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Abstract

We study the question of uniqueness of minimisers of the standard Ginzburg-Landau functional for $\mathbb{R}^n$-valued maps with a $H^{1/2} \cap L^\infty$ boundary data that is non-negative in a fixed direction $e \in \mathbb{S}^{n-1}$. We link the question of uniqueness on the one hand with the "escaping" phenomenon of minimizers, and on the other hand with a stability condition for critical points of the Ginzburg-Landau functional. In particular, we show that, when minimisers are not unique, they "escape" out of the range of the boundary condition and the set of minimisers is generated from any of its elements using appropriate orthogonal transformations of $\mathbb{R}^n$.

Keywords: uniqueness, minimisers, Ginzburg-Landau.
MSC: 35A02, 35B06, 35J50.

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Model. We consider the following Ginzburg-Landau type energy functional

$$E_\varepsilon(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2)\right] dx,$$

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with $\varepsilon > 0$ being a fixed parameter, $\Omega \subset \mathbb{R}^m$ ($m \geq 1$) is a bounded domain (i.e., open connected set) with smooth boundary $\partial\Omega$ and the potential $W \in C^1((-\infty, 1]; \mathbb{R}_+)$ satisfies

$$W(0) = 0,\ W(t) > 0 \text{ for all } t \in (-\infty, 1] \setminus \{0\},\ W \text{ is strictly convex.}$$

(The prototype of the nonlinear potential is $W(t) = t^2/2$.) We focus on minimisers of the energy E_ε over the following set

$$\mathscr{A} := \{u \in H^1(\Omega; \mathbb{R}^n) : u = u_{bd} \text{ on } \partial\Omega\}, \quad n \geq 1,$$

consisting of H^1 maps with a given boundary data (in the sense of $H^{1/2}$-trace on $\partial\Omega$):

$$u_{bd} \in H^{1/2} \cap L^\infty(\partial\Omega; \mathbb{R}^n).$$

The direct method in the calculus of variations yields existence of minimizers u_ε of E_ε over $\mathscr{A}$ for all range of $\varepsilon > 0$; moreover, any minimizer u_ε belongs to $C^1 \cap L^\infty(\Omega; \mathbb{R}^n)$ and satisfies the system of PDEs

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon W'(1 - |u_\varepsilon|^2) \quad \text{distributionally in } \Omega. \tag{0.1}$$

Aim. We are interested in the question of uniqueness (or its failure) for the minimisers of E_ε in $\mathscr{A}$ for all range of $\varepsilon > 0$. If ε is large (i.e., $\varepsilon \geq \varepsilon_0 := (|W'(1)|/\lambda_1(\Omega))^{1/2}$ where $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta)$ on Ω with zero Dirichlet data), then E_ε is strictly convex and thus, there exists a unique solution $u_\varepsilon \in \mathscr{A}$ of (0.1) which is the minimizer of E_ε over $\mathscr{A}$. If $\varepsilon < \varepsilon_0$, the problem is more delicate and it was intensively studied in the last thirty years (for details, see the references in [3]). We provide results for this problem in the special case where the boundary data is non-negative in a (fixed) direction $e \in \mathbb{S}^{n-1}$, i.e.,

$$u_{bd} \cdot e \geq 0 \quad \mathcal{H}^{m-1}\text{-a.e. in } \partial\Omega. \tag{0.2}$$

Example 1. In the scalar case $n = 1$ with zero boundary data $u_{bd} = 0$ on $\partial\Omega$, if $\varepsilon \geq \varepsilon_0$, then $\tilde{u}_\varepsilon = 0$ is the unique solution of (0.1) in $\mathscr{A}$ (so, the unique minimizer of E_ε over $\mathscr{A}$). If $\varepsilon < \varepsilon_0$, then there exists a unique positive solution $u_\varepsilon \in \mathscr{A}$ (i.e., $u_\varepsilon > 0$ in Ω) of (0.1) with zero boundary data, see e.g. [1]; as a consequence of Theorems 0.1 and 0.3 (see below), we have that u_ε and $-u_\varepsilon$ are the only two minimizers of E_ε over $\mathscr{A}$ and moreover, the trivial solution $\tilde{u}_\varepsilon = 0$ is unstable (i.e., the second variation of E_ε at $\tilde{u}_\varepsilon$ is negative in a certain direction).

Example 2. For $m = 2$ and $n = 3$, we consider the unit disk $\Omega \subset \mathbb{R}^2$ and the boundary data carrying a given winding number $k \in \mathbb{Z} \setminus \{0\}$ on $\partial\Omega$:

$$u_{bd}(\cos\varphi, \sin\varphi) = (\cos(k\varphi), \sin(k\varphi), 0) \in \mathbb{S}^1 \times \{0\} \subset \mathbb{R}^3, \quad \forall \varphi \in [0, 2\pi).$$

(Note that u_{bd} satisfies (0.2) in the vertical direction e_3.) As a consequence of Theorem 0.1 (see below), there exists $\varepsilon_k > 0$ such that

a) if $\varepsilon \geq \varepsilon_k$, the unique minimizer of E_ε over $\mathscr{A}$ is given by

$$\tilde{u}_\varepsilon := \tilde{f}_\varepsilon(r)(\cos(k\varphi), \sin(k\varphi), 0),\ \ r \in (0,1),\ \varphi \in [0, 2\pi),$$

where the radial profile $\tilde{f}_\varepsilon$ is the unique solution of the ODE (see e.g. [2])

$$\begin{cases} -\tilde{f}_\varepsilon'' - \frac{1}{r}\tilde{f}_\varepsilon' + \frac{k^2}{r^2}\tilde{f}_\varepsilon = \frac{1}{\varepsilon^2}\tilde{f}_\varepsilon\, W'(1-\tilde{f}_\varepsilon^2) \quad \text{in } (0,1), \\ \tilde{f}_\varepsilon(0) = 0, \tilde{f}_\varepsilon(1) = 1; \end{cases}$$

b) if $\varepsilon < \varepsilon_k$, then E_ε admits exactly two minimizers $u_\varepsilon^\pm$ over $\mathscr{A}$ that have the form

$$u_\varepsilon^\pm := f_\varepsilon(r)(\cos(k\varphi), \sin(k\varphi), 0) \pm g_\varepsilon(r)(0,0,1),\ g_\varepsilon(r) > 0,\ \ r \in (0,1),\ \varphi \in [0, 2\pi),$$

where the couple $(f_\varepsilon, g_\varepsilon)$ of radial profiles is the unique solution of the system

$$\begin{cases} -f_\varepsilon'' - \frac{1}{r}f_\varepsilon' + \frac{k^2}{r^2}f_\varepsilon = \frac{1}{\varepsilon^2}f_\varepsilon\, W'(1-f_\varepsilon^2 - g_\varepsilon^2) \quad \text{in } (0,1), \\ -g_\varepsilon'' - \frac{1}{r}g_\varepsilon' = \frac{1}{\varepsilon^2}g_\varepsilon\, W'(1-f_\varepsilon^2 - g_\varepsilon^2) \quad \text{in } (0,1), \\ f_\varepsilon \geq 0, g_\varepsilon > 0 \quad \text{in } (0,1), \\ f_\varepsilon(0) = 0, f_\varepsilon(1) = 1, g_\varepsilon'(0) = 0, g_\varepsilon(1) = 0. \end{cases}$$

Moreover, the solution $\tilde{u}_\varepsilon$ of (0.1) (given at point a) above) is unstable if $\varepsilon < \varepsilon_k$.

These examples suggest the following phenomenology: if $V = \operatorname{Span} u_{bd}(\partial\Omega)$ has co-dimension ≥ 1 in $\mathbb{R}^n$, then non-uniqueness of minimizers of E_ε over $\mathscr{A}$ is equivalent with the existence of "escaping" solutions $u_\varepsilon \in \mathscr{A}$ of (0.1) (i.e., $u_\varepsilon(\Omega) \not\subset V$). This is highlighted by the following result:

THEOREM 0.1 ([3]). *Let $u_\varepsilon \in H^1 \cap L^\infty(\Omega; \mathbb{R}^n)$ be an "escaping" critical point of the energy E_ε over $\mathscr{A}$ such that $u_\varepsilon \cdot e > 0$ a.e. in Ω in some direction $e \in \mathbb{S}^{n-1}$ for some $\varepsilon > 0$. Then u_ε is a minimiser of E_ε over $\mathscr{A}$ and we have the following dichotomy:*
a) If $u_{bd}(x_0) \cdot e > 0$ for some Lebesgue point $x_0 \in \partial\Omega$, then u_ε is the unique minimiser of E_ε over $\mathscr{A}$.
b) If $u_{bd}(x) \cdot e = 0$ for $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega$, then all minimisers of E_ε in $\mathscr{A}$ are given by Ru_ε where $R \in O(n)$ is an orthogonal transformation of $\mathbb{R}^n$ satisfying $Rx = x$ for all $x \in Span\, u_{bd}(\partial\Omega)$.

Using the above theorem, we prove the following result which completely characterises uniqueness and its failure for minimisers of the energy E_ε over $\mathscr{A}$ under the assumption (0.2) for the boundary data u_{bd}.

THEOREM 0.2 ([3]). *Let $\varepsilon > 0$. If (0.2) holds in direction $e \in \mathbb{S}^{n-1}$ and $V = \operatorname{Span} u_{bd}(\partial\Omega)$, then there exists a unique minimiser u_ε of the energy E_ε over $\mathscr{A}$ unless both following conditions hold:*

i) $u_{bd}(x) \cdot e = 0$ $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega$, *ii) the functional E_ε restricted to the set*

$$\mathscr{A}_{res} := \{u \in \mathscr{A} : u(x) \in Span(V \cup \{e\}) \ a.e. \ in \ \Omega\}$$

has an "escaping" minimiser $\breve{u}_\varepsilon$ with $\tilde{u}_\varepsilon(\Omega) \not\subset V$.

Moreover, if uniqueness of minimisers of E_ε in $\mathscr{A}$ does not hold, then all minimisers of E_ε in $\mathscr{A}$ are given by $R\breve{u}_\varepsilon$ where $R \in O(n)$ is an orthogonal transformation of $\mathbb{R}^n$ satisfying $Rx = x$ for all $x \in V$.

The "escaping" phenomenon is closely related to stability properties of critical points if $\operatorname{codim}_{\mathbb{R}^n}(V) \geq 1$ with $V = \operatorname{Span} u_{bd}(\partial\Omega)$. Indeed, by Theorem 0.1, every "escaping" critical point u_ε of E_ε over $\mathscr{A}$ is in fact a minimiser and there are multiple minimisers as one can reflect u_ε about the orthogonal space to the escaping direction (so, non-uniqueness holds in this case). On the contrary, we show in the following that for a "non-escaping" critical point u_ε of E_ε over $\mathscr{A}$ (i.e., $u_\varepsilon(\Omega) \subset V$), its stability is equivalent with its minimality and therefore, by Theorem 0.2, u_ε is the unique minimiser.

THEOREM 0.3 ([3]). *Assume that $V = \operatorname{Span} u_{bd}(\partial\Omega) \subset e^\perp = \{v \in \mathbb{R}^n : v \cdot e = 0\}$ for a direction $e \in \mathbb{S}^{n-1}$. For any fixed $\varepsilon > 0$, if u_ε is a bounded critical point of E_ε in $\mathscr{A}$ confined in $e^\perp$, i.e., $u_\varepsilon \in L^\infty(\Omega; e^\perp)$ and u_ε is stable in direction e, i.e.,*

$$\frac{d^2}{dt^2}\Big|_{t=0} E_\varepsilon(u_\varepsilon + t\varphi e) = \int_\Omega \left[|\nabla\varphi|^2 - \frac{1}{\varepsilon^2}W'(1 - |u_\varepsilon|^2)\,\varphi^2\right] dx \geq 0 \ \text{for all} \ \varphi \in H_0^1(\Omega),$$

then u_ε is a minimiser of E_ε in $\mathscr{A}$. Moreover, if u_ε is "non-escaping", i.e., $u_\varepsilon(\Omega) \subset V$, then u_ε is the unique minimiser of E_ε in $\mathscr{A}$.

Our results hold true also for the harmonic map problem, thus covering the well-known result of Sandier and Shafrir [4] on the uniqueness of minimising harmonic maps into a closed hemisphere. In fact, our argument does not assume the smoothness of boundary data and does not use the regularity theory of minimising harmonic maps, which appears to play a role in the argument of [4].

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