A De Giorgi type conjecture for elliptic systems under the divergence constraint

Radu Ignat

A De Giorgi type conjecture for elliptic systems under the divergence constraint

Radu Ignat*

October 27, 2020

Abstract

We study the symmetry of transition layers in Ginzburg-Landau type functionals for divergence-free maps in N-dimensions. Namely, we determine a class of nonlinear potentials such that the minimal transition layers in the periodic strip domain $\Omega = \mathbb{R} \times \mathbb{T}^{N-1}$ are one-dimensional where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the 1-torus. In particular, this class includes in dimension $N = 2$ the nonlinear potentials w^2 with w being an harmonic function or a solution to the wave equation, while in dimension $N > 2$, this class contains a perturbation of the standard Ginzburg-Landau potential as well as potentials having $N + 1$ zeros with prescribed transition cost between the zeros. For that, we develop a theory of calibrations for divergence-free maps in dimension N (similar to the theory of entropies for the Aviles-Giga model when $N = 2$). We also give a necessary condition for finite energy configurations yielding the boundary condition in L^2 and almost everywhere in $\mathbb{T}^{N-1}$ as $x_1 \to \pm\infty$.

Keywords: symmetry, De Giorgi conjecture, minimisers, calibrations, Ginzburg-Landau system, divergence constraint, nonlinear Stokes equation.

This note represents the summary of the talk of the author given at the Workshop "Calculus of Variations" in Oberwolfach, 2–8 August 2020 and is based on the articles [1, 2] written in collaboration with Antonin Monteil. This report will be included in the volume Oberwolfach Reports No. 22/2020 dedicated to that workshop.

Introduction. We analyse the symmetry of transition layers in some variational models arising in physics where the order parameter is a vector field of vanishing divergence. We develop a theory of calibrations in order to prove that one-dimensional transition layers are the unique global minimisers in these models.

This question is similar to the famous De Giorgi conjecture for minimal surfaces: if $u : \mathbb{R}^N \to \mathbb{R}$ is a C^2 solution to $\Delta u = \frac{dW}{du}(u)$ in $\mathbb{R}^N$ with $W(u) = \frac{1}{4}(1 - u^2)^2$ such that $-1 < u < 1$ and $\partial_1 u > 0$ in $\mathbb{R}^N$, then u is one-dimensional $(1D)$ provided that $N \leq 8$.

*Institut de Mathématiques de Toulouse & Institut Universitaire de France, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: Radu.Ignat@math.univ-toulouse.fr

After important contributions of Ghoussoub-Gui, Ambrosio-Cabré etc., Savin proved the conjecture under the additional boundary condition

$$\lim_{x_1\to\pm\infty} u(x_1,x') = \pm 1 \quad \text{for every } x' \in \mathbb{R}^{N-1}. \tag{1}$$

Finally, Del Pino-Kowalczyk-Wei gave a counter-example satisfying (1) for $N \geq 9$. Lately, an intensive research was developed for vector-valued solutions $u : \mathbb{R}^N \to \mathbb{R}^d$ to the elliptic system $\Delta u = \nabla W(u)$ in $\mathbb{R}^N$ for potentials $W : \mathbb{R}^d \to \mathbb{R}_+$. The typical potential arising in phase separation models (such as Bose-Einstein condensates with two components, i.e., $d = 2$) is $W(u_1,u_2) = \frac{1}{2}u_1^2u_2^2 + \Lambda(1-|u|^2)^2$ for $\Lambda \geq 0$. Under certain boundary conditions, one-dimensional symmetry of solutions has been shown provided monotonicity / growth / stability conditions on solutions in certain dimensions N.

Periodic strip. In the following, we focus on the infinite strip domain

$$\Omega = \mathbb{R} \times \mathbb{T}^{N-1}$$

in x_1 direction where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the flat torus. For $d \geq 1$ and nonnegative continuous potentials $W : \mathbb{R}^d \to \mathbb{R}_+$, we set the energy functional

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u)\,dx, \quad u : \Omega \to \mathbb{R}^d.$$

In this context, the boundary condition (1) becomes a necessary condition for finite energy configurations:

Lemma 1 ([2]). *Assume that W has a finite number of zeros and $\liminf_{|u|\to\infty} W(u) > 0$. If $E(u) < \infty$, then there exist two zeros $u^\pm \in \mathbb{R}^d$ of W such that*

$$\lim_{x_1\to\pm\infty} u(x_1,\cdot) = u^\pm \quad \textit{in } L^2 \textit{ and a.e. in } \mathbb{T}^{N-1}. \tag{2}$$

Thus, for a.e. $x' \in \mathbb{T}^{N-1}$, $x_1 \in \mathbb{R} \mapsto u(x_1,x')$ represents a curve connecting $u^\pm$ in $\mathbb{R}^d$ endowed with the singular metric $g_W = 2W g_0$ where g_0 is the Euclidean metric. Let

$$\mathrm{geod}_W(u^-,u^+) = \inf\left\{ \int_{-1}^1 \sqrt{2W(\gamma(t))}\,|\dot\gamma|\,dt \,:\, \gamma \in \mathrm{Lip}(-1,1), \gamma(\pm 1) = u^\pm \right\}.$$

Corollary 2. *If a minimal geodesic connecting two zeros $u^\pm$ of W in $(\mathbb{R}^d, g_W)$ exists, then any global minimiser of E connecting $u^\pm$ is $1D$, i.e., $u = u(x_1)$.*

Proof. If $u : \Omega \to \mathbb{R}^d$ satisfies (2), we have:

$$E(u) = \int_\Omega \frac{1}{2}|\nabla' u|^2 + \frac{1}{2}|\partial_1 u|^2 + W(u)\,dx \geq \int_\Omega \frac{1}{2}|\nabla' u|^2\,dx + \mathrm{geod}_W(u^-,u^+). \tag{3}$$

It proves that optimal $1D$ transition layers connecting $u^\pm$ are global minimisers; moreover, if u is a global minimiser of E, then $\nabla' u = 0$ a.e. in Ω, so $u = u(x_1)$. $\square$

Our model. From now on, we assume that $N = d$ and $u : \Omega \to \mathbb{R}^N$ is *divergence-free*. This constraint is natural in certain asymptotic regimes in liquid crystals, elasticity, ferromagnetism etc. In particular, if

$$\bar{u}(x_1) = \int_{\mathbb{T}^{N-1}} u(x_1, x')\, dx'$$

is the x'-average of u and $E(u) < \infty$, then $\bar{u}$ is continuous with constant first component, i.e., $\bar{u}_1 = a$ in $\mathbb{R}$. Moreover, as in Lemma 1, if the set $\{W(a,\cdot) = 0\}$ is finite and $\liminf_{u_1 \to a,\, |u'| \to \infty} W(u_1, u') > 0$, then there are two zeros $u^\pm$ of $W(a,\cdot)$ such that (2) holds (see [2, Theorem 1.3]). We want to determine $W : \mathbb{R}^N \to \mathbb{R}_+$ such that

$$\inf \left\{ E(u) \,:\, u : \Omega \to \mathbb{R}^N, \nabla \cdot u = 0 \text{ with (2)} \right\} \tag{4}$$

has only one-dimensional global minimisers.[1] They satisfy the nonlinear Stokes system $-\Delta u + \nabla W(u) = \nabla p$ for some pressure p (due to the constraint $\nabla \cdot u = 0$).

Calibrations. Inspired by the beautiful paper of Jin-Kohn [3], our strategy is to construct calibrations $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\int_\Omega \nabla \cdot [\Phi(u)]\, dx \leq E(u) \quad \text{for every } u : \Omega \to \mathbb{R}^N \text{ with } \nabla \cdot u = 0 \text{ and (2)}, \tag{5}$$

and

$$\text{there exists } u^* \text{ satisfying} \int_\Omega \nabla \cdot [\Phi(u^*)]\, dx = E(u^*),\ \nabla \cdot u^* = 0 \text{ and (2)}. \tag{6}$$

Note that (5) & (6) yield u^* is a global minimiser in (4). The aim is to prove that u^* is $1D$ and that the equality in (5) is achieved only for $1D$ vector fields u. Roughly speaking, this is related with the equipartition of the energy density for any minimiser u in (4), i.e., $\frac{1}{2}|\nabla u|^2 = W(u)$ in Ω.

Results for $N = 2$. The typical example is given by the Aviles-Giga model where $W(u) = \frac{1}{4}(1 - |u|^2)^2$; note that $\{W(a,\cdot) = 0\}$ is a finite set [2] for every $a \in \mathbb{R}$. Jin-Kohn proved that optimal $1D$ transition layers are global minimizers in (4) (for the reverse implication, see Theorem A below). Moreover, they proved for the potential $W_\delta(u) = \frac{1}{4}(1 - \delta u_2^2 - u_1^2)^2$ with $\delta > 0$ small enough that $1D$ transition layers are no longer global minimizers in (4). Our main result is the following:

Theorem A ([1]). *Assume that $W = \frac{1}{2}w^2$ where $w \in C^2(\mathbb{R}^2)$ solves the Tricomi equation*

$$\partial_{11} w(u) - f(u_1)\partial_{22} w(u) = 0 \quad \textit{for every point } u = (u_1, u_2) \in \mathbb{R}^2$$

with $f \in C^1(\mathbb{R})$ satisfying $|f| \leq 1$. If $u^\pm = (a, u_2^\pm)$ are two zeros of W such that $W(a,\cdot) > 0$ in the interval (u_2^-, u_2^+), then any minimizer $u \in L^\infty$ in (4) is $1D$.

[1] Note that for divergence-free maps u, (3) holds if additionally $u_1 = a$ in Ω.

[2] Without the constraint $\nabla \cdot u = 0$, no global minimisers of E exist when $W(u) = \frac{1}{4}(1 - |u|^2)^2$ since two zeros $u^\pm$ can be connected within the curve $\{W = 0\}$, so the infimum of E vanishes.

The assumption $u \in L^\infty$ can be dropped out provided that $|\nabla w(u)| \leq Ce^{\beta|u|^2}$ for every $u \in \mathbb{R}^2$ for some $C, \beta > 0$. If $f = 1$, then w solves the wave equation (for example, the Aviles-Giga potential). If $f = -1$, then w is an harmonic function, in particular, $w(u) = u_2^2 - u_1^2$. If $f = \frac{1}{\delta}$ with $\delta \geq 1$, we recover the potential W_δ (in a different regime of δ than in [3]). The proof of Theorem A is based on constructing a calibration Φ such that

$$\nabla\Phi(u) = \begin{pmatrix} \alpha(u) & w(u) \\ f(u_1)w(u) & \alpha(u) \end{pmatrix}$$

for some function α.

Results for general $N \geq 2$. We present two strategies to construct calibrations Φ. We denote by Π_0 (resp. Π^+) the projection on traceless $N \times N$ matrices (resp. the projection on symmetric matrices). For divergence-free $u : \Omega \to \mathbb{R}^N$, we have:

$$\nabla \cdot [\Phi(u)] = \nabla\Phi(u) : \Pi_0 \nabla u^T = \Pi_0 \nabla\Phi(u) : \nabla u^T \leq \frac{1}{2}(|\nabla u|^2 + |\Pi_0 \nabla\Phi(u)|^2).$$

Strategy 1. We impose that $|\Pi_0\nabla\Phi(u)|^2 \leq 2W(u)$ for every $u \in \mathbb{R}^N$. In particular, (5) holds. This strategy yields the following result (see [1, Theorem 2.11]): Assume that $X = \{x_0, \dots, x_N\}$ is an affine basis in $\mathbb{R}^N$ and ρ is a (prescribed) metric on X. Then there exists a potential W such that $X = \{W = 0\}$, $\rho = \mathrm{geod}_W$ on $X \times X$ and any minimiser of (4) connecting $u^\pm \in X$ in a periodic strip in direction $\nu \perp (u^+ - u^-)$ is $1D$. The proof is based on the calibration $\Phi = \varphi\nu$ where $\varphi \in \mathrm{Lip}(\mathbb{R}^N)$ satisfies $|\varphi(u^+) - \varphi(u^-)| = \rho(u^+, u^-)$ for every $u^\pm \in X$, yielding the potential $W = \frac{1}{2}|\nabla\varphi|^2$.

Strategy 2. We impose that $\nabla\Phi(u)$ is symmetric and $|\Pi_0\nabla\Phi(u)|^2 \leq 4W(u)$ for every $u \in \mathbb{R}^N$ (the constant 4 is crucial here). Then for divergence-free u,

$$\nabla \cdot [\Phi(u)] = \Pi_0\nabla\Phi(u) : \nabla u^T = \Pi_0\nabla\Phi(u) : \Pi^+\nabla u^T \leq |\Pi^+\nabla u|^2 + \frac{1}{4}|\Pi_0\nabla\Phi(u)|^2.$$

It yields (5) since $\|\nabla u\|^2_{L^2(\Omega)} = 2\|\Pi^+\nabla u\|^2_{L^2(\Omega)}$ (see [1, Proposition 4.12]). This strategy yields a class of potentials W for which (4) has only $1D$ global minimisers (see [1, Theorem 2.10]). The proof is based on calibrations such that $\nabla\Phi = \nabla^2\Psi$ with Ψ solving the wave equation in any directions x_i and x_j, i.e., $\partial_{ii}\Psi = \partial_{jj}\Psi$ in $\mathbb{R}^N$; the potential is then given by $W = \frac{1}{2}\sum_{i<j}|\partial_j\Phi_i|^2$. In particular, we recover the following extension of the Aviles-Giga potential in dimension $N \geq 3$: $W(u) = \frac{1}{4}(1 - |u|^2)^2 + |u''|^2(u_1^2 + u_2^2)$ corresponding to $\Psi(u) = -\frac{u_1u_2}{\sqrt{2}}(\frac{u_1^2+u_2^2}{3} + |u''|^2 - 1)$ for every $u = (u_1, u_2, u'') \in \mathbb{R}^N$ with $u'' = (u_3, \dots, u_N)$.

Perspective. Motivated by micromagnetics, a future problem is to extend this study for divergence-free vector fields u satisfying the nonconvex constraint $|u| = 1$.

References

[1] R. Ignat, A. Monteil, *A De Giorgi-type conjecture for minimal solutions to a nonlinear Stokes equation*, Comm. Pure Appl. Math. **73** (2020), 771–854.

[2] R. Ignat, A. Monteil, *A necessary condition in a De Giorgi type conjecture for elliptic systems in infinite strips*, Special Issue Dedicated to Professor Haim Brezis on the occasion of his 75th birthday, Pure and Applied Functional Analysis **5** (2020), 981–998.

[3] W. Jin, R.V. Kohn, *Singular perturbation and the energy of folds*, J. Nonlinear Sci. **10** (2000), 355–390.