Describing biological invasions with nonlinear PDE's.

Haifa – Getting started with PDEs.
September 16 – 17 2014.

Remerciements.

Goal of these lectures: explain and discuss an important and influential result of Aronson – Weinberger (Adv. Math, 1976) on the large time behaviour of a special class of parabolic equations.

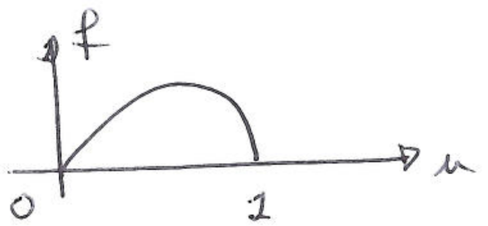
The model.

– Unknown: $u(t,x)$; $t>0$, $x \in \mathbb{R}^N$.

$$\frac{\partial u}{\partial t} - \Delta u = f(u) \quad (t>0,\ x \in \mathbb{R}^N). \qquad (1)$$

$$u(0,x) = u_0(x).$$

u_0: nonnegative, nonzero, compactly supported; $u_0 \leq 1$.



KPP - type nonlinearity
(Kolmogorov, Petrovskii, Piskunov)
1937.

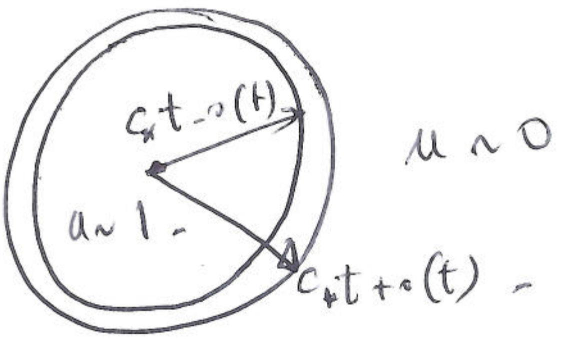
The result.

Th. (AW, 1976). Let $u(t,x)$ be the sol. of (1) (yes, (1) has a unique solution defined on $\mathbb{R}_+ \times \mathbb{R}$). Set

$$c_* = 2\sqrt{f'(0)} > 0.$$

- If $c > c_*$, then
$$\lim_{t \to +\infty} \sup_{|x| \geq ct} u(t,x) = 0.$$
- If $c < c_*$, then
$$\lim_{t \to +\infty} \inf_{|x| \leq ct} u(t,x) = 1.$$

What does it mean mathematically?



In between, there is a mushy region that we will not investigate.

Please note that this is a very nonlinear result. Classical heat equation:

$$u_t - \Delta u = 0.$$
$$u(0,x) = u_0(x).$$

Take $u_0(x) = \delta_0(x)$. Then

$$u(t,x) = \frac{e^{-|x|^2/4t}}{(2\pi t)^{N/2}}.$$

<u>level sets</u>: $|x| \sim 2\sqrt{t}$. Very different.

Also, there ~~se~~ is a unique scale. In AW's theorem, there are at least 2 scales.

And this is why, as a matter of fact, this theorem had a big impact in the physics community dealing with combustion in the 80s.

Here we will not be interested in ~~bi~~ life sciences. Instead of combustion. More precisely, we will be interested in <u>biological invasions</u>.

<u>What is this</u>? A living species (animal, vege-

erium) is released in small quantities
iven environment. It moves around (dif-
, reproduces, and uses food resources.
ot stopped by a predator or any other
-hre, it will spread, and the goal
to monitor - and possibly understand -
, it spreads.

ample given: the muskrat.

- Introduction in Europe.
- Spreading (caution w. data)

We see on AW's theorem that this is a linear spreading, in qualitative agreement with the data.

Interpretation. release a small amount some-where. Then it will thrive and develop at a speed which can easily be computed from elements that are very few.

Hence the interest of this theorem.

Plan. I). Derivation of (1).
II). Existence theory for (1).
Boring and no one does it. However this

is a school called "Getting started with PDE's".
So it is relevant to study this issue, at least in a pedestrian fashion.

III). The "hair trigger effect".

A weaker version of the theorem – 2 animals will produce an invasion (but the speed is still to be measured).

Tools: max ple, sub/super-solutions.

IV). The proof of AW's theorem.

V). Further qualitative properties.

The level sets ~~on~~ of u are located in an area of width $o(t)$, from [AW]. We will see that they are in fact in an area of width $O(1)$.

VI). ~~Recent~~ More recent results and open pbs.

when the medium is heterogeneous.

Please Do not hesitate to interrupt.

I). Derivation of (1).

One could do it w. probabilistic arguments - as in R. Pinsky's lectures. I am going to do it on the basis of very crude modelling.

Assume (which is debatable) the animal population given by a density $u(t,x)$. Then

$\frac{\partial u}{\partial t}(t,x) =$ number of animals arriving at x from outside

$-$ number of animals leaving x to go elsewhere

$+$ ~~reproduction term~~ creation / annihilation at x.

$k(x,y)$: proportion of animals leaving x to go to y. We could make it depend on t.

$R(u)$: reproduction term. We assume that it depends only on $u(t,x)$. Sometimes it depends on other $u(t,y)$ - not only does it complicate things but it produces different effects.

Number of animals arriving at x from outside:

$$\int k(y,x)\, u(t,y)\, dy .$$

Number of animals leaving x:

$$\int k(x,y)\, u(t,x)\, dy .$$

So: $$\frac{\partial u}{\partial t} = \int k(y,x)\, u(t,y) - \int k(x,y)\, u(t,x)\, dy + R(u).$$

Next set of assumptions: homogeneous medium (yes, it exists - not so much in France, though). So:

$$k(x,y) = k(y,x) = k(|x-y|). \quad \text{So:}$$

$$\frac{\partial u}{\partial t} = \int k(|x-y|)\,\big(u(t,y) - u(t,x)\big)\, dy + R(u).$$

Please note that this is a very interesting integral eq^n to study. Now, let us decide:

$$k(x) = \frac{1}{\varepsilon^N}\, \rho\left(\frac{x}{\varepsilon}\right).$$ Does it recall something? Here we are just saying that the animals do not go very far from their ini-

tial point. BTW, we assume that

$\int \rho$, $\int |x|\rho$, $\int |x|^2\rho$, ~~and~~ $\int |x|^3\rho$ are

finite (ex. ρ compactly supp. or $\rho(r) = e^{-r}$).

So we have

$$\int k(|x-y|)\,(u(t,y) - u(t,x))\,dy$$

$$= \frac{1}{\varepsilon^N} \int \rho\left(\frac{|x-y|}{\varepsilon}\right)(u(t,y) - u(t,x))\,dy$$

$$\underset{z = \frac{-x+y}{\varepsilon}}{=} \int \rho(|z|)\,(u(t, x+\varepsilon z) - u(x))\,dz.$$

$$u(t, x+\varepsilon z) = u(t,x) + \varepsilon Du(t,x)\cdot z + \frac{\varepsilon^2}{2} D^2u(t,x)\cdot z^{(2)} + O(\varepsilon^3 |z|^3).$$

$$\int k(|x-y|)\,(u(t,y) - u(t,x))\,dy$$

$$= \varepsilon \int k(|z|)\, Du(t,x)\cdot z\, dz + \frac{\varepsilon^2}{2} \int k(|z|)\, D^2u(t,x)\cdot z^{(2)}\, dz + O\left(\varepsilon^3 \int k(z)\, |z|^3\, dz\right).$$

- $\int k(|z|)\, Du(t,x) \cdot z \, dz = 0$.
- $O\left(\varepsilon^3 \int k(|z|)\, |z|^3\right) = O(\varepsilon^3)$

because k has finite 3^{rd} moment. If it was not true, or if k had finite 1^{st} moment but infinite 2^{nd} moment what we are going to say would not be true. We would find another kind of equation.

- $\int k(|z|)\, D^2 u(t,x) \cdot z^{(2)}\, dz$

$$= \int k(|z|) \sum \lambda_i(t,x)\, z_i^2 \, dz \quad (\lambda_i(t,x) : i^{th} \text{ eigenvalue of } D^2 u(t,x))$$

$$= \sum \lambda_i(t,x) \int k(|z|)\, z_i^2 \, dz$$

$$= \sum \lambda_i(t,x) \int k(|z|)\, \frac{|z|^2}{N}\, dz \quad \text{by symmetry}$$

$$= D\, \Delta u(t,x) \; ; \; D = \int k(|z|)\, |z|^2 \frac{dz}{N}.$$

So, our equation reads:

$$\frac{\partial u}{\partial t} - \varepsilon^2 D \Delta u = R(u) + \sout{O(\varepsilon^3)}$$

$$\frac{\partial u}{\partial t} - \varepsilon^2 D \Delta u = R(u).$$

Now we say that we want to see something. Because of the spatial scale at which the phenomenon occurs, we need to accelerate the process if we wish to see something. So, set $\tau = \varepsilon^2 t$. $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} \frac{\partial \tau}{\partial t} = \varepsilon^2 \frac{\partial}{\partial \tau}$.

$$\tau = 1 \Leftrightarrow t = \frac{1}{\varepsilon^2}.$$

The equation becomes

$$\frac{\partial u}{\partial \tau} - D \Delta u = \frac{1}{\varepsilon^2} R(u).$$

To balance the RHS we choose: $R(u) = \varepsilon^2 f(u)$.

Now, one word about the modelling of $R(u)$. Once again it is going to be heuristic. BUT: note that there is here no 1st principle, no 2nd principle ... only comparison between model and data is relevant.

For this we introduce the notion of carrying capacity (I am serious, this notion is quite important in ecology).

<u>def</u>. The carrying capacity of a given species is the maximum density above which the reproduction rate decreases. In other words if we call ~~a~~ θ the carrying capacity we have

$$f(u) > 0 \text{ if } u < \theta \quad \text{and } f(u) < 0 \text{ if } u > \theta.$$

Here we assume that it is uniform and, normalising u to u_0 ~~we have~~ the normalised carrying capacity is 1.

<u>Example</u>. $f(u) = u - u^2$.

u ↑ reproduction (includes births - deaths)

u^2 ← deaths induced by limitation of food resources.

<u>Conclusion on the assumption</u>: not irrelevant but can be challenged ~~at~~ in some given occasions.

<u>Remarks</u>. One may find this derivation in the introduction of a paper by KPP " Etude de l'é-

quation de la chaleur avec augmentation de la quantité de matière". No, I, am not fed up w. English... the paper is written in French. This is the type of paper where, every time you read it, you learn something.

So, we are now faced with our equation. The 1st thing to do is to make sure it has a solution.

II). Existence theory for (1).

what we need to do is to prove that, under reasonable assumptions, (1) has solutions that are as nice as possible. This is given by the

Th. Assume u_0 continuous, $\lim_{|x|\to+\infty} u_0(x) = 0$.

(so that u_0 is uniformly continuous). Assume just : f C^∞, globally lipschitz. Then (1) has a unique solution $u(t,x)$ ~~which~~ such that

- u is C^∞ on $\mathbb{R}_+^* \times \mathbb{R}^N$.
- $\forall t \geq 0$, $\lim_{|x|\to+\infty} u(t,x) = 0$, uniformly on every compact in t.

I am not going to give a detailed proof. Rather, I will show the 2 main steps and explain where the difficulties are.

Step 1. Existence for a fixed point formulation.

Step 2. Regularity of the fixed point.

Fixed point. Forget the equation and look at the ODE

$$\begin{cases} \dot X + AX = f(X). & X : \mathbb{R}_+ \longrightarrow \mathbb{R}^d. \\ X(0) = X_0. & A \in \mathcal{M}_+ (\mathbb{R}) \text{ for instance.} \end{cases}$$

A fixed point formulation is

$$X(t) = e^{-tA} X_0 + \int_0^t e^{-(t-s)A} f(X(s))\, ds.$$

(Duhamel formula – just the elaborate form of solution of $\dot X + AX = f(t)$ = a particular solution + a solution of the homogeneous pb).

Back to $\begin{cases} u_t - \Delta u = f(u) \\ u(0,x) = u_0(x). \end{cases}$

If we set $A = -\Delta$

and $e^{+t\Delta} u_0(x) = \int G(t, x-y) u_0(y)\, dy$

with $G(t,x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x-y|^2}{4t}}$ then

(exercise) we have

$$u(t,x) = e^{t\Delta} u_0 + \int_0^t \int e^{(t-s)\Delta} f(u)\, ds\, dy$$

$$= \underbrace{e^{t\Delta} u_0 + \int_0^t \int G(t-s, x-y) f(u(y))\, ds\, dy}_{\mathcal{T} u(t,x)}.$$

Fixed point in the space.

$X = C([0,T], C^0(\mathbb{R}^N))$.

We have, for $(u,v) \in X^2$:

$$\|\mathcal{T}u - \mathcal{T}v\|_{L^\infty([0,t]\times\mathbb{R}^N)} \leq \|f\|_{Lip} \int_0^t \underbrace{\int G(t-s, x-y)\, ds\, dy}_{=1} \|u-v\|_\infty$$

$$\leq \underbrace{\|f\|_{Lip} T}_{<1 \text{ if } T \text{ small enough.}} \|u-v\|_\infty$$

Extension: start from $t = T$, go to $2T$ replace u_0 by $u(T)$. One checks that we still have the initial weak formulation between T and $2T$.

~~Iter~~ Then go from $2T$ to $3T$, ..., etc.

OK, we have a fixed point but what does it have to do with the equation? This is why we need the step

<u>Regularity</u>.

<u>prop</u>. $\forall \varepsilon > 0$, $u \in C^\infty([\varepsilon, +\infty[\times \mathbb{R}^N)$.

<u>Exercise</u> Prove that $\partial_t u - \Delta u = f(u)$ with the aid of the proposition.

<u>Why do we have the proposition?</u>

- $e^{t\Delta} u_0$ is smooth. So we should worry about $V(t,x) = \int_0^t \int G(t-s, x-y) f(u(s)) \, ds \, dy$.

(we omit the variable y in u for short).

~~We are going to assume~~

- Try to differentiate wrt x_i: $\Pi = 2\Pi = \frac{1}{2\Pi} = 1 \dots$

$$\partial_{x_i} V(t,x) \text{"="} \int_0^t \int \partial_{x_i} G(t-s, x-y) f(u(s,y)) \, ds \, dy$$

$$= \int_0^t \int \frac{x_i - y_i}{(t-s)^{\frac{N}{2}+1}} e^{-\frac{|x-y|^2}{4(t-s)}} f(u) \, ds \, dy -$$

$$\left(z = \frac{x-y}{2\sqrt{t-s}} \right) = \int_0^t \int \underbrace{\frac{1}{\sqrt{t-s}} z_i e^{-|z|^2} f(u) \, ds \, dz}_{\in L^1} -$$

So, one proves that ∇V exists and is continuous.

Try a second derivative:

$$\partial_{x_i x_j} V \text{ " = " } \int \frac{(x_i - y_i)(x_j - y_j)}{(t-s)^2} \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{(t-s)^{N/2}} f(u)\, ds\, dy$$

$$\text{" = " } \left(z = \frac{x-y}{2\sqrt{t-s}}\right) \int \frac{1}{t-s} \frac{z_i z_j e^{-|z|^2}}{(t-s)^{N/2}} f(u)\, ds\, dy.$$

Lost! what can we do?

- be a bit more careful: look at

$$V_\varepsilon(t,x) = \int_0^{t-\varepsilon} \int \frac{(x_i - y_i)(x_j - y_j)}{(t-s)^2} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{N/2}} f(u)\, ds.$$

* $$\int_{\mathbb{R}^N} \frac{(x_i - y_i)(x_j - y_j)}{(t-s)^2} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{N/2}} dy = 0. \text{ So:}$$

$$V_\varepsilon(t,x) = \int_0^{t-\varepsilon} \int \frac{(x_i - y_i)(x_j - y_j)}{(t-s)^2} \frac{e^{-(\ )}}{(\ )^{N/2}} \big(f(u(s,y)) - f(u(t,x))\big)\, ds\, dy$$

We have a bit of smoothness in u: u is Lipschitz in x.

* Prove a bit of smoothness for V in t. $t' > t$:

$$V(t,x) - V(t',x) = \iint_0^t \big(G(t-s, x-y) - G(t'-s, x-y)\big) f(u)\, ds\, dy + \underbrace{\int_t^{t'} \int G(t'-s, x-y) f(u)\, ds\, dy}_{\le C|t-t'|}.$$

$$\int_0^t \left(G(t' - s, x - y) - G(t - s, x - y) \right) f(u)\, ds\, dy$$

$$= \int_0^t \iint_0^1 \partial_t G\left(\sigma t' + (1-\sigma)t - s, x - y\right) f(u)\, ds\, dy \cdot (t' - t).$$

<u>Typical term</u>: $\dfrac{e^{-\frac{(x-y)^2}{\sigma t' + (1-\sigma)t - s}}}{\left(\sigma t' + (1-\sigma) t - s\right)^{N/2 + 1}}$.

$(\ )^{N/2}$ goes into the volume element and we need to control

$$\int_0^1 \int_0^t \frac{d\sigma\, ds}{\left(\sigma t' + (1-\sigma)t - s\right)} = \int_0^1 \log \frac{\sigma(t'-t)}{t + \sigma(t'-t)}\, d\sigma$$

$$\leq \log t + \int_0^1 \log \sigma (t' - t)\, d\sigma$$

$$= \log t + \int_0^{t'-t} \log \sigma\, d\sigma \cdot \frac{1}{t'-t}.$$

In the end we estimate by $C\sqrt{t'-t}$. (no need to go further).

* Now, use that $|u(t,x) - u(s, y)| \leq C(\sqrt{|t-s|} + |x-y|)$ in the representation of V_ε. ~~Cut the integral into pieces: $|x-y| \lessgtr \frac{1}{2}|x - x'|$, $|t-s| \lessgtr \frac{1}{2}|t - t'|$~~ We are now facing terms of the form

$$\frac{\left(\sqrt{t-s} + |x-y|\right)(x_i - y_i)(x_j - y_j)}{(t-s)^2}$$

~~and prove that~~
and we win at each time

Now, we are in business:

- $\partial_{x_i x_j} V$ exist, so does $\partial_t V$.
- Iterate.

We have constructed a solution.

Note. Nothing in what I showed is optimal. There is a very elaborate regularity of parabolic PDEs — that is just not useful to us here.

Rq. All derivatives of u are bded. Why?

III). The "hair-trigger effect".

As said before: weaker form of AW, but with weaker assumptions on f. Here is the result.

Th.
$$\begin{cases} u_t - \Delta u = f(u) & t>0,\ x\in\mathbb{R}^N \\ u(0,x) = u_0(x) \geq 0, & \text{comp. supp}^d. \end{cases}$$

Assume:
- $f>0$ in $(0,1)$, $f'(0)>0$.
- $f(0)=f(1)=0$.

Then ~~there is $\alpha>0$ such that~~

$$\forall x\in\mathbb{R}^N,\quad \lim_{t\to+\infty} u(t,x) = 1.$$ ~~(kind of u_0)~~.

This says: put a small amount of animals. Then even if this is a small amount, we will be surrounded by these animals. Not so optimistic in terms of muskrats...

To prove such a result, we need comparison between solutions. We are going to develop sub and super solutions argument – once again rather simple – but quite powerful here.

Sub and super-solutions ~~===~~ may be viewed as simplified models for the true solution of a parabolic PDE. In other words one needs to know how to exhibit them. ~~This is why~~ One way to do it is by looking at special functions: the eigenfunctions of the Laplacian. Precisely, the 1^{st} one.

In this section, and before proving the theorem, we will successively review

- The 1^{st} eigenvalue of the Laplacian.
- Comparison theorems for parabolic eq^{ns}
- The theorem on sub and super-solutions
- The ~~hair~~ proof of the hair trigger effect.

1°). <u>The 1^{st} eigenvalue of the Laplacian.</u>

Ω: bdd open subset of $\mathbb{R}^N$ such that one can define the trace operator on $\partial\Omega$.

Let $$\lambda_1(\Omega) = \inf_{\phi \in H_0^1(\Omega)} \frac{\int |\nabla \phi|^2 \, dx}{\int \phi^2 \, dx}.$$

<u>Exercise</u> : $\lambda_1(\Omega)$ is attained by some fc$^{\underline{t}}$ $\phi_1(x) \neq 0$.

- We have : $\forall v \in H^1_0(\Omega)$, $\int \nabla\phi_1 \nabla v = \int \lambda_1 \phi_1 v$.

Indeed look at $t \overset{H}{\longmapsto} \dfrac{\int |\nabla(\phi_1 + tv)|^2}{\int (\phi_1 + tv)^2}$.

Write $\left.\dfrac{dH}{dt}\right|_{t=0} = 0$.

By the regularity theory for the laplacian, we have (as soon as Ω is smooth enough)

$$\begin{cases} -\Delta\phi_1 = \lambda_1(\Omega)\,\phi_1 & (\Omega) \\ \phi_1 = 0 & (\partial\Omega) \end{cases}.$$

- We may choose $\phi_1 \geq 0$. Indeed $\int |\nabla\phi_1|^2 = \int |\nabla|\phi_1||^2$

- By the strong MP, $\phi_1 > 0$ in Ω and $\partial_\nu \phi_1 > 0$ on $\partial\Omega$.

- Finally, if ψ_1 is another eigenfc$^{\underline{t}}$, ~~have~~ then $\exists k \in \mathbb{R} / \psi_1 = k\phi_1$. Indeed : $\exists K > 0 / \psi_1 \leq K\phi_1$. Away from $\partial\Omega$ it is obvious, near $\partial\Omega$ compare $\partial_\nu\psi_1$ to $\partial_\nu\phi_1$.

- Take the least K such that $\psi_1 \leq K\phi_1$. Then either $\exists x_0 \in \Omega / \psi_1 = K\phi_1$ at x_0, or there is $x_0 \in \partial\Omega$ s.t. $\partial_\nu\psi_1(x_0) = \partial_\nu\phi_1(x_0)$. Both cases are impossible.

Finally, if $\Omega \subset \Omega'$, we have $\lambda_1(\Omega) \geq \lambda(\Omega')$.
Indeed, $\tilde{\Phi}_1(x) = \begin{cases} \Phi_1(x) & x \in \Omega \\ 0 & x \in \Omega' \setminus \Omega \end{cases} \in H^1_0(\Omega')$.

So, it is an admissible test function.
We did all this for the following

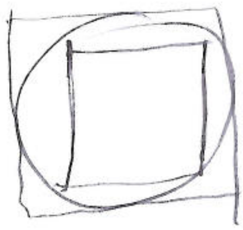
<u>Lemma</u>. Set $\lambda_1(R) := \lambda_1(B_R(0))$.

We have $\lim_{R \to +\infty} \lambda_1(R) = 0$.

$\lim_{R \to 0} \lambda_1(R) = +\infty$.

<u>Proof</u>. A much more sophisticated version was proved by Cenya. However, here we have the very simple situation: $\exists\, c > 0$ such that

$$[-cR, cR]^N \subset B_R(0) \subset [-R, R]^N.$$



<u>N = 1</u>. We are solving

$$\begin{cases} -u'' = \lambda u & [-L, L] \\ u(\pm L) = 0. \end{cases}$$ ~~This gives~~

We have ~~By Poincaré~~, $\lambda_1 \geq 0$. So, ~~$u(x) = A \sin \frac{\pi}{2L}(L+x)$~~
$u(x) = \sin\sqrt{\lambda}(L+x)$. Want: $2\sqrt{\lambda}\, L = \pi$, $\lambda = \frac{\pi^2}{4L^2}$.

<u>N ≥ 1</u>. Separation of variables work here. We have

$$~~\lambda_1(\Omega) =~~ \lambda_1\left((-L, L)^N\right) = \frac{N\pi^2}{4L^2}.$$

Now, try successively $L = cR$ and $L = R$. ∎

2°) <u>Comparison theorems for parabolic equations.</u>

For simplicity we look at solutions of

$$\partial_t u - \Delta u + q(t,x)\cdot\nabla u + r(t,x)\,u \leq 0 \qquad t>0,\ x \in \mathbb{R}^N$$

~~$\lim_{|x|\to+\infty} u(t,x) = 0$~~.

$u(t,x)$ smooth.

There are much more sophisticated versions. But this will serve our purpose well.

<u>Prop.</u> (weak max. principle) $u(0,x) \geq 0 \Rightarrow u(t,x) \geq 0$.

<u>Proof.</u> $v(t,x) = e^{-\lambda t} u(t,x)$, $\lambda > \|r\|_\infty$.

$$v_t - \Delta v + q\cdot\nabla v + (\lambda + r)\,v = 0.$$

$$v(t,x) \geq 0.$$

At a negative minimum: $\partial_t v \leq 0$, $-\Delta v \leq 0$, $q\cdot\nabla v = 0$, $(\lambda + r)\,v < 0$. ~~if~~ ~~λ~~ $\Rightarrow$ Absurd. ∎

If no minimum: work w. $v(t,x) + \varepsilon(1+|x|^2)$. ∎

<u>Prop.</u> (strong max pple). Supp. $\exists\, t_0 > 0$, $x_0 \in \mathbb{R}^N$ s.t. $u(t_0, x_0) = 0$. Then $u(t,x) \equiv 0$.

Very much in the spirit of elliptic eq^{ns} - see Lenya's course.

Consequence. $u_1(t,x)$ and $u_2(t,x)$ solve:

$$\begin{cases} \partial_t u - \Delta u = f(u). \\ u_1(0,x) \leq u_2(0,x). \end{cases}$$

Then $u_1(t,x) \leq u_2(t,x)$. (write an eq$^{\underline{n}}$ for $v(t,x) = u_2(t,x) - u_1(t,x)$.

3°). The theorem on sub and super-solutions.

As promised, we are now going to find "simplified models" for seemingly ~~look~~ hard eq$^{\underline{ns}}$. We are only going to treat sub/super sol$^{\underline{ns}}$ for the steady pb

$$~~q(x)\nabla u~~ - \Delta u = f(u) \quad (x \in \mathbb{R}^N). \quad (2')$$

def. $\underline{u}(x)$ ~~is a sub. sol.~~ is a sub. sol. to (2) iff
- $\underline{u}$ is lipschitz.
- we have, for all $\phi \in C_0^\infty(\mathbb{R}^N)$:

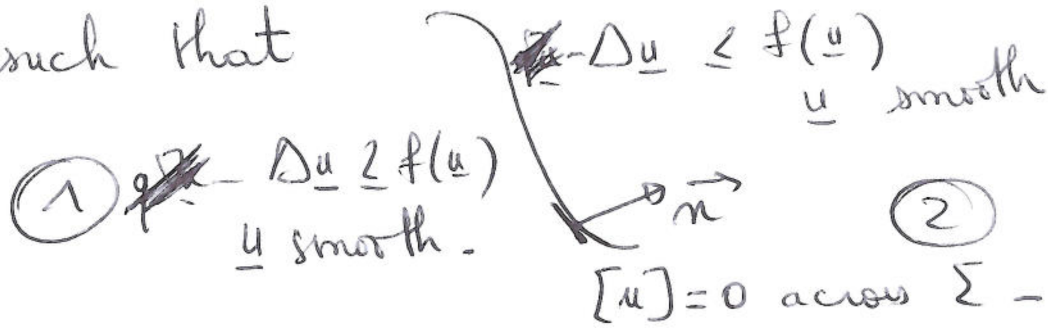
$$\int q \nabla u\, \phi + \int \nabla \underline{u}\, \nabla\phi \leq \int f(\underline{u})\,\phi.$$

In other words $-\Delta \underline{u} \leq f(\underline{\phi})$ in the distributional sense.

$\overline{u}(x)$ is a super-solution if $\leq$ is replaced by $\geq$.

Criterion for a sub-solution.

prop. Assume the existence of a hypersurface Σ such that



Then $\underline{u}$ is a sub-sol. to $-\Delta \underline{u} = f(u)$ iff

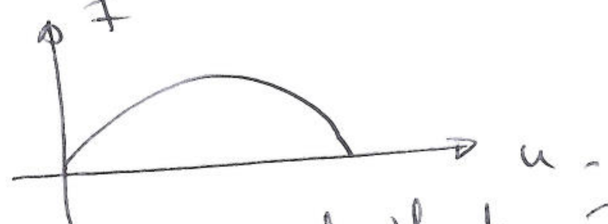
$\left[\frac{\partial u}{\partial n}\right] \geq 0$. super-sol $\longrightarrow \left[\frac{\partial u}{\partial n}\right] \le 0$.

Proof. Start from $\int \nabla \underline{u} \nabla \phi \leq \int f(\underline{u})\, \phi$.

Take $\phi \in C_0^\infty(①)$, $\phi \in C_0^\infty(②)$, & then obtain supp ϕ centred at $x_0 \in \Sigma$.

$$\int_\Sigma \left[\frac{\partial u}{\partial n}\right]\phi \geq 0 .$$

Application.

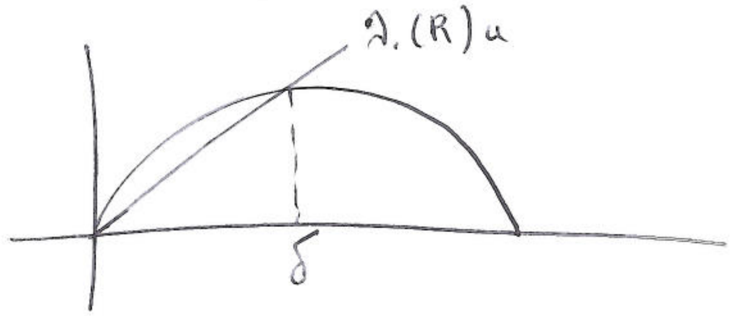


choose $R>0$ such that $\lambda_1(R) < f'(0)$.

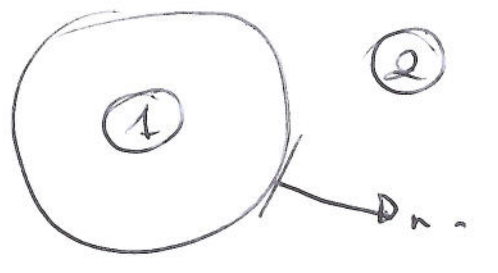
prop. $\underline{u}(x) = \begin{cases} \delta\phi_1(x) & |x| \le R, \\ 0 & |x| \ge R. \end{cases}$ $\delta > 0$

~~There is $\delta>0$ such that~~ If δ is small enough, $\underline{u}(x)$ is a sub-sol. to $-\Delta u = f(u)$.

Proof. This is a computation that will be used many times under several disguises.



Assume $\|\phi_1\|_\infty = 1$. Then



- $-\Delta \underline{u} \leq f(\underline{u})$ in ①.
- $\left[\frac{\partial u}{\partial n}\right] < 0$ by Hopf Lemma (in fact we do not even need this).

- In ①: $-\Delta \underline{u} = -\delta \Delta \phi_1$

$$= +\delta \lambda_1(R) \phi_1$$
$$\leq f(\underline{u}).$$ ∎

Finally, let us mention the

prop. The inf of a family of supersol is a supersol.
sup sub sub.

Proof. Exercise.

We are now ready to state the main result of this subsection.

Back to (1): $u_t - \Delta u = f(u)$.

Th. ~~(i)~~ Assume $f(0) = f(1) = 0$ and nothing else. Let $u_0(x)$ be an initial datum and $u(t,x)$ be the solution starting from u_0. Then

(i) - If $0 \leq u_0 \leq 1$, then $0 \leq u(t,x) \leq 1$.

(ii) - If $\underline{u}$ is a sub-solution of $-\Delta u = f(u)$, and if $u_0 = \underline{u}$, then:

- we have $\partial_t u \geqslant 0$ (in fact > 0).
- $u(t,\cdot)$ converges, as $t \to +\infty$, uniformly on compact sets to a solution of $-\Delta u = f(u)$ which is $\geq \underline{u}$.

(iii). If $\overline{u}$ is a supersol, $\partial_t u \leq 0$.

Why? (i): Apply the consequence of the comparison theorems. 0 and 1 are sol^{ns} of (i).

(ii). Assume $\underline{u}$ to be smooth. Then

$$\partial_t u(0,x) = \Delta \underline{u}(x) + f(u(0,x))$$
$$= \Delta \underline{u}(x) + f(\underline{u}(x))$$
$$\geqslant 0.$$

Now, $\partial_t u$ solves $\begin{cases} v_t - \Delta v - f'(u)v = 0 \\ v(0,x) \geqslant 0. \end{cases}$

Therefore $v \geqslant 0$, and > 0 by the SMP.

Convergence on every compact set is then a consequence of Ascoli's theorem (all derivatives are uniformly bded in time).

If $\underline{u}$ is only lipschitz, first prove $u(h,x) \geq \underline{u}(x)$, then $u(t+h,x) \geq u(t,x)$. Do it by examining $\left(\frac{\underline{u}}{u+\varepsilon(1+|x|)}\right)$

Let us end this tool section by the following simple consideration, that will be very useful in the sequel.

<u>Def.</u> $u(x)$ and $v(x)$ are continuous fct^s. We say that u touches v from below at $x_0 \Leftrightarrow$

- $u \leq v$
- $u(x_0) = v(x_0)$

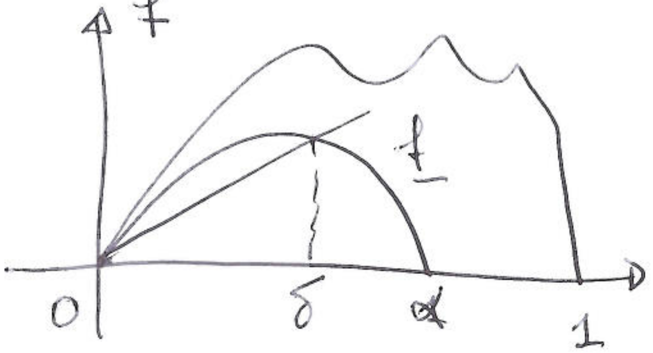
<u>Prop.</u> u sol. of (1).
$\underline{u}$ sub-sol of $-\Delta u = f(u)$.
Then $\underline{u}$ cannot touch u from below.

We will only need the results for the subsol^ns of the type ① / ②. But then

- $\underline{u}$ and u cannot touch at a regular point of $\underline{u}$ by the SMP.
- At a nonregular point of $\underline{u}$ we have

which cannot touch a smooth fct.

4°). The hair-trigger effect.



$\underline{f}'(0) = \frac{f'(0)}{2}$.

We have $\partial_t u - \Delta u - \frac{f(u)}{u} u = 0$, $u \geq 0$.
By the STP we have $u > 0$. Let $R > 0$ large enough so that $\lambda_1(R) < \frac{f'(0)}{4}$ for instance.
Assume $\|\phi_1\|_\infty = 1$. Then:

$$\underline{u}_0(x) = \begin{cases} \delta \phi_1(x) & |x| < R \\ 0 & |x| \geq R \end{cases}$$

is a sub-solution to $-\Delta u = \underline{f}(u)$, so to $-\Delta u = f(u)$. By lowering δ we have

$$\underline{u}_0(x) \leq u(0, x) \quad \text{for } x \in B_R.$$

Consequently: $u(t, x) \geq \underline{u}(t, x)$ solution of

$$\begin{cases} \partial_t \underline{u} - \Delta \underline{u} = f(\underline{u}) \\ \underline{u}(0, x) = \underline{u}_0(x). \end{cases}$$

Moreover there is u^∞, sol. of $-\Delta u^\infty = f(u^\infty)$, such that $\underline{u} \longrightarrow u^\infty$ as $t \to +\infty$. We have $\underline{u}_\infty \geq \underline{u}_0$.

However, now, move around $\underline{u}_0$. In other words, for every $e \in S^{N-1}$, consider

$$\underline{u}_0(x + \sigma e), \quad \sigma > 0.$$

Then: – either $\underline{u}_0(x+\sigma e) \leq \underline{u}_\infty(x)$ for all x.

– or ~~the~~ $\underline{u}_0(x+\sigma e)$ touches $\underline{u}_\infty$ for some σ, from below. But this cannot happen.

<u>Conclusion</u>: $\underline{u}_\infty \geq \max \underline{u}_0 := \delta$.

However: $\underline{u}_\infty$ solves $\begin{cases} \partial_t u - \Delta u = f(u) \\ u(0,x) = \underline{u}_\infty(x). \end{cases}$

So, $\underline{u}_\infty(x) \geq \tilde{u}(t)$ solution of

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = f(\tilde{u}) \\ \tilde{u}(0) = \delta. \end{cases}$$

But we have $\tilde{u}(t) \to 1$. So, $\tilde{u}_\infty \geq 1$. And this is the end of the HTE.

IV). <u>The proof of the Aronson-Weinberger theorem</u>.

Recall the theorem: $\begin{cases} u_t - \Delta u = f(u). \\ u(0,x) = u_0(x). \end{cases}$

<u>Th.</u> Assume

|| $0 \leq u_0 \leq 1$; u_0 comp. supp$^{\text{d}}$. Then

Set $c_* = 2\sqrt{f'(0)}$.

- If $c > c_*$, $\lim_{t \to +\infty} \sup_{|x| \geq ct} u(t,x) = 0$.
- If $c < c_*$, $\lim_{t \to +\infty} \inf_{|x| \leq ct} u(t,x) = 1$.

This is a more sophisticated version of the HTE, but it is important. So, we are going to repeat it.

1°). <u>The upper bound.</u>

The important observation - that was not present in the HTE is that we have now

$$f(u) \leq f'(0)u.$$

So, $u(t,x) \leq \bar{u}(t,x)$ where

$$\begin{cases} \partial_t \bar{u} - \Delta \bar{u} = f'(0)\bar{u} \\ \bar{u}(t,x) = u_0(x). \end{cases}$$

We have: $\bar{u}(t,x) = e^{f'(0)t} \int e^{-\frac{|x-y|^2}{4t}} u_0(y)\,dy$.

If $\operatorname{supp} u_0 \subset B_R$, we have

$$\bar{u}(t,x) \leq \frac{e^{f'(0)t}}{(4\pi t)^{N/2}} \left(\int_{[-R,R]^N} e^{-\sum \frac{|x_i - y_i|^2}{4t}} dy \right)$$

So,

$$u(t,x) \leq \underbrace{\frac{e^{f'(0)t - \frac{||x|-R|^2}{4t}}}{(4\pi t)^{N/2}} (2R)^N}_{\varphi(t,x)} .$$

$$\varphi(t,x) \leq \varepsilon \iff f'(0)t - \frac{||x|-R|^2}{4t} - \log t - C \leq \log \varepsilon$$

$$\iff |x|^2 \gg 4 f'(0) t^2 .$$

So, if one sends $\frac{|x|^2}{4f'(0)t^2}$ to $+\infty$, we have $u(t,x) \to 0$. But $4f'(0) = c_*^2$.

2°). <u>The lower bound</u>.

We are going to show the following result: for a direction $e \in \mathbb{R}^N$ and $c < c_*$, we have

$$\lim_{t \to +\infty} u(t, x + cte) = 1$$

on every compact subset in x. The result will be proved first for $N=1$, then for $N > 1$.

a) <u>$N = 1$</u>

It is sufficient to prove it for $e = -1$ (propagation from right to left). And so, $v(t,x) := u(t, x-ct)$ solves

$$v_t - v_{xx} + c v_x = f(v).$$

Take $c < c_*$. We look at <u>plane wave solutions</u> of the linear equation:

$$\phi_t - \phi_{xx} + c\phi_x = f'(0)\phi ;$$

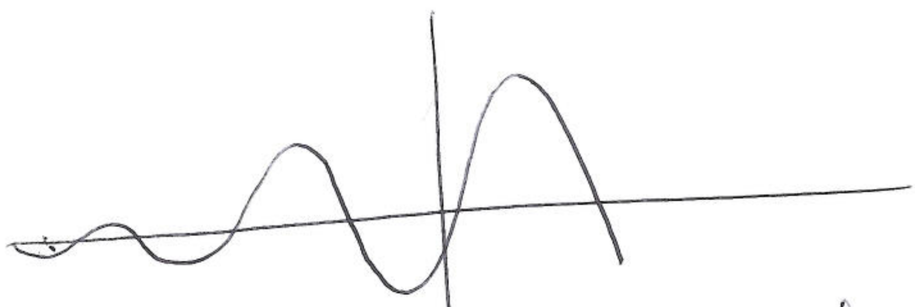
i.e. sol. of the form $\phi(x) = e^{rx}$.

The answer is $-r^2 + cr - f'(0) = 0$, hence

$$r = \frac{c \pm \sqrt{c^2 - 4f'(0)}}{2} = \frac{c \pm \sqrt{c^2 - c_*^2}}{2}.$$

Hence $r \notin \mathbb{R}$.

Take real parts: $\phi(x) = e^{\frac{cx}{2}} \cos \frac{\sqrt{c_*^2 - c^2}}{2} x$.



Call $\underline{\phi}(x)$ any positive arch:

→ check!

It is a subsolution of the linear equation, but not of the nonlinear equation! Is it hopeless? No, just replace $f(u)$ by $(f'(0) - \delta)u$; for $u \leq u_\delta$ we have $(f'(0) - \delta)u \leq f(u)$.

Consequence: if $\varepsilon > 0$ is small enough $(\varepsilon = \varepsilon(\delta))$ then $\varepsilon \underline{\phi}(x)$ is a subsol. to

$$-v'' + cv' = f(v).$$

Now, the argument becomes the same as for the HTE:

- ~~for the~~ lowering ε we have
$$\varepsilon \underline{\phi}(x) \leq v(1,x) \quad \text{for all } x.$$

- $w(t,x)$ solves $\begin{cases} w_t - w_{xx} + cw_x = f(w) \\ w(0,x) = \varepsilon \underline{\phi}(x). \end{cases}$

Existence of w? $w(t, x+ct)$ solves $w_t - w_{xx} = f(w)$. So we are on a safe ground.

Then $w \to w_\infty$, a solution of
$$-w_\infty'' + cw_\infty' = f(w_\infty)$$
which is $> \varepsilon \underline{\phi}$.

Move $\varepsilon \underline{\phi}$ around: $w_\infty \geq \bar{\varepsilon} := \max \varepsilon \underline{\phi}$.

So, for all $t > 0$:
$$w_\infty > \tilde{w}(t), \quad \frac{d\tilde{w}}{dt} = f(\tilde{w}).$$

We have $\tilde{w} \to 1$, so $w_\infty \equiv 1$. Since ~~wt~~ $v \leq 1$, we have the result.

b). <u>$N > 1$</u>.

It is enough to set $e = -e_1 = (-1, 0, \dots, 0)$.

Then $v(t,x) = u(t, x - cte_1)$ solves

$$\partial_t v - \Delta v + c\partial_{x_1} v = f(v).$$

This time we are not going to repeat everything, we know that our job is to find a compactly supported sub-solution of

~~If~~ $$-\Delta\phi + c\partial_{x_1}\phi = (f'(0) - \delta)\phi.$$

__Test 1__. $\phi(x) = \varepsilon \underline{\phi}(x_1)$. Solves the eq^{n}: but not compactly supported.

__Test 2__. Set $x = (x_1, x')$. Let $\underline{\phi}$ be a small such for $-\phi'' + c\phi' = (f'(0) - 2\delta)\phi$.

$\phi_1(x) = 1^{st}$ eigenfunction of $-\Delta_{x'}$ in B_R ($N-1$ dim ball). Take $R > 0$ large enough so that $\lambda_1(R) < \delta$.

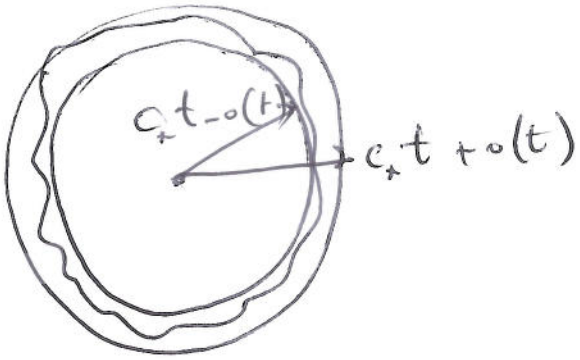
Then $\underline{v}(x) = \varepsilon \underline{\phi}(x)\, \phi_1(x')$ does the job.

So, we have proved the Aronson–Weinberger theorem. The problem that arises now is: what does the transition zone look like?

This is a very interesting pb whose complete details – although everyone agrees on them – have not to my knowledge been written entirely. We are going to explain one of them and state ~~other~~ some other results.

V]. Further qualitative properties.

Recall that we want to understand what happens in



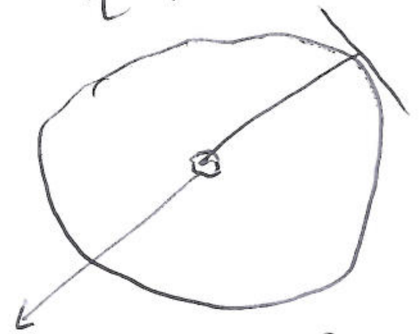
The level sets may wiggle happily within a region which is $o(t)$-wide – could be $\frac{t}{\log t}$ for instance.

Here is an amazing theorem.

<u>Th.</u> (CKRT Jones, 1983) (Rocky Mountain Math Journal). Assume u_0 compactly supp^d and $u(t,x)$ the solution of (1). Let λ be a non-critical value of $u(t,\cdot)$. Then, ~~the~~ for each $x \in \{u(t,x)=\lambda\}$, the normal ~~vector~~ line to

$\{u = \lambda\}$ at x intersects the convex hull of supp(u_0).

<u>Consequence of the theorem.</u> We know that, if t is very large, $\{u = \lambda\}$ is roughly a circle of radius $c_* t$. Renormalise by t and look at $\frac{1}{t}\{u = \lambda\}$.

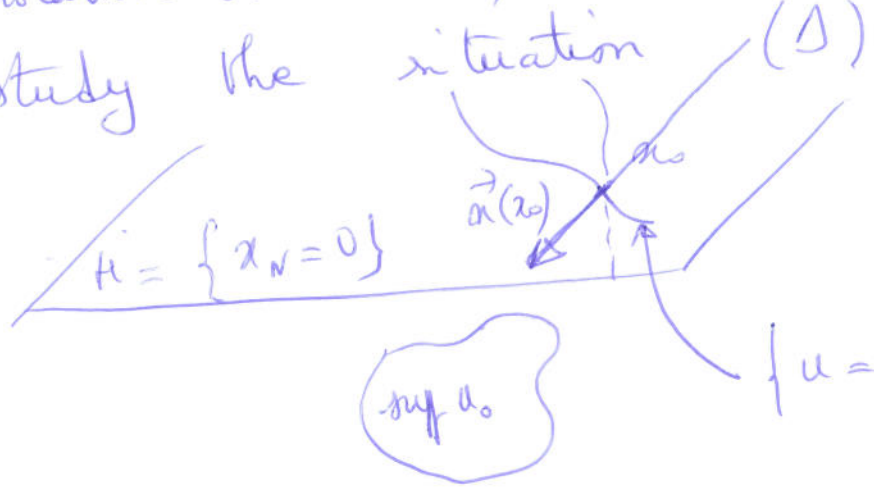


convex hull of supp u_0 of size $\frac{1}{t}$.

We are saying that angle $(\vec{n}(x), \frac{x}{|x|}) = O(\frac{1}{t})$.
This implies that $\frac{1}{t}\{x = \lambda\}$ is $\frac{1}{t}$-away from a sphere. Reverting to initial coordinates: $\{u = \lambda\}$ is $O(1)$-away from a sphere. In other words, the oscillations are of ~~O(~~ order 1. ~~and not~~ And, in other words, it is enough to look at what happens for radial solutions - which we are not going to do now. Just state it.

<u>Proof of Jones's theorem.</u> Assume the existence of (t_0, x_0) such that the conclusion does not

hold, and let Δ be the line passing through x_0 and carried by the normal to $\{u = \lambda\}$ at x_0. By Hahn-Banach theorem, there is a hyperplane H containing Δ such that the convex hull of $\mathrm{supp}(u_0)$ lies on one side of H. By rotation one can, without loss of generality, study the situation



$\mathbb{R}^N = \{(x', x_N)\}$.

Set $w(t,x) = u(t,x',x_N) - u(t,x',-x_N)$ ← reflexion about H.

We have $\partial_t w - \Delta w = c(t,x)\, w$.

$$c(t,x) \sim \int_0^1 u_{x_N}(t, x', \sigma x_N - (1-\sigma)x_N)\, d\sigma$$

$$\begin{cases} w(0,x',x_N) = u(0,x',x_N) - u(0,x',x_N) \geq 0 \text{ if } x_N < 0, \\ w(t,x',0) = 0. \end{cases}$$

Conclusion: $w(t,x',x_N) \geq 0$ if $x_N \leq 0$.

In particular, the Hopf lemma states:

$$\frac{\partial w}{\partial x_N}(t, x', 0) < 0.$$

On the other hand, (Δ) being normal to H, then e_N belongs to the tangent plane to $\{u = \lambda\}$ at x_0 at time t_0. So we have $\frac{\partial u}{\partial x_N}(t_0, x'_0, 0) = 0$. But this is a contradiction, since $\frac{\partial w}{\partial x_N}(t, x', 0) = 2\frac{\partial u}{\partial x_N}(t_0, x', 0)$.

This implies the theorem. ☒
The pf here is due to H. Berestycki.

<u>Conclusion</u>. It is enough to worry about what happens when solving the pb with u_0 radially symmetric. ~~The level sets of u will be trapped between~~

V) <u>More general pbs – open problems.</u>

1°). <u>Homogeneous pb</u>.

Thanks to Jones' theorem, it is sufficient to consider a radial expansion.

- $N = 1$. $\quad u_t - u_{xx} = f(u)$ $\quad f$ concave.

<u>Th</u> (Bramson) $u(t, x(t)) = \lambda$. Then

$$x(t) = c_* t - \frac{3}{2c_*} \log t + q(\lambda) + o(1).$$

- <u>$N > 1$</u>.

<u>Th</u>. (Gärtner). $\quad u(t, r(t)) = \lambda$. Then

$$r(t) = c_* t - \frac{N+2}{c_*} \log t + O(1).$$

The story of $O(1)$ remains to be solved, we have many clues for 

2°). <u>Inhomogeneous coefficients</u>.

$$u_t - \Delta u = c(x) u - u^2. \qquad t > 0, \ x \in \mathbb{R}^N.$$

This already carries all the difficulties already.

- c periodic, $c > 0$.
- Steady states $\begin{cases} -\Delta u = c(x) u - u^2 & x \in \mathbb{R}^N. \\ u > 0 \end{cases}$

<u>Th</u>. (Berestycki - Hamel - Roques 05).

There is a unique steady solution. Moreover it is periodic.

Spreading given by Freidlin - Gärtner. Velocity in direction e: $w_*(e) = \inf_{e'.e>0} \frac{c_*(e')}{e.e'}$; $c_*(e')$ minimal speed of a planar wave.

Of course the shift is interesting. $N=1$: HNRR. $x(t) = c_* t - k \log t + O(1)$; k expressed in terms of the pb.

BUT a lot needs to be understood if c(x) is not periodic anymore.

- Bounds? (BR).
- Examples of c where spreading can be explicitly computed.