Propagation enhancement by a line of fast diffusion -

Banff - Fronts and particle systems.
01/09 to 05/09 2014.

Remerciements -

Introduction. The system we will deal with in this minicourse is the model, introduced by Henri, Luca and I:

$$u_t + Lu = v(t,x,0) - \mu u$$

$$d\partial_y v = \mu u - v.$$

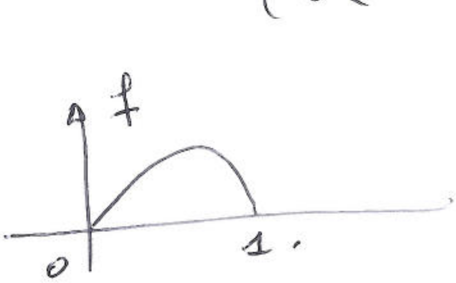
$$v_t - d\Delta v = f(v). \qquad (1)$$

Unknowns : $u(t,x)$, $v(t,x,y)$.

Parameters : $d > 0$, $\mu > 0$.

L : diffusive operator ; $L = -D\partial_{xx}$ or $(-\partial_{xx})^\alpha$.
$(0 < \alpha < 1)$.

$f(u)$: of KPP type.



is a model for heterogeneities, although it more looks like a dynamical BVP. Indeed let us consider the more general model:

$$\partial_t v_+ - d_+ \Delta v_+ = f_+(v_+). \qquad \mathbb{R}^2_+$$

$$d_+\partial_y v_+ = \mu_+ u - v_+.$$

$$u_t + Lu = (v_+ + v_-)(t, x, 0) - (\mu_+ + \mu_-)\, u.$$

$$\partial_t v_- - d_- \Delta v_- = f_-(v_-) \qquad \mathbb{R}^2_-$$

$$d_-\partial_y v_- = \mu_- u - v_-$$

The road is a beautiful heterogeneity, embedded in the medium, and the main difference is that the medium has an action on it. If $+ = -$ every where we recover our model - up to coefficients.

We take an initial datum $(u(0,x), v(0,x,y))$ $= (u_0(x) \geq 0, \cancel{(x,y)})$ and u_0 compactly supported.

And we study the behaviour $t \to +\infty$.

Plan.

I) . Basic properties (to get acquainted)

II) . $L = -D\partial_{xx}$ (with HB, LR)

III) . $L = (-\partial_{xx})^\alpha$. (with HB, ACC, LR)

If time permits we will study other nonlinearities.

Goal and motivation: give a mathematical content to the following statement: "transportation networks speed up biological invasions".

NB. When we talk about a biological invasion, we may be referring to quite a peaceful phenomenon, which may be occurring on quite a large time scale (a species colonises its environment. However, here, the examples we have in mind are a little less peaceful and a little more depressing.

Here are three inspirational examples -

- Wolves.
- Pine processionary moth + tiger mosquito.
- Black plague.

In view of the above examples,

- the line $\{y=0\}$ will be referred to as "the road". $u(t,x)$: density on the road.
- the lower half-plane will be referred to as "the field". $v(t,x,y)$: density in the field.
- L expresses the fact that the species moves quickly on the road.

– The model says that reproduction – or mass creation – only occurs in the field.

<u>Question</u>. Does the road enhance the ~~invasion of~~ overall invasion of the species?

In all the examples that we are going to treat, the answer is yes. More precisely we will show that

- when $L = -D\partial_{xx}$, the spreading velocity may go to infinity as $D \to +\infty$.
- when $L = (-\partial_{xx})^\alpha$, the spreading velocity on the road is exponential in time.

All this will of course be defined more precisely. In view of the above examples, this is depressing.

<u>Link with Henri's lectures</u>. We both agreed that we would talk about the role of heterogeneities in propagation driven by reaction-diffusion. In Henri's talk the heterogeneities are embedded in the coefficients. or in the domain. Ex: $\partial_t u - \Delta u = c(x)u - u^2$. This is not, however, the only way to model them and I declare that (1) (yes, let us call it (1))

I) Basic features.

1°). Mass conservation.

Set $f \equiv 0$ in (1). Integrate:

$$\frac{d}{dt}\int u(t,x)\,dx + \frac{d}{dt}\int v(t,x,y)\,dx\,dy = 0.$$

The intuition is confirmed: no hidden mass creation if there is no reproduction in the field.

2°). Comparison principle.

I swear that, when we wrote the model, we did not aim at obtaining a comparison principle. However, here it is.

prop. $(u_{01}, v_{01}) \leq (u_{02}, v_{02})$. (u_i, v_i): solution starting from (u_{0i}, v_{0i}). Then

$$(u_1, v_1) \leq (u_2, v_2).$$

The reason is that (1) has the structure of a monotone system. What is it?

$$\frac{\partial U}{\partial t} + \begin{pmatrix} L_1 u_1 \\ \vdots \\ L_d u_d \end{pmatrix} = F(U)$$

$F = (f_1, \ldots, f_d)$ with $\dfrac{\partial f_i}{\partial u_j} \geq 0$ if $i \neq j$.

Let us check: the equations may be rewritten as
$$\begin{cases} \partial_t u + Lu = v(t,x,0) - \mu u. \\ \partial_t v - d\Delta v = f(v) + 2d\,\delta_{y=0}\,(\mu u - v). \end{cases}$$

The ~~or~~ monotone system assumptions are satisfied. And because the comparison principle is there we are going to use it abundantly and shamelessly.

3°). Steady states. ($\partial_t = 0$).

This is a little less trivial than ~~the~~ what was before. Many steady states may result in quite complicated propagation patterns. However this is not going to be the case here.

- Trivial steady states: $(0,0)$ and $(\frac{1}{\mu}, 1)$.

Prop. They are the only nonnegative and bounded steady states.

There are a few things to say for the proof. The main step is the

Lemma. $(u(x), v(x,y))$ ~~st~~ nonnegative bded steady state. Then there is $\delta > 0$ such that
$$u(x),\ v(x,y) \geq \delta.$$

This lemma is a particular case of an observation made by Henri, François, Lionel Roques on problems of the form

$$\begin{cases} -\Delta u = c(x)u - u^2 \\ u \geq 0 \end{cases} \quad (\mathbb{R}^N).$$

<u>Th.</u> Assume $c(x)$ periodic. Then $u \not\equiv 0 \Rightarrow \exists \delta > 0$ such that $u \geq \delta$.

This implies that there is only one steady state, which can easily be proved to be periodic.

It remains to say that, if $u_0(x) \leq \frac{1}{\mu}$, then $u(t,x) \leq \frac{1}{\mu}$ and $0 \leq v \leq 1$. This implies the existence of a unique smooth solution (details may be painful in the case $L = (-\partial_{xx})^\alpha$).

II). The case $L = -D\partial_{xx}$.

We will discuss the KPP case in a reasonably detailed fashion. If time permits, we will say a word about different nonlinearities.

1°). <u>f is of the KPP type</u>

We said several times that the road enhances

We need a benchmark to assess how much propagation is indeed enhanced. So, recall the classical

<u>Th.</u> (Aronson - Weinberger, 1976). Consider the equation
$$u_t - d\Delta u = f(u) \quad (\mathbb{R}^N).$$

f: of KPP type. $u(0,x) \geq 0$, compactly supported. Then, if $c_{KPP} = 2\sqrt{d f'(0)}$:

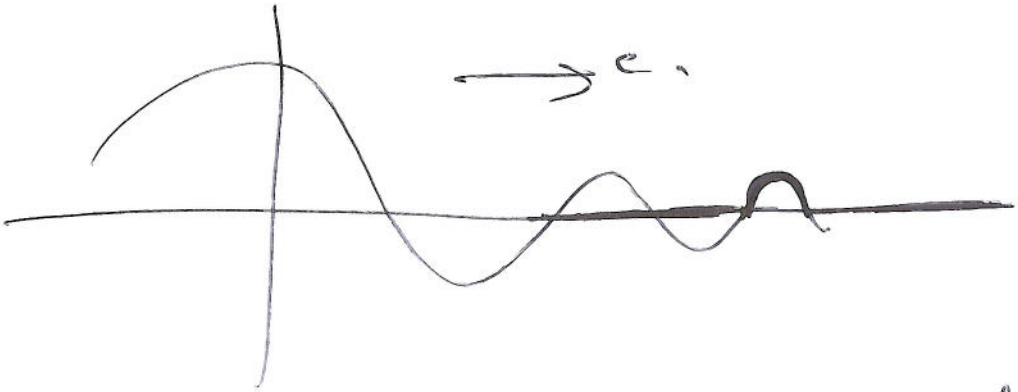
- If $c > c_{KPP}$, $\lim_{t\to+\infty} \sup_{|x|\geq ct} u(t,x) = 0$.
- If $c < c_{KPP}$, $\lim_{t\to+\infty} \inf_{|x|\leq ct} u(t,x) = 1$.

One word of the proof: assume $f(u) = f'(0)u$ on $[0,\theta]$; θ small.

$f(u) \leq f'(0)u$ is the main feature.

* <u>$N=1$</u>. The linear problem
$$\partial_t u - d\partial_{xx} u = f'(0)u$$
has positive plane wave solutions of the form $e^{r(x\pm ct)}$. They serve as super-solutions and ensure that nothing will propagate faster than c_{KPP}.

For $c < c_{KPP}$ the linear problem has <u>complex</u> plane waves $e^{r(x\pm ct)}$. Real part:



Take a small arch moving with speed c : this is a subsolution.

<u>N>1</u>. • <u>Supersolution</u> : one can do the same as N = 1 in every direction e.

$$u(t,x) \leq \inf_{|e|=1} e^{r(x.e - ct)}$$

* <u>Subsolution</u>. The difficulty is to have a subsolution which is compactly supported in every direction. ~~Multiply~~ $\underline{u}(t,x)$ = small arch moving in the direction $e_1 \times \varphi_R(x_2, -, x_N)$;

φ_R : $1^{\underline{st}}$ Dirichlet eigenvalue in $B_R(0)$.

<u>Exercise</u> : Complete the details.

There are 2 reasons why I gave this result :

- plane waves will play an important role in the study of (1).
- This is the type of results that we will be looking for. In particular, we

will not look for a precise asymptotic location of the front. It will always be located with $o(t)$ precision.

Back to (1). We will first be interested in the behaviour on the road, then in the field.

Th. (the road) ~~The~~ There is $c_*(D) > 0$ such that

- if $c > c_*(D)$, then

$$\lim_{t\to+\infty} \sup_{|x|\geq ct} u(t,x) = 0.$$

$$\lim_{t\to+\infty} \sup_{|x|\geq ct} v(t,x,y) = 0$$ uniformly on every compact in y.

- if $c < c_*(D)$, then

$$\lim_{t\to+\infty} \inf_{|x|\leq ct} u(t,x) = \frac{1}{\mu}.$$

$$\lim_{t\to+\infty} \inf_{|x|\leq ct} v(t,x,y) = 1$$ uniformly on every compact in y.

Moreover:

- If $2d \geq D$, then $c_*(D) = c_{KPP} = 2\sqrt{d f'(0)}$.
- If $2d < D$, then $c_*(D) > c_{KPP}$.
- There is $c_\infty > 0$ such that

$$c_*(D) \sim c_* \sqrt{D}\,.$$
$$D \to +\infty\,.$$

Remarks.

(i). No discussion, we have enhancement at least on the road.

(ii). The threshold $D = 2d$: nothing to do with dimension or symmetry. ~~the~~ The threshold becomes $D = d$ when the equation for u is replaced by

$$\partial_t u - D\partial_{xx} u = v(t,x,0) - \mu u + f(u)$$

(the same f).

But notice once again that we do not need reproduction on the road to get enhancement.

(iii). The fact that there is an exchange term on the road does not change the order of magnitude of c w.r.t. D. Notice indeed that, for the KPP equation: ~~if~~ ~~is~~ $c_{KPP} \propto \sqrt{d}$. So, only the proportionality coefficient changes.

- Properties as $D \to +\infty$.

  - $\lim_{D \to +\infty} \theta_{crit}(D) = 0$.

  - $\lim_{D \to +\infty} w_*(\theta, D) = \dfrac{c_{KPP}}{\cos\theta}$.

Interpretation of some results.

- $w_*(\theta, D) \xrightarrow[D \to +\infty]{} \dfrac{c_{KPP}}{\cos\theta}$: take what happens on the road for granted. Then, as D approaches $+\infty$ the picture is close to

$$v \sim 1$$

$$v_t - d\Delta v = f(v).$$

Things propagate downwards with speed c_{KPP} and the front is just the line that is $c_{KPP}t$ distant from the x-axis.

- Existence of the critical angle. The influence of the road is felt well into the field, and not only in the vicinity of the road. The larger D is, the further the influence is felt. This was not completely intuitive.

- <u>Numerical simulation</u>. I will present, today and tomorrow, a rather long list of numeri-cal experiments carried out by A.-C. Coulon in her thesis. What is interesting about them is that they are not only an illus-tration of the results. She conducted them with sufficient care and determination to see further very likely properties, that the intuition would perhaps have missed.

Here we see that the simulation con-firms essentially the theorems on the expansion set, but shows an a priori surprising behaviour of the level sets of v in the vicinity of the road.

No panic: these bumps are only of $O(1)$ size, so they do not contradict our picture. Recall that we are only seeking for $o(A)$ precision.

So, what is $\mathcal{W}$? (or what is $w_*(\theta, D)$).

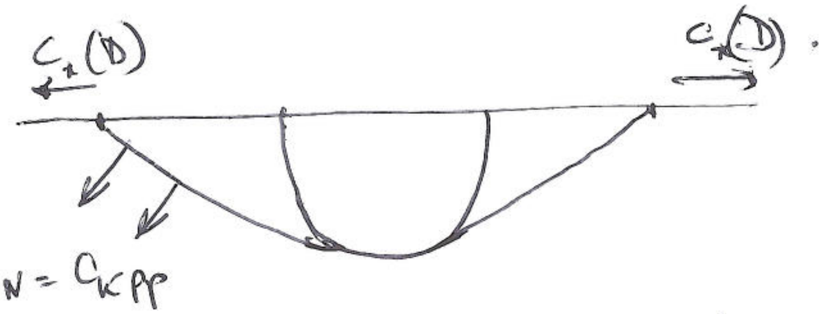
1^{st} guess: consider all trajectories in the lower half plane, of duration 1, comprising:

- a trip on the road ~~with~~ at speed $c_*(D)$ during a time λ
- a straight trip in the field with speed c_{KPP} during time λ.

and take those who lead you the furthest.

This is

In other words:



curve whose end points are slaved to the horizontal velocity c_{KPP} and whose normal velocity is c_{KPP}.

In other words: solve the eikonal equation

$$\partial_t \varphi = c_{KPP} |D\varphi| \quad (x, y) \in \mathbb{R}^2_-.$$

$$\varphi(t, x, 0) = \mathbb{1}_{\{|x| \leq c_*(D) t\}}$$

Exercise: prove that we are talking of the same object.

In fact we have the

<u>prop</u>. Call $\underline{\mathcal{W}}$ the so-constructed set.
Then $\underline{\mathcal{W}} \subsetneq \mathcal{W}$.

To construct $\mathcal{W}$ or $w_\alpha(\theta, D)$, back to basics: the planar waves.

<u>def</u>. (φ, ψ) is a planar wave in the direction e_θ w. speed c iff (i) (φ, ψ) solves (1) with $f(\psi)$ replaced by $f'(0)\psi$.

$$(ii)\ \begin{pmatrix} \varphi(x) \\ \psi(x,y) \end{pmatrix} = \begin{pmatrix} e^{(\alpha,\beta)\cdot((x,0)-cte_\theta)} \\ \lambda e^{(\alpha,\beta)((x,y)-cte_\theta)} \end{pmatrix}.$$

with $\lambda > 0$. The parameters α, β, λ are to be determined.

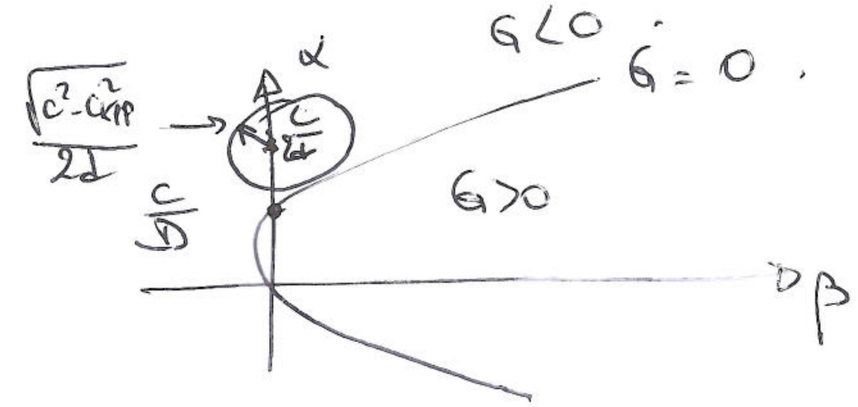
<u>Th</u>. $w_*(\theta, D)$ is the smallest $c > 0$ such that (1) has a planar wave of speed c in the direction θ.

<u>The proof in a nutshell</u>.

- Planar waves are supersolutions. This estimates $w_\alpha(\theta, D)$ from above.
- <u>Subsolutions</u>. let us $1^{\underline{st}}$ deal with $\theta = \frac{\pi}{2}$

(and assume $f(u) = f'(0)u$, $u \le u_0$ small). Plug
$\begin{pmatrix} e^{\alpha(x+ct)} \\ \lambda e^{\alpha(x+ct)+\beta y} \end{pmatrix}$ in the linear system to obtain

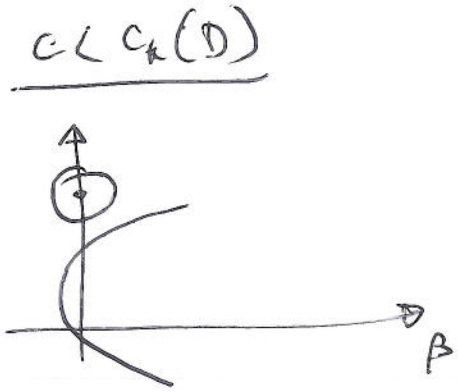
$$\begin{cases} F(c,\alpha,\beta) = 0 . \text{ (the road)}. \\ G(c,\alpha,\beta) = 0 \text{ (the field)} = -d(\alpha^2+\beta^2)+c\alpha-f'(0). \end{cases}$$

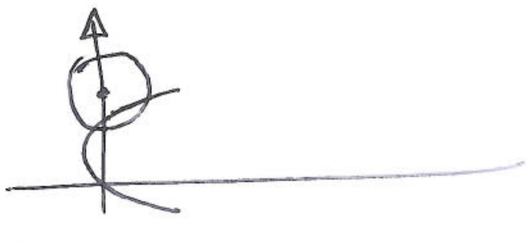


$D > 2d$.

$c = 0 \Rightarrow$ no contact point. $c \gg 1 \Rightarrow 2$ intersection points $\Rightarrow$ there is $c_*(D)$ so that the curves are just tangent.

$c < c_*(D)$ $\qquad$ $c > c_*(D)$

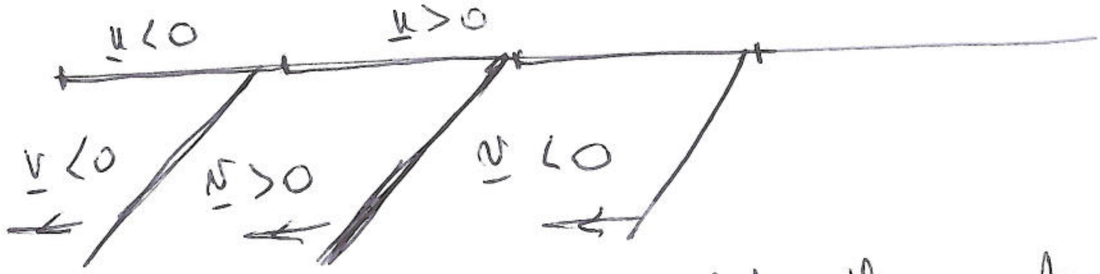




For c just below $c_*(D)$: $F = G = 0$ has two conjugate complex roots (α,β), $(\bar\alpha,\bar\beta)$.
Take, ~~$(\overline{\phi}, \phi$~~ ~~$(\phi$~~, as a 1^{st} guess:

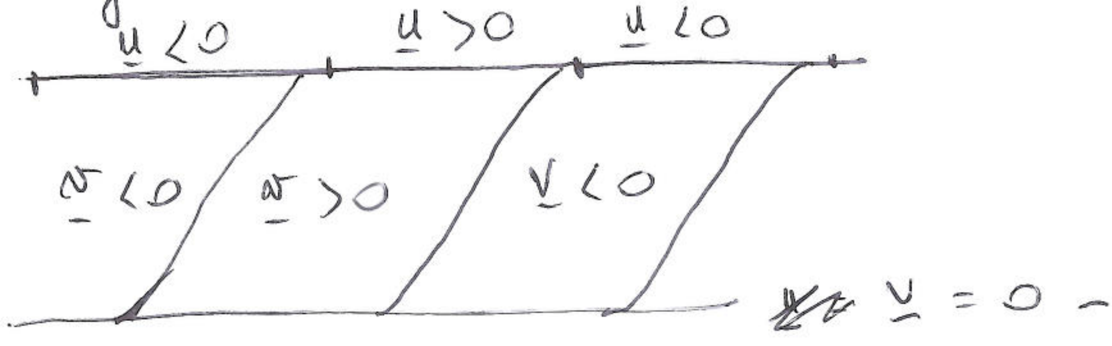
$$(\underline{u}(t,x), \underline{v}(t,x,y)) = \mathrm{Re}\left(e^{\alpha(x-ct)}, \lambda e^{\alpha(x-ct)+\beta y}\right).$$

Level sets of $(\underline{u}, \underline{v})$:



Take one arch and delete the rest. Are we happy? No, because $(\underline{u}, \underline{v})$ is not compactly supported! Are we desperate? No, solve the linear system in the strip $\{x \in \mathbb{R}, -L \leq y < 0\}$ with Dirichlet condition at $y = L$.

For $c < c_*(D)$ close to $c_*(D)$, there is a large enough y so that the above picture perturbs to

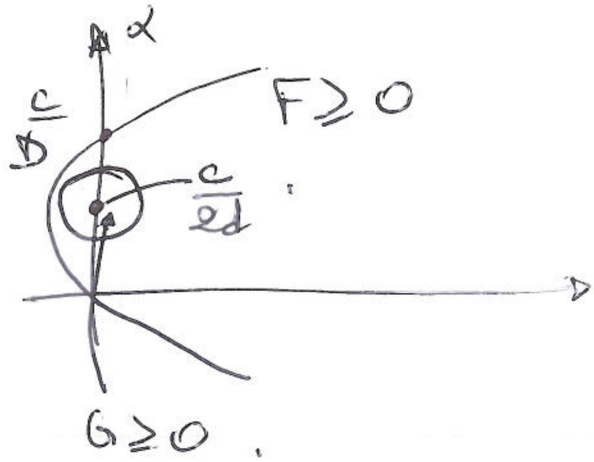


- What happens for $\theta \neq \frac{\pi}{2}$?

$$\begin{cases} F(\theta, \alpha, \beta, c) = 0 \longrightarrow \text{curve similar to } F. \\ G(\theta, \alpha, \beta, c) = 0 \longrightarrow \text{circle rotated by } \frac{\pi}{2} - \theta. \end{cases}$$

- So: there is ~~an~~ $w_*(\theta, D)$ such that $F = G = 0$ has a double root.
- For c slightly below $w_*(\theta, D)$, we do the same analysis but ~~in the strip~~ instead of putting a Dirichlet condition on a line parallel to Ox, we put a Dirichlet condition on a line // to e_θ. ~~And the~~ This works. I do not know why, but this works.

One last thing: why do we have $c = c_{KPP}$ for small D? Well, we read it again on the diagram ~~F~~ $\begin{cases} F = 0 \\ G = 0 \end{cases}$. The picture is



So, for c close to c_{KPP}, ~~we have~~ any wave of the form

$$\begin{pmatrix} e^{\alpha(x+ct)} \\ \lambda e^{\alpha(x+ct)+\beta y} \end{pmatrix}$$

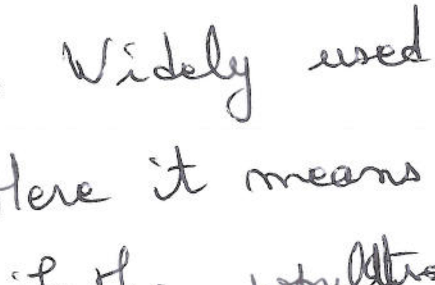
such that $F \geq 0$, $G \geq 0$ is a super-solution. Thus it is impossible to go past this speed.

2°). f is not of the KPP type -

When f is of the KPP type, the proof of enhancement comprises two steps : (i). to prove that one does not make a mistake in reducing the pb to planar waves, (ii) to compute the velocity of planar waves. Step (ii) is almost explicit and allows to prove $c_*(D) \underset{D \to +\infty}{\sim} c_0 \sqrt{D}$ -

It is also known that (i). KPP nonlinearities are not the only nonlinearities in life, (ii). the results that they yield are quite specific. So, it is good to have their conclusions confirmed in other situations.

We will consider here



"ignition type" nonlinearity. Widely used in flame propagation theory. Here it means that reproduction cannot happen if the population density is too small ("weak Allee effect").

Model under study

$$u_t - Du_{xx} = v(t,x,0) - \mu u \qquad y = 0.$$

$$dv_y = \mu u - v$$

$$v_t - d\Delta v = f(v) \qquad (1)$$

$$y = -L$$

$$v_y = 0.$$

The point of view that we are going to take is that of travelling waves.

Benchmark. $u_t - du_{xx} = f(u) \quad x \in \mathbb{R}.$

A travelling wave is a fct $\phi(x-ct)$ which, in the present case, connects 0 to 1. So:

$$\begin{cases} -\phi'' - c\phi' = f(\phi). \\ \phi(-\infty) = 1, \quad \phi(+\infty) = 0. \end{cases}$$

So, it is legitimate to ask whether, in the case of (1), the road enhances the propagation.

Steady states for (1): $(0,0)$ and $(\frac{1}{\mu}, 1)$ are still steady states.

Travelling waves: $(u(t,x), v(t,x,y))$
$= (\phi(x-ct), \psi(x-ct, y))$.

$$D\Phi'' - c\Phi' = \psi(x,0) - \varphi$$

$$d\psi_y = \mu\Phi - \psi$$

$$-d\Delta\psi - c\psi_x = f(\psi) \qquad \text{(TW)}$$

$$(\Phi,\psi) \to (\tfrac{1}{\mu},1) \longleftarrow \qquad \longrightarrow (\Phi,\psi)\to(0,0)$$

$$\psi_y = 0 .$$

The following analysis is due to L. Dietrich.

<u>prop</u>. There is a unique $c_*(D) > 0$ such that (TW) has a travelling wave solution connecting ~~to~~ $(\frac{1}{\mu},1)$ to $(0,0)$. Moreover $\Phi' < 0$, $\psi_x < 0$ and the TW profile is unique.

Not a big surprise but he explored various continuations.

<u>Th</u>. (i). There is a unique $c_\infty > 0$ such that the problem

$$-\Phi'' - c\Phi' = \psi(x,0) - \Phi$$

$$d\psi_y = \mu\Phi - \psi$$

$$-c\psi_x - d\psi_{yy} = f(\psi)$$

$$\psi_y = 0 .$$

has a solution connecting $(\frac{1}{\mu},1)$ to $(0,0)$.

Please note that this is not completely obvious: the problem is parabolic inside the strip, elliptic at the boundary. In other words: in the strip, the solution would only like to see what is to the right, whereas at the boundary the solution ~~would~~ wants to see what happens in the whole domain. The proof consists in making the two talk together.

~~Difficulty in the proof of (ii).~~

Method for the proof of (ii). Of course one sets $x := x\sqrt{D}$, $c := c\sqrt{D}$ and we arrive at

$$-\phi'' - c\phi' = \psi - \mu\phi$$

$$d\psi_y = \mu\phi - \psi$$

$$-\frac{d}{D}\psi_{xx} - d\psi_{yy} - c\psi_y = f(\psi).$$

$$\psi_y = 0.$$

The term $-\frac{d}{D}\psi_{xx}$ does not help, and one has to prove that it does not harm. The key is a lower estimate on c. Without it things break down (and we may end up with quite ugly solutions).

So, enhancement seems to be a universal phenomenon.

III). The case $L = (-\partial_{xx})^\alpha$.

Back to KPP.

Recall that if $u(x)$ $(x \in \mathbb{R}^N)$ is smooth enough and bounded,

$$(-\Delta)^\alpha u(x) = c_\alpha \int \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}}\, dy \quad (0 < \alpha < 1).$$

and c_α is adjusted so that $\widehat{(-\Delta)^\alpha u}(\xi) = |\xi|^{2\alpha}$.

The heat kernel $\begin{cases} \partial_t p_\alpha + (-\Delta)^\alpha p_\alpha = 0. \\ p_\alpha(t=0) = \delta_0 \end{cases}$

satisfies $\frac{1}{C} \frac{t}{|x|^{N+2\alpha}} \le p_\alpha(t,x) \le \frac{Ct}{|x|^{N+2\alpha}}$

for t, $|x|$ large enough. Notice that there is a big difference with $\alpha = 1$, where the heat kernel decays in a Gaussian fashion.

Just as in the case $L = -D\partial_{xx}$ we will need a benchmark. Maybe not to see by how much ~~the~~ the road enhances the propagation, but rather to see how bad things can be (especially if one refers to tiger mosquitoes, or worse).

Plan - 1°). Reaction - diffusion with fractional diffusion - (long digression).

2°) - Results for (1) -

1°). Reaction - diffusion with fractional diffusion.

Let us for one moment forget about (1) and consider the model

$$\begin{cases} u_t + (-\Delta)^\alpha u = f(u) \quad (t>0,\ x \in \mathbb{R}^N). \\ u(0,x) = u_0(x) \geq 0, \text{ comp. supp}^t. \end{cases}$$

<u>Th</u>- (Cabré, R, 2009) - $c_* = \dfrac{f'(0)}{N+2\alpha}$. Then

- If $c < c_*$, $\liminf\limits_{t\to+\infty} \inf\limits_{|x| \leq e^{ct}} u(t,x) = 1$.
- If $c > c_*$, $\lim\limits_{t\to+\infty} \sup\limits_{|x| \geq e^{ct}} u(t,x) = 0$.

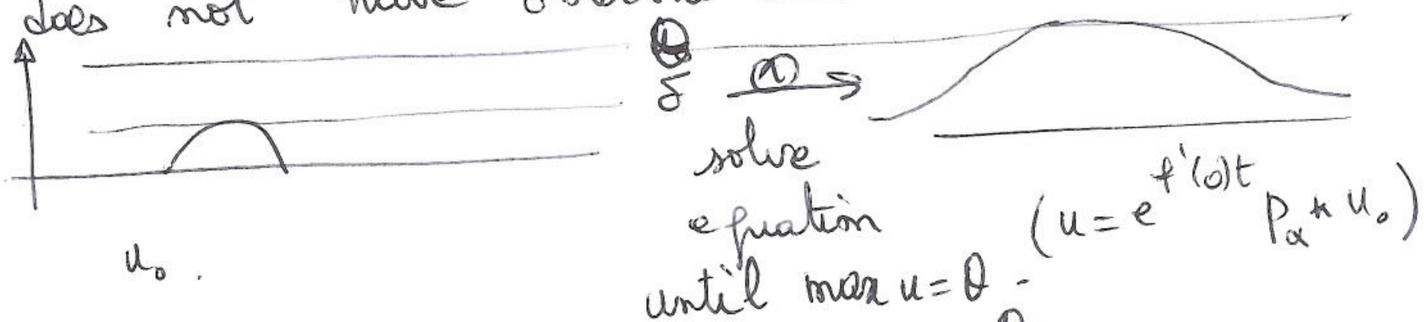
<u>Idea of proof</u>. Of course, here, planar waves do not help anymore. However a crude upper bound can be obtained as

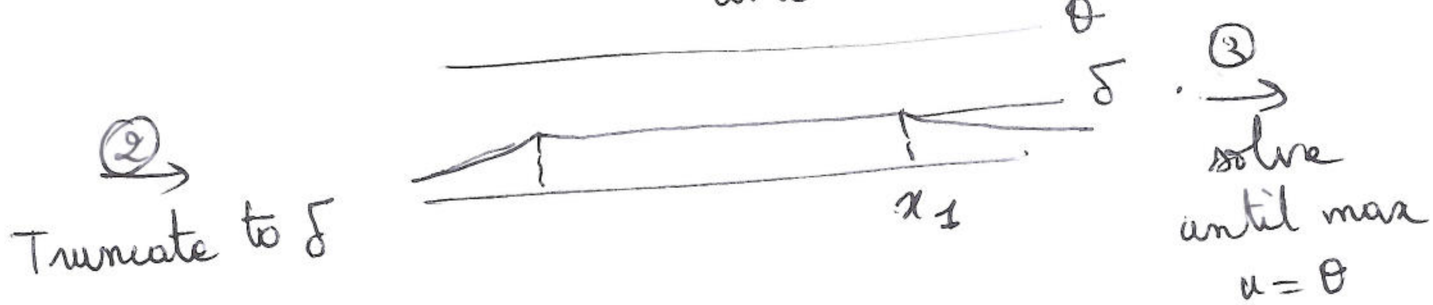
$$u \leq \bar{u}, \left(\partial_t + (-\Delta)^\alpha \bar{u} - f'(0)\right)\bar{u} = 0.$$

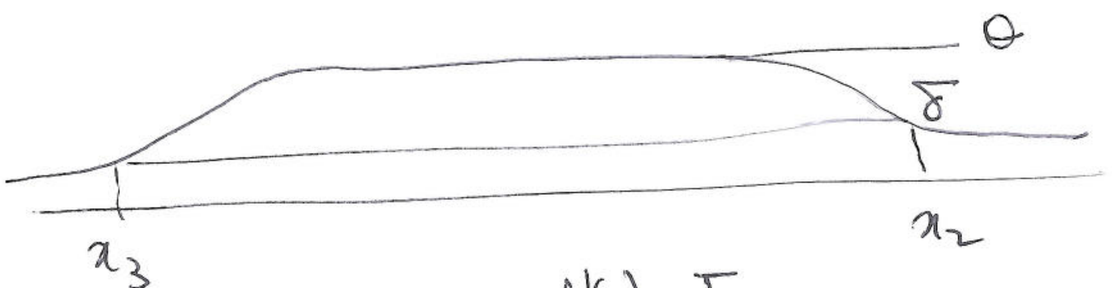
$$\bar{u}(t,x) \sim \frac{ct}{|x|^{N+2\alpha}} e^{f'(0)t}.$$

One just has to locate the $\frac{1}{2}$-level set of $\bar{u}$.

- <u>lower bound</u>. Assume $f(u) = f'(0)u$ for $u \leq \theta$. But, even with this commodity assumption, one does not have obvious subsolutions. So:







and so on.

<u>prop</u>. $x_n \sim C e^{\frac{f'(0)nT}{N+2\alpha}}$; T: time for the sol. to go from δ to θ (one has to prove that it is uniform).

<u>Exercise</u>: fill in the details, - tedious but elementary.

This result was reproved by Méléard and Mirrahimi with a proof that is (to my taste) more

transparent than the original one.
This proof has another drawback: it does not estimate precisely the location of the level sets. In other words, ~~if~~ if $u(t,x)$ is spherically symmetric and

$$u(t, R(t)) = \frac{1}{2}$$

we have: $\log R(t) = \frac{f'(0)t}{N+2\alpha} + o(t)$.

Can we make the $o(t)$ precise? We had no idea of how we could start working on it before the year 2015, until Anne-Charline Coulon started her thesis with us. And we obtained the following result:

<u>Th.</u> (Cabré - Coulon - R.). For all $\gamma \in (0,1)$, there is $C_\gamma > 0$ such that

$$\{u = \gamma\} \subset \left\{ \frac{1}{C_\gamma} e^{\frac{f'(0)t}{N+2\alpha}} \le |x| \le C_\gamma e^{\frac{f'(0)t}{N+2\alpha}} \right\}.$$

In other words, the ~~$\log R(t)$ is $\frac{f'(0)t}{}$~~ $o(t)$ in $\log R(t)$ is $O(1)$.

<u>Idea of the proof</u>: we have the idea that $e^{\lambda t}$ is the correct time scale, for

some λ. So, we set

$$u(t,x) = v(t, e^{-\lambda t} x)$$

with the idea that ∇v will be bounded.

So we get

$$\partial_t v - \lambda \xi . \nabla v + e^{-2\alpha\lambda t} (-\Delta)^\alpha v = f(v).$$

How can we make a sub or a super-solution out of this? Two terms look more important than others, ie the $\xi . \nabla$ one and the $f(v)$ one. ~~And,~~ For all λ, $-\lambda \xi . \nabla v = f(v)$ has a ~~solution~~ radially symmetric solution $V(\xi)$ that satisfies $V(\xi) \sim \frac{\cancel{\quad}}{|\xi|^{f'(0)/\lambda}}$.

<u>Idea</u>: look for sub/super solutions of the form $\underline{u}(t,x) = \frac{a}{1+b(t)|x|^{f'(0)/\lambda}}$; with ~~$b(t)$~~

$$b(t) \underset{t \to +\infty}{\sim} e^{-\cancel{f'(0)} t \lambda}.$$

And this works! It is only a matter of computation.

<u>Problem</u>: how does one choose λ? At time $t=1$,

$$\frac{c^{-1}}{1+|x|^{N+2\alpha}} \leq u(1,x) \leq \frac{c}{1+|x|^{N+2\alpha}}$$

Let me ~~say~~ describe one result that one can prove – not the most complete, but the whole study of the spreading can be deduced from it.

<u>Th.</u> (Cabré – Coulon – R.). If $\gamma > 0$ is small enough, then $\exists\, C_\gamma > 0$ such that

$$\{u = \gamma\} \subset \left\{ \frac{e^{\frac{\mu_1 t}{N+2\alpha}}}{C_\gamma} \leq |x| \leq e^{\frac{\mu_1 t}{N+2\alpha}} C_\gamma \right\}.$$

Same idea as before, but computationally much more delicate.

This result was presented by each of us, and we advertised one specific feature: the spreading velocity does not depend on the direction – in sharp contrast with what happens with linear propagation. One of these presentations was in front of Henri, who did not want to believe us. The question that was asked to us was that of a universal prefactor in front of the exponential – in which case spreading actually depends on the direction.

If there is a universal prefactor, then it must be constant in the $c(x) = 1$ case. ~~[illegible]~~

This imposes ~~[illegible]~~ $N + 2\alpha = \dfrac{f'(0)}{\lambda}$ if we want to put $\dfrac{1}{1+|x|^{f'(0)/\lambda}}$ above or below $u(1,.)$.

<u>Conclusion</u>. The effect of the fractional diffusion is felt only at small times. After that what takes over the phenomenon is the transport eq^n $-\lambda \xi v_\xi = f(v)$ - hence the ODE $v_t = f(v)$.

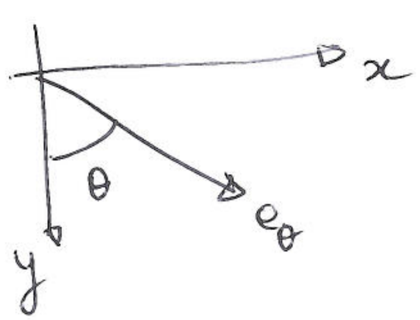
This, BTW, ~~also~~ comes out quite clearly from the proof of Méléard and Sepideh.

It turns out that this very simple idea can be applied to inhomogeneous equations. Consider indeed the model

$$u_t + (-\Delta)^\alpha u = c(x)u - u^2 \quad \begin{pmatrix} t>0 \\ x \in \mathbb{R}^N \end{pmatrix}$$

$c(x)$: 1-periodic, >0. This implies that the steady state $u \equiv 0$ is unstable and that $\mu_1^{per}((-\Delta)^\alpha - c(x) I) < 0$. ($1^{st}$ ~~eigen~~- periodic eigenvalue of $(-\Delta)^\alpha$). Call it $-\mu_1$.

Turn to the field.

Th. 

$D \geq 2d$

There is a function $w_*(\theta, D)$ such that

- $c > w_*(\theta, D) \Rightarrow \lim_{t \to +\infty} \sup_{r \geq ct} v(t, r e_\theta + (x,y)) = 0$

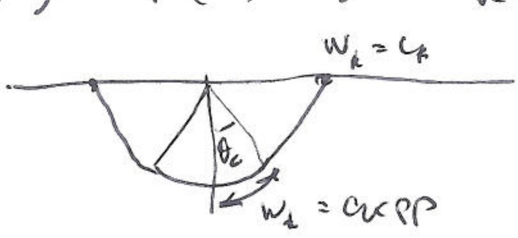
  uniformly in (x,y) on compact sets in (x,y)
- $c < w_*(\theta, D) \Rightarrow \lim_{t \to +\infty} \inf_{r \leq ct} v(t, r e_\theta + (x,y)) = 1$

  uniformly on compact sets in (x,y).

<u>Properties of w_*.</u>

- The curve $\theta \longmapsto w_*(\theta, D)$ is C^1. The set $\mathcal{W}$ limited by the curve $r = w_*(\theta, D)$ is convex.
- $w_*(\frac{\pi}{2}, D) = c_K(D)$
- There is $\theta_{crit}(D) \in (0, \frac{\pi}{2})$ such that:
  - if $\theta \in [\theta_{crit}(D), \frac{\pi}{2}]$: $w_*(\theta, D) = c_{KPP}$.
  - if $\theta \in [0, \theta_{crit}(D))$: $w_*(\theta, D) > c_{KPP}$.

~~we have to~~



This implies that the solution symmetrises. I did not want to believe it, so I asked Anne-Charline to do numerical simulations for $c(x)=1$. The challenge was to prove that Henri was wrong.

Numerical experiments: Beautiful symmetrisation holds in the compactly supported case.

Theorem (Tarfulea, R.). There is $\varphi_1(t) \leq \varphi_2(t)$, $\lim_{t\to+\infty} \varphi_i(t) = 0$: $\{u=\lambda\} \subset \{C_\lambda e^{t+k(u_0)+\varphi_1(t)} \leq |x| \leq C_\lambda e^{t+k(u_0)+\varphi_2(t)}\}$

provided that u_0 is compactly supported.

~~So: Henri was~~ A word of the pf:

Lemma $\exists C>0$: $|\nabla u(t,x)| \leq Ce^{-\gamma t} u(t,x)$ for some $\gamma > 0$.

This uses CCR + some new estimates. We carry this to the fractional laplacian and end up w

$$u_t = \left(\frac{f(u)}{u} + O(e^{-\gamma t})\right) u.$$

This is obvious if $f(u) = f'(0)u$ for $u \leq \theta$ ...

So
- Henri was not wrong.
- But: there is a non-universal $O(1)$ term that might, in the periodic case, compete w a possible universal term.

2°). Spreading for System (1).

Th. (Berestycki, Coulon, Roquejoffre, R.).

- u and v spread like $e^{\frac{f'(0)t}{1+2\alpha}}$ on the road.
- In the field, v spreads like $\frac{c_{KPP}}{\cos\theta}$ in the direction θ.

- Spreading in the field: we saw this yesterday. The picture is

$$\frac{v \sim 1.}{v_t - d\Delta v = f(v).}$$

which implies downward propagation at velocity c_{KPP}.

- Spreading on the road. 1^{st} idea: try to implement the CCR idea. This necessitates the construction of a global solution to $-\lambda\xi\phi' = -\mu\phi + \psi$.

$$d\psi_y = \mu\phi - \psi$$

$$-\xi\psi_\xi - d\psi_{\xi\xi} = f(\psi) \qquad \xrightarrow{\psi \to 0}$$

$$\xleftarrow{\psi \to 0}$$

We were not able to do it – and it is not clear to me that there is one.

So: back ~~ses~~ to results of weaker type.

1. Super-solution.

We simply solve the Cauchy problem for (1) with $f(v) \longrightarrow f'(0)v$.

Th. (Coulon). $u(t,x) \sim C_\alpha \dfrac{e^{f'(0)t}}{t^{3/2}|x|^{1+2\alpha}}$

|| for large t and large x.

This is a perfectly nontrivial computation, which involves Polya-type integrals. So,

$$\left\{u = \frac{1}{2\mu}\right\} \subset \left\{|x| \leq \frac{e^{\frac{f'(0)t}{1+2\alpha}}}{t^{\frac{3}{2(1+2\alpha)}}}\right\}.$$

Remark 1. This is not a pure exponential, ~~just~~ as in the pb in the free space.

2. Sub-solution.

Replace the domain by a strip:

$$u_t + (-\partial_{xx})^\alpha u = v(t,x,0) - \mu u$$

$$v_t - d\Delta v = f(v). \qquad dv_y = \mu u - v$$

$$v = 0.$$

To devise a subsolution, we ~~may~~ need to understand the pb in ξ with the Dirichlet boundary condition. But here, setting $\xi = e^{-\lambda t} x$ we reduce the pb to finding a global solution of

$$\begin{cases} \frac{du}{dt} = \cancel{k} v(t,0) - \mu u(t). & (y=0). \\ v_t - dv_{yy} = f(v) & (-L < y < 0) \\ d v_y(t,0) = \mu u(t) - v; & v(t,-L) = 0. \end{cases}$$

(We suppress one variable: the x one).

Such solutions in a bounded domain are not so difficult to construct.

Putting everything together we obtain a subsolution of the form $\cancel{\dfrac{\Phi_L(y)}{1+ e^{\left(\frac{f'(0)}{1+2\alpha} - O\left(\frac{1}{L^2}\right)\right)t}}}$

$$\frac{\Phi(y)}{1 + e^{\left(\frac{f'(0)}{1+2\alpha} - O\left(\frac{1}{L^2}\right)\right)t} |x|^{1+2\alpha}}.$$

which allows to conclude by sending $L \to +\infty$.

3°) Numerical simulations.

OK, we have the results but the numerics tell us much more.

- Confirmation of the theorem. This is the least. However the front in the field is not completely flat. There is an $O(1)$ bump that deserves to be understood.
- Possibly related is the precise location of the level sets of u. They seem to follow very closely those of the super-solution and the guess is now that the sharp asymptotics is $\left(\frac{e^{f'(0)t}}{t^{3/2}}\right)^{\frac{1}{1+2\alpha}}$.

This is currently under study.