u}(x,y)\le\overline{u}(x,y)<\overline{u}(x,y+b-a-\lambda^*)=u^{\lambda^*}(x,y),$$

which is impossible. If $y=a$, then

$$\underline{u}(x,y)=\underline{u}(x,a)<\overline{u}(x,b-\lambda^*)=u^{\lambda^*}(x,a)=u^{\lambda^*}(x,y),$$

which is again impossible. Similarly, the case where $y=a+\lambda^*$ is impossible.

Therefore, $(x,y)\in\Omega_{\lambda^*}$. The function $z=u^{\lambda^*}-\underline{u}$ is nonnegative and continuous in $\overline{\Omega_{\lambda^*}}$, of class C^2 in Ω_{λ^*}. Furthermore, z satisfies an inequation of the type

$$\Delta z-c\partial_y z+\zeta(x,y)z\le 0\quad\text{in }\Omega_{\lambda^*}$$

for some bounded function ζ. Since z vanishes at the interior point $(x,y)\in\Omega_{\lambda^*}$, the strong maximum principle then yields $z\equiv 0$ in $\overline{\Omega_{\lambda^*}}$. But one can check as above that $z>0$ on $\partial\Omega_{\lambda^*}$. One has then reached a contradiction.

Therefore, $\lambda^*=b-a$, whence $\underline{u}\le\overline{u}$ in $\overline{\Omega}$. □

The arguments used in the proof above imply the following

Corollary 3.5 *Let ω be a bounded connected open subset of $\mathbb{R}^{N-1}$ and let $a<b$ be two real numbers. Call $\Omega=\omega\times(a,b)$. Let $c\in\mathbb{R}$ and let $f:\mathbb{R}\to\mathbb{R}$ be a Lipschitz-continuous function. Let u be a function of class C^2 in Ω and continuous in $\overline{\Omega}$. Assume that*

$$\Delta u-c\partial_y u+f(u)=0\quad in\ \Omega$$

and that $u(x,y)$ is increasing in the variable $y \in (a,b)$ for all $x \in \partial\omega$. Lastly, assume that $u(x,a) < u(x,y) < u(x,b)$ for all $(x,y) \in \Omega$.

Then u is increasing in the variable y.

Let us now come back to our problem (1.1) and let us construct approximated solutions u in a bounded cylinders. To be more precise, let R and L be two positive real numbers and call $\Omega_{R,L} = B_R \times (-L, L)$, where B_R is the open euclidean ball of $\mathbb{R}^{N-1}$ with radius R and center 0.

Fix $\alpha \in (0, \pi/2)$ and call

$$c = \frac{c_0}{\sin\alpha},$$

where c_0 is the unique planar front velocity, given in (1.4).

From [5], it is known that, for all $a > 0$, there exists a unique solution (c_a, u_a) of

$$u_a'' - c_a u_a' + f(u_a) = 0 \ \text{ in } [-a,a], \quad u_a(-a) = 0,\ u_a(0) = \theta,\ u_a(a) = 1,$$

where u_a is of class $C^2([-a,a])$, $0 < u_a < 1$ in $(-a,a)$, $u_a' > 0$ in $[-a,a]$. Furthermore, as $a \to +\infty$, $c_a \to c_0$ and $u_a \to U$ (the unique solution of (1.4) such that $U(0) = \theta$). The convergence $u_a \to U$ holds in $C^{2,\beta}_{loc}(\mathbb{R})$ for all $0 \le \beta < 1$.

Consider now the following problem

$$\left\{ \begin{array}{rcl} \Delta u - c\partial_y u + f(u) &=& 0 \ \text{ in } \Omega_{R,L} \\ u(x,y) &=& u_L(y) \ \text{ for all } (x,y) \in \partial\Omega_{R,L} \end{array} \right. \tag{3.6}$$

The constant function 0 is clearly a subsolution of this problem. On the other hand, the function $u_L(x,y) = u_L(y)$ satisfies

$$\Delta u_L - c\partial_y u_L + f(u_L) = (c_L - c)u_L' < 0 \ \text{ in } \Omega_{R,L}$$

for L large enough (indeed $u_L' > 0$ in $[-L,L]$ and $c_L - c \to c_0 - c_0/\sin\alpha < 0$ as $L \to +\infty$).

In the sequel, one assumes that $L > 0$ is large enough so that $c_L - c < 0$. There exists then a classical solution u of (3.6) such that

$$0 \le u(x,y) \le u_L(y) \ \text{ for all } (x,y) \in \overline{\Omega_{R,L}}.$$

The strong maximum principle then yields

$$0 < u(x,y) < u_L(y) \ (< 1) \ \text{ for all } (x,y) \in \Omega_{R,L}. \tag{3.7}$$

Lemma 3.6 *Under the above notations, the function u (solving (3.6) and (3.7)) is unique and increasing in the variable y.*

Proof. Since $u_L' > 0$ in $[-L,L]$ and u satisfies (3.7), the monotonicity of u in y is a consequence of Corollary 3.5.

If v is another solution of (3.6) satisfying (3.7), then call $\underline{u} = u$ and $\overline{u} = v$ and apply Lemma 3.4. It follows that $u \le v$ in $\overline{\Omega_{R,L}}$. Reversing the roles of u and v yields $v \le u$, whence $u = v$, in $\overline{\Omega_{R,L}}$. □

Lemma 3.7 *Under the assumptions of Lemma 3.6, the function u only depends on $|x|$ and y, namely $u(x,y) = \tilde{u}(|x|, y)$, and $\partial_{|x|}\tilde{u}(|x|, y) > 0$ for all $0 < |x| < R$ and $y \in (-L, L)$.*

Proof. Fix a unit vector e in $\mathbb{R}^{N-1}$ and, for $a \in [0, R)$, call $\omega_a = \{x \in B_R,\ x \cdot e > a\}$. Let u_a be the function defined in $\overline{\omega_a} \times [-L, L]$ by

$$u_a(x, y) = u(x + 2(a - x \cdot e)e, y).$$

The function u_a is the orthogonal reflection, for any given y, of the function u with respect to the hyperplane $H_a = \{x \in \mathbb{R}^{N-1},\ x \cdot e = a\}$.

Let $a \in [0, R)$. The function u_a is still a solution of $\Delta u_a - c\partial_y u_a + f(u_a) = 0$ in $\omega_a \times (-L, L)$. Furthermore, because of (3.7) and since u and u_L are increasing in y, it is easy to check that all the assumptions of Lemma 3.4 are satisfied for $\underline{u} = u_a$ and $\overline{u} = u$ in $\overline{\omega_a} \times [-L, L]$. Therefore, $u_a \leq u$ in $\overline{\omega_a} \times [-L, L]$.

Moreover, if $a > 0$ and $(x, y) \in (\partial\omega_a \backslash H_a) \times (-L, L)$, one has $(x+2(a-x\cdot e)e, y) \in \Omega_{R,L}$, whence

$$u_a(x, y) = u(x + 2(a - x \cdot e)e, y) < u_L(y) = u(x, y).$$

The strong maximum principle then yields $u_a < u$ in $\omega_a \times (-L, L)$. But since $u_a = u$ on $(B_R \cap H_a) \times (-L, L)$, it follows from Hopf lemma that

$$e \cdot \nabla_x u_a < e \cdot \nabla_x u \ \text{ on } (B_R \cap H_a) \times (-L, L).$$

Owing to the definition of u_a, one has $e \cdot \nabla_x u_a = -e \cdot \nabla_x u$, whence $e \cdot \nabla_x u > 0$ on $(B_R \cap H_a) \times (-L, L)$.

On the other hand, the case $a = 0$ implies that $u_a \leq u$ in $\overline{\omega_0} \times [-L, L]$. By choosing $-e$ instead of e, one gets that

$$u(x, y) = u(x - 2(x \cdot e)e, y) \ \text{ for all } (x, y) \in \overline{\Omega_{R,L}}.$$

Since e was an arbitrary unit vector in $\mathbb{R}^{N-1}$, one concludes that u only depends on $|x|$ and y. The monotonicity in $|x|$ follows from the above arguments. □

Remark 3.8 Since u_L is a supersolution of (3.6) and $u < u_L$ in $\Omega_{R,L}$ with equality on $\partial\Omega_{R,L}$, the Hopf lemma actually implies that $\partial_{|x|}\tilde{u}(R, y) > 0$ for all $y \in (-L, L)$.

Next, one shall pass to the limit as $L \to +\infty$.

Lemma 3.9 *Call $u^{R,L}$ the unique solution of (3.6) and (3.7) given in Lemma 3.6. There exists a sequence $(L_n)_{n\in\mathbb{N}} \to +\infty$ such that $u^{R,L_n} \to u^R$ in $C^{2,\beta}_{loc}(\overline{B_R} \times \mathbb{R})$ for all $0 \leq \beta < 1$, where u^R solves*

$$\text{(3.8)} \qquad \begin{cases} \Delta u^R - c\partial_y u^R + f(u^R) = 0 & \text{in } B_R \times \mathbb{R} \\ u^R(x, y) = U(y) & \text{for all } (x, y) \in \partial B_R \times \mathbb{R} \end{cases}$$

Furthermore,

$$\text{(3.9)} \qquad \begin{cases} 0 < u^R(x, y) \leq U(y) < 1 \ \text{ for all } (x, y) \in \overline{B_R} \times \mathbb{R} \\ u^R \text{ is increasing in } y \\ u^R(x, y) = \tilde{u}^R(|x|, y) \text{ and } \partial_{|x|}\tilde{u}^R(|x|, y) > 0 \text{ for all } 0 < |x| \leq R \\ u(x, -\infty) = 0 \text{ and } u(x, +\infty) = 1 \ \text{ for all } x \in \overline{B_R}. \end{cases}$$

Proof. The convergence, for some sequence $L_n \to +\infty$, is a consequence of standard elliptic estimates and the diagonal extraction process. The limiting function u^R immediately satisfies (3.8). Furthermore, $0 \le u^R \le U(y)$ in $\overline{B_R} \times \mathbb{R}$ because of (3.7) and because $u_L \to U$ in $C^2_{loc}(\mathbb{R})$ as $L \to +\infty$. Since $u^R = U(y) > 0$ on $\partial B_R \times \mathbb{R}$, the strong maximum principle then yields $u^R > 0$ in $\overline{B_R} \times \mathbb{R}$. Similarly,

$$u^R(x,y) < U(y) \quad \text{for all } (x,y) \in B_R \times \mathbb{R} \tag{3.10}$$

because U is a strict supersolution of (3.8).

By passage to the limit, the function u^R is nondecreasing in y and since u^R is increasing in y on $\partial B_R \times \mathbb{R}$, it follows from the strong maximum principle that u^R is increasing in y in the whole cylinder $\overline{B_R} \times \mathbb{R}$.

Similarly, the function u^R is a function of $|x|$ and y only, namely $u^R(x,y) = \tilde{u}^R(|x|,y)$ in $\overline{B_R} \times \mathbb{R}$ and $\tilde{u}^R$ is nondecreasing in $|x|$. Let e be a given unit direction of $\mathbb{R}^{N-1}$. Under the same notations as in Lemma 3.7, one has

$$u^R(x + 2(a - x\cdot e)e, y) \le u^R(x,y) \quad \text{for all } (x,y) \in \overline{\omega_a} \times \mathbb{R}$$

and for all $0 \le a < R$. Furthermore, if $a > 0$, the above inequality is strict on $(\partial\omega_a \backslash H_a) \times \mathbb{R}$ because of (3.8) and (3.10). The strong maximum principle and the Hopf lemma then imply that $u^R(x+2(a-x\cdot e)e, y) < u^R(x,y)$ in $\omega_a \times \mathbb{R}$ and $e \cdot \nabla u^R > 0$ on $(B_R \cap H_a) \times \mathbb{R}$, provided $a > 0$. Therefore, as in Lemma 3.7, one concludes that $\partial_{|x|}\tilde{u}^R > 0$ for all $0 < |x| < R$, and also for $|x| = R$ as in Remark 3.8.

From the monotonicity of u^R in y, there exist two functions $u^R_\pm$ defined in $\overline{B_R}$ such that $u^R(x,y) \to u^R_\pm(x) = \tilde{u}^R_\pm(|x|)$ as $y \to \pm\infty$. Furthermore, the convergence holds in $C^{2,\beta}_{loc}(\overline{B_R})$ (for all $0 \le \beta < 1$) from standard elliptic estimates. The functions $u^R_\pm$ satisfy

$$\Delta u^R_\pm + f(u^R_\pm) = 0 \quad \text{in } \overline{B_R}$$

and $0 \le u^R_- \le u^R_+ \le 1$ in $\overline{B_R}$. Since $u^R(x,y) \le U(y)$ in $\overline{B_R} \times \mathbb{R}$ and $U(-\infty) = 0$, one immediately gets that $u^R_- \equiv 0$ in $\overline{B_R}$. On the other hand, $u^R_+(x) = U(+\infty) = 1$ for all $x \in \partial B_R$. The function $v(r) := \tilde{u}^R_+(r)$ satisfies

$$v''(r) + \frac{N-2}{r} v'(r) + f(v(r)) = 0, \quad 0 < r \le R,$$

$0 \le v \le 1$, $v' \ge 0$ in $[0,R]$, $v'(0) = 0$ and $v(R) = 1$. Multiply the above equation by v' and integrate in $[0,R]$. It follows that

$$\frac{1}{2} v'(R)^2 + \int_{v(0)}^1 f(s)ds \le 0,$$

whence $\int_{v(0)}^1 f \le 0$. It follows from the profile of f that $v(0) = 1$. Consequently, $v \equiv 1$ and $u^R_1 \equiv 1$ in $\overline{B_R}$. That completes the proof of Lemma 3.9. □

End of the proof of Theorem 1.2. Let (R_n) be a sequence converging to $+\infty$ and let $u_n = u^{R_n}$. Up to a shift in the y variable, one can assume that $u_n(0,0) = \theta/2$. From standard elliptic estimates, the functions u_n converge in $C^{2,\beta}_{loc}(\mathbb{R}^N)$ (for all $0 \le \beta < 1$), up to extraction of some subsequence, to a solution u of (1.1) such that $0 \le u \le 1$, $u(0,0) = \theta/2$, $\partial_y u \ge 0$, $u(x,y) = \tilde{u}(|x|,y)$ with $\partial_{|x|}\tilde{u}(|x|,y) \ge 0$ for all $(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R}$.

The function u then satisfies Hypothesis 1.5 and is not constant (because $f(\theta/2) \neq 0$). It follows from Theorem 1.6 that u satisfies all the properties listed in this theorem as well as in Theorem 1.2. □

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