The viscosity solutions approach to the regularity of minimal surfaces.

KAMFAIBLE project – Fall 2009 meeting.
CIRM – October 19–24.

Remerciements.

Introduction. When I was asked to do this course, I also got the message that what was really required from me was the exposition of some analytical ideas and methods in the domain of viscosity solutions. My main competence in the domain is the large-time behaviour in HJ equations, a subject largely covered by Olivier Ley last year. On the other hand nothing was really said about the regularity of viscosity solutions: an important topic that is a full part of the KAMFAIBLE project. Of course 1st order (HJ) equations are already largely covered, but 2nd order equations have not been treated yet. This is the reason why I will in this lecture concentrate on the regularity of second-order equations in the context of viscosity solutions.

The story begins with Jensen's lemma (an estimate on the volume of lower contact points for a semiconcave function) and the extension by Krylov of the Harnack inequality to elliptic equations with coefficients with no real regularity. Major results are also due to Crandall, Evans, Lions, Ishii. And the theory comes to an almost complete point with the works of Caffarelli on fully nonlinear equations. A lot can be found in the Caffarelli-Cabré book (CC 1). One of the problems is that it is not really possible to give a reasonable account of the many ideas of this book in 7 hours and a half. Therefore I have chosen to treat the subject through the <u>regularity of minimal surfaces</u>.

This is not a new subject. The main result (that minimal surfaces are smooth) was established by de Giorgi in 1969 after the efforts of many authors. A fresh impulsion was given to the subject by Caffarelli-Cordoba [CC2] by throwing viscosity solutions in the picture. These new ideas led to important progress on the de Giorgi conjecture (a question asked by de Giorgi on the stable configurations of a phase transition model) and to its proof by Savin [S1]. Even more recently these ideas enabled us (in a work where I was involved) to prove a de Giorgi type result on a nonlocal model [CafRS]. And this is why I can explain things on minimal surfaces.

The program is therefore the following:

- explain a proof of the de Giorgi theorem, as is presented in [CC2] and a very nice paper of Savin [S2] where he explains how his ideas on the de Giorgi conjecture a fresh look on the regularity of minimal surfaces. Because not all the details are written in [S2], I think that it is interesting to work out the whole proof in a reasonably complete fashion,

- explain very briefly the de Giorgi conjecture, how it was proved, and explain (if time permits) how the ideas described in the whole lecture can helps in the understanding of an a priori completely different model.

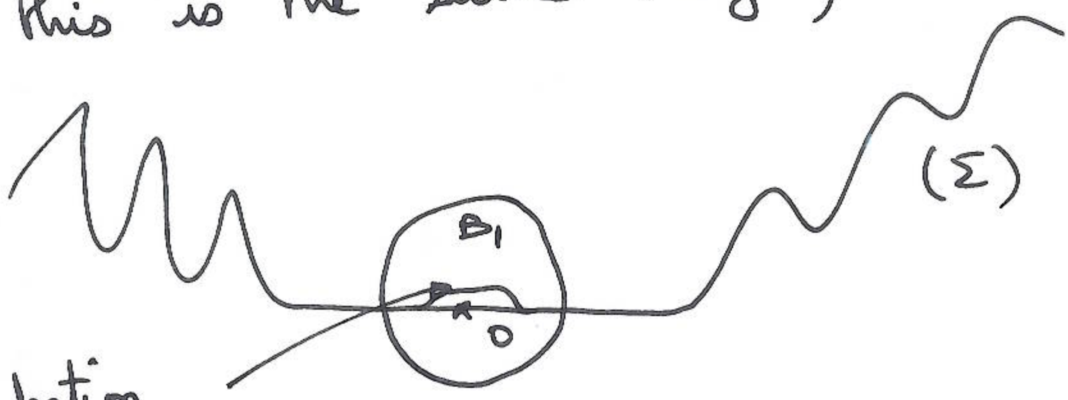
- If time permits, talk a bit about the de Giorgi conjecture (that was, by the way, discussed at length in the previous meeting).

Let me now be more specific.

1. Minimal surfaces -

Roughly speaking, we will say that a surface Σ (to fix ideas, think of a reasonably smooth (Lipschitz, for instance) graph in some coordinates) is minimal in a given domain Ω if any perturbation of Σ within Ω will increase its area (sometimes I will also talk about its perimeter: this is the same thing).

Example



A perturbation of Σ.

The question is: how smooth is such a thing? Can it have corners, cusps? At least in 2 space dimensions the situation looks clear, but in more dimensions?

Here is an account of the situation.

- The 1st serious result is due to Bernstein (1915): an entire 2D minimal graph is a plane,
- Solutions to the Plateau problem in 3D (can we find surfaces of least area bounded by a given curve?) by Douglas and Radò in the 30's,
- in the 60's, a massive attack of the problem by geometric measure theory methods (Federer, Almgren, Fleming ...)
- The major blow was struck by de Giorgi (1969). The main ideas was to consider a surface Σ as the boundary of a set E, and argue on the indicator function of E.

In the whole lecture, if E is a subset of $\mathbb{R}^N$ with boundary S, we will almost always assume that $0 \in S$.

<u>def</u>. If E is a subset of $\mathbb{R}^N$ with $0 \in \partial E$, E is a <u>Caccioppoli set</u> in B_1 iff $\mathbb{1}_{E \cap B_1}$ is a BV function in B_1.

<u>def</u>. $f \in L^1(B_1)$;

$$\|f\|_{BV(B_1)} = \sup \left\{ \int_{B_1} f \operatorname{div} g , \; g \in C^1_0(B_1), |g| \leq 1 \right\}.$$

<u>Remark</u>. $f \in BV(B_1) \iff Df$ is a Radon measure with finite mass in B_1. And $\|Df\|_{BV(B_1)}$ is the total mass of the measure $|Df|$.

<u>Remark</u>. If E is a bounded open subset of B_1 with smooth boundary, $D\mathbb{1}_E = \vec{n}\, d\sigma_E$ where $d\sigma_E$ is the classical surface measure on ∂E, and $\vec{n}$ the outer normal vector field.

<u>def</u>. E is a minimal set in $B_1 \iff$

[i]. E is a Caccioppoli set,

(ii). For every Caccioppoli sets F in B_1, coinciding with E outside B_1, we have

$$\int_{B_1} |D\mathbb{1}_E| \leq \int_{B_1} |D\mathbb{1}_F|.$$

We will denote:

$$\mathrm{Per}(E, B_1) = \int_{B_1} |D \mathbb{1}_E| .$$

The following can be proved:

- $\mathrm{Per}(E \cup F, B_1) \leq \mathrm{Per}(E, B_1) + \mathrm{Per}(F, B_1)$.

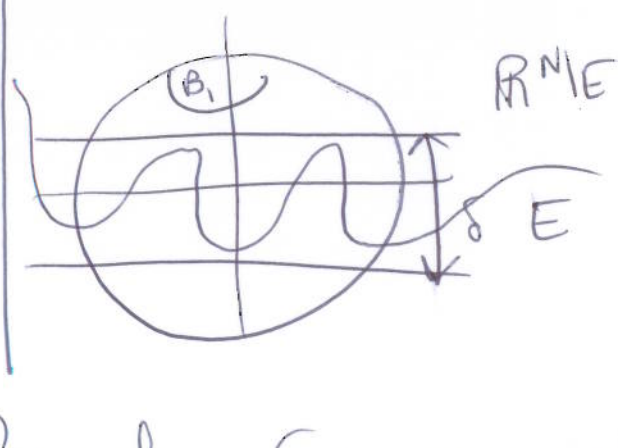
Equality holds if $d(E, F) > 0$.

The main result that we will discuss is the

<u>Th.</u> (de Giorgi, 1969). E: minimal set in B_1, $0 \in \partial E$. Assume the existence of $\delta > 0$ such that:

$$E \cap B_1 \subset \{|x'| \leq 1, x_N \leq \delta\}.$$

$$E \cap B_1 \supset \{|x'| \leq 1, x_N \leq -\delta\}.$$



If δ is small enough, then $\partial E \cap B_{1/2}$ is a $C^{1,\alpha}$ graph.

<u>Remark</u>. One may easily prove, by compactness, that there are minimal sets.

The standard theory of elliptic equations implies that $\partial E \cap B_{1/2}$ is an analytic piece of surface.

Let us say a word about how the regularity theory ends.

<u>Th</u> (Federer) [F] E : minimal set in B_1 ; which is <u>not</u> analytic ~~in in a neighbourhood~~ at O. Then there is an N-1-dimensional minimal cone with vertex at O.

<u>Th</u>. (Simons) [Sim] If $N \leq 7$, there is no minimal cone.

<u>Th</u>. (Bombieri - de Giorgi - Giusti). ([BGG]) the Simons cone $\{ x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2 \}$ is minimal.

<u>Final theorem</u>. E minimal set.

• If $N \leq 7$, a minimal set is smooth.

• If $N > 7$, the Hausdorff dimension of the singular set is N-8.

As said before, what we will discuss is the de Giorgi regularity theorem. The de Giorgi proof is discussed at length in the book of Giusti [G].

2. The plan of the notes.

We will discuss in detail the proof of the de Giorgi theorems with elements of viscosity solutions, taken from Caffarelli - Cordoba [CC2] and Savin [S2]. Thus, we will go through the following steps.

I). General properties of minimal ~~sets~~.

II). Viscosity properties.

III). The geometric Harnack inequality.

IV). Proof of the de Giorgi theorem and applications.

V). (if time permits) Nonlocal minimal surfaces.

I). Some general properties of minimal sets

To get acquainted with minimal surfaces and minimal sets, let us prove the

<u>Prop.</u> E minimal set in B_1, $0 \in \partial E$.

There is $q > 0$, universal, such that

$$\forall r < 1, \quad \frac{|B_r \cap E|}{|B_r|} \geq q .$$

<u>Interpretation</u>. A cusp (i.e. 2 C^1 hypersurfaces meeting at a cusp 

)

is forbidden.

Of course ∂E may have a much wilder imagination than 2 hypersurfaces meeting at a cusp!

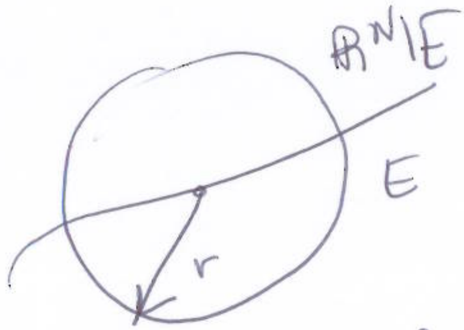
<u>Proof of the proposition</u>. We use the Sobolev imbedding: $\exists\, C > 0$ such that, for all $f \in BV(B_1)$, f compactly supported in B_1:

$$\|f\|_{L^{\frac{N}{N-1}}(B_1)} \leq C \int_{B_1} |Df| .$$

We claim that it is sufficient to prove this for $r = 1$, the general result being obtained by rescaling.

Apply the Sobolev inequality to $f := \mathbb{1}_{E\cap B_r}$,

$$\|\mathbb{1}_{E\cap B_r}\|_{L^{\frac{N}{N-1}}} \le C\int_{B_1} |D\mathbb{1}_{E\cap B_r}|.$$



- $\|\mathbb{1}_{E\cap B_r}\|_{L^{N/N-1}} = |E\cap B_r|^{\frac{N-1}{N}}$.

- $\displaystyle\int_{B_1} |D\mathbb{1}_{E\cap B_r}| = \int_{\cancel{E\cap} B_r} |D\mathbb{1}_E| + \int_{B_1} |D\mathbb{1}_{E\setminus B_r}|$

$$= \mathrm{Per}(E, B_r) + \int_{\partial B_r} \mathbb{1}_E \, d\sigma.$$

$$\underset{(\text{by minimality})}{\le} 2\int_{\partial B_r} \mathbb{1}_E \, d\sigma$$

$$= 2|E\cap S_r|$$

$$= 2\frac{d}{dr}|E\cap B_r|.$$

We have used here the fact that $r \mapsto |E\cap B_r|$ is lipschitz, ~~hence differentiable~~.

Hence differentiability almost everywhere.
Therefore there is $C>0$ such that

$$\frac{\varphi'}{\varphi^{\frac{N-1}{N}}} \geqslant C.$$

Integrate between $\frac{1}{2}$ and 1.

$$\frac{d}{dr}\varphi^{\frac{1}{N}} \geqslant \frac{C}{N} \quad \text{and thus}$$

$$\varphi\left(\tfrac{1}{2}\right) \leq \varphi(1) - \frac{C}{2N}.$$

Thus, if $\varphi(1) \leq \frac{C}{4N}$, we have

$$\varphi\left(\tfrac{1}{2}\right) < 0.$$

This does not look good. Hence $|B_1 \cap E| \geqslant \frac{C}{4N}$ and the result is proved. $\boxtimes$

The next result is a very important property which, in particular, allows to look at closures without worrying too much.

<u>Corollary</u>. If E is minimal in B_1, with $0 \in \partial E$, there is $q \in (0,1)$ universal such that: $\forall r \in (0,1)$, there exists a ball

B_{qr} of radius qr such that

$$B_{qr} \subset E \cap B_r.$$

Moreover there is a ball B'_{qr} of radius qr such that

$$B'_{qr} \subset (\mathbb{R}^N \setminus E) \cap B_r.$$

This property is known as the "clean ball property".

Consequence. E cannot have a bunch of filaments in a vicinity of O:

The proof of the clean ball property relies on the following consequence of the positive density property:

Corollary. E minimal in B_1, $O \in \partial E$.

There is $q>0$, universal, such that

$$\mathrm{Per}\,(E \cap B_r) \geq q r^{N-1}.$$

We may now go to the

<u>Proof of the corollary.</u>

This time we use the Sobolev inequality

$$\| f - \int f \|_{L^{\frac{N}{N-1}}(B_1)} \leq C \int_{B_1} |Df| ,$$

if $f \in BV(\mathbb{R}^N)$. If $f = \mathbb{1}_E$, then

$$\left\| \mathbb{1}_E - \frac{|E \cap B_1|}{|B_1|} \right\|_{L^{\frac{N}{N-1}}(B_1)} \leq C \, \mathrm{Per}(\partial E \cap B_1) .$$

$$\left\| \mathbb{1}_E - \frac{|E \cap B_1|}{B_1} \right\|_{L^{\frac{N}{N-1}}} = \left(\int_{B_1} \left(\mathbb{1}_E - \frac{|E \cap B_1|}{|B_1|} \right)^{\frac{N}{N-1}} \right)^{\frac{N-1}{N}}$$

$$= \left(\int_{E \cap B_1} \left(1 - \frac{|E \cap B_1|}{|B_1|} \right)^{\frac{N}{N-1}} + \int_{B_1 \setminus (E \cap B_1)} \left(\frac{|E \cap B_1|}{|B_1|} \right)^{\frac{N}{N-1}} \right)^{\frac{N-1}{N}}$$

$$= \left(\frac{|E \cap B_1| \, |B_1 \setminus (E \cap B_1)|^{\frac{N}{N-1}} + |E \cap B_1|^{\frac{N}{N-1}} |B_1 \setminus (E \cap B_1)|}{|B_1|^{\frac{N}{N-1}}} \right)^{\frac{N-1}{N}}$$

We have $|E \cap B_1|, \ |B_1 \setminus (E \cap B_1)| \geq c$.

Hence $\int_{B_1} |D \mathbb{1}_E| = \mathrm{Per}(\partial E \cap B_1) \geq c$.

~~We may now go to the~~

~~Proof of the corollary. Once again we may assume $r = 1$. The Sobolev inequality yields~~

$$\left\| 1\!\!1_E - \frac{|E\cap B_1|}{|B_1|} \right\|_{L^{\frac{N}{N-1}}(B_1)} \leq C \int_{B_1} |D 1\!\!1_E|$$

~~(Here we have used $\| f - f_{B_1} \|_{L^{\frac{N}{N-1}}(B_1)} \leq C \int_{B_1} |Df|$). and we use the proposition.~~

<u>Proof of the clean ball corollary</u>. Once again, assume $r = 1$. For small $\delta > 0$, cover $B_{1/2}$ with balls $\left(B_{3\delta}(x_i)\right)_{i\in I}$ such that

$$i \neq j \Longrightarrow B_\delta(x_i) \cap B_\delta(x_j) \neq 0.$$

Let us then assume that

$$\forall i \in I, \quad B_{\frac{\delta}{4}}(x_i) \cap \partial E \neq \emptyset .$$

Let $y_i \in B_{\frac{\delta}{4}}(x_i) \cap \partial E$, then

$$B_{\frac{\delta}{8}}(y_i) \subset B_\delta(x_i).$$

Thus, $i \neq j \Longrightarrow B_{\frac{\delta}{8}}(y_i) \cap B_{\frac{\delta}{8}}(y_j) \neq \emptyset$.

On the other hand we have

$\forall i \in I, \; \mathrm{Per}\left(B_{\frac{\delta}{8}}(y_i) \cap \partial E\right) \geq C\delta^{N-1}.$

Hence $\quad \mathrm{Per}\left(\partial E \cap B_{1/2}\right) \geq C N_\delta \delta^{N-1}$

where N_δ is the cardinal of I, $N_\delta \sim \delta^{-N}$.
This cannot be if δ is too small. $\boxtimes$

We ~~have not~~ also mention the following monotonicity property, that we will admit (because we will not use it):

<u>Prop.</u> E minimal in B_1, $0 \in \partial E$. The quantity $r \longmapsto \dfrac{\mathrm{Per}(E \cap B_r)}{r^{N-1}}$ is decreasing.

We will, however, need to integrate by parts in Caccioppoli sets. In particular, if Ω is a smooth bounded open subset of $\mathbb{R}^{N-1}$ we have the Green's formula: if f and g are smooth.

$$-\int_\Omega f\Delta g = \int_\Omega |\nabla f|^2 + \int_{\partial\Omega} f\frac{\partial g}{\partial n}.$$

n: outer normal to Ω. And also:

$$\int_\Omega g\Delta f - f\Delta g = \int_{\partial\Omega} f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}.$$

These formulae come from

$$\int_\Omega \operatorname{div} f \, dx = \int_{\partial\Omega} Df \cdot \vec{n} \, d\sigma(x) .$$

We are not worried here by the smoothness of f and g, we are worried by the smoothness of Ω.

We could develop a whole theory of the subject; however let us just write down the result that will be useful to us:

<u>Th.</u> (Approximation of Caccioppoli sets). E bounded Caccioppoli set; there is a sequence of C^∞ bounded subsets $(E_n)_n$ such that:

$$\lim_{n \to +\infty} \int |\mathbb{1}_{E_n} - \mathbb{1}_E| = 0$$

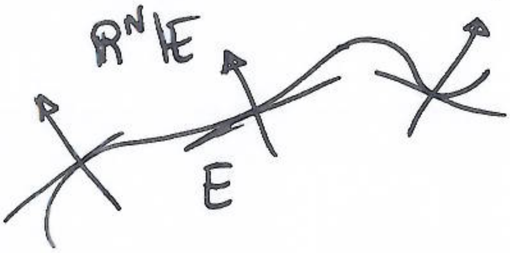
$$\lim_{n \to +\infty} \int |D\mathbb{1}_{E_n}| = \int |D\mathbb{1}_E| .$$

All the topics touched here are very much developped in the book of Giusti [G].

II). Viscosity properties of minimal surfaces.

1°). Basic notions: Geometry for dummies.

Let E be a smooth open subset of $\mathbb{R}^N$, i.e. ∂E is a C^∞ graph in the vicinity of each of its points. Let $d(\cdot, \partial E)$ be



the <u>signed</u> distance function to ∂E, i.e.

- if $x \in E$, then $d(x, \partial E) = -$(the classical distance from x to ∂E),
- if $x \in \mathbb{R}^N \setminus E$, $d(x, \partial E) =$ the classical distance from x to ∂E.

Assume, as always, that $0 \in \partial E$. There is $\delta_0 > 0$ such that the function d is C^∞ in the strip

$$E_{\delta_0} = \{x : |d(x, \partial E)| \leq \delta_0\}.$$

In particular this means that each $x \in E_{\delta_0}$ has a ~~pro~~ unique projection on ∂E, i.e. a point $\pi(x)$ such that

$$|x - \pi(x)| = |d(x, \partial E)|.$$

Let $n(x) = Dd(x, \partial E)$ for all $x \in E_{\delta_0}$, one checks easily that $|Dd(x)| = 1$ and that, if $x \in \partial E$, then $n(x)$ is the outer normal to ∂E at x, i.e. the one pointing towards $\mathbb{R}^N \setminus E$.

If $x_0 \in E_{\delta_0} \cap B_1$, and $\pi(x_0)$ its projection onto ∂E, there is a basis $\mathcal{B}$ of $T_{\pi(x_0)}(\partial E)$ such that, ~~if we complement this basis~~ in the basis $(\mathcal{B}, \vec{n}(\pi(x_0)))$, the matrix of $D^2 n(x_0)$ is

$$\begin{pmatrix} +\frac{\kappa_1(x_0)}{1+d\kappa_1(x_0)} & & & 0 \\ & \ddots & & \\ & & +\frac{\kappa_{N-1}(x_0)}{1+d\kappa_{N-1}(x_0)} & \\ 0 & & & 0 \end{pmatrix}$$

with $d = d(x_0, \partial E)$. The κ_i's are the main curvatures of ∂E at $\pi(x_0)$. The mean curvature of ∂E at $x_0 \in \partial E$ is

$$\kappa = \frac{1}{N-1} \sum_{i=1}^{N-1} \kappa_i(x_0).$$

We also have $\kappa(x_0)$ ~~$= \Delta d($~~

$$= \operatorname{tr} D^2 n(x_0)$$
$$= \operatorname{tr} D^2 d(\cdot, \partial E)\big|_{x = x_0}$$
$$= \Delta d(x_0).$$

Finally, assume (without loss of generality) that ∂E is a graph ~~with~~

$$\{x_N = u(x')\}, \quad (x' = (x_1, \dots, x_{N-1})).$$

in the vicinity of $x_0 = 0 \in \partial E$. Then

$$n(0) = \begin{pmatrix} -\dfrac{Du(0)}{\sqrt{1+|Du(0)|^2}} \\ \dfrac{1}{\sqrt{1+|Du(0)|^2}} \end{pmatrix}$$

provided that $\mathbb{R}^N \setminus E$ is, in the vicinity of 0, in the region $\{x_N > 0\}$.

Let us now consider B'_1, the $(N-1)$ dimensional unit ball, and

$$E = \{(x', x_N) : x_N < u(x')\}$$

where $u \in$, say, $C^\infty(B'_2)$. Assume that $E \cap B_1$ is minimal. One may easily check that (exercise)

$$\mathrm{Per}(E \cap B_1, B) = \int_{B_1} \sqrt{1 + |Du(x')|^2}\, dx'.$$

Perturb E into the set

$$F = \{x_N < u(x') + t\varphi(x')\}$$

φ compactly supported in B_1. We have

$$\int_{B_1} \sqrt{1+|Du(x')|^2}\, dx' \leq \int_{B_1} \sqrt{1+(Du+t D\varphi)^2}\, dx'$$

hence, expanding and letting $t \to 0$:

$$\int_{B_1} \frac{Du \cdot D\varphi}{\sqrt{1+|Du|^2}}\, dx = 0.$$

Hence the equation $\operatorname{div} \dfrac{Du}{\sqrt{1+|Du|^2}} = 0$.

This says that the mean curvature of ∂E is zero.

We may extract some more information by looking outside ∂E. Let $x_0 \in E_{\delta_0}$ such that $d(x_0; \partial E) \neq 0$, we have:

$$(N-1)\Delta d(x_0) = \sum_{i=1}^{N-1} \frac{k_i(x_0)}{1+d(x_0)k_i(x_0)}$$

$$= \sum_{i=1}^{N-1}\left(\frac{k_i(x_0)}{1+d(x_0)k_i(x_0)} - k_i(x_0)\right)$$

$$= -d(x_0)\sum_{i=1}^{N-1} \frac{k_i^2(x_0)}{1+d(x_0)k_i(x_0)}.$$

Hence we have: • $-\Delta d(x_0) < 0$ inside E.

• $-\Delta d(x_0) > 0$ in $\mathbb{R}^N \setminus E$.

We will see that these properties also hold in the (not yet defined) viscosity sense.

BTW, why do we insist in putting a $-$ sign in front of the laplacian? Well, it is a French habit related to the Green's formula: we like to have positive operators.

2º). <u>The Caffarelli - Cordoba theorem for the distance function.</u>

First, a definition.

<u>def. 1</u>. $u : \overline{B}_1 \longrightarrow \mathbb{R}$ is lower semi-continuous if, for all $x \in \overline{B}_1$:

$$\liminf_{x' \xrightarrow{\overline{B}_1} x} u(x') \geq u(x).$$

$u : \overline{B}_1 \longrightarrow \mathbb{R}$ is upper semi-continuous if, for all $x \in \overline{B}_1$:

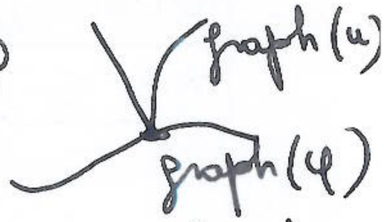
$$\limsup_{x' \xrightarrow{\overline{B}_1} x} u(x') \leq u(x).$$

<u>Thus</u>: • infima of a lsc function are minima.

• suprema of an usc function are maxima.

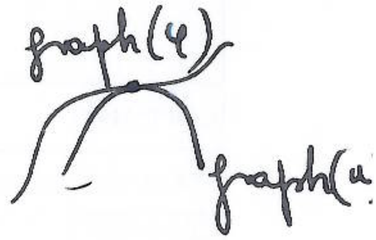
def. 2. • $u : \overline{B}_1 \longrightarrow \mathbb{R}$, lsc, is a viscosity super-solution of $-\Delta u = 0$ in B_1 if and only if: for every $\varphi \in C^2(B_1)$, $x_0 \in B_1$ is a minimum of $u - \varphi$

$$\Longrightarrow -\Delta \varphi(x_0) \geq 0 .$$



• $u : \overline{B}_1 \longrightarrow \mathbb{R}$, u.s.s. is a viscosity sub-solution of $-\Delta u = 0$ in B_1 if and only if: for every $\varphi \in C^2(B_1)$, x_0 is a maximum of $u - \varphi$

$$\Longrightarrow -\Delta \varphi(x_0) \leq 0 .$$



Remark. a classical super-solution of $-\Delta u = 0$ (i.e. ~~for~~ u ~~s.t.~~ $\in C^2(B_1)$ s.t. $-\Delta u \geq 0$ in B_1) is a viscosity super-solution, a classical sub-solution of $-\Delta u = 0$ (i.e. u such that $u \in C^2(B_1)$ and $-\Delta u \leq 0$ in B_1) is a viscosity sub-solution.

This, in contrast with 1^{st} order equations, cannot be taken for granted. ~~Here, just~~

In 1^{st} order equations $(H(Du) \geq 0)$:

$u - \varphi$ has a min at $x_0 \Rightarrow Du(x_0) = D\varphi(x_0)$,

hence $H(D\varphi(x_0)) \geq 0$.

Here: $u - \varphi$ has a min at $x_0 \Rightarrow D^2u(x_0) \geq D^2\varphi(x_0)$ in the sense of quadratic forms. Hence

$$\Delta u(x_0) \geq \Delta \varphi(x_0).$$

$$-\Delta u(x_0) \leq -\Delta\varphi(x_0)$$

and so, $-\Delta\varphi(x_0) \geq 0$. This would NOT have worked it we had replaced $-\Delta$ by Δ!

Exercise. Prove the same result for a uniformly elliptic 2^{nd} order operator

$$a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$$

with $a_{ij}(x)\xi_i\xi_j \geq 0, \ \forall \xi \in \mathbb{R}^N$.

We will see a more general definition in the next paragraph.

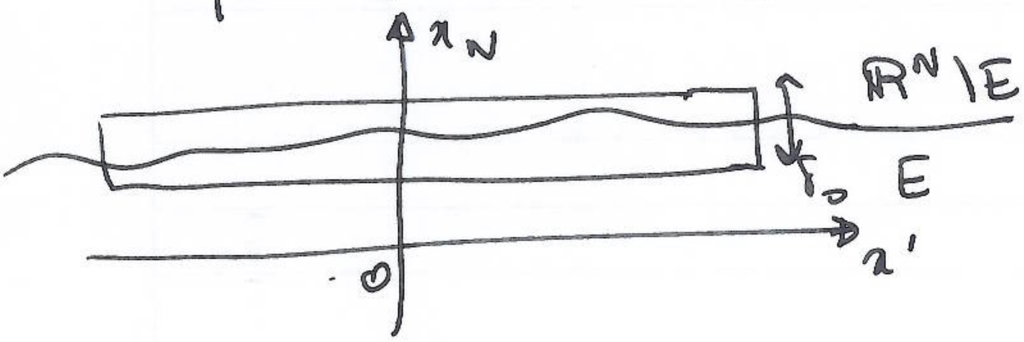
Th. (Caffarelli-Cordoba). E minimal in B_1, with $0 \in \partial E$. Then the distance function $d(\cdot, \partial E)$ satisfies, in the viscosity sense:

- $-\Delta d \cancel{\leq} 0$ in E

|| • $-\Delta d \geq 0$ in $\mathbb{R}^N \setminus (E \cup \partial E)$.

In fact, we will only need the following weaker version of Caffarelli-Córdoba's theorem. Assume ∂E to satisfy the de Giorgi assumption, i.e. there is $\delta_0 > 0$ such that

$$\{|x'| \leq 1,\ x_N \leq +\delta_0\} \subset E.$$
$$\{|x'| \leq 1,\ x_N \geq 2\delta_0\} \subset \mathbb{R}^N \setminus E.$$



For $x' \in B_1^{N-1}$, let

~~$u^-(x') = \sup\{x_N : (x', x_N) \in \partial E\}$.~~

~~$u^+(x') = \sup\{$~~ $\inf\{x_N :$

$$u^-(x') = \sup\{x_N : \forall\, \tilde{x}_N \leq x_N,\ (x', \tilde{x}_N) \in E\}.$$
$$u^+(x') = \inf\{x_N : \forall\, \tilde{x}_N \geq x_N,\ (x', \tilde{x}_N) \in \mathbb{R}^N \setminus E\}.$$

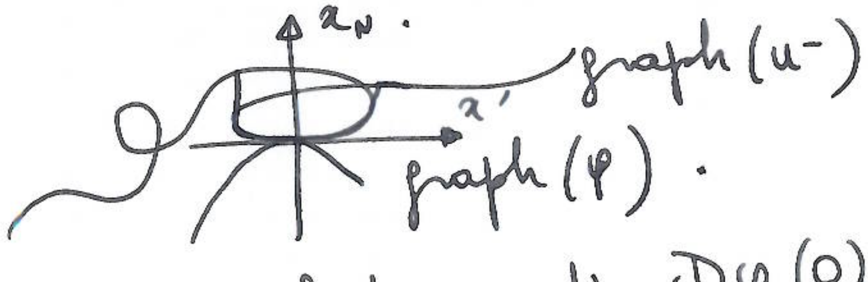
<u>Then</u>: u^- is an lsc function of B_1^{N-1}.
u^+ is an usc function B_1^{N-1}.

<u>Th.</u> (weak Caffarelli-Córdoba). u^- is a viscosity super-solution of $-\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0$ in B_1.
u^+ is a viscosity sub-solution of $-\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0$ in B_1.

The definition of a viscosity super (sub) solution is analogous to that of $-\Delta u = 0$. We notice that, once again, ~~at~~ a classical super (sub) solution is a viscosity super (sub) solution. We will develop this after the proofs.

<u>Proof of the weak Caffarelli-Córdoba.</u>

Let $\varphi \in C^2(B_1^{N-1})$ be such that $u^- - \varphi$ has a maximum at $x'_0 \in B_1^{N-1}$, We may assume $x'_0 = 0$. We may also assume that 0 is a strict minimum, and that $u^- - \varphi(0) = 0$.

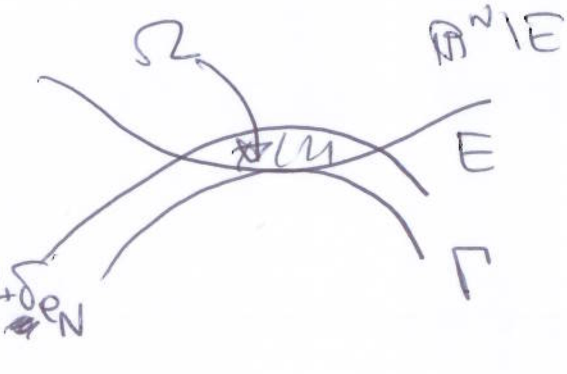


Assume that $-\operatorname{div} \frac{D\varphi(0)}{\sqrt{1+|D\varphi(0)|^2}} < 0$.

Let Γ be the graph of φ; then

$$+\Delta d(x,\Gamma)\Big|_{x=0} = -\operatorname{div} \frac{D\varphi(0)}{\sqrt{1+|D\varphi(0)|^2}}.$$

Because $x \mapsto d(x,\Gamma)$ is smooth in a neighbourhood of 0, we may find a small $\delta_0>0$ such that the following properties are true, for $\delta \le \delta_0$:



(i). $\Gamma+\delta e_N \cap B_1$ is a perturbation of E within B_1,

(ii) $(\Gamma+\delta e_N)_{\text{subgraph}} \cap (\mathbb{R}^N\setminus E)$ has nonempty interior (clean ball property)

(iii). $-\operatorname{div}\dfrac{D\varphi}{\sqrt{1+|D\varphi|^2}} < 0$ in $\{d(x,\Gamma)<\delta\}\cap(\mathbb{R}^N\setminus E)$

However, $-\operatorname{div}\dfrac{D\varphi(0)}{\sqrt{1+|D\varphi(0)|^2}} = \Delta d(\cdot,\Gamma)\Big|_{x=0}$.

So, by further restricting δ: $-\Delta d>0$ in Ω.
Integrate this inside Ω, and assume that everything is smooth. We have

$$\int_\Omega -\Delta d\,dx = -\int_\Omega \operatorname{div}(Dd)\,dx$$

$$= -\int_{\partial\Gamma} Dd\cdot n\,d\sigma - \int_{\partial E} Dd\cdot n\,d\sigma.$$

>0

Hence :

$$\int_{\partial\Gamma} d\sigma(x) < - \int_{\partial E} Dd(x) \cdot n(x)\, d\sigma(x) \leq \mathrm{Per}\,(E \cap \Omega).$$

Contradiction with the minimality of ~~∂E~~ E.

To make this rigorous, approximate E by a sequence $(E_n)_n$ as in Section 1. Integrate $-\Delta d > 0$ in $\Omega_n = (\mathbb{R}^N \setminus E_n) \cap \{d(x', \Gamma) \leq \delta\}$

and obtain :

$$\int_{\partial\Gamma} d\sigma \leq - \int_{\partial E_n} Dd(x) \cdot n(x)\, d\sigma(x)$$

$$\leq \int_{\partial E_n} d\sigma(x) = \mathrm{Per}\,(E_n \cap \Omega).$$

Send $n \to +\infty$, and obtain the contradiction. ☒

<u>Exercise</u>. Prove the strong Caffarelli - Cordoba.

<u>Hints</u>. Assume that $x_0 \in \mathbb{R}^N$ is a minimum point for $d - \varphi$. Then :

(i). φ can be taken as a quadratic polynomial.

(ii). if $d(x_0) = h$, take a coordinate system $(e_1, -, e_N)$ such that

$x_0 = h e_n$. If $h \neq 0$, then $d(x)$ varies in a linear fashion in the vicinity of $h e_n$.

(iii). Prove that $\varphi(x) = h - x_N + Q(x)$ where $Q(x)$ is a quadratic form w. ~~$a_{NN} \neq 0$~~, Assume that $-\Delta Q(x_0)$ has the wrong sign.

(iv). Argue as in the weak Caffarelli – Cordoba theorem.

<u>Remarks</u>. A ~~viscosity~~ classical sub (super) solution of $-\operatorname{div} \dfrac{Du}{\sqrt{1+|Du|^2}} = 0$ is also a viscosity solution. Indeed, expand the operator:

$$-\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = \frac{-\Delta u}{\sqrt{1+|Du|^2}} - \frac{D^2u\, Du \cdot Du}{(1+|Du|^2)^{3/2}}.$$

We claim that, for a given $p \in \mathbb{R}^N$, the operator $-\dfrac{\Delta u}{\sqrt{1+|p|^2}} - \dfrac{D^2 u\, p \cdot p}{(1+|p|^2)^{3/2}}$ is elliptic.

Indeed: $$\sum \frac{\xi_i^2}{\sqrt{1+|p|^2}} - \frac{\sum \xi_i \xi_j p_i p_j}{\sqrt{(1+|p|^2)^3}}$$

$$= \left(\sum \xi_i^2 - \sum\sum \left(\frac{\xi_i p_i}{\sqrt{1+|p|^2}} \right)^2 \right)$$

$$= |\xi|^2 - \frac{(\xi . p)^2}{1+|p|^2} .$$

$$\geq |\xi|^2 \left(1 - \frac{|p|^2}{1+|p|^2} \right).$$

We notice that this operator

- degenerates as $|p| \to +\infty$.
- is a well-behaved elliptic one as soon as $|p|$ is finite.

This suggests that, for the PDE problem

$$- \operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0$$

the main point is to establish lipschitz estimates.

We also notice that the claim "(classical super-solution") $\Rightarrow$ (viscosity super-solution)" is true.

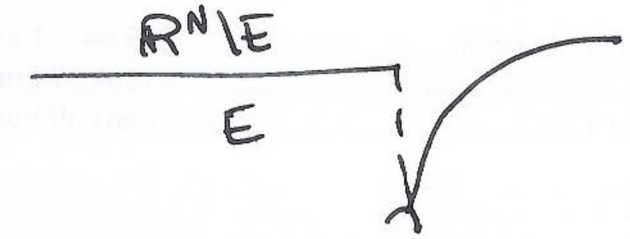
Just because the quadratic form

$$q(\xi) = \sum \frac{\xi_i^2}{\sqrt{1+|p|^2}} - \sum \frac{\xi_i \xi_j p_i p_j}{(1+|p|^2)^{3/2}}$$

is positive.

<u>Remark.</u> (Thanks to P. Cardaliaguet for pointing that out). One needs to be more careful in the definition of $u^{\pm}$. Indeed our proof that they satisfy the curvature equation in the viscosity sense is impeccable. However, because they do not refer to a closed set, they need not be lsc or usc.

<u>Example</u>:



One could instead work with ∂E. However, one needs to observe that E is in general obtained by a compactness argument, and that adding a set of measure 0 to it does not change its perimeter. In particular we could work with $\partial E \cup \mathbb{Q}$ without changing anything to the theory. However, taking the closure of $\partial E \cup \mathbb{Q}$ would be disastrous!

In any case, the perimeter is not very friendly with taking the closure.

The good compromise is the following. Remember that $D\mathbb{1}_E$ is a Radon measure and, as such, has a support, defined as the smallest set S such that: for all $g \in C_0^1(\mathbb{R}^N)$,

(supp g is included in an open set not meeting S)

$\Rightarrow \int g \, D\mathbb{1}_E = 0$.

In particular, we have removed all the useless sets of measure 0.

Now, have a closer look at the clean ball property and replace ∂E by $\mathrm{supp}(D\mathbb{1}_E)$. The crux of the argument is in saying that $\mathrm{Per}(E \cap B_\delta) \geq C\delta^{N-1}$. Now we may write

$$\mathrm{Per}(E \cap B_1) = \int_{\mathrm{supp}(D\mathbb{1}_E) \cap B_1} |D\mathbb{1}_E|$$

$$\geq \sum_{i=1}^{N_\delta} \int_{\mathrm{supp}(D\mathbb{1}_E) \cap B_{\frac{\delta}{3}}(x_i)} |D\mathbb{1}_E|$$

(because $|D\mathbb{1}_E|$ is a positive measure)

$$= \sum_{i=1}^{N_\delta} \mathrm{Per}(E \cap B_\delta(x_i))$$

$$\geq C N_\delta \delta^{N-1}.$$

And we may proceed to the end.

<u>Conclusion</u>. In what follows, the notation "∂E" will be a short notation for $\mathrm{supp}(D\mathbb{1}_E)$. One may also view it as a measure-theoretic boundary.