Partial differential equations with fractional diffusion

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Remerciements.

Introduction. The goal of this course is to investigate different topics (large-time behaviour, qualitative properties, regularity...) concerning reaction-diffusion equations involving a nonlocal diffusion operator. A typical model that will occupy us has the form:

$$u_t + \mathcal{A} u = f(x, u). \qquad (1)$$

f : some nonlinearity, sometimes giving raise to an asymptotic model.

$\mathcal{A}$: some nonlocal operator of "dissipative" type. The archetypal model for $\mathcal{A}$ is the fractional Laplacian:

$$\mathcal{A} u = c_\alpha \; \text{p.v.} \left(\int \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} dy \right)$$

where $\alpha \in (0,1)$. Much more nonlinear operators will be studied as the course develops.

u: some scalar unknown.

The limiting case: $\alpha = 1$ is a very largely studied model

$$u_t - \Delta u = f(x, u). \qquad (2)$$

It arises in countless situations in physics, chemistry, population dynamics, ecology... and even social sciences. One of the oldest models of that sort will be evoked as a comparison basis for the case $\alpha < 1$.

Why should we study the case $\alpha < 1$? There are many papers, mainly in physics (plasma physics, turbulence) where the diffusive processes at work cannot be modelled by a usual laplacian (or any second order operator). If we assume that, for instance, $u(t,x)$ is a particle density, the laplacian operator expresses that the particles move, at the microscopic level, according to the Brownian motion: $\langle x^2 \rangle \sim t$, x^2 being the ~~sp~~ fluctuation w.r.t. an average position. Some experiments show that it is sometimes definitely not the case, we may sometimes make experiments and find out: $\langle x^2 \rangle \sim t^\alpha$, $\alpha < 1$. No underlying theory, just experiments.

However, there are processes where the above relation is satisfied. They are called Lévy processes, the particles have jumps of non infinitesimal lengths and the generator of such a process is precisely the fractional laplacian.

Thus there are many heuristic models of the form (1). However this may not be considered as satisfactory, and we will see a more rigorous derivation of (1).

There are many works on models like (1) generalising the usual reaction-diffusion model. Indeed, sometimes, (1) mimicks (2) very closely. Sometimes the behaviour of (1) is radically different from that of (2). We will see one example in this course.

The plan of the work is the following.

I). Motivation of Model (1).

II). Some properties of the fractional laplacian. The periodic patch model for population dynamics.

III). Speed of propagation in the ~~periodic~~ heterogeneous patch model for population dynamics.

IV). More properties of the fractional laplacian. One phase free boundary problem involving $(-\Delta)^\alpha$.

V). Nonlocal minimal surfaces.

I). Motivation of Model (1).

We are going to show, with no real worry of mathematical rigour, that Models (1) and (2) can be obtained as limiting cases of a common integral equation. The underlying motivation is to describe, at least qualitatively, the interaction of a population with its environment. It is inspired by the introduction of an extraordinary paper of Kolmogorov, Petrovskii, Piskunov: "Etude de l'équation de la chaleur avec croissance de la quantité de matière", 1937.

Consider a population, we assume that it can be modelled by a density function: $u(t,x)$, $t>0$: time.

$$x \in \mathbb{R}^N.$$

We wish to account for its evolution according to migrations, births, deaths.

We make an account of the variations of the population density at time t and position x.

- <u>Migrations</u> : $J(x,y)$ is the fraction of the population which, in the interval $(t, t+\Delta t)$, will migrate from x to y. We could make it depend on t.

- <u>Births and deaths</u> : $g(x,u)$ = rate of growth.

- <u>Evolution of the population</u> :

$$\frac{u(t+\Delta t, x) - u(t,x)}{\Delta t}$$

$$= + \underbrace{\int J(y,x)\, u(t,y)\, dy}_{\text{arrivals}} - \underbrace{\int J(x,y)\, u(t,x)\, dy}_{\text{departures}} + g(x,u).$$

- <u>Specification of J</u>. We assume : $J \geq 0$,

$$J(x,y) = \frac{1}{\varepsilon^N}\, j\left(\frac{x-y}{\varepsilon}\right)$$ (the range of migration is not too large).

<u>Additional assumption</u> : $j(y) = j(|y|)$, $j \in L^1(\mathbb{R}^N)$.

We notice that we are preserving the L^1 norm of J.

Case 1. j is "very integrable", e.g. compactly supported or, at least, has ~~third~~ bounded third moments.

$$\int_{\mathbb{R}^N} J(y,x)\,u(y) - J(x,y)\,u(x)$$ (t does not matter)

$$= \frac{1}{\varepsilon^N}\int j\left(\frac{|x-y|}{\varepsilon}\right)\left(u(y) - u(x)\right)dy$$

$$= \int j(|z|)\left(u(x+\varepsilon z) - u(x)\right)dz$$

$$= \int j(|z|)\left(\varepsilon Du(x).z + \frac{\varepsilon^2}{2} D^2u(x).z^{(2)} + O(|z|^3)\varepsilon^3\right)dz$$

$$= \frac{\varepsilon^2}{2}\sum_{i,j}\left(\int j(|z|)\,z_i z_j\,dz\right)\partial_{ij}u(x) + O(\varepsilon^3)$$

because j has bded $2^{\underline{nd}}$ and $3^{\underline{rd}}$ moments.

$$= \frac{N\varepsilon^2}{2}\left(\int j(|z|)\,|z|^2\,dz\right)\partial_{ij}u(x) + O(\varepsilon^3).$$

Conclusion: discard $O(\varepsilon^3)$ terms, go to the limit $\Delta t \to 0$:

$$\partial_t u = \varepsilon^2 D\Delta u + g(x,u), \quad D = \int j(|z|)\,|z|^2.$$

Now: choose the time scale $\tau = \varepsilon^2 t$, this

also imposes $g(x,u) = O(\varepsilon^2)$: $g(x,u) = \varepsilon^2 f(x,u)$.
Hence the equation $u_t - D\Delta u = f(x,u)$.

<u>Exercise</u>. What equation do we obtain if

$$J(x,y) = \frac{1}{\varepsilon^N} j\left(x, \frac{|x-y|}{\varepsilon}\right) \ ?$$

<u>Case 2</u>. j is not so integrable, e.g.

$$j(z) = O\left(\frac{1}{|z|^{N+2\alpha}}\right) \text{ for large } |z|.$$

For definiteness we take: $\alpha > \frac{1}{2}$,

$$j(z) = \mathbb{1}_{\{|z| \geq 1\}} \cdot \frac{1}{|z|^{N+2\alpha}}.$$

We have: $\displaystyle\int |z| j(z) dz = \int_{r \geq 1} \frac{dr}{r^{2\alpha}} = O(1)$

$$\int |z|^2 j(z) dz = \int_{r \geq 1} \frac{dr}{r^{2\alpha - 1}} = +\infty.$$

Hence j has unbounded 2^{nd} moments.

We have:

$$\int \left(J(y,x) u(y) - J(x,y) u(x)\right) dy$$

$$= \frac{1}{\varepsilon^N} \int_{|x-y| \geq \varepsilon} \frac{u(y) - u(x)}{\left|\frac{x-y}{\varepsilon}\right|^{N+2\alpha}} dy$$

$$= \varepsilon^{2\alpha} \int_{|x-y|\geq\varepsilon} \frac{u(y)-u(x)}{|x-y|^{N+2\alpha}}\, dy .$$

Take the limit $\varepsilon \to 0$:

$$\partial_t u = \varepsilon^{2\alpha} (-\Delta)^{\alpha} u + g(x, u) .$$

<u>Time scale</u>: $\tau = \varepsilon^{2\alpha} t$, $f(x,u) = \varepsilon^{2\alpha} g(x,u)$.

$$\partial_\tau u = (-\Delta)^{\alpha} u + f(x,u) .$$

Hence both limits come from the same models.

<u>Remarks</u>. - As said before, the case "j has bded 3^{rd} moments" come from KPP. Although they wrote it in a more litterary fashion, the idea is that exposed in Case 1. Case 2 is a remark that I made w. Berestycki and Rossi.

- The expression of the population dispersal emphasises that there is a much shorter time scale with fractional diffusion than with standard diffusion. This will be observed when we look at spreading velocities.

II). Some properties of the fractional laplacian. Application to inhomogeneous models in population dynamics.

1°). Basic properties of the fractional laplacian.

Everything will not be proved here. Some proofs given later.

<u>definition</u>. Recall: if $u \in C^2(\mathbb{R}^N)$ with $|u(x)| = O(|x|^s)$ as $|x| \to +\infty$, $s < 2\alpha$:

$$(-\Delta)^\alpha u(x) = c_\alpha \, \mathrm{p.v.}\left(\int \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} \, dy\right) \qquad (\alpha \in (0,1))$$

$$= c_\alpha \int \frac{u(x) - u(y) + Du(x).(y-x)\,\mathbb{1}_{\{|y| \leq 1\}}}{|x-y|^{N+2\alpha}} \, dy$$

$$= c_\alpha \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} \, dy.$$

Remarks on the singularity of the operator.

- For $\frac{1}{2} \leq \alpha < 1$, $u \in C^{1,\beta}$ ($\beta > 2\alpha - 1$) is enough.

Indeed the integrand in the neighbourhood of $y = x$ is
$$\frac{|x-y|^{1+\beta}}{|x-y|^{N+2\alpha}} = \frac{r^{N+\beta}}{r^{N+2\alpha}} = r^{-(2\alpha - \beta)}.$$

- For $\alpha < \frac{1}{2}$, u lipschitz is enough: the singularity is $\frac{1}{r^{2\alpha}}$.

• Fourier transform.

$\mathcal{F}[(-\Delta)^\alpha u](\xi) = d_\alpha c_\alpha |\xi|^{2\alpha}$. The $|\xi|^{2\alpha}$ factor can be found out by symmetry and homogeneity, the constant ~~d_α~~ is d c_α is then chosen such that $c_\alpha d_\alpha = 1$.

Exercise. c_α has of course an explicit expression but, if we do not like to compute it, we may look at its limit when $\alpha \to 1$. Looking at the n- expansion of $(-\Delta)^\alpha u(x)$ we should have:

$(1-\alpha)c_\alpha$ has a finite limit.

The main feature of the fractional Laplacian is that it is a nonlocal operator. To know $(-\Delta)^\alpha u(x)$ one needs to know u everywhere.

Consequence.

Maximum principle. If $u \in C^2(\Omega) \cap C(\bar\Omega)$ satisfies

$$\begin{cases} (-\Delta)^\alpha u(x) \geq 0 & \text{in } (\Omega). \\ u \geqslant 0 & \text{outside } \Omega \end{cases} \qquad \text{then } u(x) \geq 0.$$

Proof. Take a point of ~~[illegible]~~ ~~[illegible]~~ minimum for u, assume that it is attained ~~and~~ inside Ω, compute : $\displaystyle\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\,dy > 0.$

This implies
* this minimum cannot be attained in Ω unless u is constant.
* $u \geq 0$ everywhere. $\boxtimes$

<u>We have for free</u> : strong maximum principle.

Of course the same holds if we perturb $(-\Delta)^\alpha$ by lower-order operators, e.g.

$$(-\Delta)^\alpha + \sum_i b_i \partial_{x_i} + c(x)$$

(~~provided~~ but then, for existence purposes it is better to have $\alpha > \frac{1}{2}$ ...).

<u>Let us anticipate on session 3, and state the Hopf lemma.</u>

<u>Th.</u> Ω smooth, $u \geq 0$ in Ω, $(-\Delta)^\alpha u(x) \geq 0$.

Then, if $x_0 \in \partial\Omega$ is such that $u(x_0) = 0$ we have, for some $q > 0$:

$$u(x) \geq q\,|(x-x_0).\nu(x_0)|^\alpha.$$

This will be the basis of Session 3.

Let us finally state some more properties, without any proof. Some of them will be much clearer in session 3.

- <u>Fundamental solution</u>. If $\alpha \in (0,1)$, and $(-\Delta)^\alpha E_\alpha = \delta_0$, we have $E_\alpha(x) = \dfrac{d_\alpha}{|x|^{N-2\alpha}}$.

Once again, consistent with $\alpha = 1$. Notice that we do not have the ~~pro~~ log problem.

- <u>Harnack inequality</u>: smoothing + strong max pple.
- <u>The evolution problem</u>.

We first wish to solve

$$\begin{cases} u_t + (-\Delta)^\alpha u = 0. & t > 0,\ x \in \mathbb{R}^N. \\ u(0,x) = u_0(x). \end{cases}$$

Let us admit that this problem has a solution $u(t,x)$, as soon as u_0 is (for instance) bounded and uniformly continuous, and that moreover $u(t,.) \in C^\infty(\mathbb{R}^N)$.

The deep reason is that, if $G(t,x)$ is the fundamental solution of $\partial_t + (-\Delta)^\alpha$, we have:

$$G(t,x) = \mathcal{F}^{-1}\left(e^{-t|\xi|^{2\alpha}}\right).$$

Moreover, one can prove: (exercise):

- $u_0 \geq 0 \Rightarrow u(t,x) \geq 0$. Strong max pple is also available.

• $u_0 \in L^1 \Rightarrow$ * $u(t,x) \in L^1$.

* $\displaystyle\int_{\mathbb{R}^N} u(t,x) = \int_{\mathbb{R}^N} u_0$.

(this in particular is useful to guess what the fundamental solution can be).

We may also state the problem in a bounded domain Ω:

$$\begin{cases} u_t + (-\Delta)^\alpha u = 0. & t>0,\ x\in\Omega. \\ u(t,x) = 0 & (t>0,\ x\notin\bar\Omega). \\ u(0,x) = 0. & \end{cases}$$

Here again we have a unique solution which satisfies the strong max. principle.

<u>Consequences</u>. Take f a bounded lipschitz nonlinearity. Then the pb

$$\begin{cases} u_t + (-\Delta)^\alpha u = f(x,u) \\ u(0,x) = u_0(x) \longleftarrow \text{bded, unif}^{t} \text{ cont.} \end{cases}$$

has a unique solution.

<u>Exercise</u>. If $0 = f(x,0) \geq f(x,1)$. then
$u(0,x) \in [0,1]$ $\quad 0 \leq u \leq 1$.

I do not want to dwell on this material too much, but this is just to say that there is no pb in solving nonlinear evolution pbs with this operator.

2°). An inhomogeneous model in population dynamics.

Let us come back to the original population dynamics model and solve:

$$u_t + \mathcal{A}u = \mu(x)\,u - u^2. \qquad \begin{matrix} t>0 \\ x \in \mathbb{R}^N. \end{matrix} \qquad (1)$$

where $\mathcal{A} = (-\Delta)^\alpha$ or $(-\Delta)$.

$\mu(x)$: birth rate. It will be assumed to be bounded and continuous.

⚠ It may not be always positive.

If the initial datum $u_0 := u(0,\cdot)$ is uniformly continuous, and takes its values in $[0, \sup\mu]$, then (1) has a unique smooth solution defined for all $t>0$. We wish to understand what happens as $t \to +\infty$.

<u>One word about the model</u>. Very crude model, but which gives very nice qualitative information. The fact that $\mu(x)$ can be positive or negative expresses that the environment can be very favourable at places (ex. a city with a lot of jobs and commodities) or

highly unfavourable (a desert ...).

<u>We will assume</u> : . $u_0 \in [0, \sup \mu]$ compactly supported.

. $\mu(x)$ 1-periodic in x.

The dynamics of the model depends on a single parameter : the 1^{st} eigenvalue of

$$\mathcal{A} - \mu(x) I$$

taken over the set of continuous, 1-periodic functions. Expresses the stability of the zero solⁿ.

<u>Remainder</u>. For large $\Lambda > 0$, $\mathcal{A} + \Lambda I - \mu(x) I$ is invertible and smoothing. The strong max. ple implies that the positive cone is mapped into its interior by $(\mathcal{A} + (\Lambda - \mu) I)^{-1}$, hence Krein-Rutman applies : there is a least eigenvalue, real, with a simple (algebraic and geometric) eigenfunction.

Let $\lambda_1^{per}(\mathcal{A} + \mu(x))$ be this first eigenvalue.

<u>Th.</u> . Assume $\lambda_1^{per} \geq 0$. Then any solution of (1) goes to 0 as $t \to +\infty$.

. Assume $\lambda_1^{per} < 0$. There is a unique bounded positive solution u_s of

$$\mathcal{A} u_s = \mu(x) u_s - u^2, \quad x \in \mathbb{R}^N .$$

Moreover, if u_0 is compactly supported and ≥ 0, we have $u(t, x) \longrightarrow u_s(x)$ uniformly on compact sets.

$\mathcal{A} = -\Delta$: Berestycki, Hamel, Roques.
$\mathcal{A} = (-\Delta)^\alpha$: ———————, R, . Rossi.

If one knows ~~abou~~ a bit about elliptic theory, one is aware of a very similar result,

$$\mathcal{A} u = f(x, u)$$

$f(x, \cdot)$ concave. Actually we are going to use it. The novelty here is that we are working here in the whole space and we do not assume any periodicity on the solutions of (1). [Result By Amann, Berestycki, – 1975 ± ε].

The proof of the theorem is based on the following

Proposition. Let v be a ~~solu~~ bded solution of $\mathcal{A} v = \mu(x) v - v^2$, $x \in \mathbb{R}^N$, $v > 0$.
Then $\inf_{x \in \mathbb{R}^N} v(x) > 0$.

In the periodic framework, this is essen–

tially trivial.

The reason why I wanted to treat this example is that it allows some nice manipulations of the operators at stake. First, let us state the

<u>Exercise</u>. Ω : bounded open subset of $\mathbb{R}^N$, smooth. Then $(-\Delta)^\alpha - \mu(x)I$, with Dirichlet conditions outside Ω, has a least eigenvalue, uniquely attained at a positive eigenfunction. Positivity characterises the 1^{st} eigenfunction.

<u>Hint</u>. 1. Redo the exercise for the Laplacian.

2. Use the Hopf lemma.

<u>Notation</u>. If $\Omega = B_R(0)$, $\lambda_1^R(\mathcal{A} - \mu(x)I)$ is the 1^{st} eigenvalue of $\mathcal{A} - \mu(x)I$ with Dirichlet conditions outside Ω.

<u>Basis of the result</u>:

<u>Th.</u> (approximation of λ_1^{per}). We have

$$\lim_{R \to +\infty} \lambda_1^R(\mathcal{A} - \mu(x)I) = \lambda_1^{per}(\mathcal{A} - \mu(x)I)$$

$\mathcal{A} = -\Delta$: noticed independently by Capdeboscq, Berestycki – Hamel – Roques.

Here the result for $\mathcal{A} = (-\Delta)^\alpha$ very much relies on the method for $\mathcal{A} = -\Delta$, it is really a generalisation. So, let us investigate first what happens for $\mathcal{A} = -\Delta$.

<u>Proof for $\mathcal{A} = -\Delta$</u>. Relies on Rayleigh quotients. It would be wrong with nonsymmetric operators.

We have

$$\lambda_1^{per} = \inf_{u \in H^1_{per}(\mathbb{R}^N)} \frac{\int_{[0,1]^N} \frac{1}{2}|\nabla u|^2 - \mu(x) u^2}{\int_{[0,1]^N} u^2}$$

(i). We have, for all n:

$$\lambda_1^{per} = \inf_{\substack{u \in H^1(\mathbb{R}^N) \\ u\ n\text{-periodic}}} \frac{\int_{[-n,n]^N} \left(\frac{1}{2}|\nabla u|^2 - \mu(x) u^2\right) dx}{\int_{[-n,n]^N} u^2\, dx}.$$

Indeed, the RHS is uniquely attained on a positive n-periodic function. But, if ϕ_1 is the 1^{st} periodic eigenfunction, it is also an n-periodic eigenfunction.

Hence: let ϕ_n be the eigenfunction associated to

$$\lambda_1^n(-\Delta - \mu(x) I)$$

Extend ϕ_n by 0 outside $B_n(0)$, and inside $[-2n, 2n]^N$.

We have therefore:

$$\lambda_1^{per} = \inf \frac{\int_{[-2n,2n]^N} \left(\frac{1}{2}|\nabla u|^2 - \mu(x) u^2\right) dx}{\int u^2 \, dx}.$$

$$\leq \frac{\int_{B_n(0)} \left(\frac{1}{2}|\nabla \phi_n|^2 - \mu(x) \phi_n^2\right) dx}{\int \phi_n^2 \, dx}$$

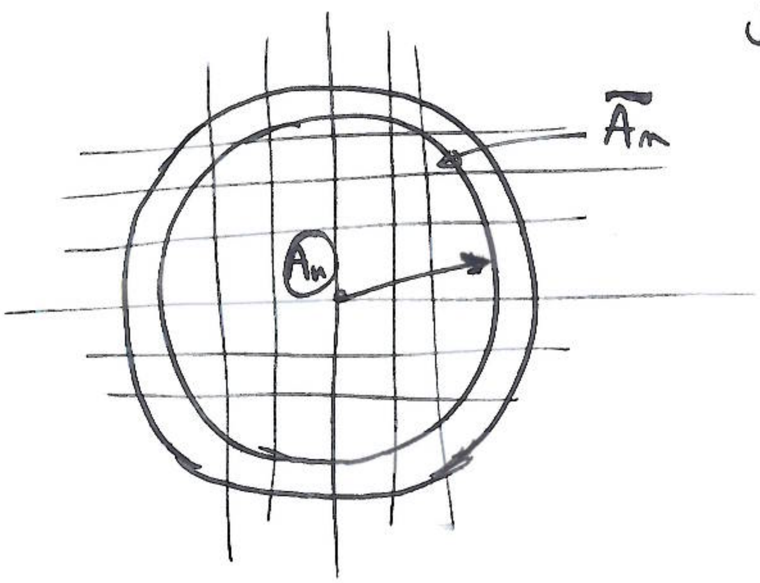
$$= \lambda_1^n \left(-\Delta - \mu(x) \, I\right).$$

(ii). Let χ_n be smooth, ≥ 0, with

$$\chi_n(x) = \begin{cases} 1 & \text{if } |x| \leq n-1 \\ 0 & \text{if } |x| \geq n. \end{cases}$$

Set $\psi_n(x) = \chi_n(x) \phi_1(x)$, ϕ_1: 1^{st} periodic eigenfunction of $(-\Delta)$. We have:

$$\lambda_1^n(-\Delta - \mu(x)) \leq \frac{\int_{B_n} \frac{|\nabla \psi_n|^2}{2} - \mu(x) \psi_n^2}{\int_{B_n} \psi_n^2}.$$



A_n = set of cubes of unit side-length inside $B_{n-1}(0)$.

$\tilde{A}_n$ = set of cubes of unit side-length with points in $B_n(0) \setminus B_{n-1}(0)$.

 $C_n = \bigcup C$.

We have:

$$\int_{B_n} \psi_n^2 = \int_{[0,1]^N} \phi_1(x)^2\,dx \,.\, |A_n| + \int_{C_n} \psi_n^2 .$$

<u>Number of cubes in $\hat{A}_n$</u>: at most $c\,n^{N-1}$.

<u>Size of $|A_n|$</u> : at least $q\,n^N$.

Hence $\displaystyle\int_{B_n} \psi_n^2 = |A_n| \int_{[0,1]^N} \phi_1(x)^2 \; (1 + o(1))$ as $n \to +\infty$

In the same fashion:

$$\int \mu(x)\, \psi_n^2 = |A_n| \int_{[0,1]^N} \mu(x)\, \phi_1^2 + O(n^{N-1}).$$

$$\int |\nabla \psi_n|^2 = |A_n| \int_{[0,1]^N} |\nabla \phi_1|^2 + O(n^{N-1}).$$

Thus
$$\frac{\displaystyle\int_{B_n} \left(\frac{|\nabla \psi_n|^2}{2} - \mu(x)\, \psi_n^2 \right) dx}{\displaystyle\int_{B_n} \psi_n^2} = \frac{\displaystyle\int_{B_1} \frac{|\nabla \phi_1|^2}{2} - \mu(x) \phi_1^2}{\displaystyle\int \phi_1^2} + O\left(\frac{1}{n}\right).$$

And thus $\lambda_1^{per} \le \lambda_1^n(-\Delta - \mu(x)) + O\left(\frac{1}{n}\right)$.

(iii). Notice that $R \longmapsto \lambda_1^R(-\Delta - \mu(x))$ is nonincreasing.

Gather (i) $\to$ (iii) ~~via~~ and get the result. ∎

<u>What about $(-\Delta)^\alpha$</u> ? It turns out that there are also Rayleigh quotients. Consider, for $u \in C^1(\mathbb{R}^N)$ 1-periodic :

$$\mathcal{E}^{per}_\alpha[u] = \frac{1}{4}\int_{[0,1]^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy.$$

<u>Notice that</u> : – the integral converges: only one of the two integration domains is bdd

– One may differentiate $\mathcal{E}_\alpha$:

$$\mathcal{E}_\alpha[u+h] - \mathcal{E}_\alpha[u] = \frac{1}{4}\int_{[0,1]^N\times\mathbb{R}^N} \frac{2(u(x)-u(y))(h(x)-h(y))}{|x-y|^{N+2\alpha}}\,dy\,dx$$

$$+ \int \frac{(h(x)-h(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy.$$

<u>Clearly</u> : $D\,\mathcal{E}_\alpha[u].h = 2\int_{[0,1]^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(h(x)-h(y))}{|x-y|^{N+2\alpha}}.$

$$= \int_{[0,1]^N} h(x)\,(-\Delta)^\alpha u(x)\,dx - \int_{[0,1]^N} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}} \int_{\mathbb{R}^N} h(y)\,dy.$$

(Ignore the singularities).

$$\int_{[0,1]^N} \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} dx \int h(y)\, dy$$

$$= \sum_{e \in \mathbb{N}^N} \int_{[0,1]^N} \frac{u(x) - u(y)}{|x-e-y|^{N+2\alpha}} \int_{[0,1]^N} h(y)\, dy$$

$$= \int_{[0,1]^N} h(y) \, \cancel{dy} \int \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} dx = - \int_{[0,1]^N} h(y) (-\Delta)^\alpha u(y)\, dy$$

<u>Conclusion</u> : $D\mathcal{E}_\alpha[u]. h = \int_{[0,1]^N} h(y) (-\Delta)^\alpha u(y)\, dy$.

Similarly, we have, for $u \in C^1(\Omega) \cap C_0(\overline{\Omega})$: if $\mathcal{E}_\alpha^{\Omega}[u] = \frac{1}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2\alpha}} dy$, then, for every $h \in C^1(\Omega) \cap C_0(\overline{\Omega})$ (extended by 0 outside Ω):

$$D\mathcal{E}_\alpha[u].h = \int_{\mathbb{R}^N} h(y) (-\Delta)^\alpha u(y)\, dy .$$

And thus :

$$\lambda_1^{per}\left((-\Delta)^\alpha - \mu(x)\right) = \inf_{u \in H^\alpha_{per}(\mathbb{R}^N)} \frac{\mathcal{E}_\alpha^{per}[u] + \int_{[0,1]^N} u^2}{\int_{[0,1]^N} u^2}$$

$H^\alpha_{per}(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^2),\ 1\text{-periodic} : \mathcal{E}_\alpha^{per}[u] < +\infty \}$.

$$\lambda_1^{\Omega}\left((-\Delta)^\alpha - \mu(x)\right) = \inf_{u \in H^\alpha_0} \frac{\mathcal{E}_\alpha^{\Omega}[u] + \int_\Omega u^2}{\int_\Omega u^2}.$$

To prove the approximation theorem for the fractional laplacian, the only new thing that we have to perform is the study of the nonlocal energies. Indeed, we have for free that:

- $R \mapsto \lambda_1^R\left((-\Delta)^\alpha - \mu(x)\right)$ is a decreasing fct.
- $\lambda_1^{per}\left((-\Delta)^\alpha - \mu(x)\right) \leq \lambda_1^n\left((-\Delta)^\alpha - \mu(x)\right)$.

We need to prove:

$$\lambda_1^n\left((-\Delta)^\alpha - \mu(x)\right) \leq \lambda_1^{per}\left((-\Delta)^\alpha - \mu(x)\right) + o(1)$$

To do this, the tool is the

<u>Lemma.</u> $u \in C^\infty(\mathbb{R}^N)$, 1-periodic. Then,

$$\lim_{n \to +\infty} \mathcal{E}_\alpha^n[\chi_n u] = \mathcal{E}_\alpha^{per}[u].$$

<u>Proof.</u> Tedious exercise. $\boxed{XXX}$

Let us come back to the steady solutions of

$$\mathcal{A} v = \mu(x) v - v^2; \quad x \in \mathbb{R}^N.$$
$$v > 0.$$

<u>Proof that $\inf_{\mathbb{R}^N} v > 0$.</u> Assume the contrary,

There is a sequence $(x_n)_n$ such that

$$\lim_{n \to +\infty} v(x_n) = 0 .$$

We always may write : $x_n = k_n + y_n$, $k_n \in \mathbb{Z}^N$ and $y_n \in [0,1]^N$. Thus, setting $v_n(y) = v(x_n + y)$ we always may work with v_n, such that

$$\lim_{n \to +\infty} \inf_{[0,1]^N} v_n = 0 .$$

From the Harnack inequality, we have :

$$\forall R>0, \quad \lim_{n \to +\infty} \sup_{B_R(0)} v_n = 0 .$$

For definiteness, set

$$\lambda_1 = \lambda_1^{per}(\mathcal{A} - \mu(x) I) < 0 .$$

choose then $R>0$ large enough so that

$$\lambda_1^R(\mathcal{A} - \mu(x) I) \leq \frac{\lambda_1}{2} < 0 .$$

It is now time to use what we know about (1) in bounded domains. Let us look at

$$\begin{cases} u_t + \mathcal{A} u = \mu(x) u - u^2 & (B_R) \\ u = 0 & (\text{outside } B_R) \end{cases} \qquad (1)_R$$

Th. (Berestycki, Amann, ... 70's). Assume

$\lambda_1^R(\mathcal{A} - \mu(x) I) < 0$. Then $(1)_R$ has a unique positive steady solution u_R, which attracts all bounded ≥ 0 solutions of $(1)_R$ (except 0).

This was formulated for laplacian-like operators, but works for abstract order-preserving operators, such as the fractional laplacian.

Choose $n \in \mathbb{N}$ such that

$$\sup_{B_R} v_n \leq \frac{1}{2} \sup_{B_R} u_R .$$

Let us solve $(1)_R$ with $u(0,x) = w(x) \leq v_n$, $w(x) > 0$ on $B_{R/2}$. This is possible from the strong max. p[rincip]le. And thus:

$$\lim_{t \to +\infty} u_w(t,x) = u_R(x) \quad \text{uniformly on } \overline{B_R}$$

On the other hand: we have

$$v_n(x) \geq u_w(t,x) \quad \text{for all } t, \text{ because } w \leq v_n .$$

Contradiction. $\boxtimes$

<u>Proof of the main theorem</u>. What really has to be proved is the uniqueness of the positive

steady solution. Notice that Berestycki, Amann... 70 give a periodic solution u_s to

$$\mathcal{A}u = \mu(x)\,u - u^2, \quad x \in \mathbb{R}^N, \quad u > 0.$$

Suppose the existence of a second one: v_s.

(i). We claim the existence of $k > 0$ such that $u_s \leq k v_s$. This is because $\inf v_s > 0$. Let k_0 be the smallest constant such that this holds.

* If there is a contact point between u_s and $k_0 v_s$, the strong maximum principle implies that $u_s = k_0 v_s$. This implies $k_0 = 1$; if not we have 2 periodic solutions: contradiction.

* If there is no contact point, there is a sequence $x_n = p_n + y_n$, $p_n \in \mathbb{Z}^N$, $y_n \in [0,1]^N$, such that

$$\lim_{n \to +\infty} \left(k\, v_s(x_n) - u_s(x_n) \right) = 0.$$

Set $v_n(y) = v_s(p_n + y)$, the sequence $(v_n)_n$ is relatively compact in $C(B)$ for every ball B. If v_∞ is one of its limits, then: v_∞ is a solution of $A v_\infty = \mu v_\infty - v_\infty^2$. Thus $k_0 v_\infty = u_s$. Once again impossible unless $k_0 = 1$. Thus: $u_s \leq v_s$.