Uniform Hölder estimates in a class of elliptic systems and applications to singular limits in models for diffusion flames

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ABSTRACT. – The main result of this paper is a general Hölder estimate in a class of singularly perturbed elliptic systems. This estimate is applied to the study of the well-known Burke-Schuman approximation in flame theory. After reviewing some classical cases (equidiffusional case; high activation energy approximation) we turn to the non-equidiffusional case, and to nonlinear diffusion models. The limiting problems are nonlinear elliptic equations; they have Hölder or Lipschitz maximal global regularity. A natural question is then whether this regularity is kept uniformly throughout the approximation process, and we show that this is true in general.

Key words: Hölder continuity, singular perturbed elliptic systems, diffusion flames.

1. Introduction and main results

The goal of this paper is the proof of a Hölder estimate for a special class of singularly perturbed elliptic systems, and its applications to the mathematically rigorous study of singular limits in the theory of diffusion flames.

1.1. The estimate

Consider the following elliptic system, with unknowns $(Y_1, ..., Y_p)$:

$$\forall k \in [0,p], \qquad L_k Y_k = -\frac{A_k(x)}{\varepsilon} F(Y), \tag{1.1}$$

where the notations are the following:
• the variable x is in an open set of $\mathbb{R}^N$, $N \geq 3$, and the functions A_k are smooth and nonnegative.
• The function $F(Y)$ - here the vector Y is the vector of all components $(Y_1, ..., Y_p)$ is smooth, nonnegative, and nonzero except if one of the components of Y is zero. Moreover, F is homogeneous: there is a p-uple $(a_k)_{1 \leq k \leq p}$ such that

$$\forall (Y_1, ..., Y_p) \in (\mathbb{R}_+)^k, \quad F(Y_1, ..., Y_p) = \prod_{k=1}^{p} Y_k^{a_k}. \tag{1.2}$$

In the applications we will have, most of the time: $p = 2$ and $F(Y) = Y_1Y_2$.

• The operator L_k is the operator $L_k = -\partial_i(a^k_{ij}\partial_j) + b^k_i\partial_i$ where the a^k_{ij} and b^k_i are bounded, measurable, and satisfy the usual ellipticity condition $a_{ij}\xi_i\xi_j \geq C|\xi|^2$.

The parameter ε may go to 0; in the applications to diffusion flames it will represent the inverse of the Damköhler number, a parameter accounting for the strength of the chemical reaction. Our main result is the

Theorem 1.1 *Assume $Y = (Y_1, ..., Y_p)$ to solve (1.1) in B_1, and that $0 \leq Y_1, ..., Y_p \leq 1$. There is $\alpha \in]0, 1[$ and $C > 0$, uniform in ε, such that*

$$\|Y\|_{C^\alpha(B_{1/2})} \leq C \ \mathrm{osc}_{B_1} Y. \tag{1.3}$$

As usual, we denote by $B_r(x)$ the ball with centre x and radius r, and B_r the same ball but with centre $x = 0$. We note here that $N \geq 3$ is a commodity assumption that could easily be relaxed.

Estimates of the type of Theorem 1.1 can be found in [10], where a singularly perturbed system of the form

$$-\Delta u_i = -\frac{u_i}{\varepsilon}\sum_{j\neq i} u_j$$

is analysed. The singular limit is a free boundary problem, and the solution $U^\varepsilon = (u_1, ..., u_p)$ is shown to be Lipschitz independently of the small parameter ε. Liouville type results, similar to the ones in Section 4 below, are a main step in the proof of the result.

The difference between Theorem 1.1 and [10] is in the fact that a uniform Lipschitz regularity needs not be true in system 1.1, unless some compatibility assumption between the matrices $A^k := (a^k_{ij})$ is made; see Section 5 below. As a consequence Hölder and Lipschitz regularity - the former always being true - are two distinct steps, contrary to what happens in [10].

1.2. Diffusion flames and Burke-Schuman approximation

As opposed to premixed flames, where the oxidizer and reactant are considered as mixed, a diffusion flame is characterized by the fact that oxidizer and reactant mix on a thin sheet, where the flame precisely occurs. This is the basis of the celebrated Burke-Schumann assumption - [5]; see also Fendell [12]. A way to justify it is to introduce a large Damköhler number - the parameter that measures the intensity of the reaction- in the reaction term. Then, a chemical reaction is described by

$$O \text{ (Oxidizer)} + F \text{ (Fuel)} \to P \text{ (Products)}.$$

Let $\Omega \in \mathbb{R}^N$ be a bounded smooth open subset; let us consider velocity field in Ω denoted by $v(x)$ - known, as smooth as needed. The simplest description is as follows: the mass fraction of the oxidizer, Y_O, and the fuel mass fraction, Y_F, satisfy the system

$$(-\Delta + v(x).\nabla)Y_O = (-\Delta + v(x).\nabla)Y_F = -DaY_OY_F \quad (x \in \Omega) \tag{1.4}$$

with, for instance, Dirichlet conditions of $\partial\Omega$. Here, Da is the Damköhler number that we assume to be large. Let us check the Burke-Schumann assumption: the equation for T uncouples from the rest of the system, and the relevant quantity to introduce is the (commonly called) mixture function: $\beta(x) = Y_O(x) - Y_F(x)$.

Set $\Omega_+ = \{\beta > 0\}$. In Ω_+ the fuel mass fraction Y_F is a then subsolution to the equation $LY = -Da\beta Y$; a super-solution of which being $y \mapsto e^{-Da^{1/3}b(x)(\rho(x)-|y-x|)}$ with $\rho(x) > 0$ chosen so that $B(x, \rho(x)) \subset \Omega_+$, and $b(x) = \inf_{B(x,\rho(x))} \beta$. Hence

$$(E) \qquad 0 \leq Y_F(x) \leq e^{-Da^{1/3}b(x)\rho(x)}.$$

Therefore, assuming that $\{\beta = 0\}$ is a smooth hypersurface - which is generically true - we have the convergence $DaY_0Y_F \to |\beta_\nu|\delta_{\{\beta=0\}}$. This very simple argument will be called the *complete combustion principle* in the sequel.

The Burke-Schuman assumption has been made very sophisticated, for instance by the introduction of the high activation energy assumption. The fundamental paper is Linãn [19]. A summary can be found in Williams [21]. For more recent applications to triple flames, see Dold [11] and the references therein.

1.3. Further results

In the remainder of the paper, we not only want to prove convergence result to the Burke-Schuman equations - in most cases, standard weak convergence arguments would do the job - but we also wish to keep track of the maximal global regularity available. This is a question of mathematical interest, but also an issue as far as numerical simulations are concerned: the maximal available regularity indeed assesses the quality of the approximation of the free boundary problem by the singularly perturbed one.

In the very simple example given in the preceding pargraph, the maximal global regularity is Lipschitz. This regularity, however, is not really difficult to prove because the function β does not depend on ε. This does not seem to be so obvious in more general cases, where β truly depends on ε, although the limiting β is nice - it usually satisfies a nonlinear elliptic equation with possibly discontinuous coefficients. The obvious *a priori* estimates for Y_O^ε are indeed H^1 and L^∞ ones; this yields - at most - C^α regularity for β^ε. Although this is quite sufficient to pass to the (pointwise and strong L^2) limit, it is not enough to keep track of the Lipschitz regularity.

A first nontrivial case is given by the following configuration. Let Ω be a smooth bounded subset of $\mathbb{R}^N$ - $N \geq 3$ - and, for $d > 0$, let L_d be the linear operator

$$(1.5) \qquad L_d = -d\Delta + v(x).\nabla$$

We are dealing with a reacting mixture in which we also want to observe the temperature variations. Consider a Lipschitz continuous function f of the 'ignition temperature' type:

$$(1.6) \qquad \begin{array}{l} f = 0 \text{ on } \mathbb{R}_- \text{ and } f > 0 \text{ on } \mathbb{R}_+. \\ \text{There is } \alpha > 0 \text{ such that } f(u) \sim \alpha u^+ \text{ around } 0. \end{array}$$

The second condition is a nonessential commodity assumption. Consider some real number $\theta > 0$; the system under consideration is

$$L_1 T = -L_{d_1} Y_0 = -L_{d_2} Y_F = \frac{1}{\varepsilon} Y_O Y_F f(\frac{T-\theta}{\varepsilon^{1/2}}) \tag{1.7}$$

As for boundary conditions, partition $\partial\Omega$ as $\partial\Omega = \Sigma_1 \cup \Sigma_2$, and impose

$$\begin{aligned} (T, Y_O, Y_F) = & \quad ((T_0(x), Y_{O,0}(x), Y_{F,0}(x)) \qquad \text{on } \Sigma_1, \\ \partial_\nu (T, Y_O, Y_F) = & \quad 0 \qquad \text{on } \Sigma_2 \end{aligned} \tag{1.8}$$

Also impose the following condition:

$$Y_{O,0} - Y_{F,0} \neq 0 \qquad \text{on } \Sigma_1 \cap \Sigma_2. \tag{1.9}$$

For fixed $\varepsilon > 0$ and bounded reaction term f, a solution $(T^\varepsilon, Y_O^\varepsilon, Y_F^\varepsilon)$ to (1.7-1.8) can easily be found. An H^1 bound for $(T^\varepsilon, Y_O^\varepsilon, Y_F^\varepsilon)$ is also easily found, allowing therefore for L^2 convergence.

Theorem 1.2 *Consider a sequence of solutions $(T^\varepsilon, Y_O^\varepsilon, Y_F^\varepsilon)$ to (1.7-1.8).*
[i]. Let $(T^\infty, Y_O^\infty, Y_F^\infty)$ be a possible limit. There are two Hölder functions $\beta(x)$ and $\gamma(x)$, and two measures μ_1 and μ_2, respectively carried by the sets $\{\beta = 0\} \cap \{\gamma > 2\theta + |\beta|\}$ and $\{\gamma = 2\theta + |\beta|\}$, such that

$$L_1 T^\infty = -L_{d_1} Y_0^\infty = -L_{d_2} Y_F^\infty = \mu_1 + \mu_2$$

Moreover the functions β and γ are $C^{1,\alpha}$ away from the (possibly empty) set

$$F_0 := \{\beta = 0\} \cap \{\gamma = 2\theta + |\beta|\}. \tag{1.10}$$

[ii]. The sequence $(T^\varepsilon, Y_O^\varepsilon, Y_F^\varepsilon)$ is uniformly Lipschitz with respect to ε, on every compact not intersecting F_0.

We see that two types of flames coexist: a classical diffusion flame, supported by $\{\beta = 0\}$, and where the temperature is above θ - just as in the classical Burke-Schuman setting - and a premixed flame, where (i) at least one of the two species is in excess, and (ii) the temperature has exactly the value θ. This effect is well-known in the context of high activation energies - [19], but does not seem to have been rigorously described in this setting. The set F_0 is nongeneric, it corresponds to possible interactions between the diffusion flame and the premixed flame.

We revert for our second application to a pure Burke-Schuman system

$$\begin{aligned} -\div\Big(A_O(Y_O, Y_F)\nabla Y_O\Big) + v(x).\nabla Y_O = & \quad -\frac{1}{\varepsilon} Y_O Y_F \\ -\div\Big(A_F(Y_O, Y_F)\nabla Y_F\Big) + v(x).\nabla Y_F = & \quad -\frac{1}{\varepsilon} Y_O Y_F \end{aligned} \tag{1.11}$$

with the conditions (1.8). The matrices A_O and A_F have smooth, symmetric entries, and satisfy the usual uniform ellipticity condition. Dirichlet-Neumann conditions (1.8) are imposed, and it is once again easy to obtain a - uniformly H^1 - sequence $(Y_O^\varepsilon, Y_F^\varepsilon)$ of solutions.

Theorem 1.3 *The following properties hold.*
[i]. The sequence $(Y_O^\varepsilon, Y_F^\varepsilon)$ *is uniformly Hölder with respect to* ε.
[ii] For any converging sequence $(Y_O^{\varepsilon_n}, Y_F^{\varepsilon_n})$ *there is a (weak) solution* β *of the following problem:*

$$-\div(\bar{A}(\beta)\nabla\beta) + V(x).\nabla\beta = 0 \qquad (\Omega) \tag{1.12}$$

with the boundary conditions (1.8) such that $Y_O^\varepsilon \to \beta^+$, $Y_F^\varepsilon \to \beta^-$. *We have denoted by* $\bar{A}(\beta)$ *the matrix function equal to* $A_O(\beta, 0)$ *if* $\beta > 0$ *and* $A_F(0, -\beta)$ *if* $\beta < 0$.
[iii]. Assume the existence of $d > 0$ *such that* $dA_0(0,0) = A_F(0,0)$. *Then the sequence* $(Y_O^\varepsilon, Y_F^\varepsilon)$ *is uniformly Lipschitz with respect to* ε.

The paper is organized as follows. In Section 2 below, we prove Theorem 1.1. We review in Section 3 the classical diffusion flame approximations: equidiffusional case, where we give an 'elementary' proof; high activation limit. Sections 4 and 5 are devoted to applications of Theorem 1.1: proof of Theorem 1.2 in Section 4, proof of Theorem 1.3 in Section 5. We conclude with some general remarks on Properties [i] and [ii], which are indeed optimal in general.

2. Uniform C^α regularity

We are studying in this section a general singularly perturbed system of the form

$$L_k Y_k = -\frac{\rho}{\varepsilon} f(Y) := f_\varepsilon, \tag{2.1}$$

with L_k any uniformly elliptic operator under divergence form:

$$L_k = -\partial_i(a_{ij}^k \partial_j) + b_i^k(x).\nabla$$

with a_{ij}^k bounded, measurable, satisfying the usual ellipticity condition; and b_i^k as smooth as needed. The function f is nonnegative, smooth, and vanishes only if one of the components vanishes. We assume that we have managed to construct a nonnegative vector solution Y to (2.1).

Note that, because the only assumption on the A_k in (1.1) is their positivity and boundedness, system (2.1) contains (1.1) with

$$\forall k \in [1, p], \qquad A_k(x) \equiv 1, \tag{2.2}$$

and with an additional parameter: the constant ρ might indeed tend to 0 as well, independently of ε. We introduce it in order to allow ourselves rescalings.

The key result to the proof of Theorem 1.1 is the following lemma.

Lemma 2.1 *If* Y *solves (2.1) in* B_1*, and if* $\mathrm{osc}_{B_1} Y_k := \lambda_k$ *for all* k*, there exist* $\mu_0 < 1$ *and* $r_0 < 1$ *such that, for some* $k_0 \in \{1, ..., p\}$ *we have:*

$$\mathrm{osc}_{B_{r_0}} Y_{k_0} \le \mu_0 \lambda_{k_0}. \tag{2.3}$$

Remark 2.1 *In the above lemma, the* λ_k *can be of arbitrary size; in particular they may depend on* ε. *Thus we will in particular keep to ourselves the possibility of renormalizing the* Y_k*'s in small balls.*

The following three facts about the operator

$$L = -\partial_i(a_{ij}\partial_j)$$

with bounded measurable coefficients, satisfing the ellipticity condition, will be of constant use. Note that the above form encompasses the operators L_k above, at the only expense of modifing the diffusion matrix into a - still uniformly elliptic - nonsymmetric problem.

- (*de Giorgi's oscillation lemma*). If $|Lu| \le C$ in B_1, then there is $\alpha > 0$ depending only on C and ellipticity, such that $u \in C^\alpha(B_1)$. Moreover there is $\lambda \in]0,1[$ such that

$$\forall r \in]0, \frac{1}{2}[, \qquad \mathrm{osc}_{B_r} u \le \lambda \mathrm{osc}_{B_{2r}} u + r^\alpha \tag{2.4}$$

- (*Littman-Stampacchia-Weinberger [18] capacitary estimate*). If $G(x,y)$ is the (Dirichlet) fundamental solution of L in B_1 there is $C > 0$ such that we have, for $(x,y) \in B_{1/2} \times B_{1/2}$:

$$\frac{1}{C}|x-y|^{2-N} \le G(x,y) \le C|x-y|^{2-N}. \tag{2.5}$$

- (*Mean value theorem*) Let u solve $Lu = 0$ in B_1. There is $K > 0$, depending only on ellipticity, such that: for all $r \in]0, \frac{1}{2}[$ and $x \in B_{1/2}$, there is a Borel set $S_r(x)$ and $K > 0$ such that

$$B_r(x) \subset S_r(x) \subset B_{Kr}(x)$$

and such that

$$u(x) = \frac{1}{|S_r(x)|}\int_{S_r(x)} u. \tag{2.6}$$

See [17]. The equality is replaced by a $\le$ sign if $Lu \le 0$.

The proof of Lemma 2.1 requires the following intermediate step.

Lemma 2.2 *Denote by λ_{min} (resp. λ_{max}) the minimum (resp. the maximum) of the λ_j's. Let k_{min} and k_{max} denote the corresponding indices. There is $\sigma_0 > 0$ depending only on ellipticity such that the conclusion of Lemma 2.1 is true for $k_0 = k_{max}$ as soon as $\lambda_{max} \ge \sigma_0 \lambda_{min}$.*

Proof. For every $k \in \{1, ..., p\}$ we write $Y_k = v_k + w_k$, where

- if $G_{k,r}(x,y)$ is the Dirichlet fundamental solution of L_k in B_r, then:

$$v_k = \int_{B_{1/2}} G_{k,1}(x,y) f_\varepsilon(y)\ dy;$$

- and w_k solves $L_k w_k = 0$ in $B_{1/2}$ with Dirichlet datum Y_k at the boundary.

Because $L_k Y_k \leq 0$ for all k, the de Giorgi oscillation lemma plus the capacitary estimate imply:

$$\begin{aligned}
\forall k \in \{1, ..., p\}, \quad \mathrm{osc}_{B_r} v_k &\leq \int_{B_{1/2}} G_{k,1}(x,y) f_\varepsilon(y)\, dy \\
&\leq C \int_{B_{1/2}} G_{k_{min},1}(x,y) f_\varepsilon(y)\, dy \\
&\leq C\lambda_{min}
\end{aligned}$$

Hence we have $\mathrm{osc}_{B_{1/2}} v_{k_{max}} \leq \frac{1}{4}\lambda_{max}$, as soon as

$$C\sigma_0^{-1} \leq \frac{1}{4}. \tag{2.7}$$

Take now $r_0 \leq \frac{1}{2}$; we have, from (2.7) and the de Giorgi oscillation lemma:

$$\mathrm{osc}_{B_{r_0}} Y_{k_{max}} \leq \frac{\lambda_{max}}{4} + C\lambda_{max} r_0^\alpha.$$

Choosing r_0 small enough yields

$$\mathrm{osc}_{B_{r_0}} Y_{k_{max}} \leq \frac{\lambda_{max}}{2}, \tag{2.8}$$

which ends the proof. $\square$

Corollary 2.1 *If $\lambda_{max} \geq \sigma_0 \lambda_{min}$, and if r satisfies (2.7), then*
(i). The conclusion of Lemma 2.1 holds for all k such that $\lambda_k \geq \sigma_0\lambda_{min}$.
(ii) For such a λ_k: either the quantity $\mathrm{osc}_{B_{r_0}} Y_k$ is comparable to $\lambda_{min} = 1$, or there is $C > 0$ independent of ε such that

$$\frac{\max_{B_{r_0}} Y_k}{\min_{B_{r_0}} Y_k} \leq C. \tag{2.9}$$

Proof. The proof of (i) is obvious; the proof of (ii) comes from the fact that the oscillatory part of Y_k, namely the quantity

$$v_k = \int_{B_{r_0}} G_{k,r_0}(x,y) f_\varepsilon(y)\, dy$$

is comparable to 1. Consequently, either $Y_k - v_k$ is comparable to 1, or the Harnack inequality applied to $Y_k - v_k$ yields (2.9). $\square$

Proof of Lemma 2.1. We may now assume the existence of σ_0 independent of ε such that, with the notations of Lemma 2.2:

$$\frac{\lambda_{max}}{\lambda_{min}} \leq \sigma_0.$$

Moreover we may normalize the Y_k's so that $\lambda_{min} = 1$. Let K be the dilation constant in the mean value theorem; for $\Lambda > 0$ (to be large) and $k \in \{1, ..., p\}$ we define

$$\Omega_{k,\Lambda} = \{x \in S_{1/K} : \ Y_k(x) \leq \frac{1}{\Lambda}\}. \tag{2.10}$$

See the definition of $S_{1/K}$ in the above statement of the mean value theorem. We consider an integer m that will be chosen in due time, once again large. The proof is broken into two cases.

Case 1. There is $k_0 \in \{1, ..., p\}$ such that

$$|\Omega_{\Lambda,k_0}| \geq \frac{1}{m}. \tag{2.11}$$

We have, if x_{max} (resp. x_{min}) is a point in $B_{1/K}$ where Y_k reaches its maximum (resp. its minimum) over $B_{1/K}$:

$$\begin{aligned}
\mathrm{osc}_{B_{1/K}} Y_{k_0} &= Y_{k_0}(x_{max}) - Y_{k_0}(x_{min}) \\
&\leq (Y_{k_0}(x_{max}) - \frac{1}{\Lambda}) + (\frac{1}{\Lambda} - Y_{k_0}(x_{min})) \\
&\leq \frac{1}{|S_{1/K}(x_{max})|} \int_{S_{1/K}(x_{max})} (Y_{k_0}(x) - \frac{1}{\Lambda}) \, dx + \frac{1}{\Lambda} \\
&\leq (1 - \frac{1}{m} + \frac{1}{\Lambda \lambda_{k_0}}) \lambda_{k_0}
\end{aligned}$$

Hence, because $\lambda_{k_0} \geq 1$, we only have to choose Λ and m large enough.

Case 2. For all $k \in \{1, ..., p\}$, we have - Λ and m are now chosen at least large enough so that the above case holds:

$$|\Omega_{\Lambda,k}| \leq \frac{1}{m}. \tag{2.12}$$

Set

$$Z = B_{1/K} \backslash (\bigcup_{k=1}^{p} \Omega_{k,\lambda}); \tag{2.13}$$

if m is large enough we have

$$|Z| \geq |B_{1/K}| - \frac{p}{m} \geq c > 0. \tag{2.14}$$

The upshot is that

- the quantity $\min_k \min_{\Omega_{k,\Lambda}} Y_k$ is uniformly controlled from below, and is comparable to σ_0. Hence we may throw it into the parameter ρ, and, due to the homogeneity assumption, assume that $f(Y)$ is bounded;
- an $O(1)$ fraction of the total mass of f_ε is bounded independently of ρ and ε.

The first point is obvious. As for the second one, by the theorem of the mean we have

$$\forall k \in \{1, ..., p\}, \quad \int_{B_{1/K} \times B_{1/K}} G_{k,1}(x, y) f_\varepsilon(y) \, dxdy \leq \mathrm{osc}_{B_1} Y_k \leq \sigma_0;$$

on the other hand we have, by the capacitary estimate and assuming without loss of generality that $K \geq 2$:

$$\begin{aligned}
\int_{B_{1/K}\times B_{1/K}} G_{k,1}(x,y) f_\varepsilon \, dxdy \geq & \; C \int_{B_{1/K}\times B_{1/K}} \frac{f_\varepsilon(y)}{|x-y|^{N-2}} \, dxdy \\
\geq & \; \frac{C|Z\backslash\{|x-y| \leq \delta\}|}{(\mathrm{diam} Z)^{N-2}} \rho\varepsilon^{-1} \inf_{Z^p} f \\
\geq & \; C\rho\varepsilon^{-1}
\end{aligned} \tag{2.15}$$

as soon as $\delta > 0$ is small enough: indeed, $Z \cap \{|x-y| \geq \delta\}$ is nonempty and its measure is at least $C(1-\delta^N)$. By (2.15) we have

$$\rho\varepsilon^{-1} \leq C \text{ independent of } \varepsilon \text{ and } \rho \tag{2.16}$$

Hence $\rho\varepsilon^{-1}$ is bounded, and we get once again an oscillation decrease. □

Remark 2.2 *The above case* **2** *is the only place where we use the particular structure of the nonlinearity in order to get the oscillation decay. The other oscillation decays only result from the fact that we have* $L_k Y_k \leq 0$.

Proof of Theorem 1.1. The goal is to get the conclusion of Lemma 2.1 true for *all* $k \in \{1,...,p\}$. To reach this conclusion, we use the following induction argument, whose n^{th} step is

- If the assumptions of Lemma 2.2 are true, then apply it: there is a universal $r_0 \in]0,1[$ and a universal $\rho_0 \in]0,1[$ such that

  $$\max_{1\leq k\leq p} \mathrm{osc}_{B_{r_0^{-n-1}}} Y_k \leq \rho_0 \max_{1\leq k\leq p} \mathrm{osc}_{B_{r_0^{-n}}} Y_k. \tag{2.17}$$

  Also,
  $$\max_{1\leq k\leq p} \mathrm{osc}_{B_{r_0^{-n-1}}} Y_k \sim \sigma_0 \min_{1\leq k\leq p} \mathrm{osc}_{B_{r_0^{-n}}} Y_k. \tag{2.18}$$

  Stop.

- If the assumptions of Lemma 2.2 are not satisfied, there is a $k_{n+1} \in \{1,...,p\}$ and a universal constant $\lambda \in]0,1[$ such that

  $$\mathrm{osc}_{B_{2^{-(n+1)}}} Y_{k_{n+1}} \leq \lambda \mathrm{osc}_{B_{2^{-n}}} Y_{k_{n+1}}. \tag{2.19}$$

  Assuming that, at step 1, a real number $\rho_1 \in]0,1[$, close to 1, has been chosen so that
  $$\rho_1^{-p}\lambda < 1, \tag{2.20}$$
  check the inequality

  $$\max_{k\in\{1,...,p\}} \mathrm{osc}_{B_{r_0^{-(n+1)}}} Y_k \leq \rho_1 \max_{k\in\{1,...,p\}} \mathrm{osc}_{B_{r_0^{-n}}} Y_k. \tag{2.21}$$

  If true, then stop.

- If (2.21) is incorrect, go to the first point above point and make a step $n+1$.

We claim that, in a finite number of steps n_{max}, then

- either an inequality of the type (2.21) occurs; in which case we have

$$\max_{k\in\{1,...,p\}} \mathrm{osc}_{B_{r_0^{-(n_{max}+1)}}} Y_k \leq \rho_1 \max_{k\in\{1,...,p\}} \mathrm{osc}_{B_1} Y_k,$$

  (this does not express Hölder continuity yet),

- or the assumptions of Lemma 2.2 become true in $B_{r_0^{-n_{max}}}$.

Indeed, if the algorithm still runs at step $n = lp$, there is at least one $k_0 \in \{1, ..., p\}$ that will be have been concerned at least l times by the second point of the algorithm. For that particular k_0 we have:

$$\begin{aligned} \mathrm{osc}_{B_{r_0^{-lp}}} Y_{k_0} \leq\ & \lambda^l \mathrm{osc}_{B_1} Y_{k_0} \\ \leq\ & \sigma_0 \lambda^l \max_{k\in\{1,...,p\}} \mathrm{osc}_{B_1} Y_k \\ \leq\ & \sigma_0 (\rho_1^{-p}\lambda)^l \max_{k\in\{1,...,p\}} \mathrm{osc}_{B_{r_0^{-lp}}} Y_k \end{aligned}$$

Consequently, if

$$l = l_{max} = \left[\frac{|\mathrm{Log}\sigma_0|}{|\mathrm{Log}(\rho_1^{-p}\lambda)|}\right]+1 \tag{2.22}$$

the assumptions of Lemma 2.2 are true in $B_{r_0^{-lp}}$. Moreover, all oscillations in $B_{r_0^{-lp}}$ become controlled by 1 - the minimum oscillation in B_1 - by virtue of Corollary 2.1, (ii). Moreover, the smallest oscillation in $B_{r_0^{-lp}}$ is a small constant.
Now, we claim that the oscillation decay for all the components of the vector Y is obtained by applying our algorithm one more time. Indeed, by corollary 2.1, the first application of this algorithm has yielded a - possibly empty - subset I_1 of $\{1, ..., P\}$ such that (2.9) holds for all $k \in I_1$. Also, all $k \in I_1$ satisfy the oscillation decay property. Set

$$Y^1 = (Y_k)_{k\notin I_1} \quad , \quad \rho^1 = \rho \prod_{k\in I_1} \min_{B_1} Y_k^{a_k}$$

$$A_k^1(x) = \prod_{k\in I_1} \frac{Y_k^{a_k}}{\min_{B_1} Y_k^{a_k}}$$

and consider now the system for the new unknown Y^1. Recall that, by Corollary 2.1, the functions A_k - that depend on ε - are uniformly bounded and bounded away from 0.

Renormalize the unknowns once again to have the minimum oscillation in $B_{r_0^{-lp}}$ to be 1. This means changing ρ^1, but our lemmas work at all scales of ρ and ε.

So we may apply our algorithm a second time. This yields a subset I_2 of $\{1, ..., P\}\backslash I_1$, enjoying the same properties as I_1, outside which the maximum oscillation in $B_{r_0^{-2lp}}$ is controlled by $\min_k \mathrm{osc}_{B_{r_0^{-lp}}} Y_k$. By construction, this is an $O(1)$. Undoing the normalization, this leads to

$$\max \mathrm{osc}_{B_{r_0^{-2lp}}} Y_k << 1.$$

Hence Lemma 2.1 is true for all the components of the vector Y, and Theorem 1.1 follows by a standard iteration argument. $\square$

3. The special case of equidiffusion

3.1. Large Damköhler number

In this section the set Ω is the cylinder $\{(x,y) \in (-L,L)\times\omega\}$; we are interested in the solutions (T, Y_0, Y_F) of

$$LT = -LY_O = -LY_F = Y_O Y_F f(T), \quad x \in \Omega \tag{3.1}$$

with the mixed conditions (1.8) where $\Sigma_1 = \{-a,a\}\times\omega$, $\Sigma_2 = (-a,a)\times\partial\omega$. The function f has an ignition temperature θ; see (1.6). The functions

$$\beta = Y_O - Y_F, \qquad \gamma = 2T + Y_O + Y_F \tag{3.2}$$

solve $L\beta = L\gamma = 0$.

We will in this section make the assumption

(H) The sets $\{\beta = 0\}$ and $\{\gamma = 2\theta + |\beta|\}$ are nonempty smooth connected hypersurfaces that do not intersect Σ_1. Moreover we have
$$\{\beta = 0\} \subset \{\gamma > 2\theta + |\beta|\}.$$

A lot of easy cases where (H) is satisfied are available. For instance, take $v(x_1, y) := v(y)$ and the following assumptions:

$$\begin{aligned} T_0(-a,y) <\ & \theta << T_0(a,y), \\ 0 < Y_O(-a,y) <\ & Y_F(-a,y), \qquad Y_O(a,y) > 0,\ Y_F(a,y) = 0 \end{aligned}$$

Then we have $\partial_{x_1}\beta > 0$, $\partial_{x_1}(\gamma - \beta) > 0$ in $\{\beta < 0\}$ hence, if $T_0(a,.)$ is large enough, $\{\beta = 0\} \subset \{\gamma > 2\theta + |\beta|\}$.

We now replace f by $\varepsilon^{-1} f$, and our goal is to prove Theorem 1.2. Here the measures μ_1 and μ_2 can be computed explicitely.

Proposition 3.1 *Under Assumption (H), the conclusions of Theorem 1.2 hold true. Moreover, let $\Omega^+ = \{\gamma > 2\theta + |\beta|\}$, $\Omega^+ = \{\gamma > 2\theta + |\beta|\}$. Let $T_\pm$ be the unique solutions of*

$$LT_\pm = 0 \quad in\ \Omega_\pm, \qquad T_\pm = \theta \quad on\ \{\gamma = 2\theta + |\beta|\}$$

with the conditions (1.8) on $\partial\Omega_\pm \cap (\Sigma_1 \cup \Sigma_2)$. The measures μ_1 and μ_2 read

$$\mu_1 = (\partial_\nu T_+ - \partial_\nu T_-)\delta_{\{\gamma = 2\theta + |\beta|\}}, \qquad \mu_2 = |[\beta_\nu]|\delta_{\{\beta=0\}}$$

where the vector ν points is chosen such that $\nu.e_1 > 0$.

The sequence $(Y_O^\varepsilon, Y_F^\varepsilon)_\varepsilon$ is, as usual, bounded in $H^1_{loc}(\Omega)$. Hence, because $2T^\varepsilon = \gamma - Y_O^\varepsilon - Y_F^\varepsilon$, there is a subsequence - that we relabel $(T^\varepsilon, Y_O^\varepsilon, Y_F^\varepsilon)$ converging strongly in $L^2_{loc}(\Omega)$ to a triple (T, Y_O, Y_F) satisfying

$$LT \geq 0, \quad LY_O \leq 0, \quad LY_F \leq 0. \tag{3.3}$$

Let us introduce the relaxed semi-limit

$$\overline{T}(x) = \liminf_{\varepsilon\to 0,\ x'\to x} T^\varepsilon(x). \tag{3.4}$$

From the Barles-Perthame lemma [1] the function $\overline{T}$ satisfies $L\overline{T} \geq 0$ in the viscosity sense. This implies $\overline{T} = T$: take $r > 0$ small; for all $x \in \Omega$ such that $d(x, \partial\Omega) > r$ and all x' close to x we have $T^\varepsilon(x') \geq |B_r(x')^{-1}| \int_{B_r(x')} T^\varepsilon + O(r)$; hence due to the L^2_{loc} convergence of T^ε to T:

$$\begin{aligned}
\overline{T}(x) \geq & \liminf_{\varepsilon \to 0,\ x' \to x} \frac{1}{|B_r(x')|} \int_{B_r(x')} T^\varepsilon + O(r) \\
= & \lim_{\varepsilon \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} T^\varepsilon + O(r) \\
= & \frac{1}{|B_r(x)|} \int_{B_r(x)} T^\varepsilon + O(r)
\end{aligned}$$

Lebesgue's Lemme implies $\overline{T} \geq T$ almost everywhere. On the other hand the definition of $\overline{T}$ implies $\overline{T} \leq T$. This implies in turn that (i) the set $\{T > \theta\}$ is an open subset, (ii) if $x_0 \in \{T > \theta\}$, then there exists $\delta > 0$ such that $T^\varepsilon(x_0) \geq \theta + \delta$, for all $\varepsilon > 0$ small enough. As a result we have the

Lemma 3.1 *Let $x_0 \in \Omega$ be such that $\beta(x_0) > 0$ and $T(x_0 > \theta$ (resp. $\beta(x_0) > 0$ and $T(x_0 > \theta$). Then $Y_F(x_0) = 0$ (resp. $Y_O(x_0) = 0$).*

PROOF. Consider, as is allowed by the above remark, $\rho > 0$ and $\delta > 0$ such that

$$\forall x \in B_r(x_0), \quad f(T(x)) \geq \delta,\ \beta(x) \geq \delta.$$

Hence $Y_O^\varepsilon \geq Y_F^\varepsilon + \delta \geq \delta$ and we have

$$LY_F^\varepsilon + \delta^2 Y_F^\varepsilon \leq 0 \ \text{ in } B_r(x_0), \qquad 0 \leq Y_F^\varepsilon \leq 1 \ \text{ on } \partial B_r(x_0).$$

The complete combustion principle leads to $Y_F(x_0) = 0$. □

Proposition 3.2 *Let $x_0 \in \Omega$ be such that $T(x_0) > \theta$. Then $\gamma(x_0) = 2T(x_0) + |\beta(x_0)|$.*

PROOF. If $\beta(x_0) > 0$, the above lemma shows that $Y_F(x_0) = 0$, hence the result; same argument if $\beta(x_0) < 0$. It remains to see what happens if $\beta(x_0) = 0$.

To prove the proposition, it is enough to prove the continuity of Y_F and Y_O across the surface $\{\beta = 0\}$ - which is, by the way, the core of the Burke-Schumann assumption. Parametrize a point x in a tubular neighbourhood by its projection y_x on $\{\beta = 0\}$ and its (signed) distance $t(x)$ to y_x; for definiteness assume that $t(x) > 0$ if $\beta(x) > 0$. Consider a strip

$$S_{L,\delta} = \{x \in \Omega : \quad |y_x| < L,\ -L < t(x) < \delta\}$$

parallel to the surface $\{\beta > 0\}$ enclosing the point x_0 - with $t(x_0) = 0$ and $y_{x_0} = 0$, with δ having a vocation to become small. Then $Y_F \leq \overline{Y}$, where

$$\begin{aligned}
L\overline{Y} = 0 \text{ in } \quad & S_{L,\delta} \\
\overline{Y} = 1 \text{ in } \quad & (\{|y_x| = L,\ -\delta \leq t(x) \leq \delta\} \cup \{t(x) = -L\}), \qquad \overline{Y} = 0 \text{ in } (\{t(x) = \delta\})
\end{aligned}$$

The function $\overline{Y}$ is Lipschitz in $\overline{S}_{L/2,\delta}$, independently of δ. Hence we have

$$Y_F(x) \le C\delta \ \text{ if } |y_x| \le \frac{L}{2}, |t(x)| \le \delta.$$

This implies the (Lipschitz) continuity of Y_F across the interface $\{\beta = 0\}$. □

Corollary 3.1 *We have $\partial(\{T > \theta\}) = \{\gamma = 2\theta + |\beta|\}$, and $T < \theta$ in $\{\gamma = 2\theta + |\beta|\}$.*

Proof. That $\partial(\{T > \theta\}) \subset \{\gamma = 2\theta + |\beta|\}$ comes from the preceding proposition. For the converse statement we argue as follows: if it was not so, there would be two points x and y and a continuous path η connecting x and y such that

- $T(x) < \theta < T(y)$
- η does not meet $\{\gamma = 2\theta + |\beta|\}$.

However η does have to meet $\partial(\{T > \theta\} \subset \{\gamma = 2\theta + |\beta|\}$, a contradiction.

Assume now that a point x_0 is such that $\gamma(x_0) < 2\theta + |\beta(x_0)|$. Then, because $|\beta| \le Y_O + Y_F$ we have $T(x_0) < \theta$. □

Proof of Proposition 3.1. There is a nonnegative measure μ such that

$$\frac{1}{\varepsilon} Y_O^\varepsilon Y_F^\varepsilon f(T^\varepsilon) \to \mu \ \text{ in the measure sense.}$$

The theorem will be proved when we have proved that $\mu - |[\beta_\nu]|\delta_{\{\beta=0\}}$ is supported on $\{\gamma = 2\theta + |\beta|\}$. To see this, recall that the function $\gamma + \beta$, considered as a function in Ω_- - the notation is given in Proposition 3.1, is maximal and equal to 2θ on $\partial(\{\beta > 0\} \cap \{\gamma = 2\theta + \beta\})$; by the Hopf Lemma it has a nontrivial linear decay in the vicinity of this portion of boundary. Consequently, there is $\delta > 0$ such that

$$\forall x \in O^+ \text{ s.t. } d(x, O_+^0) \le \delta, \qquad T^\varepsilon(x) \le \max_{O_+} T^\varepsilon - Cd(x, O_+^0);$$

we use the fact that T^ε goes to θ on ∂O_+ - as a consequence of the preceding corollary - to infer: $f^\varepsilon(T(x)) = 0$ for $\varepsilon > 0$ small enough; hence the result.

From then on, we have $LT = LY_O = LY_F = 0$ in $(\Omega^+ \backslash \{\beta = 0\}) \cup \Omega^-$; the intensity of μ is therefore the jump of derivatives of T - resp. Y_0 across $\{\gamma = 2\theta + |\beta|\}$ - resp. $\{\gamma > 2\theta + |\beta|\} \backslash \{\beta = 0\}$. □

3.2. The high activation energy limit

We are still solving (3.1)-(1.8) where Ω is still the cylinder $(-L, L) \times \omega$, but we now replace the nonlinearity f in (3.1) by

$$f_\varepsilon(T) = \frac{\delta}{\varepsilon^3} \phi\left(\frac{T-1}{\varepsilon}\right), \tag{3.5}$$

where $\delta > 0$ will be made to run through $\mathbb{R}_+$. Introduce the additional assumptions

(H1) ϕ is Lipschitz-continuous and has -1 as an ignition temperature.
(H2) We have $\partial_{x_1}\beta > 0$ in $\bar{\Omega}$.

In this model, as opposed to the previous one, the temperature is not supposed to exceed the value 1 too much. In some sense, model (3.1)-(1.8)-(3.5) may be understood as the limit, as the burnt gas temperature tends to the ignition temperature, of model (3.1)-(1.8). Therefore we assume that the Dirichlet data are chosen so that γ is close to 2; namely:

$$\gamma = 2 + \varepsilon\gamma_1(x); \qquad \text{with } \gamma_1 > -1. \tag{3.6}$$

where γ_1 is smooth. We have implicitely assumed 0 to be in Ω; the particular values of γ_1 will be of no importance; the assumption $\gamma_1 > -1$ is meant to ensure that the temperature may rise above the ignition temperature.

Theorem 3.1 *There are two real numbers $\delta_1^\varepsilon > \delta_0^\varepsilon > 0$, such that:*

- *the sequences $(\delta_i^\varepsilon)_\varepsilon$ converges, as $\varepsilon > 0$, to some constants $\delta_i > 0$;*
- *for $\delta < \delta_0^\varepsilon$ the only solution to (3.1)(1.8) is the solution of $LT = LY_O = LY_F = 0$ in Ω with the conditions (1.8); we call this problem the 'extinguished problem'.*
- *for $\delta_0 < \delta < \delta_1$, there exists a family of stable solutions $(T_+^\varepsilon, Y_{O,+}^\varepsilon, Y_{F,+}^\varepsilon)$ such converging to the unique solution $(T_+, Y_{O,+}Y_{F,+})$ of*

$$\begin{array}{rll} LT & = -LY_0 = -LY_F = 0 & (\Omega\backslash\{\beta=0\}) \\ T & = 1;\ Y_0 = Y_F = 0 & (\{\beta = 0\}) \end{array} \tag{3.7}$$

  with the conditions (1.8). We call this problem the 'burning problem'.
- *There is a family of stable solutions $(\tilde{T}_+^\varepsilon, \tilde{Y}_{O,+}^\varepsilon, \tilde{Y}_{F,+}^\varepsilon)$ converging to the unique solution of the extinguished problem, and there is a family of unstable solutions $(T_-^\varepsilon, Y_{O,-}^\varepsilon, Y_{F,-}^\varepsilon)$ converging to the unique solution of the burning problem.*

Notice that $(\tilde{T}_+^\varepsilon, \tilde{Y}_{O,+}^\varepsilon, \tilde{Y}_{F,+}^\varepsilon)$ is really an extinguished solution: the strong maximum principle indeed implies that, if $\tilde{T}_+$ is the limiting solution, we have $\tilde{T}_+ < 1$ in Ω.

Corollary 3.2 *There is $\delta_2^\varepsilon > \delta_1^\varepsilon$ for which there is a unique solution to (3.1), (1.8) as soon as $\delta > \delta_2^\varepsilon$. Moreover we have $\displaystyle\lim_{\delta\to+\infty,\varepsilon\to 0} \frac{\delta}{\varepsilon^3}\phi(\frac{T_+^\varepsilon - 1}{\varepsilon}) = |\beta_\nu|\delta_{\{\beta=0\}}$*

This corollary really stems from Theorem 3.1 and Proposition 3.1.

Remark 3.1 *[i]. We believe that $\delta_1^\varepsilon = \delta_2^\varepsilon$, and also that the found solutions are the only ones. Because this is not the main topic of the paper, this aspect will not be investigated any further.*
[ii]. The apparition of the unstable solution comes from a classical topological degree argument.

[iii]. We have retrieved (a sketch of) the classical picture [19], [21]: for small Damkhöler numbers, only the extinguished solution exists. For large Damkhöler numbers, only the burning solution exists. To prove this, just use Proposition 3.1.
[iv]. The fact that the burning solution is equal to 1 on the whole set $\{\beta = 0\}$ is not obvious, except from physical arguments. It will be checked in Proposition 3.3 below.
[v]. A slightly non-equidiffusional model is allowed; and could be treated by the same arguments pertaining to Theorem 3.1. Namely, the system would be

$$L_{1+\varepsilon d_1} T = -L_{1+\varepsilon d_2} Y_O = -L_{1+\varepsilon d_2} Y_F = Y_O Y_F f_\varepsilon(T).$$

If we were interested in a truly non-equidiffusional model, with the same features, we would have to define a possibly inhomogeneous flame temperature. If not, the system under consideration behaves as the one studied in Section 4 (System (1.7). Such a model, however, seems to have more physical relevance to us.

As is usual, the term $f_\varepsilon(T)$ is nontrivial only when $T - 1$ is of order ε. Because the function γ is of order 2, the temperature is ε-close to 1 only if $Y_O + Y_F$ is of order ε; this may only occur near the set $\{\beta = 0\}$. Hence we meet again the Burke-Schumann assumption.

If we believe that the limiting problems in Theorem 3.1 are the right ones, then we will have to match the derivatives of the solution of (3.7) with the ones of the inner problem described below. This implies in particular that

$$|T_\nu| \geq |\beta_\nu| \quad \text{on } \{\beta = 0\}. \tag{3.8}$$

The hypersurface $\{\beta = 0\}$ being a graph, there exists $b > 0$ such that any point x in the tubular neighbourhood

$$S_b = \{\mathrm{dist}(x, \{\beta = 0\}) \leq b\} \tag{3.9}$$

is described by its projection x_1 on $\{\beta = 0\}$ and its projection along the normal $\nu(x_1)$, namely $x - x_1 = x_2 \nu(x_1)$. Without loss of generality assume that $\beta(0) = 0$, and blow up the coordinates: set

$$\begin{gathered} x_1 = \varepsilon\xi, \qquad x_2 = \varepsilon\zeta \\ p(\xi,\zeta) = \frac{T(\varepsilon(\xi,\zeta)) - 1}{\varepsilon}, \qquad q_{F,O}(\xi,\zeta) = \frac{Y_{F,O}(\varepsilon(\xi,\zeta))}{\varepsilon} \end{gathered} \tag{3.10}$$

In the new coodinates (ξ, ζ) the surface $\{\beta = 0\}$ is a smooth open subset $\mathcal{O}^\varepsilon$ of $\mathbb{R}^{N-1}$, with diameter of order $O(\varepsilon^{-1})$.

In the coordinate system (ξ, ζ), System (3.1) reads

$$L^\varepsilon p = -L^\varepsilon q_O = -L^\varepsilon q_F = q_0 q_F \phi(p), \tag{3.11}$$

where

$$L = -\Delta_\xi - \frac{d^2}{d\zeta^2} + \varepsilon M(\varepsilon) \tag{3.12}$$

and where the operator $M(\varepsilon)$ is a first-order differential operator with smooth coefficients and no zero-order terms. Take any point $\xi_0 \in \mathcal{O}^\varepsilon$; in the vicinity of ξ_0 we have

$$\beta(\xi,\zeta) = \beta_0(\varepsilon\xi_0, \varepsilon\zeta)\zeta, \quad \gamma(\xi,\zeta) = 2 + \varepsilon\gamma_1(\varepsilon(\xi,\zeta))$$

where β_0 is smooth and $\beta_0(\xi_0, 0) \neq 0$. We always may assume

$$\beta_0(\xi_0, 0) > 0. \tag{3.13}$$

Hence we have, at the order 0 in ε, and omitting the arguments:

$$q_O q_F \phi(p) = -(\gamma_1 + \beta_0\zeta - p)(\gamma_1 - \beta_0\zeta + p)\phi(p) := f_{\xi_0}(\zeta, p). \tag{3.14}$$

This leads to the family of one-dimensional problems, parametrized by δ, β_0 and α

$$\begin{aligned} -p''(\zeta) = & \ f_{\xi_0}(\zeta, p) \qquad (\zeta \in \mathbb{R}) \\ p'(-\infty) = & \ \alpha, \quad p'(+\infty) = -\alpha \end{aligned} \tag{3.15}$$

This problem is analyzed in the Appendix below.

Let $(T^\varepsilon)_\varepsilon$ be a sequence of solutions of (3.1). We have $0 \leq LT^\varepsilon \leq \frac{C}{\varepsilon}\mathbf{1}_{T\geq 1-\varepsilon}$; hence by [6], the Lipschitz constant of T^ε is bounded independently of ε, and the sequence $(T^\varepsilon)_\varepsilon$ is relatively compact in $C(\overline{\Omega})$. Let T be the limit of a subsequence; obviously T satisfies $LT = 0$ in $\Omega \backslash \{\beta = 0\}$.

Proposition 3.3 *Assume the existence of a nontrivial closed subset of $\{\beta = 0\}$ on which $T = 1$. Then $T = 1$ on $\{\beta = 0\}$.*

Proof. We work in the original coordinates. Assume the result is false, i.e. there exists a closed set set F of $\{\beta = 0\}$ such that

- we have $T < 1$ outside F,
- the open set of $\{\beta = 0\}$: $\{\beta = 0\}\backslash F$ is non void.

Let us therefore choose a (geodesic) ball B_0 of $\{\beta = 0\}$ touching ∂F at some point x_0. Call F_0 its complement in $\{\beta = 0\}$. Consider the function $\overline{T}(x)$ solution of

$$LT = 0 \text{ in } \Omega\backslash F_0 \qquad T = T_0 \text{ on } \partial\Omega \qquad T = 1 \text{ on } F_0 \tag{3.16}$$

We have, for all x in B_0 close to x_0 [14]: $\overline{T}(x) \leq 1 - C|x|^{1/2}$. On the other hand we have $T \leq \overline{T}$ and $T(x_0) = \overline{T}(x_0) = 1$. Hence T cannot be Lipschitz. $\square$

Any sequence $(T^\varepsilon)_\varepsilon$ of solutions of (3.1)-(1.8) converges to the extinguished state as soon as δ is small. The next proposition leads us to the characterization of the first δ when a truly burning flame appears, and implies Theorem 3.1 as an easy consequence.

Proposition 3.4 *Set*

$$\delta_1 = \sup_{\xi_0 \in \mathbf{R}^{N-1}} \delta_{cr}(\beta_0(\xi_0, 0), T_\nu(\xi_0, 0)). \tag{3.17}$$

where $T_\nu(\xi_0, 0)$ is the normal derivative of the limiting burning solution at the point $(\xi_0, 0)$. Then, for all $\delta < \delta_1$, any sequence $(T^\varepsilon)_\varepsilon$ of solutions of (3.1)-(1.8) converges to the extinguished state. For all $\delta > \delta_1$, there is a sequence $(T^\varepsilon)_\varepsilon$ of solutions of (3.1)-(1.8) converging to the upper burning state.

Proof. Assume first that $\delta < \delta_1$, and assume the convergence of a - possibly relabelled - subsequence $(T^\varepsilon)_\varepsilon$ to the upper burning state. Then, in the straightened out and rescaled coordinates (ξ,ζ), there is a ball B^ε of $\mathcal{O}^\varepsilon$, of radius $O(\varepsilon^{-1})$, such that

$$\forall \xi \in B^\varepsilon, \quad \delta < \delta_{cr}(\beta_0(\xi,0)).$$

Let p^ε be the rescaled function. Then, for $\varepsilon > 0$ small enough, we have $\inf\limits_{\xi \mathcal{O}^\varepsilon} p^\varepsilon(\xi,0) > -1$. If this were not true, then $(T^\varepsilon)_\varepsilon$ would indeed converge to the extinguished state by the preceding proposition. Therefore the function $\underline{p}(\zeta) = \max\limits_{\xi \in \overline{B}^\varepsilon} p(\xi,\zeta)$ is a nontrivial subsolution to (3.15), with the corresponding α. Since 0 is a super-solution, there exists a solution to (3.15), a contradiction.

If $\delta > \delta_1$, a sub-solution $\underline{T}^\varepsilon(x)$ to (3.11) is constructed in the following fashion. Let us choose $\eta > 0$ so small that

$$\delta = \sup_{\xi_0 \in \mathbf{R}^{N-1}} \delta_{cr}(\beta_0(\xi_0,0), T_\nu(\xi_0,0)+\eta). \tag{3.18}$$

Define $\Sigma^\varepsilon = (-A,A) \times \mathcal{O}^\varepsilon$; the implicit functions theorem argument yields the existence of a solution $\underline{p}^\varepsilon_+(\xi,\zeta)$ to

$$\begin{aligned} \tilde{L}^\varepsilon p = &\ f_\xi(\zeta,p) \quad (\Sigma^\varepsilon) \\ \partial_\zeta p(\xi,-A) = &\ T_\nu(\xi_0,0)+\eta, \quad \partial_\zeta p(\xi,A) = T_\nu(\xi_0,0)-\eta \\ p(\xi,\zeta) = &\ p^+_{\delta,\beta(\xi,0),T_\nu(\xi_0,0)+\eta)}(\zeta) \quad (\xi \in \partial\mathcal{O}^\varepsilon) \end{aligned} \tag{3.19}$$

A sub-solution $\underline{T}^\varepsilon$ is defined in the following way:

- for $(\xi,\zeta) \in \Sigma^\varepsilon$ we set $\underline{p}(\xi,\zeta) = \underline{p}^\varepsilon_+(\xi,\zeta)$;
- if S^ε is the image of Σ^ε in undoing the change of coordinates let $\underline{T}^+(x)$ represent the function $\underline{p}^\varepsilon_+$ in S^ε. We take for $\underline{T}^\varepsilon$ the unique solution of $LT = 0$ outside S^ε, with values $\underline{T}^\varepsilon$ on ∂S^ε.

For η small enough and $\varepsilon \to 0$, the function $\underline{T}^\varepsilon$ is a sub-solution. And we know that 1 is a super-solution. □

The proof of Theorem 3.1 is now straightforward: once the upper solution is constrcuted, one only has to prove that a sequence of intermediate solutions will converge to the limiting burning one; this, however, can be proved by blowing up the solution around a burning point: the profile can then only be the 1D stable, or 1D unstable solution constructed in Proposition A1 of the appendix. The temperature has to be ε-close to 1 on the whole set $\beta = 0$; Proposition 3.3 implies that the temperature is strictly below 1 everywhere.

4. Non-equidiffusional models with constant diffusion

4.1. Pure Burke-Schuman non-equidiffusional models

We mean by the above appellation diffusion flame models where the chemical production term is assumed to be temperature-independent. An immediate application of Theorem 1.1 is indeed the following system: investigate systems of the form

$$L_1 Y_O = L_d Y_F = -\frac{1}{\varepsilon} Y_O Y_F \qquad (x \in \Omega) \tag{4.1}$$

with the mixed conditions (1.8). A sequence of solutions $(Y_O^\varepsilon, Y_F^\varepsilon)$ to (4.1,1.8) can easily be constructed. Theorem 1.1 yields a uniform Hölder bound for $(Y_O^\varepsilon, Y_F^\varepsilon)$; we will see that it implies a uniform Lipschitz bound.

Let us introduce the function β solution of

$$\begin{array}{rl} L_1\beta + (d-1)v(x).\nabla\beta^- = & 0,\ (x \in \Omega) \\ \beta = Y_{O,0} - dY_{F,0}, & (x \in \Sigma_1) \quad \partial_\nu \beta = 0 \quad (x \in \Sigma_1) \end{array} \tag{4.2}$$

It is in every $W^{2,p}(\Omega)$; moreover - see for instance [16] - the set $\{\beta = 0\}$ is a smooth hypersurface in the vicinity of all its nondegenerate points.

Theorem 4.1 *[i] We have, in the measure sense :*$\frac{1}{\varepsilon} Y_O^\varepsilon Y_F^\varepsilon \to \Delta|\beta|$*. If* Y_O *and* Y_F *are the respective limiting value of* Y_O^ε *and* Y_F^ε*, then* $Y_0 = \beta^+$ *and* $Y_F = \frac{1}{d}\beta^-$*. In particular, at a nondegenerate point of* $\{\beta = 0\}$ *we have the Burke-Schuman jump condition*

$$[\partial_{\nu^+} Y_0] = d[\partial_{\nu^-} Y_F].$$

[ii]. The family $(Y_O^\varepsilon, Y_F^\varepsilon)$ *is uniformly Lipschitz - independently of* ε *and the function* $\beta^\varepsilon := Y_O^\varepsilon - dY_F^\varepsilon$ *is - independently of* ε *- in each* $W^{2,p}(\Omega)$*.*

The singular limit is once again almost obvious and does not need Theorem 1.1: the sequence $(Y_O^\varepsilon, Y_F^\varepsilon)$ is uniformly in $H^1(\Omega)$; the equation

$$L_1 \beta^\varepsilon = -V(x).\nabla(Y_O^\varepsilon + Y_F^\varepsilon) \tag{4.3}$$

implies a Hölder bound on β^ε, hence the compactness of the sequence $(\beta^\varepsilon)_\varepsilon$ in $C(\overline{\Omega})$. Part [ii], i.e. the Lipschitz bound, relies - as is classical - on a Liouville type result.

Lemma 4.1 *There is no positive locally bounded solution of*

$$-\Delta p + p^2 = 0, \qquad x \in \mathbb{R}^N. \tag{4.4}$$

There is no positive locally bounded solution of

$$\begin{array}{rl} -\Delta p + p^2 = & 0 \quad (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}_+ \\ \partial_{x_N} p = & 0, \quad (x' \in \mathbb{R}^{N-1},\ x_N = 0) \end{array} \tag{4.5}$$

Proof. Let us first study (4.4). If p is such a solution, then

$$q(r) = \sup_{x\in S_r(0)} p(x) \tag{4.6}$$

is a Lipschitz sub-solution to (4.4); integrating once between 0 and r we have:

$$q_r \geq \int_0^r (\frac{\rho}{r})^{n-1} q^2 \, d\rho \geq (1 - \frac{n-1}{r}) \int_0^r q^2 \, d\rho. \tag{4.7}$$

the last inequality being obtained by an integration by parts. Multiplying by q^2 and integrating from $r_0 > 0$ large enough to r yields, for some $\lambda < 1$ and $C > 1$ - the latter depending on $p(0)$, which is assumed to be positive:

$$q^2 \geq \lambda \Big(\int_{r_0}^r q^2 \, d\rho + C \Big)^2 ;$$

hence we have $\displaystyle\int_{r_0}^r q \, d\rho \geq Q(r)$, where Q is the solution of

$$\dot{Q} = \lambda(Q^2 + C), \qquad Q(r_0) = 0$$

which blows up for finite $r > r_0$.
As for Problem (4.5) we introduce the sub-solution

$$q(r) = \sup_{x\in S_r(0)\cap\{x_N>0\}} p(x)$$

and the proof follows the above lines. $\square$

Lemma 4.1 may be strengthened into the following

Lemma 4.2 *Consider a bounded function $V(x)$. For $\delta > 0$ small enough, there is no positive C^2 solution of*

$$\begin{aligned} -\Delta p + \delta V(x).\nabla p + (p-\delta)p \leq \;& 0, \qquad x \in \mathbb{R}^N \\ p(0) = \;& 1 \end{aligned} \tag{4.8}$$

The same result is valid for

$$\begin{aligned} -\Delta p + \delta V(x).\nabla p + (p-\delta)p \leq \;& 0 \quad (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}_+ \\ \partial_{x_N} p = \;& 0, \quad (x' \in \mathbb{R}^{N-1},\ x_N = 0) \\ p(0) = \;& 1 \end{aligned} \tag{4.9}$$

Proof. It is enough to prove the result for the inequation

$$-\Delta p - \|V\|_\infty |\nabla p| + (p-\delta)p \leq 0, \quad p(0) = 1. \tag{4.10}$$

The upper envelope q, defined by (4.6) satisfies inequalities of the following form, for all $\delta > 0$ small enough

$$q_r \geq (1 - \mu_\delta) \int_0^r \frac{\rho^{n-1} + \delta\|V\|_\infty e^{\delta\|V\|_\infty \rho}}{r^{n-1} + \delta\|V\|_\infty e^{\delta\|V\|_\infty r}} q(q-\delta) \, d\rho. \tag{4.11}$$

where $\mu_\delta > 0$ is close to 0. If $\delta > 0$ is small enough, the RHS of (4.11) becomes larger than

$$(1-2\mu_\delta) \geq (1-\frac{n-1}{r})\int_0^r q^2 \, d\rho \quad \text{as long as} \quad p \geq \frac{1}{2}.$$

Then - simple modification of the computations - the new function q_r blows up at r close to the radius r_0 of Lemma 4.1. □

Proof of Theorem 4.1. Only Point [ii] needs a proof. For this, notice that equation (4.3) implies a uniform $C^{1,\alpha}$ bound on β^ε; hence this function becomes amenable to the natural scaling of the equation. Consider $A > 0$ large, and let x^ε be such that

$$\beta^\varepsilon(x^\varepsilon) \in [-A\varepsilon^{1/3}, A\varepsilon^{1/3}]. \tag{4.12}$$

We point out that it is enough to do so: indeed, outside this set, the complete combustion principle implies the Lipschitz bound trivially.

Rescale around x^ε:

$$p_O^\varepsilon(\xi) = \frac{1}{\varepsilon^{1/3}} Y_O^\varepsilon(x^\varepsilon + \varepsilon^{1/3}\xi), \quad p_F^\varepsilon(\xi) = \frac{1}{\varepsilon^{1/3}} Y_F^\varepsilon(x^\varepsilon + \varepsilon^{1/3}\xi). \tag{4.13}$$

Our main task will be to prove that the family $(p_O^\varepsilon, p_F^\varepsilon)$ is uniformly bounded. We have

$$-\Delta p_O^\varepsilon + \varepsilon^{1/3} V.\nabla p_O^\varepsilon = -(b^\varepsilon + O(\xi) + p_O^\varepsilon) p_O^\varepsilon, \quad p_O^\varepsilon(0) = \mu^\varepsilon. \tag{4.14}$$

Case 1. The sequence $\mathrm{dist}(x^\varepsilon, \partial\Omega)_\varepsilon$ goes to $+\infty$, and assume that $\lim_{\varepsilon\to 0} \varepsilon^{-1/3}\beta^\varepsilon(x^\varepsilon) = +\infty$. Due to Theorem 1.1 we have, for some $\alpha < 1$:

$$p_O^\varepsilon(\xi) - p_O^\varepsilon(\xi') = O(\varepsilon^{(\alpha-1)/3})|\xi - \xi'|^\alpha; \tag{4.15}$$

therefore we may assume

$$\varepsilon^{-1/3}\beta^\varepsilon(x^\varepsilon) = O(\varepsilon^{(\alpha-1)/3}). \tag{4.16}$$

Indeed, if μ^ε is above that order of magnitude, (4.15) implies that it is so in a large neighbourhood of 0. Examine the equation for p_F^ε: because $\xi \mapsto \dfrac{\beta^\varepsilon(x_0 + \varepsilon^{1/3}\xi) - b^\varepsilon}{\varepsilon^{1/3}}$ is Lipschitz - due to the invariance of Lipschitz norms under the scaling (4.13) - and because of the equality $Y_F^\varepsilon = Y_O^\varepsilon - \beta^\varepsilon$ we have $p_F^\varepsilon = O(\mu^\varepsilon)$ in $B_1(0)$. On the other hand we have, in $B_1(0)$:

$$-\Delta p_F^\varepsilon + \varepsilon^{1/3} V.\nabla p_F^\varepsilon \leq -\mu^\varepsilon p_F^\varepsilon.$$

This implies - complete combustion principle - $p_F^\varepsilon(0) = O(\mu^\varepsilon e^{-(\mu^\varepsilon)^{1/3}})$; contradicting the boundedness of $\beta^\varepsilon(x_0)$. Hence (4.16) is true.
Rescale now p_O^ε and ξ as

$$\xi = \sqrt{\mu^\varepsilon}\zeta, \quad q_O^\varepsilon(\zeta) = \frac{1}{\mu^\varepsilon} p_O^\varepsilon(\sqrt{\mu^\varepsilon}\zeta). \tag{4.17}$$

Equation (4.14) becomes

$$-\Delta q_O^\varepsilon + \varepsilon^{1/3}\sqrt{\mu^\varepsilon} V.\nabla p_O^\varepsilon = -(\frac{b^\varepsilon + O(\xi)}{\mu^\varepsilon} + p_O^\varepsilon)p_O^\varepsilon, \quad q_O^\varepsilon(0) = 1. \tag{4.18}$$

Recall that $\varepsilon^{1/3}\sqrt{\mu^\varepsilon} \to 0$ due to the uniform Hölder bound for Y_O^ε. By Lemma 4.2 there is ξ^ε, with norm controlled from above, such that $q_O^\varepsilon(\xi^\varepsilon) = +\infty$. This is of course impossible.
Case 2. $\text{dist}(x^\varepsilon, \partial\Omega)_\varepsilon$ is bounded. This can only occur near a boundary point with either a Neumann condition, or a point where we have exactly $Y_O - Y_F = 0$. In the first case, one may work exactly as in Case 1, and arrive - once a suitable change of coordinates has been performed - to equation (4.5); this is also an impossibility. In the second case, remember that we have chosen a Lipschitz boundary datum; hence the sequence $p_O^\varepsilon(0)$ is globally bounded.

From then on, Point [i] is easy: we may as well assume that $b^\varepsilon = 0$ and set, up to a rotation of the coordinates: $\nabla\beta(x_0) = le_1$ with $l \geq 0$. The family $(p_0^\varepsilon)_\varepsilon(\xi)$ converges to a solution of

$$-\Delta p_0 + (p_0 - b - l\xi_1)p_0 = 0, \tag{4.19}$$

a continuous family of sub-solutions of which being: $\xi \mapsto a\xi_1^+$, $0 \leq a \leq l$. Hence we have, by the maximum principle: $u(\xi) \geq l\xi_1^+$. However, the function $p \mapsto p - l\xi_1$ is strictly increasing as soon as $p \geq l\xi_1^+$; hence (4.19) has a unique nontrivial solution - to see this, just examine the difference between two possible solutions and realize that the usual maximum principle applies - namely the function $u(\frac{l}{\xi_1})$, where u is the unique nonzero solution of the ODE $-u'' = -(u - l\xi_1)u$. This ends the proof of the theorem. $\square$

4.2. Putting the temperature equation back in

This section is basically devoted to the proof of Theorem 1.2. Having Section 3.1 in mind, let us immediately deal with the functions β and γ. The function β was already accounted for; define γ^ε as

$$\gamma^\varepsilon = 2T^\varepsilon + d_1 Y_0^\varepsilon + d_2 Y_F^\varepsilon. \tag{4.20}$$

An equation for γ is

$$-\Delta\gamma^\varepsilon + V.\nabla\gamma^\varepsilon = -V.\nabla\Big((1-d_1)Y_0^\varepsilon + (1-d_2)Y_F^\varepsilon\Big). \tag{4.21}$$

An immediate consequence is a uniform estimate of γ^ε in Hölder norms; hence, in particular, a bound on the temperature that is also independent of ε. To go to $C^{1,\alpha}$ we need to improve the regularity of at least two of the three unknowns. This is given by the following

Theorem 4.2 *The triple $(T^\varepsilon, Y_O^\varepsilon, Y_F^\varepsilon)$ is uniformly Hölder away from the set F_0 defined by (1.10).*

PROOF. The assumptions of Theorem 1.1 concerning the functions A_k can be relaxed a bit: namely, we may assume them to be nonnegative - Lemma 2.2 is still true under that assumption -; however, in order to keep Lemma 2.1 true we need to say a few more words. Scanning the proof of this lemma we see that only Case 2 has to be re-examined; and only around the points where T^ε is close to θ. Then we set $u^\varepsilon = T^\varepsilon - \theta$; the situation is the following: in a set of $B_1(0)$ that we call Ω_Λ with measure as close to $|B_1(0)|$ as we wish, we have

$$\begin{aligned} L_1 T^\varepsilon = -L_{d_1} Y_O^\varepsilon = -L_{d_2} Y_F^\varepsilon = &\ \rho\varepsilon^{-1} Y_O^\varepsilon Y_F^\varepsilon f_\varepsilon(T^\varepsilon) \\ Y_O^\varepsilon,\ Y_F^\varepsilon \geq &\ \frac{1}{\Lambda^2} \\ 1 \leq \operatorname{osc}_{B_1} Y_O^\varepsilon,\ \operatorname{osc}_{B_1} Y_F^\varepsilon \leq &\ \sigma_0 \end{aligned} \tag{4.22}$$

where $\rho > 0$ accounts for some possible normalisation.

Because the functions β^ε and γ^ε are uniformly Hölder, they converge - up to a subsequence - to Hölder functions β and γ. Also, the results of Corollary 3.1 are true in our case.

1. Assume that we have

$$\gamma(0) > 2\theta + |\beta(0)|. \tag{4.23}$$

Then there is $\delta > 0$ such that

$$T^\varepsilon \geq \theta + \delta \text{ in } B_1.$$

We are then in the situation of Lemma 2.1, and the oscillation decay is automatically valid for Y_O^ε and Y_F^ε. The oscillation decay of T^ε follows from the expression of T^ε in terms of γ and $Y_{O,F}$.

2. Assume that we have

$$\gamma(0) = 2\theta + |\beta(0)| \text{ and } \beta(0) > 0. \tag{4.24}$$

Let us set

$$v^\varepsilon = \frac{u^\varepsilon}{\sqrt{\varepsilon}}. \tag{4.25}$$

2.1. Assume that the sequence $(\max_{B_{1/2}} v^\varepsilon)_\varepsilon$ is unbounded. Then we may find $\mu_{\varepsilon,\rho}$, that we denote by μ, and $\delta > 0$ independent of ε and ρ, such that we have, in B_1:

$$(-\Delta + \rho V.\nabla) T^\varepsilon \geq \frac{1}{\mu}(\theta + \mu - T^\varepsilon)\Big(\frac{T^\varepsilon - \theta - \mu}{\mu}\Big)_+.$$

In particular we have arranged that

$$\begin{aligned} \forall T \in [0, \theta + \sqrt{\mu}],\ \forall x \in B_1, \quad & \frac{\rho}{\varepsilon} Y_F\Big(\frac{\gamma^\varepsilon + \beta^\varepsilon}{2} - T\Big) f\Big(\frac{T - \theta}{\varepsilon}\Big) \\ \geq &\ \frac{1}{\mu}(\theta + \mu - T)\Big(\frac{T - \theta}{\mu}\Big)_+ \end{aligned} \tag{4.26}$$

and that $(T^\varepsilon - \theta)(0) \geq \sqrt{\mu}$. Setting

$$w^\varepsilon(\xi) = \frac{1}{\mu}(T^\varepsilon(\mu\xi) - \theta).$$

we have

$$-\Delta w^\varepsilon \geq C\mu(-w^\varepsilon(1-w^\varepsilon)_+ + \rho|\nabla w^\varepsilon|). \tag{4.27}$$

Now, argue as in lemma 4.1: set

$$\underline{p}^\varepsilon(r) = \inf_{B_r} w^\varepsilon; \tag{4.28}$$

then $\underline{p}^\varepsilon$ satisfies (4.27) in the viscosity sense - except that ∇ should be replaced by ∂_r. This implies (just integrate the ODE as in Lemma 4.1) that $\underline{p}^\varepsilon$ decays from 0 at most like $-\mu r$. Scaling back, we end up with a nontrivial ball B_{r_0}, r_0 bounded away from 0 independently of ρ and ε, such that $\inf\limits_{B_{r_0}} v^\varepsilon$ blows up. This implies that $\rho\varepsilon^{-1}$ is bounded; hence the oscillation decay.

2.2 Assume now the existence of a uniform constant C such that v^ε is bounded by C in B_1. We might as well assume that $C = 1$. Let us set

$$\rho^\varepsilon = \int_{B_1} (v^\varepsilon)^+. \tag{4.29}$$

Define the sets Ω_k, $k \geq 0$, as

$$\Omega_k = \{\frac{1}{2^{k+1}} \leq v^\varepsilon < \frac{1}{2^k}\} \tag{4.30}$$

and

$$\mu_k = |\Omega_k|. \tag{4.31}$$

Equality (4.29) implies existence of a constant C such that:

$$\frac{\rho}{\varepsilon^{3/2}}(\mu_0 + \sum_{k\geq 1} \frac{\mu_k}{2^k}) \geq C\rho^\varepsilon. \tag{4.32}$$

In Ω_k, rescale the coordinate x as $x = \mu_k^{1/N}\xi$; if $\tilde{\Omega}_k$ is the so obtained transformed set we have $|\tilde{\Omega}_k| = 1$. If $\tilde{v}_k^\varepsilon$ is the rescaled function we have, by assumption of f:

$$(-\Delta + \rho V.\nabla)\tilde{v}_k^\varepsilon \geq \frac{\mu_k^{2/N}\rho}{\varepsilon^{3/2}2^{k+1}}\mathbf{1}_{\tilde{\Omega}_k}$$

This implies in turn, taking into account the values of u^ε in $\tilde{\Omega}_k$ - here we have $k \geq 1$:

$$\alpha\frac{\mu_k^{2/N}\rho}{\varepsilon^{3/2}2^{k+1}} \leq \frac{C}{2^k}. \tag{4.33}$$

The constant α is defined in (1.6). Inequality (4.33) implies, together with (4.32):

$$\rho^\varepsilon \leq C\Big(\frac{\varepsilon^{3/2}}{\rho}\Big)^{N/2}. \tag{4.34}$$

Now, let us come back to the equation

$$(-\Delta + \rho V.\nabla)u^\varepsilon = O(\alpha\frac{\rho}{\varepsilon}v^\varepsilon).$$

From (4.34) there is $p > N$ such that

$$\frac{\rho}{\varepsilon}\|v^\varepsilon\|_{L^p(B_1)} \text{ is controlled independently of } \varepsilon.$$

Therefore u^ε has a uniform Hölder bound in B_1, hence the oscillation decay once again.

3. Assume that

$$\gamma(0) = 2\theta. \tag{4.35}$$

This implies, due to the uniform Hölder bound on γ^ε, that $\gamma^\varepsilon \sim 2\theta$ in B_1. Assume that $|\beta| \neq 0$; then we have, once again from the Hölder bound for β:

$$Y_0^\varepsilon + Y_F^\varepsilon \geq \delta_0 \text{ in } B_1$$

for some uniform positive constant δ_0. As a consequence, T^ε is uniformly below θ in B_1; and we conclude, for the last time, to the uniform Hölder bound for the triple $(T^\varepsilon, Y_O^\varepsilon, Y_F^\varepsilon)$ in B_1. The remaining case is the set F_0, that we have excluded. □

Proof of Theorem 1.2. Let x_0 be a point outside F_0; once again we may translate it to 0. the functions β^ε and γ^ε are now uniformly $C^{1,\alpha}$ in a vicinity of this set. Let (T, Y_O, Y_F) be a possible limit for $(T^\varepsilon, Y_O^\varepsilon, Y_F^\varepsilon)$.

1. Assume that $\gamma(0) > 2\theta + \beta(0)$. We already saw that there was nothing to prove if $\beta(0) \neq 0$; if $\beta(0) = 0$ we are in the situation of Theorem 4.1.

2. We have $\gamma(0) = 2\theta$ and $\beta(0) \neq 0$. If $T(0) < \theta$, this evidently implies the Lipschitz bound in a vicinity of 0. The remaining case to discuss is $T(0) = \theta$; however the argument essentially the same as in Theorem 4.1: the only change to be made is the equation for which we prove the Liouville theorem. Let us sketch the argument, leaving the complete proof to the reader.

Assume for definiteness that we have $\beta(0) > 0$. Thus Y_O^ε is bounded away from 0, uniformly in ε, in a - uniform - vicinity of 0. The equation for T^ε writes

$$L_1 T = \frac{1}{4}(\gamma^\varepsilon + \beta^\varepsilon - 2T^\varepsilon)(\gamma^\varepsilon - \beta^\varepsilon - 2T^\varepsilon) f(\frac{T-\theta}{\sqrt{\varepsilon}}).$$

The natural scaling is

$$x = \sqrt{\varepsilon}\xi, \quad T(\sqrt{\varepsilon}\xi) = \sqrt{\varepsilon}p^\varepsilon(\xi).$$

Under this scaling, and due to the $C^{1,\alpha}$ character of β^ε and γ^ε, and because $\beta(0) > 0$, there is $\delta > 0$ independent of ε such that the function p^ε satisfies

$$-\Delta p^\varepsilon + \sqrt{\varepsilon}V.\nabla p^\varepsilon \leq -\delta p f(p) \tag{4.36}$$

in an $\varepsilon^{-1/2}$ neighbourhood of 0. Arguing as in the proof of Lemma 4.1 we first prove that there is no nonzero solution of

$$-\Delta p + p f(p) = 0,$$

then finish the proof as in the proof of Theorem 4.1. □

Remark 4.1 *We point out here the similarity between the above case* **2** *and the singular perturbation problems studied in, for instance, [2] and [6]. This emphasizes the fact that we are studying a point where the flame is premixed.*

5. Nonlinear diffusion

In this section we concentrate on pure Burke-Schuman models, but the additional effect that we want to observe is nonlinear diffusion. This is, as a matter of fact, the natural assumption in flame theory; see [21]. We need, however, maximum principles that we are not able to ensure in systems with cross-diffusion effects.

Part [i] of Theorem 1.3 follows from Theorem 1.1; hence the uniform convergence of $(Y_O^\varepsilon, Y_F^\varepsilon)$ is granted. Part [ii] follows from standard weak convergence arguments. Hence we only need to say something about Part [iii], and this the object of the next paragraph.

5.1. Uniform Lipschitz regularity

In this case, it seems to be less easy to derive a global equation for the function β^ε, that would imply a $C^{1,\alpha}$ estimate for it. Let us, however, notice that the limiting function β - due to (1.12) - satisfies an equation of the following type:

$$-\sum_{i,j} \bar{A}_{ij}(\beta)\partial_{ij}\beta + \sum_i B_i(\beta)\partial_i\beta = 0.$$

We have set $\bar{A} = (\bar{A}_{ij})_{1\le i,j\le N}$; because $A_O(0,0) = A_F(0,0)$ the functions $\bar{A}_{ij}(\beta)$ are Hölder in ξ. The functions B_i are L^∞; consequently we have $\beta \in W^{2,p}(\Omega)$ for all $p \in (1, +\infty)$. Hence $\beta \in C^{1,\alpha}(\bar{\Omega})$.

In the sequel, σ will be a small number; in any case much smaller than $\dfrac{1}{3}$. Choose any point x_0 in Ω.

Case 1. We have $Y_O^\varepsilon(x_0) \ge \varepsilon^\sigma$. Therefore there is $C > 0$ such that

$$Y_O^\varepsilon(x) \ge \frac{\varepsilon^\sigma}{2} \quad \text{in } B_{\varepsilon^{\sigma/\alpha}/C}(x_0).$$

The complete combustion principle implies

$$Y_F^\varepsilon(x) \le Ce^{-\varepsilon^{-1/4}} \quad \text{in } B_{\varepsilon^{\sigma/\alpha}/2C}(x_0);$$

hence a bound of the same sort in ∇Y_F^ε. This implies, by rescaling, a uniform C^α bound for ∇Y_O^ε and $D^2 Y_O$.

The same argument would hold if we had $Y_F^\varepsilon(x_0) \ge \varepsilon^\sigma$.

Case 2. We have $Y_O^\varepsilon(x_0), Y_F^\varepsilon(x_0) \le \varepsilon^\sigma$. Set, once again

$$\beta^\varepsilon(x) = Y_O^\varepsilon(x) - \frac{1}{d} Y_F^\varepsilon(x);$$

rescale x once again as

$$x_0 + \varepsilon^{1/3}\xi, \qquad (p_O^\varepsilon, p_F^\varepsilon(\xi), b^\varepsilon(\xi)) = \varepsilon^{-1/3}(Y_O^\varepsilon, Y_F^\varepsilon, \beta^\varepsilon)(x_0 + \varepsilon^{1/3}\xi).$$

Case 2.1. We have $\lim\limits_{\varepsilon\to 0} b^\varepsilon(0) = \pm\infty$. Same argument as in Case 1.

Case 2.2. The sequence $(b^\varepsilon(0))_\varepsilon$ is bounded. Then, an equation for b^ε in the vicinity of x_0 can then be written as - we set $B_{O,F} = A_{O,F} - A_{O,F}(0,0)$:

$$-\div (A_O(0,0)\nabla b^\varepsilon) = \div(B_O(Y_O^\varepsilon, Y_F^\varepsilon)\nabla p_O^\varepsilon + B_F(Y_O^\varepsilon, Y_F^\varepsilon)\nabla p_F^\varepsilon). \tag{5.1}$$

The matrices B_O and B_F are denoted by $(B_O^{ij})_{1\leq i,j\leq N}$ and $(B_F^{ij})_{1\leq i,j\leq N}$. Set

$$\Omega_1^\varepsilon = \{Y_O > \varepsilon^\sigma\} \cup \{Y_F > \varepsilon^\sigma\}; \quad \Omega_2^\varepsilon = \Omega \backslash \bar{\Omega}_1^\varepsilon.$$

Let $(\gamma_1(\xi), \gamma_2(\xi))$ be a partition of unity in $\varepsilon^{-1/3}(\Omega - x_0)$: $\gamma_1 + \gamma_2 \equiv 1$ in $\varepsilon^{-1/3}(\Omega - x_0)$; $\mathrm{supp}\gamma_2 \subset \{Y_O > \varepsilon^{\sigma/2}\}$. In particular, γ_1 and γ_2 have uniform $C^{2,\alpha}$ estimates - in ξ.

Next, we set

$$\begin{aligned} \Lambda_O^{ij}(p_O, b) = & \int_0^{p_O} B_O^{ij}(\varepsilon^{1/3}p'_O, \varepsilon^{1/3}(p'_O - b))\, dp'_O \\ \Lambda_F^{ij}(p_F, b) = & \int_0^{p_F} B_F^{ij}(\varepsilon^{1/3}(p'_F + b), \varepsilon^{1/3}p'_F)\, dp'_F \end{aligned}$$

We have then

$$\begin{aligned} B_O^{ij}(Y_0, Y_F)\partial_j p_O^\varepsilon = & \partial_{\xi_j}\Lambda_O^{ij} - \varepsilon^{1/3}\partial_\beta \Lambda_O^{ij}\partial_j b^\varepsilon \\ B_F^{ij}(Y_0, Y_F)\partial_j p_F^\varepsilon = & \partial_{\xi_j}\Lambda_F^{ij} - \varepsilon^{1/3}\partial_\beta \Lambda_F^{ij}\partial_j b^\varepsilon \end{aligned} \tag{5.2}$$

In the functions $\Lambda_{O,F}^{ij}$ we have - in order to alleviate the notations - omitted the arguments (Y_O, Y_F). Equation (5.1), together with (5.2), implies, with the Einstein summation convention:

$$-\div(A_O(0,0)\nabla b^\varepsilon = \varepsilon^{1/3}\partial_i(\partial_\beta(\Lambda_O^{ij} + \Lambda_F^{ij})\partial_j b^\varepsilon) - \partial_{ij}(\Lambda_O^{ij} + \Lambda_F^{ij}) \tag{5.3}$$

Let b_1^ε be the unique solution of

$$-\div(A_O(0,0)\nabla b_1^\varepsilon = -\partial_{ij}\Big((\Lambda_O^{ij} + \Lambda_F^{ij})\gamma_1)\Big)$$

in $\varepsilon^{-1/3}(\Omega - x_0)$, with the corresponding boundary conditions. If $G_0(\xi)$ is the Green function of the operator $-\div(A_O(0,0)\nabla)$ in $\mathbb{R}^N$, we have

$$b_1^\varepsilon \sim -G_0 * \partial_{ij}\Big((\Lambda_O^{ij} + \Lambda_F^{ij})\gamma_1)\Big) \tag{5.4}$$

in an $\varepsilon^{-\sigma/2}$-neighbourhood of 0. We notice that b_1^ε is uniformly $C^{2,\alpha}$ on ε, in an $\varepsilon^{-\sigma/2}$-neighbourhood of 0.

Now, we set

$$b_2^\varepsilon = b^\varepsilon - b_1^\varepsilon.$$

There are N^2 functions $c_{ij}^\varepsilon(\xi)$, such that $\|c_{ij}^\varepsilon\|_{C^\alpha(B_{\varepsilon^{-\sigma/2}})}$ is uniformly bounded with respect to ε.

$$-\div(A_O(0,0)\nabla b_2^\varepsilon = -\varepsilon^{1/3}\partial_i(\partial_\beta(\Lambda_O^{ij} + \Lambda_F^{ij})\partial_j b_2^\varepsilon) + \varepsilon^\sigma \partial_{ij}c_{ij}^\varepsilon.$$

Hence we have

$$\|c_{ij}^\varepsilon\|_{C^\alpha(B_{\varepsilon^{-\sigma/2}})} \leq C\varepsilon^{\sigma/2}. \tag{5.5}$$

Consequently, the function b^ε decomposes into a uniformly $C^{2,\alpha}$ function and a small one. The proof of Theorem 4.1 from then on applies. $\square$

5.2. Concluding remarks

5.2.1. Error estimates

The theory developped in the preceding section readily implies readily error estimates. For the sake of simplicity we formulate them in the context of Problem (1.11), accounted for in Theorem 1.3. Also assume

- that we are in the cylinder case, i.e $\Omega = \{(x,y) \in (-L,L) \times \omega\}$;
- the velocity field V only depends on y:

Thus the limiting steady problem for β has a unique solution. This can be seen by the sliding method [3]. Let (Y_O^∞, Y_F^∞) be the limiting solution.

Theorem 5.1 *Assume $A_O(0,0)$ and $A_F(0,0)$ to be proportional. For every $\alpha < \dfrac{1}{3}$ there is $C_\alpha > 0$ such that*

$$|Y_O^\varepsilon - \beta^+| + |Y_F^\varepsilon - \beta^-| \leq C_\alpha \varepsilon^\alpha. \tag{5.6}$$

Proof. Consider any $\alpha < \dfrac{1}{3}$.

1. In the area $\{\beta^\varepsilon \leq \varepsilon^\alpha\}$ we notice that $\partial_x \beta^\varepsilon \geq \delta$ for some $\delta > 0$ independent of ε. This, together with the uniform Lipschitz bound for $Y_{O,F}^\varepsilon$, implies the error bound.

2. In the area $\{\beta^\varepsilon \geq \varepsilon^\alpha\}$ we have $Y_F^\varepsilon \leq Ce^{-\varepsilon^{-\alpha/2}}$. Hence

$$\frac{1}{\varepsilon} Y_O^\varepsilon Y_F^\varepsilon << \varepsilon^\alpha, \quad \|Y_F\|_{C^{2,\alpha}(\{\beta^\varepsilon > \varepsilon^\alpha\})} << \varepsilon^\alpha.$$

The area $\partial(\{\beta^\varepsilon > \varepsilon^\alpha\})$ has a nonempty intersection with Σ_0 - i.e. where the values of Y_O and Y_F are imposed. Setting

$$\tilde{A}^\varepsilon(x) = \int_0^1 \partial_Y A_O(Y_O^\varepsilon + tY_O^\infty, 0)\ dt$$

we have

$$|-\div(A^\varepsilon \nabla(Y_O^\varepsilon - Y_O^\infty)) + v.\nabla(Y_O^\varepsilon - Y_O^\infty)| << \varepsilon^\alpha.$$

with Dirichlet conditions on Σ_0, Neumann boundary conditions on Σ_1 and $Y_O^\varepsilon - Y_O^\infty = O(\varepsilon^\alpha)$ on $\partial(\{\beta^\varepsilon > \varepsilon^\alpha\}) \cap \Omega$. From the global Lipschitz bound for Y_O^ε, the operator $-\div(A^\varepsilon \nabla)$ can be written as a nondivergence elliptic operator with L^∞ coefficients; hence [4] it has a first eigenvalue that is bounded away from 0 independently of ε. This implies once again the error bound in the considered region.

3. In the area $\{\beta^\varepsilon \leq -\varepsilon^\alpha\}$ we argue as above. $\square$

5.2.2. Monotonicity formula

Another tool to prove the global Lipschitz regularity - once Hölder is known - in the cases where it is possible is the monotonicity formula, as presented in [7] and used in [8]. Let us consider System 1.11 with the assumption that $A_O(0,0)$ and $A_F(0,0)$

are proportional. Once the function b^ε is proved to be a small quantity plus a $C^{1,\alpha}$ function we may prove an equality of the form

$$(Y_O - A\varepsilon^{1/3})^+(Y_F - A\varepsilon^{1/3})^+ = 0. \tag{5.7}$$

Also, recall that Y_0 and Y_F are uniformly α-Hölder, for some $\alpha > 0$: this implies that, up to a change of coordinates, we are in the conditions of application of Lemma 2.1 of [7]. The function

$$r \mapsto \frac{1}{r^4 e^{-r^\alpha}} \int_{B_r(0)} \frac{|\nabla(Y_O - A\varepsilon^{1/3})^+|}{|x|^{N-2}}\, dx \int_{B_r(0)} \frac{|\nabla(Y_F - A\varepsilon^{1/3})^+|}{|x|^{N-2}}\, dx$$

is nonincreasing. Arguing as in Step 1 of Theorem 5.1 of [8] implies the boundedness of $\varepsilon^{-1/3}(Y_{O,F} - A\varepsilon^{1/3})$; hence the result in a straightforward way.

In the general case - i.e. $A_O(0,0)$ and $A_F(0,0)$ are not proportional - the monotonicity formula is not guaranteed. See a counter-example in [9].

APPENDIX: the flame layer in the high activation energy model

Proposition A1. *Assume $\gamma_1 > -1$ and $\alpha \geq \beta_0$. There is $\delta_{cr}(\beta_0, \alpha) > 0$ such that Problem (3.15) has at least two solutions $p^-_{\delta,\beta_0,\alpha} < p^+_{\delta,\beta_0,\alpha} \leq 0$ for $\delta > \delta_{cr}(\beta_0,\alpha)$, and no solution for $\delta \leq \delta_{cr}(\beta_0,\alpha)$. Moreover*

- *We have $p^\pm(\zeta) + \gamma_1 < -\beta_0|\zeta|$,*
- *the branche $\delta \mapsto p^\pm_{\delta,\beta_0,\alpha}$ are smooth,*
- *consider $\delta > \delta_{cr}(\beta_0,\alpha) > 0$, and $A > 0$ large enough so that $p^\pm_{\delta,\beta_0,\alpha}(\pm A) < -1$. Also consider the operator*

$$\mathbb{L}^\pm_{\delta,\beta_0,\alpha} = -\frac{d^2}{d\zeta^2} - \partial_p f_{\xi_0}(., p^\pm_{\delta,\beta_0,\alpha}), \tag{5.8}$$

defined for all $H^2([-A,A])$ functions with zero derivative at $\pm A$. Then its first eigenvalue $\mu_1(\mathbb{L}^\pm_{\delta,\beta_0,\alpha})$ is positive - resp. negative.

Proof. First, notice that it is enough to assume $\gamma_1 = 0$. Then, assume $\alpha = \beta_0$ until step **4**.

1. Estimates. Let us first carry out the proof for the stable branch. For all $\alpha \in [0,1]$, the function $\zeta \mapsto -\alpha|\zeta|$ is a super-solution to (3.15). Hence any solution of (3.15) is below $\zeta \mapsto -\beta_0|\zeta|$.

2. Existence and stability for large δ: a standard geometric singular perturbation argument (see Jones [15]) shows that $p_{\beta_0,\delta}(\zeta) \sim -\beta_0|\zeta|$. Moreover, it yields uniqueness for large δ's.

On the other hand, the function $q_{\beta_0,\delta} := \partial_\delta p_{\beta_0,\delta}$ satisfies, because of Step **1**:

$$\mathbb{L} q_{\beta_0,\delta} > 0, \quad q'_{\beta_0,\delta}(\pm A) = 0.$$

Hence $\mu_1(_{\beta_0,\delta}) > 0$.
3. Continuation. The implict functions theorem implies the existence of a smooth branch $p_{\beta_0,\delta}$ down to some $\delta_{cr}(\beta_0) \geq 0$. Let us prove that $\delta_{cr}(\beta_0) > 0$; to see this let us notice that any solution p of (3.15) satisfies

$$p'' \geq -\delta\|\phi\|_\infty,$$

hence $p'(+\infty)$ cannot reach $-|\beta_0|$ if δ is too small. This also proves that we cannot have $\lim_{\delta\to\delta_{cr}} p_{\beta_0,\delta} = 0$.
4. Uniqueness. If $\delta_0 > \delta_{cr}(\beta_0) \geq 0$ is given, any solution p of (3.15) is below p_{δ,β_0} if δ large. Then decrease δ to get a contact point between p and p_{δ,β_0}.
5. $\alpha > \beta_0$**.** Let us notice that any solution $p_{\delta,\beta_0,\alpha}$ is below $-\beta_0|\zeta|$. It is certainly true for large negative ζ; consider then the first point ζ_0 when $p_{\delta,\beta_0,\alpha}$ and $-\beta_0|\zeta|$ meet; at that point (i) the two functions are differentiable - strong maximum principle - and (ii) the derivatives are strictly ordered. Hence $p'_{\delta,\beta_0,\alpha}(\zeta)$ is increasing; hence is always above $-\beta_0$. This contradicts the condition at $+\infty$.

Once we have this estimate, we may follow steps **1** to **4**.
6. Unstable branch. We first notice that the solution is now decreasing with respect to δ. We may get some help from the singular profile $\min(-\alpha|\zeta|, -1)$. What may occur is that the branch is degenerate at some $\delta_0 > \delta_c r(\beta_0, \alpha)$ and hence might lose its smoothness. However, for $\delta \in (\delta_{cr}, \delta_0)$ there has to be a solution, by a standard - see [20] - topological degree argument. □

Remark 5.1 *We believe that the solutions $p_\pm$ constructed in Proposition A1 are the only ones, and that the branch p_- is smooth - at least in most cases of interest. Should this fact be true, this would open the way to a much more precise description of the multi-D solutions constructed in Section 3.2, and would render the complete S-shaped solution curve claimed in the classical physics textbooks.*

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