Vortex sheet solutions for the Ginzburg-Landau system in cylinders: symmetry and global minimality

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Abstract

We consider the Ginzburg-Landau energy E_ε for $\mathbb{R}^M$-valued maps defined in a cylinder shape domain $B^N \times (0,1)^n$ satisfying a degree-one vortex boundary condition on $\partial B^N \times (0,1)^n$ in dimensions $M \geq N \geq 2$ and $n \geq 1$. The aim is to study the radial symmetry of global minimizers of this variational problem. We prove the following: if $N \geq 7$, then for every $\varepsilon > 0$, there exists a unique global minimizer which is given by the non-escaping radially symmetric vortex sheet solution $u_\varepsilon(x,z) = (f_\varepsilon(|x|)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N}})$, $\forall x \in B^N$ that is invariant in $z \in (0,1)^n$. If $2 \leq N \leq 6$ and $M \geq N+1$, the following dichotomy occurs between escaping and non-escaping solutions: there exists $\varepsilon_N > 0$ such that

- if $\varepsilon \in (0, \varepsilon_N)$, then every global minimizer is an escaping radially symmetric vortex sheet solution of the form $R\tilde{u}_\varepsilon$ where $\tilde{u}_\varepsilon(x,z) = (\tilde{f}_\varepsilon(|x|)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N-1}}, g_\varepsilon(|x|))$ is invariant in z-direction with $g_\varepsilon > 0$ in $(0,1)$ and $R \in O(M)$ is an orthogonal transformation keeping invariant the space $\mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$;
- if $\varepsilon \geq \varepsilon_N$, then the non-escaping radially symmetric vortex sheet solution $u_\varepsilon(x,z) = (f_\varepsilon(|x|)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N}})$, $\forall x \in B^N, z \in (0,1)^n$ is the unique global minimizer; moreover, there are no bounded escaping solutions in this case.

We also discuss the problem of vortex sheet $\mathbb{S}^{M-1}$-valued harmonic maps.

Keywords: vortex, uniqueness, symmetry, minimizers, Ginzburg-Landau equation, harmonic maps.
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Contents

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1 Introduction and main results

In this paper, we consider the following Ginzburg-Landau type energy functional

$$E_\varepsilon(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1-|u|^2)\right] dX, \tag{1}$$

where $\varepsilon > 0$, $X = (x, z) \in \Omega = B^N \times (0,1)^n$ is a cylinder shape domain with B^N the unit ball in $\mathbb{R}^N$, $n \geq 1$, $N \geq 2$ and the potential $W \in C^2((-\infty, 1]; \mathbb{R})$ satisfies

$$W(0) = 0,\ W(t) > 0 \text{ for all } t \in (-\infty, 1] \setminus \{0\} \text{ and } W \text{ is convex.} \tag{2}$$

(The prototype potential is $W(t) = \frac{t^2}{2}$ for $t \leq 1$.) We investigate the global minimizers of the energy E_ε in the set of $\mathbb{R}^N$-valued maps:

$$\mathscr{A}_N := \{u \in H^1(\Omega; \mathbb{R}^N) : \ u(x,z) = x \text{ for every } x \in \partial B^N = \mathbb{S}^{N-1}, z \in (0,1)^n\}.$$

The boundary assumption $u(x,z) = x$ for every $x \in \mathbb{S}^{N-1}$ and every $z \in (0,1)^n$ is referred in the literature as the degree-one vortex boundary condition.

The direct method in the calculus of variations yields the existence of a global minimizer u_ε of E_ε over $\mathscr{A}_N$ for all range of $\varepsilon > 0$. Moreover, any minimizer u_ε satisfies $|u_\varepsilon| \leq 1$ in Ω, u_ε belongs to $C^1(\overline{\Omega}; \mathbb{R}^N)$ and solves the system of PDEs (in the sense of distributions) with mixed Dirichlet-Neumann boundary conditions:

$$\begin{cases} -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon\, W'(1-|u_\varepsilon|^2) \quad \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial z} = 0 \quad \text{on } B^N \times \partial(0,1)^n, \\ u(x,z) = x \quad \text{on } \partial B^N \times (0,1)^n. \end{cases} \tag{3}$$

1.1 Minimality of the $\mathbb{R}^N$-valued vortex sheet solution

The first goal of this paper is to prove the uniqueness and radial symmetry of the global minimizer of E_ε in $\mathscr{A}_N$ for *all* $\varepsilon > 0$ in dimensions $N \geq 7$ and $n \geq 1$. In fact, in these dimensions, we show that the global minimizer of E_ε in $\mathscr{A}_N$ is unique and given by the following radially symmetric critical point of E_ε that is invariant in z: [1]

$$u_\varepsilon(x,z) = f_\varepsilon(|x|)\frac{x}{|x|} \quad \text{for all } x \in B^N \text{ and } z \in (0,1)^n, \tag{4}$$

[1] If $n = 0$ and $N \geq 2$, then $SO(N)$ induces a group action on $\mathscr{A}_N$ given by $u(x) \mapsto R^{-1}u(Rx)$ for every $x \in B^N$, $R \in SO(N)$ and $u \in \mathscr{A}_N$ under which the energy E_ε and the vortex boundary condition are invariant. Then every bounded critical point of E_ε in $\mathscr{A}_N$ that is invariant under this $SO(N)$ group action has the form (4), see e.g. [8, Lemma A.4].

where the radial profile $f_\varepsilon : [0,1] \to \mathbb{R}$ in $r = |x|$ is the unique solution to the ODE:

$$\begin{cases} -f_\varepsilon'' - \frac{N-1}{r} f_\varepsilon' + \frac{N-1}{r^2} f_\varepsilon = \frac{1}{\varepsilon^2} f_\varepsilon W'(1-f_\varepsilon^2) \quad \text{for } r \in (0,1), \\ f_\varepsilon(0) = 0, f_\varepsilon(1) = 1. \end{cases} \tag{5}$$

We recall that the unique radial profile f_ε satisfies $f_\varepsilon > 0$ and $f_\varepsilon' > 0$ in $(0,1)$ (see e.g. [7, 9, 8]). Note that the zero set of u_ε is given by the n-dimensional vortex sheet $\{0_{\mathbb{R}^N}\} \times (0,1)^n$ in Ω (in particular, if $n = 0$, it is a vortex point, while for $n = 1$, it is a vortex filament); therefore, u_ε in (4) is called (radially symmetric) *vortex sheet solution* to the Ginzburg-Landau system (3).

Theorem 1. *Assume that W satisfies* (2) *and $n \geq 1$. If $N \geq 7$, then u_ε given in* (4) *is the unique global minimizer of E_ε in $\mathscr{A}_N$ for every $\varepsilon > 0$.*

The proof is reminiscent of the works of Ignat-Nguyen-Slastikov-Zarnescu [12, 11] studying uniqueness and symmetry of minimizers of the Ginzburg-Landau functionals for $\mathbb{R}^M$-valued maps defined on smooth N-dimensional domains, where M is not necessarily equal to N. The idea is to analyze $E_\varepsilon(u)$ for an arbitrary map u and to exploit the convexity of W to lower estimate the excess energy w.r.t. $E_\varepsilon(u_\varepsilon)$ by a suitable quadratic energy functional depending on $u - u_\varepsilon$. This quadratic functional comes from the linearized PDE at u_ε and can be handled by a factorization argument. The positivity of the excess energy then follows by a Hardy-type inequality holding true only in high dimensions $N \geq 7$. This is similar to the result of Jäger and Kaul [14] on the minimality of the equator map for the harmonic map problem in dimension $N \geq 7$ that is proved using a certain inequality involving the sharp constant in the Hardy inequality.

We expect that our result remains valid in dimensions $2 \leq N \leq 6$:

Open problem 2. *Assume that W satisfies* (2)*, $n \geq 1$ and $2 \leq N \leq 6$. Is it true that for every $\varepsilon > 0$, u_ε given in* (4) *is the unique global minimizer of E_ε in $\mathscr{A}_N$?*

It is well known that the uniqueness of u_ε holds true for large enough $\varepsilon > 0$ in any dimension $N \geq 2$. Indeed, denoting by λ_1 the first eigenvalue of $-\Delta_x$ in B^N with zero Dirichlet boundary condition, then for any $\varepsilon > \sqrt{W'(1)/\lambda_1}$, E_ε is strictly convex in $\mathscr{A}_N$ (see e.g., [1, Theorem VIII.7], [12, Remark 3.3]) and thus has a unique critical point in $\mathscr{A}_N$ that is the global minimizer of our problem. We improve this result as follows: for the radial profile f_ε in (5), we denote by $\ell(\varepsilon)$ the first eigenvalue of the operator

$$L_\varepsilon = -\Delta_x - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2) \tag{6}$$

acting on maps defined in B^N with zero Dirichlet boundary condition. It is proved in [8, Lemma 2.3] that if $2 \leq N \leq 6$ and $W \in C^2((-\infty, 1])$ satisfies (2), then the first eigenvalue $\ell(\varepsilon)$ is a continuous function in ε and there exists $\varepsilon_N \in (0, \infty)$ such that

$$\ell(\varepsilon) < 0 \text{ in } (0, \varepsilon_N), \quad \ell(\varepsilon_N) = 0 \quad \text{and} \quad \ell(\varepsilon) > 0 \text{ in } (\varepsilon_N, \infty). \tag{7}$$

Note that[2] $0 = \ell(\varepsilon_N) > \lambda_1 - \frac{1}{\varepsilon_N^2} W'(1)$ yielding

$$\varepsilon_N < \sqrt{W'(1)/\lambda_1}.$$

THEOREM 3. *Assume that W satisfies* (2)*, $n \geq 1$ and $2 \leq N \leq 6$. If $\varepsilon \geq \varepsilon_N$, then u_ε given in* (4) *is a global minimizer of E_ε in $\mathscr{A}_N$. Moreover, if either $\varepsilon > \varepsilon_N$, or ($\varepsilon = \varepsilon_N$ and W is in addition strictly convex), then u_ε is the unique global minimizer of E_ε in $\mathscr{A}_N$.*

The case $\varepsilon < \varepsilon_N$ is still not solved as stated in Open Problem 2. Let us summarize some known results:

I. The case of $n = 0$ and $\Omega = B^N$ (we also discuss here the problem for $\Omega = \mathbb{R}^N$). In this case, the above question was raised in dimension $N = 2$ for the disk $\Omega = B^2$ in the seminal book of Bethuel, Brezis and Hélein [1, Problem 10, page 139], and in general dimensions $N \geq 2$ and also for the blow-up limiting problem around the vortex point (when the domain Ω is the whole space $\mathbb{R}^N$ and by rescaling, ε can be assumed equal to 1) in an article of Brezis [3, Section 2]. For *sufficiently small* $\varepsilon > 0$ and for the disk domain $\Omega = B^2$, Pacard and Rivière [20, Theorem 10.2] showed that E_ε has a unique critical point in $\mathscr{A}_2$ and so, it is given by the radially symmetric solution u_ε in (4) (for $n = 0$). For $N \geq 7$, $\Omega = B^N$ and *any* $\varepsilon > 0$, it is proved in [11] that E_ε has a unique minimizer in $\mathscr{A}_N$ which is given by the radially symmetric solution u_ε in (4) (for $n = 0$). For $2 \leq N \leq 6$ and $\Omega = B^N$, Ignat-Nguyen [8] proved that for any $\varepsilon > 0$, u_ε is a local minimizer of E_ε in $\mathscr{A}$ (which is an extension of the result of Mironescu [18] in dimension $N = 2$). Also, Mironescu [19] showed in dimension $N = 2$ that, when B^2 is replaced by $\mathbb{R}^2$ and $\varepsilon = 1$, a local minimizer of E_ε satisfying a degree-one boundary condition at infinity is unique (up to translation and suitable rotation). This was extended in dimension $N = 3$ by Millot and Pisante [17] and in dimensions $N \geq 4$ by Pisante [21] in the case of the blow-up limiting problem on $\mathbb{R}^N$ and $\varepsilon = 1$. All these results (holding for $n = 0$) are related to the study of the limit problem obtained by sending $\varepsilon \to 0$ when the Ginzburg-Landau problem on the unit ball 'converges' to the harmonic map problem from B^N into the unit sphere $\mathbb{S}^{N-1}$. For that harmonic map problem, the vortex boundary condition yields uniqueness of the minimizing harmonic $\mathbb{S}^{N-1}$-valued map $x \mapsto \frac{x}{|x|}$ if $N \geq 3$; this is proved by Brezis, Coron and Lieb [4] in dimension $N = 3$ and by Lin [15] in any dimension $N \geq 3$; we also mention Jäger and Kaul [14] in dimension $N \geq 7$ for the equator map $x \in B^N \mapsto (\frac{x}{|x|}, 0) \in \mathbb{S}^N$.

II. The case of $n \geq 1$ and $\Omega = B^N \times (0,1)^n$. As we explain in Remark 6 below, for some $\varepsilon > 0$, if the minimality of the radially symmetric solution u_ε in (4) holds in the case $n = 0$ (so, for $\Omega = B^N$), then this implies the minimality of u_ε in $\Omega = B^N \times (0,1)^n$ also for every dimension $n \geq 1$. In particular, the result of Pacard-Rivière [20, Theorem 10.2] for $n = 0$ and $N = 2$ yields the minimality of u_ε in (4) defined in $B^2 \times (0,1)^n$ for every $n \geq 1$ if $\varepsilon > 0$ is sufficiently small. Also, the result of Ignat-Nguyen-Slastikov-Zarnescu [11, Theorem 1]

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[2] Indeed, if $v \in H_0^1(B^N)$ is a first eigenfunction of L_{ε_N} in B^N such that $\|v\|_{L^2(B^N)} = 1$ then

$$\lambda_1 \leq \int_{B^N} |\nabla_x v|^2 \, dx = \frac{1}{\varepsilon_N^2} \int_{B^N} W'(1 - f_{\varepsilon_N}^2) v^2 \, dx < \frac{W'(1)}{\varepsilon_N^2}$$

because $\ell(\varepsilon_N) = 0$, $0 < f_{\varepsilon_N} < 1$ in $(0,1)$ and (2) implies $W'(0) = 0$ and $W'(t) > 0$ for $t \in (0,1]$.

for $n = 0$, $N \geq 7$ and any $\varepsilon > 0$ generalizes to dimension $n \geq 1$ for $\Omega = B^N \times (0,1)^n$ (see the proof of Theorem 1). We also mention the work of Sandier-Shafrir [24] where they treat the case of topologically trivial $\mathbb{R}^2$-valued solutions in the domain $\Omega = \mathbb{R}^3$ (see also [5, 22] for vortex filament solutions).

1.2 Escaping $\mathbb{R}^M$-valued vortex sheet solutions when $M \geq N + 1$

In dimension $2 \leq N \leq 6$ and for $\varepsilon < \varepsilon_N$ given in (7), a different type of radially symmetric vortex sheet solution appears provided that the target space has dimension $M \geq N + 1$. More precisely, we consider the energy functional E_ε in (1) over the set of $\mathbb{R}^M$-valued maps

$$\mathscr{A} := \{u \in H^1(\Omega; \mathbb{R}^M) : u(x, z) = (x, 0_{\mathbb{R}^{M-N}}) \text{ on } \partial B^N = \mathbb{S}^{N-1} \subset \mathbb{R}^M, z \in (0,1)^n\}. \quad (8)$$

If $M \geq N + 1$, the prototype of radially symmetric critical points of E_ε in $\mathscr{A}$ has the following form (invariant in z-direction): [3]

$$\tilde{u}_\varepsilon(x, z) = (\tilde{f}_\varepsilon(r)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N-1}}, g_\varepsilon(r)) \in \mathscr{A}, \quad x \in B^N, z \in (0,1)^n, r = |x|, \quad (9)$$

where $(\tilde{f}_\varepsilon, g_\varepsilon)$ satisfies the system of ODEs

$$-\tilde{f}_\varepsilon'' - \frac{N-1}{r}\tilde{f}_\varepsilon' + \frac{N-1}{r^2}\tilde{f}_\varepsilon = \frac{1}{\varepsilon^2}W'(1 - \tilde{f}_\varepsilon^2 - g_\varepsilon^2)\tilde{f}_\varepsilon \quad \text{in } (0,1), \quad (10)$$

$$-g_\varepsilon'' - \frac{N-1}{r}g_\varepsilon' = \frac{1}{\varepsilon^2}W'(1 - \tilde{f}_\varepsilon^2 - g_\varepsilon^2)g_\varepsilon \quad \text{in } (0,1), \quad (11)$$

$$\tilde{f}_\varepsilon(1) = 1 \text{ and } g_\varepsilon(1) = 0. \quad (12)$$

We distinguish two type of radial profiles:

• the *non-escaping* radial profile ($\tilde{f}_\varepsilon = f_\varepsilon, g_\varepsilon = 0$) with the unique radial profile f_ε given in (5); in this case, we say that $\tilde{u}_\varepsilon = (u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ is a *non-escaping* (radially symmetric) vortex sheet solution where u_ε is given in (4).

• the *escaping* radial profile $(\tilde{f}_\varepsilon, g_\varepsilon)$ with $g_\varepsilon > 0$ in $(0,1)$; in this case, we call an *escaping* (radially symmetric) vortex sheet solution $\tilde{u}_\varepsilon$ in (9). In this case, $\tilde{f}_\varepsilon \neq f_\varepsilon$ and obviously, $(\tilde{f}_\varepsilon, -g_\varepsilon)$ is another radial profile to (9)-(12).

The properties of such radial profiles (e.g., existence, uniqueness, minimality, monotonicity) are analyzed in Theorem 9 below and are based on ideas developed by Ignat-Nguyen [8].

Our main result proves the radial symmetry of global minimizers of E_ε in $\mathscr{A}$. More precisely, the following dichotomy occurs at ε_N defined in (7): if $\varepsilon < \varepsilon_N$, then *escaping* radially symmetric vortex sheet solutions exist and determine (up to certain orthogonal transformations) the full set of global minimizers of E_ε in $\mathscr{A}$; if instead $\varepsilon \geq \varepsilon_N$, then the *non-escaping* radially symmetric vortex sheet solution is the unique global minimizer of E_ε in $\mathscr{A}$ and no escaping radially symmetric vortex sheet solutions exist in this case.

[3] If $M = N + 1$, then $\tilde{u}_\varepsilon(x, z) = (\tilde{f}_\varepsilon(r)\frac{x}{|x|}, g_\varepsilon(r))$ for every $x \in B^N$ and $z \in (0,1)^n$. In fact, if $n = 0$ (so, for $\Omega = B^N$), every bounded critical point of E_ε in $\mathscr{A}$ that is invariant under the action of a special group (isomorphic to $SO(N)$) has the form of $\tilde{u}_\varepsilon$, see [8, Definition A.1, Lemma A.5].

Theorem 4. *Let $n \geq 1$, $2 \leq N \leq 6$, $M \geq N+1$, $W \in C^2((-\infty, 1])$ satisfy (2) and be strictly convex. Consider $\varepsilon_N \in (0, \infty)$ such that $\ell(\varepsilon_N) = 0$ in (7). Then there exists an escaping radially symmetric vortex sheet solution $\tilde{u}_\varepsilon$ in (9) with $g_\varepsilon > 0$ in $(0,1)$ if and only if $0 < \varepsilon < \varepsilon_N$. Moreover,*

1. *if $0 < \varepsilon < \varepsilon_N$, the escaping radially symmetric vortex sheet solution $\tilde{u}_\varepsilon$ is a global minimizer of E_ε in $\mathscr{A}$ and all global minimizers of E_ε in $\mathscr{A}$ are radially symmetric given by $R\tilde{u}_\varepsilon$ where $R \in O(M)$ is an orthogonal transformation of $\mathbb{R}^M$ satisfying $Rp = p$ for all $p \in \mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$. In this case, the non-escaping vortex sheet solution $(u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ in (4) is an unstable critical point of E_ε in $\mathscr{A}$.*

2. *if $\varepsilon \geq \varepsilon_N$, the non-escaping vortex sheet solution $(u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ in (4) is the unique global minimizer of E_ε in $\mathscr{A}$. Furthermore, there are no bounded critical points w_ε of E_ε in $\mathscr{A}$ that escape in some direction $e \in \mathbb{S}^{M-1}$ (i.e., $w_\varepsilon \cdot e > 0$ a.e. in Ω).*

The result above holds also if $n = 0$, i.e., $\Omega = B^N$ and the vortex sheets corresponding to the above solutions become vortex points (see Theorem 10). It generalizes [12, Theorem 1.1] that was proved in the case $N = 2$ and $M = 3$ (without identifying the meaning of the dichotomy parameter ε_N in (7)). The dichotomy in Theorem 4 happens in dimensions $2 \leq N \leq 6$ because of the phenomenology occurring for the limit problem $\varepsilon \to 0$. More precisely, if $M \geq N+1$, then minimizing $\mathbb{S}^{M-1}$-valued harmonic maps in $\mathscr{A}$ are smooth and escaping in a direction of $\mathbb{S}^{M-1}$ provided that $N \leq 6$; if $N \geq 7$, then there is a unique minimizing $\mathbb{S}^{M-1}$-valued harmonic maps in $\mathscr{A}$, non-escaping and singular, the singular set being given by a vortex sheet of dimension n in Ω (see Theorem 11 in Appendix below). This suggests why in dimension $N \geq 7$ and for any $\varepsilon > 0$, there is no escaping radially symmetric vortex sheet critical point $\tilde{u}_\varepsilon$ of E_ε in $\mathscr{A}$ while the non-escaping vortex sheet solution $(u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ is the unique global minimizer of E_ε in $\mathscr{A}$ (see Theorem 5 and Remark 8 below).

The paper is meant to be self-contained and it is organized as follows. In Section 2, we prove the minimality and the uniqueness results for the non-escaping radially symmetric solution in Theorems 1 and 3; this is done in a more general setting by considering the target dimension $M \geq N$ for the set of configurations $\mathscr{A}$ instead of $\mathscr{A}_N$. Section 3 is devoted to characterize escaping vortex sheet solutions. First, we prove the minimality of such bounded solutions stated in Theorem 7. Second, we prove existence, minimality and uniqueness results for the escaping radial profile in Theorem 9. Finally, we prove our main result on the dichotomy between escaping / non-escaping radially symmetric vortex sheet solutions in Theorem 4. In Appendix, we prove the corresponding dichotomy result for $\mathbb{S}^{M-1}$-valued harmonic maps in Theorem 11 which again is based on the minimality of escaping $\mathbb{S}^{M-1}$-valued harmonic maps in Theorem 12.

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2 The non-escaping vortex sheet solution. Proof of Theorems 1 and 3

Theorem 1 will be obtained as a consequence of a stronger result on the uniqueness of global minimizers of the $\mathbb{R}^M$-valued Ginzburg-Landau functional with $M \geq N \geq 7$. For that, we consider the energy functional E_ε in (1) over the set $\mathscr{A}$ defined in (8). The aim is to prove the minimality and uniqueness of the vortex sheet solution $(u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ where u_ε given in (4) with the obvious identification $u_\varepsilon \equiv (u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ if $M = N$, following the ideas of Ignat-Nguyen-Slastikov-Zarnescu [12, 11].

Theorem 5. *Assume that W satisfies* (2) *and $n \geq 1$. If $M \geq N \geq 7$, then for every $\varepsilon > 0$, $(u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ given in* (4) *is the unique global minimizer of E_ε in $\mathscr{A}$.*

Proof. To simplify notation, we identify

$$u_\varepsilon \equiv (u_\varepsilon, 0_{\mathbb{R}^{M-N}}) \quad \text{when} \quad M \geq N. \tag{13}$$

The proof will be done in several steps following the strategy in [12, Theorem 1.7], [11, Theorem 1]. First, for an arbitrary competitor $u_\varepsilon + v$, we consider the excess energy $E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon)$ for the critical point u_ε defined in (4) and show a lower estimate by a quadratic energy functional $F_\varepsilon(v)$ coming from the operator L_ε in (6). Second, we show that $F_\varepsilon(v) \geq 0$ using the properties of the radial profile f_ε in (5) and a Hardy decomposition method; this proves in particular that u_ε is a global minimizer of E_ε over $\mathscr{A}$. Finally, by analyzing the zero excess energy states, we conclude to the uniqueness of the global minimizer u_ε.

Step 1: Excess energy. For any $v \in H_0^1(B^N \times \mathbb{R}^n; \mathbb{R}^M)$, we have

$$\begin{aligned} E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) = &\int_\Omega \Big[\nabla u_\varepsilon \cdot \nabla v + \frac{1}{2}|\nabla v|^2\Big]\, dxdz \\ &+ \frac{1}{2\varepsilon^2}\int_\Omega \Big[W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2)\Big]\, dxdz. \end{aligned}$$

Note that for every $u \in \mathscr{A}$, $u_\varepsilon - u$ can be extended to $v \in H_0^1(B^N \times \mathbb{R}^n; \mathbb{R}^M)$. In particular, $v(\cdot, z) \in H_0^1(B^N, \mathbb{R}^M)$ for a.e. $z \in (0,1)^n$. The convexity of W yields

$$W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \geq -W'(1 - |u_\varepsilon|^2)(|u_\varepsilon + v|^2 - |u_\varepsilon|^2). \tag{14}$$

Combining the above relations, we obtain the following lower bound for the excess energy:

$$\begin{aligned} E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq &\int_\Omega \Big[\nabla u_\varepsilon \cdot \nabla v - \frac{1}{\varepsilon^2}W'(1 - f_\varepsilon^2)u_\varepsilon \cdot v\Big]\, dxdz \\ &+ \int_\Omega \Big[\frac{1}{2}|\nabla v|^2 - \frac{1}{2\varepsilon^2}W'(1 - f_\varepsilon^2)|v|^2\Big] dxdz \\ = &\int_\Omega \frac{1}{2}|\nabla_z v|^2\, dxdz + \int_{(0,1)^n} \frac{1}{2}F_\varepsilon(v(\cdot, z))\, dz, \end{aligned} \tag{15}$$

where we used the PDE (3) and introduced the quadratic functional

$$F_\varepsilon(\Psi) = \int_{B^N} \left[|\nabla_x \Psi|^2 - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2)|\Psi|^2 \right] dx,$$

for all $\Psi \in H_0^1(B^N;\mathbb{R}^M)$. Note that the L^2-gradient of F_ε represents a part of the linearization of the PDE (3) at u_ε and it is given by the operator L_ε in (6). The rest of the proof is devoted to show that for $N \geq 3$:

$$F_\varepsilon(\psi) \geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{\psi^2}{r^2}\, dx, \quad \forall \psi \in H_0^1(B^N)$$

yielding the conclusion for $N \geq 7$ and also the inequality for the first eigenvalue $\ell(\varepsilon)$ of the operator L_ε in (6) in B^N: [4]

$$\ell(\varepsilon) \geq \frac{(N-2)^2}{4} - (N-1) > 0, \quad \forall \varepsilon > 0 \quad \text{and} \quad N \geq 7.$$

To keep the paper self-contained, we explain in the following the simple idea used in [12, 11].

Step 2: A factorization argument. As $f_\varepsilon > 0$ is a smooth positive radial profile in $(0,1)$, we decompose every scalar test function $\psi \in C_c^\infty(B^N \setminus \{0\};\mathbb{R})$ as follows

$$\psi(x) = f_\varepsilon(r) w(x), \quad \forall x \in B^N \setminus \{0\},\ r = |x|,$$

where $w \in C_c^\infty(B^N \setminus \{0\};\mathbb{R})$. Integrating by parts (see e.g. [10, Lemma A.1]), we deduce:

$$\begin{aligned} F_\varepsilon(\psi) = \int_{B^N} L_\varepsilon \psi \cdot \psi\, dx &= \int_{B^N} w^2 (L_\varepsilon f_\varepsilon \cdot f_\varepsilon)\, dx + \int_{B^N} f_\varepsilon^2 |\nabla_x w|^2\, dx \\ &= \int_{B^N} f_\varepsilon^2 \left(|\nabla_x w|^2 - \frac{N-1}{r^2} w^2 \right) dx, \end{aligned}$$

because $L_\varepsilon f_\varepsilon \cdot f_\varepsilon = -\frac{N-1}{r^2} f_\varepsilon^2$ in B^N by (5). Furthermore, we decompose

$$w = \varphi g \quad \text{in} \quad B^N \setminus \{0\}$$

with $\varphi = |x|^{-\frac{N-2}{2}}$ satisfying

$$-\Delta_x \varphi = \frac{(N-2)^2}{4|x|^2} \varphi \quad \text{ in } \mathbb{R}^N \setminus \{0\}$$

and $g \in C_c^\infty(B^N \setminus \{0\};\mathbb{R})$. Then

$$|\nabla_x w|^2 = |\nabla_x g|^2 \varphi^2 + |\nabla_x \varphi|^2 g^2 + \frac{1}{2} \nabla_x(\varphi^2) \cdot \nabla_x(g^2).$$

[4] Observe the difference between dimension $N \geq 7$ and the case of dimension $2 \leq N \leq 6$ where we have $\ell(\varepsilon) < 0$ for $\varepsilon < \varepsilon_N$ in (7); moreover, if $N \leq 6$, then $\ell(\varepsilon)$ blows up as $-\frac{1}{\varepsilon^2}$ as $\varepsilon \to 0$ (see [8, Lemma 2.3]).

As $|\nabla_x \varphi|^2 = \frac{(N-2)^2}{4|x|^2}\varphi^2$ and φ^2 is harmonic in $B^N \setminus \{0\}$ (recall that $N \geq 7$), integration by parts yields

$$\begin{aligned} F_\varepsilon(\psi) &= \int_{B^N} f_\varepsilon^2 \left(|\nabla_x g|^2 \varphi^2 + \frac{(N-2)^2}{4r^2}\varphi^2 g^2 - \frac{N-1}{r^2}\varphi^2 g^2 \right) dx - \frac{1}{2}\int_{B^N} \nabla_x(\varphi^2)\cdot\nabla_x(f_\varepsilon^2) g^2\, dx \\ &\geq \int_{B^N} f_\varepsilon^2 |\nabla_x g|^2 \varphi^2\, dx + \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{f_\varepsilon^2}{r^2}\varphi^2 g^2\, dx \\ &\geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{\psi^2}{r^2}\, dx \geq 0, \end{aligned} \tag{16}$$

where we used $N \geq 7$ and $\frac{1}{2}\nabla_x(\varphi^2)\cdot\nabla_x(f_\varepsilon^2) = 2\varphi\varphi' f_\varepsilon f_\varepsilon' \leq 0$ in $B^N \setminus \{0\}$ because $\varphi, f_\varepsilon, f_\varepsilon' > 0$ and $\varphi' < 0$ in $(0,1)$ (see e.g. [7, 9, 8]).

Step 3: We prove that $F_\varepsilon(\Psi) \geq 0$ for every $\Psi \in H_0^1(B^N;\mathbb{R}^M)$; moreover, $F_\varepsilon(\Psi) = 0$ if and only if $\Psi = 0$. Let $\Psi \in H_0^1(B^N;\mathbb{R}^M)$. As a point in $\mathbb{R}^N$ has zero H^1 capacity, a standard density argument implies the existence of a sequence $\Psi_k \in C_c^\infty(B^N \setminus \{0\};\mathbb{R}^M)$ such that $\Psi_k \to \Psi$ in $H^1(B^N,\mathbb{R}^M)$ and a.e. in B^N. On the one hand, by definition of F_ε, since $W'(1-f_\varepsilon^2) \in L^\infty$, we deduce that $F_\varepsilon(\Psi_k) \to F_\varepsilon(\Psi)$ as $k \to \infty$. On the other hand, by (16) and Fatou's lemma, we deduce

$$\begin{aligned} \liminf_{k\to\infty} F_\varepsilon(\Psi_k) &\geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \liminf_{k\to\infty} \int_{B^N} \frac{|\Psi_k|^2}{r^2}\, dx \\ &\geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{|\Psi|^2}{r^2}\, dx. \end{aligned}$$

Therefore, we conclude that

$$F_\varepsilon(\Psi) \geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{|\Psi|^2}{r^2}\, dx \geq 0, \quad \forall \Psi \in H_0^1(B^N;\mathbb{R}^M).$$

Moreover, $F_\varepsilon(\Psi) = 0$ if and only if $\Psi = 0$.

Step 4: Conclusion. By (15) and Step 3, we deduce that u_ε is a global minimizer of E_ε over $\mathscr{A}$. For uniqueness, assume that $\hat{u}_\varepsilon$ is another global minimizer of E_ε over $\mathscr{A}$. If $v := \hat{u}_\varepsilon - u_\varepsilon$, then v can be extended in $H_0^1(B^N \times \mathbb{R}^n;\mathbb{R}^M)$ and by Steps 1 and 3, we have that

$$0 = E_\varepsilon(\hat{u}_\varepsilon) - E_\varepsilon(u_\varepsilon) \geq \int_\Omega \frac{1}{2}|\nabla_z v|^2\, dxdz + \int_{(0,1)^n} \frac{1}{2} F_\varepsilon(v(\cdot,z))\, dz \geq 0,$$

which yields $\nabla_z v = 0$ a.e. in Ω and $F_\varepsilon(v(\cdot,z)) = 0$ for a.e. $z \in (0,1)^n$. In other words, $v = v(x)$ and Step 3 implies that $v = 0$, i.e., $\hat{u}_\varepsilon = u_\varepsilon$ in Ω. □

REMARK 6. *Theorem 5 reveals the following fact: if for $n = 0$ (i.e., $\Omega = B^N$) and some $\varepsilon > 0$, a (radially symmetric) critical point $\hat{u}_\varepsilon : B^N \to \mathbb{R}^M$ of E_ε in $\mathscr{A}$ is proved to be a global minimizer (and additionally, if one proves that it is the unique global minimizer), then for any dimensions $n \geq 1$ (i.e., $\Omega = B^N \times (0,1)^n$), this z-invariant solution $\hat{u}_\varepsilon$ of* (3) *in $B^N \times (0,1)^n$ is also a global minimizer (and additionally, it is the unique minimizer) of E_ε in $\mathscr{A}$. This is because for every $u : B^N \times (0,1)^n \to \mathbb{R}^M$ with $u \in \mathscr{A}$, then $u(\cdot, z)$ satisfies the degree-one vortex boundary condition on ∂B^N for every $z \in (0,1)^n$ yielding*

$$\begin{aligned} E_\varepsilon(u) &= \int_\Omega \frac{1}{2}|\nabla_z u|^2 \, dxdz + \int_{(0,1)^n} E_\varepsilon(u(\cdot, z)) \, dz \\ &\geq \int_{(0,1)^n} E_\varepsilon(\hat{u}_\varepsilon) \, dz = E_\varepsilon(\hat{u}_\varepsilon); \end{aligned}$$

the equality occurs only when u is z-invariant. Thus, if the uniqueness of the global minimizer $\hat{u}_\varepsilon$ holds in B^N (i.e., $n = 0$), then this yields uniqueness of the global minimizer $\hat{u}_\varepsilon$ in $\Omega = B^N \times (0,1)^n$ (as a map independent of z-variable) for every $n \geq 1$.

Proof of Theorem 3. We prove the result in the more general setting of $\mathbb{R}^M$-valued maps u belonging to $\mathscr{A}$ for $M \geq N$ using the same identification (13). By Step 1 in the proof of Theorem 5 (see (15)), the excess energy is estimated for every $v \in H_0^1(B^N \times \mathbb{R}^n; \mathbb{R}^M)$:

$$E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_\Omega \frac{1}{2}|\nabla_z v|^2 \, dxdz + \frac{1}{2}\int_{(0,1)^n} < L_\varepsilon v(\cdot, z), v(\cdot, z) > \, dz,$$

where L_ε is the operator in (6) and $< \cdot, \cdot >$ denotes the duality pairing (H^{-1}, H_0^1) in B^N. If $\varepsilon \geq \varepsilon_N$, then $\ell(\varepsilon) \geq 0$ (by [8, Lemma 2.3]) and therefore, [5]

$$< L_\varepsilon v(\cdot, z), v(\cdot, z) > \geq \ell(\varepsilon)\|v(\cdot, z)\|^2_{L^2(B^N)} \geq 0 \quad \text{for a.e. } z \in (0,1)^n, \tag{17}$$

where we used that $v(\cdot, z) \in H_0^1(B^N; \mathbb{R}^M)$ for a.e. $z \in (0,1)^n$. Thus, u_ε is a minimizer of E_ε over $\mathscr{A}$. It remains to prove uniqueness of the global minimizer. For that, if $\hat{u}_\varepsilon$ is another global minimizer of E_ε over $\mathscr{A}$, setting $v := \hat{u}_\varepsilon - u_\varepsilon$, then v can be extended in $H_0^1(B^N \times \mathbb{R}^n; \mathbb{R}^M)$ and

$$0 = E_\varepsilon(\hat{u}_\varepsilon) - E_\varepsilon(u_\varepsilon) \geq \int_\Omega \frac{1}{2}|\nabla_z v|^2 \, dxdz + \frac{\ell(\varepsilon)}{2}\int_{(0,1)^n}\int_{B^N} |v(x,z)|^2 \, dxdz \geq 0 \tag{18}$$

because $\ell(\varepsilon) \geq 0$ for $\varepsilon \geq \varepsilon_N$. Thus, equality holds in the above inequalities.

Case 1: $\varepsilon > \varepsilon_N$. In this case, $\ell(\varepsilon) > 0$ and we conclude that $v = 0$ in Ω, i.e., $\hat{u}_\varepsilon = u_\varepsilon$ in Ω.

[5] Indeed, for a scalar function $v \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R})$, if $\psi = \psi(r) > 0$ is a radial first eigenfunction of L_ε in B^N with zero Dirichlet data, i.e., $L_\varepsilon \psi = \ell(\varepsilon)\psi$ in B^N, then the duality pairing (H^{-1}, H_0^1) term in B^N writes (see e.g. [10, Lemma A.1]):

$$< L_\varepsilon v, v > = \int_{B^N} \psi^2 |\nabla(\frac{v}{\psi})|^2 \, dx + \int_{B^N} (\frac{v}{\psi})^2 L_\varepsilon \psi \cdot \psi \, dx = \int_{B^N} \psi^2 |\nabla(\frac{v}{\psi})|^2 \, dx + \ell(\varepsilon)\|v\|^2_{L^2(B^N)}.$$

By a density argument, Fatou's lemma yields for every scalar function $v \in H_0^1(B^N, \mathbb{R})$,

$$< L_\varepsilon v, v > \geq \int_{B^N} \psi^2 |\nabla(\frac{v}{\psi})|^2 \, dx + \ell(\varepsilon)\|v\|^2_{L^2(B^N)}.$$

Case 2: $\varepsilon = \varepsilon_N$ and W is in addition strictly convex. In this case, $\ell(\varepsilon) = 0$ and by (18), v is invariant in z, i.e., $v = v(x)$ and equality holds in (17) and in (15), thus, equality holds in (14). Note that by footnote 5 the equality in (17) holds if and only if $v = \lambda\psi$ for some $\lambda \in \mathbb{R}^M$, where $\psi = \psi(r)$ is a radial first eigenfunction of L_ε in B^N with zero Dirichlet data, in particular $\psi > 0$ in $[0, 1)$ and $\psi(1) = 0$. Also, by the strict convexity of W, the equality (14) is achieved if and only if $|u_\varepsilon + v| = |u_\varepsilon|$ a.e. in Ω, that is, $|v|^2 + 2v \cdot u_\varepsilon = 0$ a.e. in B^N. It yields

$$|\lambda|^2\psi^2 + 2f_\varepsilon(|x|)(\frac{x}{|x|}, 0_{\mathbb{R}^{M-N}}) \cdot \lambda\psi = 0 \quad \text{for every } x \in B^N. \tag{19}$$

Dividing by ψ in B^N, the continuity up to the boundary ∂B^N leads to $2f_\varepsilon(|x|)(x, 0_{\mathbb{R}^{M-N}}) \cdot \lambda = 0$ for every $x \in \partial B^N$ since $\psi = 0$ on ∂B^N. As $f_\varepsilon(1) = 1$, it follows that the first N components of λ vanish. Coming back to (19), we conclude that $|\lambda|^2\psi^2 = 0$ in B^N, i.e., $\lambda = 0$ and so, $v = 0$ and $\hat{u}_\varepsilon = u_\varepsilon$ in Ω. □

3 Properties of escaping vortex sheet solutions when $M \geq N + 1$

3.1 Minimality of escaping vortex sheet solutions

In this section, we require the additional assumption of strict convexity of W in order to determine the set of global minimizers of E_ε over $\mathscr{A}$ in (8). However, W is assumed to be only C^1 not C^2. We prove that every bounded solution to (3) escaping in some direction is a global minimizer of E_ε over $\mathscr{A}$; moreover, such global minimizer is unique up to an orthogonal transformation of $\mathbb{R}^M$ keeping invariant the space $\mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$.

Theorem 7. *We consider the dimensions $n \geq 1$ and $M > N \geq 2$, the potential $W \in C^1((-\infty, 1], \mathbb{R})$ satisfying (2) and an escaping direction $e \in \mathbb{S}^{M-1}$. Fix any $\varepsilon > 0$ and let $w_\varepsilon \in H^1 \cap L^\infty(\Omega, \mathbb{R}^M)$ be a critical point of the energy E_ε in the set $\mathscr{A}$ which is positive in the direction e inside Ω:*

$$w_\varepsilon \cdot e > 0 \text{ a.e. in } \Omega. \tag{20}$$

Then w_ε is a global minimizer of E_ε in $\mathscr{A}$. If in addition W is strictly convex, then all minimizers of E_ε in $\mathscr{A}$ are given by Rw_ε where $R \in O(M)$ is an orthogonal transformation of $\mathbb{R}^M$ satisfying $Rp = p$ for all $p \in \mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$.

This result is reminiscent from [12, Theorem 1.3]. However, it doesn't apply directly as the domain Ω is not smooth here and the boundary condition is a mixed Dirichlet-Neumann condition (w.r.t. Dirichlet boundary condition in [12]).

Proof. In the following, we denote the variable $X = (x, z) \in \Omega = B^N \times (0, 1)^n$. As a critical point of E_ε in the set $\mathscr{A}$, $w_\varepsilon : \Omega \to \mathbb{R}^M$ satisfies

$$\begin{cases} -\Delta w_\varepsilon = \frac{1}{\varepsilon^2} w_\varepsilon \, W'(1 - |w_\varepsilon|^2) & \text{in } \Omega, \\ \frac{\partial w_\varepsilon}{\partial z} = 0 & \text{on } B^N \times \partial(0, 1)^n, \\ w_\varepsilon(x, z) = (x, 0_{\mathbb{R}^{M-N}}) & \text{on } \partial B^N \times (0, 1)^n. \end{cases} \tag{21}$$

In particular, $\Delta w_\varepsilon \in L^\infty(\Omega)$ (as W' is continuous and $w_\varepsilon \in L^\infty(\Omega)$); then standard elliptic regularity for the mixed boundary conditions in (21) yields $w_\varepsilon \in C^1(\bar\Omega, \mathbb{R}^M)$. Thus, (20) implies $w_\varepsilon \cdot e \geq 0$ in $\bar\Omega$ and the vortex boundary condition in $\mathscr{A}$ implies that e is orthogonal to $\mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$. By the invariance of the energy and the vortex boundary condition under the transformation $w_\varepsilon(X) \mapsto Rw_\varepsilon(X)$ for any $R \in O(M)$ satisfying $Rp = p$ for all $p \in \mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$, we know that Rw_ε is also a critical point of E_ε over $\mathscr{A}$; thus, we can assume that

$$e := e_M = (0, \ldots, 0, 1) \in \mathbb{R}^M. \tag{22}$$

We prove the result in several steps.

Step 1: Excess energy. By Step 1 in the proof of Theorem 5, we have for any $v \in H_0^1(B^N \times \mathbb{R}^n, \mathbb{R}^M)$:

$$E_\varepsilon(w_\varepsilon + v) - E_\varepsilon(w_\varepsilon) \geq \int_\Omega \Big[\frac{1}{2}|\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - |w_\varepsilon|^2)|v|^2\Big]\, dX =: \frac{1}{2} G_\varepsilon(v) \tag{23}$$

(note that $G_\varepsilon(v)$ is larger than the integration of $F_\varepsilon(v)$ in (15) over $(0,1)^n$ as it contains also the integration of $|\nabla_z v|^2$). If in addition W is strictly convex, then equality holds above if and only if $|w_\varepsilon(X) + v(X)| = |w_\varepsilon(X)|$ a.e. $X \in \Omega$ (by (14)).

Step 2: Global minimality of w_ε. It is enough to show that the quadratic energy $G_\varepsilon(v)$ defined in (23) is nonnegative for any $v \in H_0^1(B^N \times \mathbb{R}^n, \mathbb{R}^M)$. Denoting the M-component of w_ε by $\phi := w_\varepsilon \cdot e_M$, we know that $\phi \in C^1(\bar\Omega)$, $\phi \geq 0$ in Ω (by (20)) and satisfies the Euler-Lagrange equation in the sense of distributions:

$$\begin{cases} -\Delta\phi - \frac{1}{\varepsilon^2} W'(1 - |w_\varepsilon|^2)\phi = 0 \text{ in } \Omega, \\ \phi = 0 \quad \text{on } \partial B^N \times (0,1)^n, \\ \frac{\partial\phi}{\partial z} = 0 \quad \text{on } B^N \times \partial(0,1)^n. \end{cases} \tag{24}$$

Note that by strong maximum principle, $\phi > 0$ in Ω (as ϕ cannot be identically 0 in Ω by (20)). Moreover, Hopf's lemma yields $\phi > 0$ on $B^N \times \partial(0,1)^n$ as $\frac{\partial\phi}{\partial z}$ vanishes there. Now, for any smooth map $v \in C_c^\infty(B^N \times \mathbb{R}^n; \mathbb{R}^M)$, we can define $\Psi = \frac{v}{\phi} \in C^1(\bar\Omega; \mathbb{R}^M)$ with $\Psi = 0$ in a neighborhood of $\partial B^N \times (0,1)^n$ and integration by parts yields for every component $v_j = \phi\Psi_j$ with $1 \leq j \leq M$ (as in [10, Lemma A.1.]):

$$\begin{aligned} G_\varepsilon(v_j) &= \int_\Omega \Big[|\nabla v_j|^2 - \frac{1}{\varepsilon^2} W'(1 - |w_\varepsilon|^2)\phi \cdot \phi\Psi_j^2\Big]\, dX \\ &\overset{(24)}{=} \int_\Omega \Big[|\nabla(\phi\Psi_j)|^2 - \nabla\phi \cdot \nabla(\phi\,\Psi_j^2)\Big]\, dX = \int_\Omega \phi^2 |\nabla\Psi_j|^2\, dX. \end{aligned}$$

As G_ε is continuous in strong $H^1(\Omega)$ topology (since $W'(1 - |w_\varepsilon|^2) \in L^\infty(\Omega)$), by density of $C_c^\infty(B^N \times \mathbb{R}^n; \mathbb{R}^M)$ in $H_0^1(B^N \times \mathbb{R}^n; \mathbb{R}^M)$, Fatou's lemma yields

$$G_\varepsilon(v) \geq \int_\Omega \phi^2 |\nabla(\frac{v}{\phi})|^2\, dX \geq 0, \quad \forall v \in H_0^1(B^N \times \mathbb{R}^n; \mathbb{R}^M).$$

As a consequence of (23), we deduce that w_ε is a minimizer of E_ε over $\mathscr{A}$. Moreover, $G_\varepsilon(v) = 0$ if and only if there exists a (constant) vector $\lambda \in \mathbb{R}^M$ such that $v = \lambda\phi$ for a.e. $x \in \Omega$.

Step 3: Set of global minimizers. From now on, we assume that W is strictly convex and denote $w_\varepsilon = (w_{\varepsilon,1}, \dots, w_{\varepsilon,M})$. Note that the map

$$\tilde{w}_\varepsilon := (w_{\varepsilon,1}, \dots, w_{\varepsilon,N}, 0_{\mathbb{R}^{M-N-1}}, \sqrt{w_{\varepsilon,N+1}^2 + \cdots + w_{\varepsilon,M}^2}) \tag{25}$$

belongs to $\mathscr{A}$, $|\tilde{w}_\varepsilon| = |w_\varepsilon|$ and $|\nabla \tilde{w}_\varepsilon| \leq |\nabla w_\varepsilon|$ in Ω, so $E_\varepsilon(w_\varepsilon) \geq E_\varepsilon(\tilde{w}_\varepsilon)$ and

$$\sqrt{w_{\varepsilon,N+1}^2 + \cdots + w_{\varepsilon,M}^2} \geq w_{\varepsilon,M} = \phi > 0 \quad \text{in} \quad \Omega.$$

Hence, $\tilde{w}_\varepsilon$ is a minimizer of E_ε on $\mathscr{A}$ (as w_ε minimizes E_ε over $\mathscr{A}$ by Step 2). Therefore, up to interchanging w_ε and $\tilde{w}_\varepsilon$, we may assume

$$\begin{cases} w_{\varepsilon,N+1} = \cdots = w_{\varepsilon,M-1} \equiv 0 \text{ in } \Omega \\ w_{\varepsilon,M} = \phi \overset{(20)}{>} 0 \text{ in } \Omega. \end{cases}$$

We now consider another minimizer U_ε of E_ε over $\mathscr{A}$ and denote $v := U_\varepsilon - w_\varepsilon \in H_0^1(B^N \times \mathbb{R}^n; \mathbb{R}^M)$ after a suitable extension. From Steps 1 and 2 we know that $E_\varepsilon(U_\varepsilon) = E_\varepsilon(v + w_\varepsilon) = E_\varepsilon(w_\varepsilon)$, $G_\varepsilon(v) = 0$, $|v + w_\varepsilon| = |w_\varepsilon|$ a.e. in Ω and $v = \lambda\phi$ for some $\lambda = (\lambda_1, \dots, \lambda_M) \in \mathbb{R}^M$ where we recall that $\phi = w_\varepsilon \cdot e_M$. By continuity of w_ε and ϕ, the relation $|v + w_\varepsilon| = |w_\varepsilon|$ a.e. in Ω implies $2w_\varepsilon \cdot v + |v|^2 = 0$ everywhere in Ω. Since $v = \lambda\phi$, dividing by $\phi > 0$ in Ω, we obtain

$$2\lambda \cdot w_\varepsilon + \phi|\lambda|^2 = 0 \text{ in } \Omega \tag{26}$$

and by continuity, the equality holds also on $\partial\Omega$. As for every $(x, z) \in \partial B^N \times (0,1)^n$, $\phi(x, z) = 0$ and $w_\varepsilon(x, z) = (x, 0_{\mathbb{R}^{M-N}})$, we deduce that $\lambda \cdot (x, 0_{\mathbb{R}^{M-N}}) = 0$ for every $x \in \partial B^N$. It follows that $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 0$ and therefore, recalling that $w_{\varepsilon,N+1} = \cdots = w_{\varepsilon,M-1} = 0$ in Ω, we have by (26):

$$2\lambda_M \phi + (\lambda_{N+1}^2 + \cdots + \lambda_M^2)\phi = 0 \text{ in } \Omega.$$

As $\phi > 0$ in Ω, we obtain

$$\lambda_{N+1}^2 + \cdots + \lambda_{M-1}^2 + (\lambda_M + 1)^2 = 1;$$

hence we can find $R \in O(M)$ such that $Rp = p$ for all $p \in \mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$ and

$$Re_M = (0, \dots, 0, \lambda_{N+1}, \dots, \lambda_{M-1}, \lambda_M + 1).$$

This implies $U_\varepsilon = w_\varepsilon + v = w_\varepsilon + \lambda\phi = Rw_\varepsilon$ as required. The converse statement is obvious: if w_ε is a minimizer of E_ε over $\mathscr{A}$ and $R \in O(M)$ is a transformation fixing all points of $\mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$, then Rw_ε is also a minimizer of E_ε over $\mathscr{A}$ (because E_ε and the boundary condition in $\mathscr{A}$ are invariant under such orthogonal transformation R). $\square$

Remark 8. *Note that if $n \geq 1$, $M > N \geq 7$ and W satisfies* (2) *(not necessarily strictly convex), then there are no bounded critical points of the energy E_ε in the set $\mathscr{A}$ escaping in a direction $e \in \mathbb{S}^{M-1}$. Indeed, if such an escaping critical point of E_ε in $\mathscr{A}$ exists, then by Theorem 7, this solution would be a global minimizer of E_ε in $\mathscr{A}$ which is a contradiction with the uniqueness of the global minimizer $(u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ in* (4) *(that is non-escaping) proved in Theorem 5.*

3.2 Escaping radial profile

Let $M \geq N+1$. We give a necessary and sufficient condition for the existence of an escaping radial profile $(\tilde{f}_\varepsilon, g_\varepsilon > 0)$ in $(0,1)$ to the system (9)–(12); we also prove uniqueness, minimality and monotonicity of the escaping radial profile. For that, in the context of E_ε defined over $\mathscr{A}$, we introduce the functional

$$\begin{aligned} I_\varepsilon(f,g) &= \frac{1}{|\mathbb{S}^{N-1}|} E_\varepsilon\bigg((f(r)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N-1}}, g(r)) \bigg) \\ &= \frac{1}{2}\int_0^1 \Big[(f')^2 + (g')^2 + \frac{N-1}{r^2} f^2 + \frac{1}{\varepsilon^2} W(1-f^2-g^2) \Big] r^{N-1}\, dr \end{aligned}$$

where (f,g) belongs to

$$\mathscr{B} = \Big\{ (f,g) : r^{\frac{N-1}{2}} f', r^{\frac{N-3}{2}} f, r^{\frac{N-1}{2}} g', r^{\frac{N-1}{2}} g \in L^2(0,1), f(1)=1, g(1)=0 \Big\}. \tag{27}$$

The following result is reminiscent from Ignat-Nguyen [8, Theorem 2.4] (for $\tilde{W} \equiv 0$). The proof of [8, Theorem 2.4] is rather complicated (as it is proved for some general potentials $\tilde{W}$). We present here a simple proof that works in our context:

Theorem 9. *Let $2 \leq N \leq 6$, $M \geq N+1$, $W \in C^2((-\infty, 1])$ satisfy* (2) *and be strictly convex. Consider $\varepsilon_N \in (0,\infty)$ in* (7) *such that $\ell(\varepsilon_N) = 0$. Then the system* (9)–(12) *has an escaping radial profile $(\tilde{f}_\varepsilon, g_\varepsilon)$ with $g_\varepsilon > 0$ in $(0,1)$ if and only if $0 < \varepsilon < \varepsilon_N$. Moreover, in the case $0 < \varepsilon < \varepsilon_N$,*

1. *$(\tilde{f}_\varepsilon, g_\varepsilon > 0)$ is the unique escaping radial profile of* (9)–(12) *and $\frac{\tilde{f}_\varepsilon}{r}, g_\varepsilon \in C^2([0,1])$, $\tilde{f}_\varepsilon^2 + g_\varepsilon^2 < 1$, $\tilde{f}_\varepsilon > 0$, $\tilde{f}_\varepsilon' > 0$, $g_\varepsilon' < 0$ in $(0,1)$;*
2. *there are exactly two minimizers of I_ε in $\mathscr{B}$ given by $(\tilde{f}_\varepsilon, \pm g_\varepsilon)$;*
3. *the non-escaping radial profile $(f_\varepsilon, 0)$ is an unstable critical point of I_ε in $\mathscr{B}$ where f_ε is the unique radial profile in* (5).

Recall that for $\varepsilon \geq \varepsilon_N$, the non-escaping radial profile $(f_\varepsilon, 0)$ is the unique global minimizer of I_ε in $\mathscr{B}$ (by Theorem 3 whose proof yields the minimality of $(u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ of E_ε in $\mathscr{A}$).

Proof of Theorem 9. First, we focus on the existence of escaping radial profiles of (9)–(12). Note that the direct method in calculus of variations implies that I_ε admits a minimizer

$(\tilde{f}_\varepsilon, g_\varepsilon) \in \mathscr{B}$. Since $(\tilde{f}_\varepsilon, g_\varepsilon) \in \mathscr{B}$, $(\tilde{f}_\varepsilon, g_\varepsilon) \in C((0,1])$. It follows that $(\tilde{f}_\varepsilon, g_\varepsilon)$ satisfies (10)–(12) in the weak sense, and so $\tilde{f}_\varepsilon, g_\varepsilon \in C^2((0,1])$. Since $(|\tilde{f}_\varepsilon|, |g_\varepsilon|)$ is also a minimizer of I_ε in $\mathscr{B}$, the above argument also shows that $|\tilde{f}_\varepsilon|, |g_\varepsilon| \in C^2((0,1])$ satisfies (10)–(12). Since $|\tilde{f}_\varepsilon|, |g_\varepsilon| \geq 0$ and $\tilde{f}_\varepsilon(1) = 1$, the strong maximum principle yields $|\tilde{f}_\varepsilon| > 0$ in $(0,1)$, and either $|g_\varepsilon| > 0$ in $(0,1)$ or $g_\varepsilon \equiv 0$ in $(0,1)$. It follows that $\tilde{f}_\varepsilon > 0$ in $(0,1)$, and there are three alternatives: $g_\varepsilon > 0$ in $(0,1)$, $g_\varepsilon < 0$ in $(0,1)$ or $g_\varepsilon \equiv 0$ in $(0,1)$. Clearly, when $g_\varepsilon \equiv 0$, $\tilde{f}_\varepsilon$ is equal to the unique radial profile f_ε in (5). By considering $(\tilde{f}_\varepsilon, -g_\varepsilon)$ instead of $(\tilde{f}_\varepsilon, g_\varepsilon)$ if necessary, we assume in the sequel that $g_\varepsilon \geq 0$.

Claim: if $0 < \varepsilon < \varepsilon_N$, then $g_\varepsilon > 0$ in $(0,1)$ and $(f_\varepsilon, 0)$ is an unstable critical point of I_ε in $\mathscr{B}$.

Proof of Claim: We define the second variation of I_ε at $(f_\varepsilon, 0)$ as

$$
\begin{aligned}
Q_\varepsilon(\alpha, \beta) &= \frac{d^2}{dt^2}\Big|_{t=0} I_\varepsilon\Big((f_\varepsilon, 0) + t(\alpha, \beta)\Big) \\
&= \int_{B^N} \Big[L_\varepsilon \alpha \cdot \alpha + L_\varepsilon \beta \cdot \beta + \frac{N-1}{r^2}\alpha^2 + \frac{2}{\varepsilon^2} W''(1 - f_\varepsilon^2) f_\varepsilon^2 \alpha^2\Big]\, dx,
\end{aligned}
$$

for $\alpha, \beta \in C_c^\infty((0,1))$ which extends by density to the Hilbert space

$$
\mathscr{H} = \{(\alpha, \beta) : (f_\varepsilon + \alpha, \beta) \in \mathscr{B}\} \text{ with the norm } \quad \|(\alpha, \beta)\|_{\mathscr{H}} := \|(\alpha \frac{x}{|x|}, \beta)\|_{H^1(B^N, \mathbb{R}^{N+1})}.
$$

As $\varepsilon \in (0, \varepsilon_N)$, we have $\ell(\varepsilon) < 0$ by (7). Taking $\beta \in H_0^1(B^N)$ to be any first eigenfunction of L_ε in B^N, which is radially symmetric, we have $r^{\frac{N-1}{2}}\beta', r^{\frac{N-1}{2}}\beta \in L^2(0,1)$, $\beta(1) = 0$ and

$$
Q_\varepsilon(0, \beta) = \int_{B^N} L_\varepsilon \beta \cdot \beta \, dx = \ell(\varepsilon) \int_{B^N} \beta^2 \, dx < 0.
$$

So, $(f_\varepsilon, 0)$ is an unstable critical point of I_ε in $\mathscr{B}$ if $\varepsilon < \varepsilon_N$. In particular, $(f_\varepsilon, 0)$ is not minimizing I_ε in $\mathscr{B}$ and therefore, by the above construction of the minimizer $(\tilde{f}_\varepsilon, g_\varepsilon)$ of I_ε in $\mathscr{B}$, we deduce that $g_\varepsilon > 0$. This proves the above Claim.

Moreover, by [8, Lemmas 2.7 and A.5, Proposition 2.9] (for $\tilde{W} \equiv 0$), we deduce that $\frac{\tilde{f}_\varepsilon}{r}, g_\varepsilon \in C^2([0,1])$, $\tilde{f}_\varepsilon^2 + g_\varepsilon^2 < 1$, $\tilde{f}_\varepsilon' > 0$ and $g_\varepsilon' < 0$ in $(0,1)$.

To conclude, we distinguish two cases:

Case 1: if $\varepsilon \in (0, \varepsilon_N)$, Claim yields the existence of an escaping radial profile $(\tilde{f}_\varepsilon, g_\varepsilon > 0)$. By [8, Lemmas 2.7], every escaping radial profile $(\tilde{f}_\varepsilon, g_\varepsilon > 0)$ is bounded (i.e., $\tilde{f}_\varepsilon^2 + g_\varepsilon^2 < 1$ in $(0,1)$) and therefore, by Theorem 7, the corresponding (bounded) escaping critical point $\tilde{u}_\varepsilon$ in (9) is a global minimizer of E_ε over $\mathscr{A}$ and the set of minimizers of E_ε over $\mathscr{A}$ is then given by $\{R\tilde{u}_\varepsilon : R \in O(M), Rp = p, \forall p \in \mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}\}$. Therefore, $(\tilde{f}_\varepsilon, \pm g_\varepsilon)$ are the only two minimizers of I_ε in $\mathscr{B}$. In particular, this proves the uniqueness of the escaping radial profile $(\tilde{f}_\varepsilon, g_\varepsilon > 0)$.

Case 2: if $\varepsilon \geq \varepsilon_N$, by the proof of Theorem 3, the non-escaping vortex sheet solution $u_\varepsilon(x) \equiv (f_\varepsilon(|x|)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N}})$ (by (13)) is the unique minimizer of E_ε over $\mathscr{A}$. In particular, $(f_\varepsilon, 0)$ is the unique minimizer of I_ε in $\mathscr{B}$, i.e., in the above construction of the minimizer

$(\tilde{f}_\varepsilon, g_\varepsilon)$ of I_ε in $\mathscr{B}$, we have $\tilde{f}_\varepsilon = f_\varepsilon$ and $g_\varepsilon = 0$ in $(0,1)$. We claim that no escaping radial profile $(\hat{f}_\varepsilon, \hat{g}_\varepsilon > 0)$ exists if $\varepsilon \geq \varepsilon_N$. Assume by contradiction that such an escaping radial profile $(\hat{f}_\varepsilon, \hat{g}_\varepsilon > 0)$ exists. The same argument presented in Case 1 would imply that $(\hat{f}_\varepsilon, \hat{g}_\varepsilon > 0)$ is a minimizer of I_ε in $\mathscr{B}$ which contradicts the uniqueness of the global minimizer $(f_\varepsilon, 0)$. □

3.3 Proof of Theorem 4

We now prove the main result:

Proof of Theorem 4. By Theorem 9, the existence of an escaping radially symmetric solution $\tilde{u}_\varepsilon$ in (9) is equivalent to $\varepsilon \in (0, \varepsilon_N)$. Moreover, in that case, the escaping radial profile $(\tilde{f}_\varepsilon, g_\varepsilon > 0)$ is unique and bounded, i.e., $\tilde{f}_\varepsilon^2 + g_\varepsilon^2 < 1$ in $(0,1)$.
Case 1: if $\varepsilon \in (0, \varepsilon_N)$, Theorem 7 implies that the (bounded) escaping radially symmetric critical point $\tilde{u}_\varepsilon$ in (9) is a global minimizer of E_ε over $\mathscr{A}$ and every minimizer of E_ε over $\mathscr{A}$ has the form $R\tilde{u}_\varepsilon$ for some orthogonal transformation $R \in O(M)$ keeping invariant the space $\mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$. Moreover, by Theorem 9, the non-escaping radial profile $(f_\varepsilon, 0)$ is proved to be an unstable critical point of I_ε in $\mathscr{B}$, so the non-escaping vortex sheet solution $(u_\varepsilon, 0_{\mathbb{R}^{M-N}})$ is an unstable critical point of E_ε in $\mathscr{A}$.
Case 2: if $\varepsilon \geq \varepsilon_N$, the proof of Theorem 3 implies that the non-escaping radially symmetric vortex sheet solution $u_\varepsilon(x) \equiv (f_\varepsilon(|x|)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N}})$ (by (13)) is the unique minimizer of E_ε over $\mathscr{A}$. In this case, there is no bounded critical point w_ε of E_ε over $\mathscr{A}$ that escapes in some direction $e \in \mathbb{S}^{M-1}$; indeed, if such (bounded) escaping solution w_ε satisfying (20) exists, then Theorem 7 would imply that w_ε is a global minimizer of E_ε over $\mathscr{A}$ which contradicts that the non-escaping vortex sheet solution u_ε is the unique global minimizer of E_ε over $\mathscr{A}$. □

Theorem 4 holds also for the "degenerate" dimension $n = 0$. In this case, $\Omega = B^N$ and vortex sheets are vortex points,

$$E_\varepsilon(u) = \int_{B^N} \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2)\right] dx,$$

$$\mathscr{A} := \{u \in H^1(B^N; \mathbb{R}^M) : \, u(x) = (x, 0_{\mathbb{R}^{M-N}}) \text{ on } \partial B^N = \mathbb{S}^{N-1}\}$$

and radially symmetric vortex critical points of E_ε in $\mathscr{A}$ have the corresponding form in (9):

$$\tilde{u}_\varepsilon(x) = (\tilde{f}_\varepsilon(r)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N-1}}, g_\varepsilon(r)) \in \mathscr{A}, \quad x \in B^N, r = |x|, \tag{28}$$

where the radial profiles $(\tilde{f}_\varepsilon, g_\varepsilon)$ satisfy the system (10)-(12) and are described in Theorem 9; the non-escaping radially symmetric vortex solution is given here by

$$u_\varepsilon(x) = (f_\varepsilon(|x|)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N}}) \quad \text{ for all } x \in B^N, \tag{29}$$

where the radial profile f_ε is the unique solution to (5). We obtain the following result which generalizes [12, Theorem 1.1] that was proved in the case $N = 2$ and $M = 3$ (without identifying the meaning of the dichotomy parameter ε_N in (7)).

Theorem 10. *Let $2 \leq N \leq 6$, $M \geq N+1$, $\Omega = B^N$, $W \in C^2((-\infty, 1])$ satisfy* (2) *and be strictly convex. Consider $\varepsilon_N \in (0, \infty)$ such that $\ell(\varepsilon_N) = 0$ in* (7)*. Then there exists an escaping radially symmetric vortex solution $\tilde{u}_\varepsilon$ in* (28) *with the radial profile $(\tilde{f}_\varepsilon, g_\varepsilon > 0)$ given in Theorem 9 if and only if $0 < \varepsilon < \varepsilon_N$. Moreover,*

1. *if $0 < \varepsilon < \varepsilon_N$, $\tilde{u}_\varepsilon$ is a global minimizer of E_ε in $\mathscr{A}$ and all global minimizers of E_ε in $\mathscr{A}$ are radially symmetric given by $R\tilde{u}_\varepsilon$ where $R \in O(M)$ is an orthogonal transformation of $\mathbb{R}^M$ satisfying $Rp = p$ for all $p \in \mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$. In this case, the non-escaping vortex solution u_ε in* (29) *is an unstable critical point of E_ε in $\mathscr{A}$.*

2. *if $\varepsilon \geq \varepsilon_N$, the non-escaping vortex solution u_ε in* (29) *is the unique global minimizer of E_ε in $\mathscr{A}$. Furthermore, there are no bounded critical points w_ε of E_ε in $\mathscr{A}$ that escape in a direction $e \in \mathbb{S}^{M-1}$, i.e., $w_\varepsilon \cdot e > 0$ a.e. in Ω.*

The proof follows by the same argument used for Theorem 4, the main difference is that in the ball $\Omega = B^N$, a critical point w_ε of E_ε in $\mathscr{A}$ satisfies the PDE system with Dirichlet boundary condition (instead of the mixed Dirichlet-Neumann condition in (21)):

$$\begin{aligned} -\Delta w_\varepsilon &= \frac{1}{\varepsilon^2} w_\varepsilon \, W'(1 - |w_\varepsilon|^2) \quad \text{in } B^N, \\ w_\varepsilon(x) &= (x, 0_{\mathbb{R}^{M-N}}) \quad \text{on } \partial B^N. \end{aligned}$$

A Appendix. Vortex sheet $\mathbb{S}^{M-1}$-valued harmonic maps in cylinders

In dimensions $M > N \geq 2$ and $n \geq 1$, for the cylinder shape domain $\Omega = B^N \times (0,1)^n$, we consider the harmonic map problem for $\mathbb{S}^{M-1}$-valued maps $u \in H^1(\Omega; \mathbb{S}^{M-1}) \cap \mathscr{A}$ associated to the Dirichlet energy

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx dz.$$

Any critical point $u : \Omega \to \mathbb{S}^{M-1}$ of this problem satisfies

$$\begin{cases} -\Delta u = u \, |\nabla u|^2 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial z} = 0 \quad \text{on } B^N \times \partial(0,1)^n, \\ u(x,z) = (x, 0_{\mathbb{R}^{M-N}}) \quad \text{on } \partial B^N \times (0,1)^n. \end{cases} \tag{30}$$

We will focus on radially symmetric vortex sheet $\mathbb{S}^{M-1}$-valued harmonic maps having the following form (invariant in z-direction):

$$u(x,z) = (f(r)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N-1}}, g(r)) \in \mathscr{A}, \quad x \in B^N, z \in (0,1)^n, r = |x|, \tag{31}$$

where the radial profile (f, g) satisfies

$$f^2 + g^2 = 1 \quad \text{in} \quad (0,1), \tag{32}$$

and the system of ODEs:

$$-f'' - \frac{N-1}{r}f' + \frac{N-1}{r^2}f = \Gamma(r)f \quad \text{in} \quad (0,1), \tag{33}$$

$$-g'' - \frac{N-1}{r}g' = \Gamma(r)g \quad \text{in} \quad (0,1), \tag{34}$$

$$f(1) = 1 \text{ and } g(1) = 0, \tag{35}$$

where

$$\Gamma(r) = (f')^2 + \frac{N-1}{r^2}f^2 + (g')^2$$

is the Lagrange multiplier due to the unit length constraint in (32). As for the Ginzburg-Landau system, we distinguish two type of radial profiles:

- the *non-escaping* radial profile ($\bar{f} \equiv 1, \bar{g} \equiv 0$) yielding the *non-escaping* (radially symmetric) vortex sheet $\mathbb{S}^{M-1}$-valued harmonic map (also called "equator" map):

$$\bar{u}(x,z) = (\frac{x}{|x|}, 0_{\mathbb{R}^{M-N}}) \quad x \in B^N, z \in (0,1)^n. \tag{36}$$

Note that $\bar{u}$ is singular and the singular set of this map is the vortex sheet $\{0_{\mathbb{R}^{M-N}}\} \times (0,1)^n$ of dimension n in Ω. Also, observe that $\bar{u} \in H^1(\Omega, \mathbb{S}^{M-1})$ if and only if $N \geq 3$.

- the *escaping* radial profile (f,g) with $g > 0$ in $(0,1)$; in this case, it holds $f(0) = 0$, $g(0) = 1$ and we say that u in (31) is an *escaping* (radially symmetric) vortex sheet $\mathbb{S}^{M-1}$-valued harmonic map. Note that u is smooth for every dimension $M > N \geq 2$ and $n \geq 1$ and the zero set of $(u_1, \dots, u_N)$ is the vortex sheet $\{0_{\mathbb{R}^{M-N}}\} \times (0,1)^n$ of dimension n in Ω. Obviously, $(f, -g < 0)$ is another radial profile satisfying (32)-(35).

The properties of such radial profiles are proved in [14] (see also [8, Theorem 2.6] for $\tilde{W} \equiv 0$ in those notations). More precisely,

(a) If $N \geq 7$, the non-escaping radial profile ($\bar{f} \equiv 1, \bar{g} \equiv 0$) is the unique minimizer of

$$I(f,g) = \frac{1}{|\mathbb{S}^{N-1}|} E\bigg((f(r)\frac{x}{|x|}, 0_{\mathbb{R}^{M-N-1}}, g(r)) \bigg) = \frac{1}{2}\int_0^1 \Big[(f')^2 + (g')^2 + \frac{N-1}{r^2}f^2 \Big] r^{N-1}\, dr,$$

where (f,g) belongs to $\mathscr{B} \cap \big\{(f,g) \,:\, f^2 + g^2 = 1\big\}$ with $\mathscr{B}$ defined in (27). Moreover, the system (32)–(35) has no escaping radial profile (f,g) with $g > 0$ in $(0,1)$.

(b) If $2 \leq N \leq 6$, then there exists a unique escaping radial profile (f,g) with $g > 0$ satisfying (32)–(35). Moreover, $(f, \pm g)$ are the only two global minimizers of I in $\mathscr{B} \cap \big\{(f,g) \,:\, f^2 + g^2 = 1\big\}$, $\frac{f}{r}, g \in C^\infty([0,1])$, $f(0) = 0$, $g(0) = 1$, $f > 0$, $f' > 0$ and $g' < 0$ in $(0,1)$. In addition, for $3 \leq N \leq 6$, the non-escaping solution ($\bar{f} \equiv 1, \bar{g} \equiv 0$) is an unstable critical point of I in $\mathscr{B} \cap \big\{(f,g) \,:\, f^2 + g^2 = 1\big\}$.[6]

[6]For $N = 2$, $(1,0) \notin \mathscr{B}$; however, we can define the second variation of I at $(1,0)$ along directions $(0,q)$ compactly supported in $(0,1)$:

$$Q(0,q) = \int_0^1 \Big[(q')^2 - \frac{N-1}{r^2}q^2 \Big] r^{N-1}\, dr,$$

and one can prove the existence of $q \in Lip_c(0,1)$ such that $Q(0,q) < 0$ (see e.g. [8, Remark 2.16]).

There is a large number of articles studying existence, uniqueness, regularity and stability of radially symmetric $\mathbb{S}^{M-1}$-valued harmonic maps (e.g., [13, 14, 25, 26, 23, 16, 12]). We summarize here the main result for our problem in the cylinder shape domain $\Omega = B^N \times (0,1)^n$: if $N \le 6$, then minimizing $\mathbb{S}^{M-1}$-valued harmonic maps in $\mathscr{A}$ are smooth, radially symmetric and escaping in one-direction; if $N \ge 7$, then there is a unique minimizing $\mathbb{S}^{M-1}$-valued harmonic map in $\mathscr{A}$ which is singular and given by the equator map $\bar{u}$ in (36). [7]

THEOREM 11. *Let $n \ge 1$, $N \ge 2$, $M \ge N+1$ and $\Omega = B^N \times (0,1)^n$. Then*

1. *if $2 \le N \le 6$, then the escaping radially symmetric vortex sheet solution u in* (31) *with $g > 0$ is a minimizing $\mathbb{S}^{M-1}$-valued harmonic map in $\mathscr{A}$ and all minimizing $\mathbb{S}^{M-1}$-valued harmonic maps in $\mathscr{A}$ are smooth radially symmetric given by Ru where $R \in O(M)$ satisfies $Rp = p$ for all $p \in \mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$. In this case, the equator map $\bar{u}$ in* (36) *is an unstable $\mathbb{S}^{M-1}$-valued harmonic map in $\mathscr{A}$.*

2. *if $N \ge 7$, the non-escaping vortex sheet solution $\bar{u}$ in* (36) *is the unique minimizing $\mathbb{S}^{M-1}$-valued harmonic map in $\mathscr{A}$. Moreover, there is no $\mathbb{S}^{M-1}$-valued harmonic map w in $\mathscr{A}$ escaping in a direction $e \in \mathbb{S}^{M-1}$, i.e., $w \cdot e > 0$ a.e. in Ω.*

The main ingredient is the following result yielding minimality of escaping $\mathbb{S}^{M-1}$-valued harmonic maps. This is reminiscent from Sandier-Shafrir [23] (see also [12, Theorem 1.5]).

THEOREM 12. *Let $n \ge 1$, $M > N \ge 2$ and $\Omega = B^N \times (0,1)^n$. Assume that $w \in \mathscr{A} \cap H^1(\Omega, \mathbb{S}^{M-1})$ is a $\mathbb{S}^{M-1}$-valued harmonic map satisfying* (30) *and*

$$w \cdot e > 0 \text{ a.e. in } \Omega \tag{37}$$

in an escaping direction $e \in \mathbb{S}^{M-1}$. Then w is a minimizing $\mathbb{S}^{M-1}$-valued harmonic map in $\mathscr{A}$ and all minimizing $\mathbb{S}^{M-1}$-valued harmonic maps in $\mathscr{A}$ are of the form Rw where $R \in O(M)$ is an orthogonal transformation of $\mathbb{R}^M$ satisfying $Rp = p$ for all $p \in \mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$.

Proof of Theorem 12. We give here a simple proof based on the argument in [12] that avoids the regularity results used in [23]. By the $H^{1/2}$-trace theorem applied for $w \in H^1(\Omega, \mathbb{S}^{M-1})$, (37) implies that $w \cdot e \ge 0$ on $\partial B^N \times (0,1)^n$. Combined with the vortex boundary condition in (30), we deduce that the escaping direction e has to be orthogonal to $\mathbb{R}^N \times \{0_{\mathbb{R}^{M-N}}\}$ and up to a rotation, we can assume that $e = e_M$ (as in (22)). Then $\phi = w \cdot e_M > 0$ a.e. in Ω satisfies

$$-\Delta\phi = |\nabla w|^2 \phi \text{ in } \Omega, \ \frac{\partial \phi}{\partial z} = 0 \text{ on } B^N \times \partial(0,1)^n, \ \phi = 0 \text{ on } \partial B^N \times (0,1)^n. \tag{38}$$

We consider configurations[8] $\tilde{w} = w + v : \Omega \to \mathbb{S}^{M-1}$ with $v \in H^1_0(B^N \times \mathbb{R}^n, \mathbb{R}^M)$ (in particular, $|v| \le 2$ in Ω). Then

$$2w \cdot v + |v|^2 = 0 \quad \text{a.e. in } \Omega. \tag{39}$$

[7] We mention the paper of Bethuel-Brezis-Coleman-Hélein [2] about a similar phenomenology in a domain $\Omega = (B^2 \setminus B_\rho) \times (0,1) \subset \mathbb{R}^3$ where $B_\rho \subset \mathbb{R}^2$ is the disk centered at 0 of radius ρ.

[8] Note that for any $\tilde{w} \in \mathscr{A} \cap H^1(\Omega, \mathbb{S}^{M-1})$, the map $\tilde{w} - w$ has an extension in $H^1_0(B^N \times \mathbb{R}^n, \mathbb{R}^M)$.

Using (30) and (39), we obtain

$$2\int_{\Omega}\nabla w\cdot\nabla v=2\int_{\Omega}|\nabla w|^2 w\cdot v\,dx=-\int_{\Omega}|\nabla w|^2|v|^2\,dx,$$

yielding[9]

$$\int_{\Omega}|\nabla(w+v)|^2\,dx-\int_{\Omega}|\nabla w|^2\,dx=\int_{\Omega}|\nabla v|^2-|\nabla w|^2|v|^2\,dx=:Q(v). \tag{40}$$

To show that w is minimizing, we prove that $Q(v)\geq 0$ for all $v\in H_0^1(B^N\times\mathbb{R}^n,\mathbb{R}^M)\cap L^\infty(\Omega;\mathbb{R}^M)$ (note that this is a class larger than what we need, as we do not require that v satisfy the pointwise constraint (39)). For that, we take an arbitrary map $\tilde{v}\in C_c^\infty(B^N\times\mathbb{R}^n,\mathbb{R}^M)$ of support ω and decompose it as $\tilde{v}=\phi\Psi$ in Ω. This decomposition makes sense as $\phi\geq\delta>0$ in $\omega\cap\Omega$ for some $\delta>0$ (which may depend on ω). Indeed, by (37) and (38), ϕ is a superharmonic function (i.e., $-\Delta\phi\geq 0$ in Ω) that belongs to $H^1(\Omega)$. As $\frac{\partial\phi}{\partial z}=0$ on $B^N\times\partial(0,1)^n$, ϕ can be extended by even mirror symmetry to the domain $\tilde{\Omega}=B^N\times(-1,2)^n$ so that ϕ is superharmonic in $\tilde{\Omega}$. Thus, the weak Harnack inequality (see e.g. [6, Theorem 8.18]) implies that on the compact set $\omega\cap\Omega$ in $\tilde{\Omega}$, we have $\phi\geq\delta>0$ for some δ. So, $\tilde{v}=\phi\Psi$ in Ω with $\Psi=(\Psi_1,\dots,\Psi_M)\in H^1\cap L^\infty(\Omega;\mathbb{R}^M)$ vanishing in a neighborhood of $\partial B^N\times(0,1)^n$. Then integration by parts yields for $1\leq j\leq M$:

$$\begin{aligned}Q(\tilde{v}_j)&=\int_{\Omega}|\nabla\tilde{v}_j|^2-|\nabla w|^2\phi\cdot\phi\Psi_j^2\,dx\\&\overset{(38)}{=}\int_{\Omega}|\nabla(\phi\Psi_j)|^2-\nabla\phi\cdot\nabla(\phi\,\Psi_j^2)\,dx=\int_{\Omega}\phi^2|\nabla\Psi_j|^2\,dx\geq 0\end{aligned}$$

for all $\tilde{v}\in C_c^\infty(B^N\times\mathbb{R}^n,\mathbb{R}^M)$. Then for every $v\in H_0^1(B^N\times\mathbb{R}^n,\mathbb{R}^M)\cap L^\infty(\Omega;\mathbb{R}^M)$, there exists a sequence $\tilde{v}^k\in C_c^\infty(B^N\times\mathbb{R}^n,\mathbb{R}^M)$ such that $\tilde{v}^k\to v$ and $\nabla\tilde{v}^k\to\nabla v$ in L^2 and a.e. in $B^N\times\mathbb{R}^n$ and $|\tilde{v}^k|\leq\|v\|_{L^\infty(\Omega)}+1$ in Ω for every k. In particular, by dominated convergence theorem, we have $Q(\tilde{v}^k)\to Q(v)$ thanks to (40). Thus, we deduce that for every compact $\omega\subset\tilde{\Omega}=B^N\times(-1,2)^n$,

$$Q(v)=\lim_{k\to\infty}Q(\tilde{v}^k)\geq\liminf_{k\to\infty}\int_{\omega\cap\Omega}\phi^2|\nabla\big(\frac{\tilde{v}^k}{\phi}\big)|^2\,dx\geq\int_{\omega\cap\Omega}\phi^2|\nabla\big(\frac{v}{\phi}\big)|^2\,dx\geq 0,$$

where we used Fatou's lemma. In particular, w is a minimizing $\mathbb{S}^{M-1}$-valued harmonic map by (40) and $Q(v)=0$ yields the existence of a vector $\lambda\in\mathbb{R}^M$ such that $v=\lambda\phi$ a.e. in Ω. Then the classification of the minimizing $\mathbb{S}^{M-1}$-valued harmonic maps follows by (39) as in the Step 3 of the proof of Theorem 7. $\square$

Proof of Theorem 11. 1. This part concerning the dimension $2\leq N\leq 6$ follows from Theorem 12 and the instability of the radial profile $(1,0)$ for I in $\mathscr{B}\cap\big\{(f,g)\,:\,f^2+g^2=1\big\}$ as explained above.

[9] Note that the functional Q represents the second variation of E at w, but here the map v is not necessarily orthogonal to w.

2. This part for dimension $N \geq 7$ follows the ideas in [14]. More precisely, calling $X = (x, z)$ the variable in Ω, we have as in the proof of Theorem 12 for every $v \in H_0^1(B^N \times \mathbb{R}^n, \mathbb{R}^M)$ with $|v + \bar{u}| = 1$ in Ω:

$$\begin{aligned}\int_\Omega |\nabla(\bar{u} + v)|^2 \, dX - \int_\Omega |\nabla \bar{u}|^2 \, dX &= \int_\Omega \left(|\nabla v|^2 - |\nabla \bar{u}|^2 |v|^2 \right) dX \\ &= \int_\Omega |\nabla_z v|^2 \, dX + \int_{(0,1)^n} dz \int_{B^N} \left(|\nabla_x v|^2 - \frac{N-1}{|x|^2} |v|^2 \right) dx \\ &\geq \int_\Omega |\nabla_z v|^2 \, dX + \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_\Omega \frac{|v|^2}{|x|^2} \, dX \geq 0\end{aligned}$$

where we used the Hardy inequality for $v(\cdot, z) \in H_0^1(B^N, \mathbb{R}^M)$ for a.e. $z \in (0,1)^n$. This proves that $\bar{u}$ is the unique minimizing $\mathbb{S}^{M-1}$-valued harmonic map in $\mathscr{A}$. Combined with Theorem 12, we conclude that there is no escaping $\mathbb{S}^{M-1}$-valued harmonic map w in $\mathscr{A}$. $\square$

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