Lecture 3: Asymptotics of E_ε by Γ-convergence ①

Jacobian: $u \in H^1(\Omega \subset \mathbb{R}^2, \mathbb{R}^2)$, $\mathrm{jac}(u) = \partial_1 u \wedge \partial_2 u \in L^1$

$u \in W^{1,1} \cap L^\infty(\Omega, \mathbb{R}^2)$, $\mathrm{jac}(u) = \frac{1}{2} \operatorname{curl}(\underbrace{u \wedge \nabla u}_{\in L^1}) \in \mathcal{D}'$

Ex: $\mathbb{S}^1$-harmonic canonical map

$$u_* = \underbrace{e^{i\varphi_*}}_{\text{smooth}} \left(\frac{x-a_1}{|x-a_1|}\right)^{d_1} \cdots \left(\frac{x-a_m}{|x-a_m|}\right)^{d_m} \in W^{1,1}(\Omega, \mathbb{S}^1)$$

$$\Rightarrow \mathrm{jac}(u_*) = \pi \sum_1^m d_k \delta_{a_k}$$

Thm (BMP) if $u \in W^{1,1}(\Omega, \mathbb{S}^1)$, $\mathrm{jac}(u) \in \mathcal{M}(\Omega) = \{\text{finite measure}\}$

$$\Rightarrow \mathrm{jac}(u) = \pi \sum_1^m d_k \delta_{a_k}, \quad a_1, \dots, a_m \in \Omega,\ d_1, \dots, d_m \in \mathbb{Z} \setminus \{0\}.$$

Degree: $g : \partial\Omega \to \mathbb{S}^1$ C^1, $\deg(g) = \frac{1}{2\pi} \int_{\partial\Omega} g \wedge \partial_\tau g \in \mathbb{Z}$

if $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ C^2 extension of g, then

$$\deg(g) = \frac{1}{\pi} \int_\Omega \mathrm{jac}(u)\, dx.$$

The map $\quad C^1(\partial\Omega, \mathbb{S}^1) \longrightarrow \mathbb{Z}$ can be extended uniquely

$\qquad\qquad\qquad g \longrightarrow \deg(g)$

as continuous map to

$\nearrow (C^0(\partial\Omega, \mathbb{S}^1), \|\cdot\|_\infty)$

$\searrow (H^{1/2}(\partial\Omega, \mathbb{S}^1), \|\cdot\|_{H^{1/2}})$

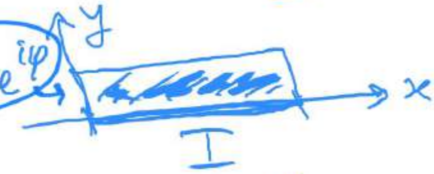
Rk: The degree can be extended to a larger class $W^{1/p,p}(\partial\Omega, S^1)$ for every $p<\infty$, even $BMO(\partial\Omega, S^1)$, but not in $BV(\partial\Omega, S^1)$.

Prop: if $g \in H^{1/2}(\partial\Omega, S^1)$, then $\left[\deg(g)=0 \iff g = e^{i\varphi}\right.$ with $\left.\varphi \in H^{1/2}(\partial\Omega, \mathbb{R})\right]$. If $\deg(g)\neq 0$, then $\exists\, \varphi_1 \in H^{1/2}$, $\varphi_2 \in BV(\partial\Omega, \mathbb{R})$ s.t. $g = e^{i(\varphi_1+\varphi_2)}$ on $\partial\Omega$.

Proof: 1) $\deg(g)=0 \Rightarrow g = e^{i\varphi}$, $\varphi \in H^{1/2}$.

Identify $g(\theta) = g(e^{i\theta})$, $g \in H^{1/2}_{per}(\mathbb{R}, S^1)$. Then

Fact 1: $\forall\, I \subset \mathbb{R}$ bdd, $\exists\, u \in H^1(I \times (0,\varepsilon), S^1)$ extension of g.



$\Rightarrow g = e^{i\varphi}$, $\varphi \in H^{1/2}_{loc}(\mathbb{R}, \mathbb{R})$

Fact 2: $t \longmapsto \frac{1}{2\pi}\left(\varphi(t+2\pi) - \varphi(t)\right) \in H^{1/2}_{loc}(\mathbb{R}, \mathbb{Z})$

is constant.

This constant is in fact $\deg(g) = 0 \Rightarrow$

$\Rightarrow \varphi$ periodic $\Rightarrow \varphi \in H^{1/2}_{per}(\mathbb{R}) \Rightarrow \varphi(\theta) = \varphi(e^{i\theta})$

satisfies $g = e^{i\varphi}$, $\varphi \in H^{1/2}(S^1, \mathbb{R})$.

Idea for Fact 1:

$$\tilde{u}(t,s)=\frac{1}{s}\int_t^{t+s} g(x)\,dx\,, \quad g\in H^{1/2}(I,S^1)$$

$\Rightarrow |\tilde{u}(\cdot,s)| \to 1$ unif as $s\to 0$

$\Rightarrow$ for $s<\varepsilon_0$, $|\tilde{u}(\cdot,s)|\geq \frac{1}{2} \Rightarrow u(t,s)=\dfrac{\tilde{u}(t,s)}{|\tilde{u}(t,s)|}$

the good extension.

Idea for Fact 2: $\varphi\in H^{1/2}_{loc}(\mathbb{R},\mathbb{Z}) \Rightarrow \varphi\equiv cte.$

Indeed, $I=(0,1)$, $\displaystyle \|\varphi\|^2_{H^{1/2}} \simeq \int_0^1\int_0^1 \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|}\,\frac{dx\,dy}{|x-y|}$

$$\geq \int_I\int_I \frac{|\varphi(x)-\varphi(y)|}{|x-y|}\,\frac{dx\,dy}{|x-y|^{\dim(I)=1}}$$

Lemma: $\Omega\subset\mathbb{R}^N$ open + connected, $\varphi:\Omega\subset\mathbb{R}^N\to\mathbb{R}$ measurable

s.t. $\displaystyle \int_\Omega\int_\Omega \frac{|\varphi(x)-\varphi(y)|}{|x-y|}\,\frac{dx\,dy}{|x-y|^N}<+\infty$

$\Rightarrow \varphi = cte.$

2) $g = e^{i\varphi}$, $\varphi \in H^{1/2} \Rightarrow \deg(g) = 0$ (4)

Indeed, $\exists\, \phi \in H^1(\Omega, \mathbb{R})$ extension of $\varphi : \partial\Omega \to \mathbb{R}$

$\Rightarrow u := e^{i\phi} \in H^1(\Omega, \mathbb{S}^1)$, $u = g$ on $\partial\Omega$

$$\Rightarrow \deg(g) = \frac{1}{\pi} \int_\Omega \underbrace{\operatorname{jac}(u)}_{= \partial_1 u \wedge \partial_2 u} dx = 0 .$$

3) $d = \deg(g) \neq 0$, $\partial\Omega = \mathbb{S}^1$, $f = e^{id\theta} \Rightarrow \deg(f) = d$

$\Rightarrow \tilde{g} := \boxed{g}\, \bar{f} \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$, $\deg(\tilde{g}) = \deg(g) + \underbrace{\deg(\bar{f})}_{= -\deg(f)} = 0 \Rightarrow \tilde{g} = e^{i\varphi_1}$, $\varphi_1 \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$

$\Rightarrow g = e^{i(\varphi_1 + \underline{d\theta})}$, $\varphi_1 + d\theta \in BV(\mathbb{S}^1, \mathbb{R})$. ∎

<u>Prop</u> Let $g \in H^{1/2}(\partial\Omega, \underline{\mathbb{S}^1})$ and $u \in H^1(\Omega, \mathbb{R}^2)$ an extension of g in Ω. If $\deg(g) \neq 0$, then $\operatorname{ess\,inf} |u| = 0$, i.e., u has "zeros" in Ω.

<u>Proof:</u> By contradiction, assume $\operatorname{ess\,inf} |u| \geq C > 0$,

$\tilde{u} = \frac{u}{|u|} \in H^1(\Omega, \underline{\mathbb{S}^1})$, $\tilde{u} = g$ on $\partial\Omega$

$\Rightarrow \deg(g) = \frac{1}{\pi} \int_\Omega \operatorname{jac}(\tilde{u})\, dx = 0$. ($\cancel{\lozenge}$)

Γ-convergence of $\frac{1}{|\log\varepsilon|}E_\varepsilon$ in $W^{1,1}$-topology

$E_0, E_\varepsilon : W^{1,1}(\Omega, \mathbb{R}^2) \to [0,+\infty]$

$$E_\varepsilon(u) = \begin{cases} \int \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 & \text{if } u \in H^1(\Omega,\mathbb{R}^2) \\ +\infty & \text{otherwise} \end{cases}$$

$$E_0(u) = \begin{cases} \|jac(u)\|_{\mathcal{M}} & \text{if } u \in W^{1,1}(\Omega, \underline{S^1}), \\ & jac(u) \in \mathcal{M} \\ +\infty & \text{otherwise.} \end{cases}$$

Thm (Jerrard–Soner) $\frac{1}{|\log\varepsilon|}E_\varepsilon \to E_0$ Γ-conv in $W^{1,1}$-top

i.e.,) 1) lower bound: If $u_\varepsilon \to u$ $W^{1,1}$, then

$$\liminf \frac{1}{|\log\varepsilon|} E_\varepsilon(\underline{u_\varepsilon}) \geq E_0(\underline{u})$$

2) upper bound: If $u \in W^{1,1}(\Omega, S^1)$, $jac(u) \in \mathcal{M}$, then $\exists\, u_\varepsilon \in H^1(\Omega,\mathbb{R}^2)$, $\frac{1}{|\log\varepsilon|}E_\varepsilon(u_\varepsilon) \to E_0(u)$.

$u_\varepsilon \to u$ $W^{1,1}$

Proof: 1) lower bound:

Step 1: $\left|\int_\Omega \zeta \, jac(u)\right| \leq \frac{1}{|\log\varepsilon|} E_\varepsilon(u)\|\zeta\|_\infty + o(1)\|\nabla\zeta\|_\infty \quad \forall \zeta \in C_c^\infty(\Omega)$

Step 2: Conclude.

Proof of Step 2: $u_\varepsilon \to u$ $W^{1,1}$ $\not\Rightarrow$ $jac(u_\varepsilon) \to jac(u)$ $\mathcal{D}'$

W.l.o.g. $E_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$ and $u_\varepsilon \to u$ a.e., $|u|=1$

Then $\underbrace{-2\int_\Omega \zeta \, \underbrace{jac(u_\varepsilon)}_{\frac{1}{2}\mathrm{curl}(u_\varepsilon \wedge \nabla u_\varepsilon)}}_{\forall \zeta \in C_c^\infty)} = \int \nabla^\perp\zeta \cdot u_\varepsilon \wedge \nabla u_\varepsilon$

$= \underbrace{\int_{\{|u_\varepsilon|\geq 10\}}}_{\text{(Fact 1) SMALL}} + \underbrace{\int_{\{|u_\varepsilon|\leq 10\}}}_{\text{(Fact 2)}}$

Fact 1:

$$\left|\int_{\{|u_\varepsilon|\geq 10\}} \nabla^\perp\zeta \cdot u_\varepsilon \wedge \nabla u_\varepsilon\right| \leq \|\nabla\zeta\|_\infty \int_{|u_\varepsilon|\geq 10} |\nabla u_\varepsilon| \cdot \underbrace{|u_\varepsilon|}_{\leq \frac{|u_\varepsilon|^2-1}{\varepsilon}\cdot\boxed{\varepsilon}}$$

$$\lesssim \|\nabla\zeta\|_\infty \, \varepsilon \underbrace{\int \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2}_{\leq C|\log\varepsilon|} = o(1)\|\nabla\zeta\|_\infty$$

$$\int_{\{|u_\varepsilon|\le 10\}} \nabla^\perp \zeta \cdot u_\varepsilon \wedge \nabla u_\varepsilon = \int_\Omega \nabla^\perp \zeta \cdot \underbrace{u_\varepsilon}_{\le 10} \wedge \nabla u_{\varepsilon}\, \mathbb{1}_{\{|u_\varepsilon|\le 10\}}$$

$u_\varepsilon \to u$ a.e., $\nabla u_\varepsilon \xrightarrow{L^1} \nabla u$, $\mathbb{1}_{\{|u_\varepsilon|\le 10\}} \to \mathbb{1}_\Omega$ since $|u|=1$ in Ω

$$\xrightarrow[DCT]{} \int_\Omega \nabla^\perp \zeta \cdot u \wedge \nabla u$$

$$\Rightarrow -2\int_\Omega \zeta \operatorname{jac}(u_\varepsilon) \xrightarrow{\varepsilon\to 0} \int_\Omega \nabla^\perp \zeta \cdot u \wedge \nabla u$$

$$\Rightarrow \left|\int \zeta \operatorname{jac}(u)\right| \underset{\varepsilon\to 0}{\leftarrow} \left|\int \zeta \operatorname{jac}(u_\varepsilon)\right|$$

$$\le \liminf_{\varepsilon\to 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \|\zeta\|_\infty .$$

Step 1

Proof of Step 1: Consider $\Omega = B^2$ and $u \in H^1 \cap C^1(B^2\setminus\{0\}, \mathbb{R}^2)$, $|u|>0$ in $B^2\setminus\{0\}$ and $|u|=1$ ∂B^2 with $\deg(u) = d > 0$. Then

Claim 1: $\pi d \le \frac{E_\varepsilon(u)}{|\log \varepsilon|}(1+o(1))$

Claim 2: $\left|\int \zeta \operatorname{jac}(u)\right| \le \underbrace{\pi d}\, \zeta(0) + o(1)\|\nabla \zeta\|_\infty$, $\forall \zeta \in C_c^\infty$, $\zeta \ge 0$.

$\Rightarrow$ conclude Step 1.

Proof of claim 1: $\Omega = B^2$

write $u = \rho v$, $\rho = |u|$, $v = \frac{u}{|u|} \in C^1(B \setminus \{0\}, S^1)$

$$|\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla v|^2$$

$$E_\varepsilon(u) \geq \int_{r \geq \varepsilon} dr \int_{\partial B_r} \frac{1}{2}\rho^2 |\nabla v|^2 + \left(\frac{1}{2}|\nabla \rho|^2 + \frac{1}{4\varepsilon^2}(1-\rho^2)^2\right) dx$$

Let $m = \min_{\partial B_r}\{\rho, 1\}$, $r > 0$.

Fact 1: $\deg(v, \partial B_r) = d$, $\forall r \in (0,1)$ ✓

Indeed, $v \in C^1(B^1 \setminus \{0\}, S^1)$, $\frac{1}{\pi}\int_{B_1 \setminus B_r} \mathrm{jac}(v) = \frac{1}{2\pi}\int_{\partial B_1} \underset{u}{\overset{\shortparallel}{v}} \wedge \underset{\partial_\theta u}{\overset{\shortparallel}{\partial_\theta v}}$ as $|u| = 1$ on ∂B_1

$$- \frac{1}{2\pi}\int_{\partial B_r} v \wedge \partial_\theta v = \underset{\substack{\shortparallel \\ d}}{\deg(u)} - \deg(v, \partial B_r).$$ ∎

Fact 2: $\int_{\partial B_r} \frac{1}{2}\rho^2 |\nabla v|^2 \geq \frac{\pi d^2 m^2}{r}$, $r > 0$.

Indeed, $$\int_{\partial B_r} \frac{1}{2}\rho^2 |\nabla v|^2 \geq \frac{m^2}{2}\int_{\partial B_r} |\nabla v|^2 \underset{(C-S)}{\geq} \frac{m^2}{2} \frac{\left(\int_{\partial B_r} v \wedge \partial_\theta v\right)^2}{|\partial B_r|}$$

$$= \frac{m^2}{2}(2\pi d)^2 \frac{1}{2\pi r}.$$ ∎

Fact 3: $\exists C>0$, $I = \int_{\partial B_r} \frac{1}{2}|\nabla \rho|^2 + \frac{1}{4\varepsilon^2}(1-\rho^2)^2 \, d\sigma \geq \frac{1}{C\varepsilon}(1-m)^2$, $\forall r \geq \varepsilon$

$$m = \min_{\partial B_r}\{\rho, 1\}$$

Indeed,

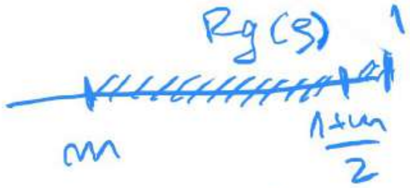
- if $\underset{|u|}{\rho} \equiv m$, then $I = \underbrace{|\partial B_r|}_{2\pi r \,\geq\, \varepsilon} \cdot \frac{1}{4\varepsilon^2} \underbrace{(1-m^2)^2}_{\geq (1-m)^2}$

  $\geq \frac{1}{C\varepsilon}(1-m)^2$.

- if $\rho \leq \frac{1+m}{2} \leq 1$ in ∂B_r, then $(1-\rho^2)^2 \geq$

  $\geq (1-\rho)^2 \gtrsim (1-m)^2 \Rightarrow$ ok.

- otherwise



$$I \gtrsim \frac{1}{\varepsilon} \int_{\partial B_r} \underbrace{|1-\rho^2|}_{\geq |1-\rho|} |\nabla \rho| \gtrsim \frac{1}{\varepsilon} \int_{Rg(\rho)} |1-\rho| \, d\rho \sim \frac{(1-m)^2}{\varepsilon} \quad \blacksquare$$

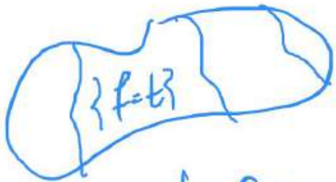
Fact 2+3: $E_\varepsilon(u) \geq \int_\varepsilon^1 dr \left(\underbrace{\min_{m \in [0,1]} \left(\frac{\pi d^2 m^2}{r} + \frac{1}{C\varepsilon}(1-m)^2 \right)}_{\geq \frac{\pi d^2}{r}\left(1 - \tilde{C}\frac{\varepsilon}{r}\right)} \right)$

$$\underset{d^2 \geq d}{\geq} \pi d \, |\log \varepsilon| + O(1).$$

Proof of Claim 2: use co-area formula:

Thm: $\Omega \subset \mathbb{R}^2$ open, $f : \Omega \to \mathbb{R}$ C^1 s.t. $Df \neq 0$ in Ω

$$\Rightarrow \forall g \in L^1(\Omega), \quad \int_\Omega g(x)\,dx = \int_{t \in \mathbb{R}} dt \int_{\{f=t\}} \frac{g}{|Df|}\, d\mathcal{H}^1$$



AIM: $\left| \int_\Omega \zeta \,\mathrm{jac}(u) \right| \le \pi d\, \zeta(0) + o(1) \|D\zeta\|_\infty$, $\forall \zeta \in C_c^\infty(\Omega)$, $\zeta \ge 0$

$\boxed{\Omega = B^2}$ (!) $\boxed{u \in C^1(B^2 \setminus \{0\}, \mathbb{R}^2),\ |u| > 0}$

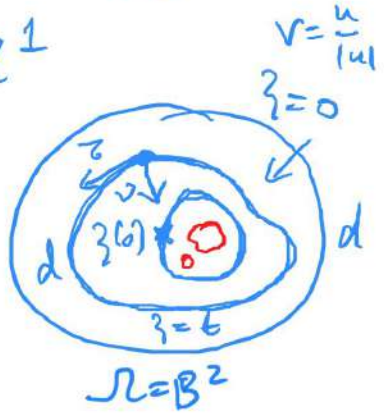
Rk: $u = \left(\frac{x}{|x|}\right)^d \Rightarrow \mathrm{jac}(u) = \pi d\, \delta_0 \Rightarrow$ AIM is ok.

In general, apply co-area for $f = \zeta$:

$$2 \int_\Omega \zeta \,\mathrm{jac}(u) = \int_\Omega \zeta \,\mathrm{curl}(u \wedge \nabla u) = -\int_\Omega \nabla^\perp \zeta \cdot u \wedge \nabla u$$

$$\overset{\text{co-area}}{=} - \int_{t \in \mathbb{R}_+} dt \int_{\{\zeta = t\}} \underbrace{\frac{\nabla^\perp \zeta}{|D\zeta|}}_{-\tau} \cdot u \wedge \nabla u \, d\mathcal{H}^1$$

$\boxed{\nu = \frac{\nabla \zeta}{|D\zeta|}, \quad \nu^\perp = \tau \quad (\tau, \nu) \text{ direct frame}}$



$\Omega = B^2$

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$$= \int_{t\geq 0} dt \int_{\{\zeta = t\}} u \wedge \partial_\tau u \, d\mathcal{H}^1 = \underbrace{\int_{t\geq 0} dt \int_{\{\zeta=t\}} \left(v \wedge \partial_\tau v\right.}_{\mathrm{I}} + \underbrace{\left.\frac{|u|^2-1}{|u|^2} u \wedge \partial_\tau u \right) d\mathcal{H}^1}_{\mathrm{II}}$$

$$\left[u = |u| v \Rightarrow u \wedge \partial_\tau u = |u|^2 v \wedge \partial_\tau v \right]$$

$$\int_{\{\zeta = t\}} v \wedge \partial_\tau v \, d\mathcal{H}^1 = \begin{cases} 2\pi \deg(v) = 2\pi d & \text{if } 0 < t < \zeta(0) \\ 0 & \text{if } t > \zeta(0) \end{cases}$$

$$\mathrm{I} = \underbrace{\int_{0 \leq t \leq \zeta(0)} 2\pi d \, dt}_{2\pi d \zeta(0)}$$

$$\int_{t\geq 0} \int_{\{\zeta=t\}} \frac{|u|^2-1}{|u|^2} u \wedge \partial_\tau u = \mathrm{II}$$

is SMALL $= o(1) \|\nabla \zeta\|_\infty$

Hint: $\mathrm{II} = - \int_\Omega \nabla^\perp \zeta \cdot \frac{|u|^2-1}{|u|^2} u \wedge \nabla u \, dx$

$$= \int_{|u| \geq \frac{1}{2}} + \int_{|u| \leq \frac{1}{2}}$$

$$\left| \int_{|u|\geq \frac{1}{2}} \nabla^\perp \zeta \cdot (|u|^2 - 1) \cdot \underbrace{\boxed{\frac{u}{|u|^2}}}_{\leq 2} \wedge \nabla u \right|$$

$$\underset{C-S}{\lesssim} \|\nabla \zeta\|_\infty \int |\nabla u| \, ||u|^2 - 1| \lesssim \|\nabla \zeta\|_\infty \underbrace{\varepsilon \cdot E_\varepsilon(u)}_{\leq \varepsilon |\log \varepsilon|}$$

The integral on $\{|u| \le \frac{1}{2}\}$, this set is small:

$$\left|\{|u| \le \tfrac{1}{2}\}\right| \lesssim \int_\Omega (1-|u|^2)^2 \lesssim \varepsilon^2 E_\varepsilon(u).$$

(see details in Jerrard-Soner).

$\Longrightarrow$ Claim 2.

2) <u>Upper bound</u>: $u \in W^{1,1}(\Omega, S^1)$, $\underline{jac(u)} = \pi \sum_1^M \delta_{a_k}$

We want $\forall \delta > 0$ small, $\exists$ sequence $u_\varepsilon \in H^1(\Omega, \mathbb{R}^2)$ s.t. $\frac{1}{|\log \varepsilon|} E_\varepsilon(u_\varepsilon) \to \|jac(u)\|_{\mathcal{M}}$ and $\|u_\varepsilon - u\|_{W^{1,1}} \lesssim \delta$.

<u>Idea</u>: Choose $0 < r < \delta$ s.t. $B_k = B(a_k, r)$ disjoint, $\int_{\cup_k B_k} |\nabla u| \le \delta$.

Let $\omega = \Omega \setminus \cup B_r \Rightarrow jac(u) = 0$ in ω.

<u>Thm (Demengel)</u> if $\omega \subset \mathbb{R}^2$ open, then $C^\infty(\omega, S^1)$ are dense in $\left(\{u \in W^{1,1}(\omega, S^1) : jac(u) = 0\}, \|\cdot\|_{W^{1,1}}\right)$. In particular,

In particular, $\exists\, v \in C^\infty(\Omega \setminus \cup B_k, S^1)$, $\|u-v\|_{W^{1,1}(\omega)} \lesssim \delta r$
and $\|v \wedge \nabla v - u \wedge \nabla u\|_{L^1(\omega)} \lesssim \delta r$

$$\Rightarrow \underbrace{\deg(u, \partial B_k)}_{= d_k} = \deg(v, \partial B_k), \quad \int_{\partial B_k} |\nabla v| \lesssim \frac{\delta}{r}$$

Claim: $\exists\, v_\varepsilon \in H^1(B_k, \mathbb{R}^2)$ s.t. $v_\varepsilon = v_{\partial B_k}$

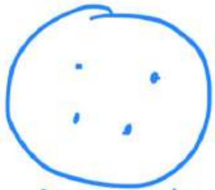
$$\frac{1}{|\log \varepsilon|} E_\varepsilon(v_\varepsilon) \to \pi |d_k| = \|\mathrm{jac}(u)\|_{M(B_k)}$$

and $\int_{B_k} |\nabla v_\varepsilon| \lesssim \delta$.

Idea: Reduce to the case $v = e^{i d_k \theta}$ and use BBH that

$$\min_{\substack{U = e^{i d_k \theta} \\ \text{on } \partial B_k}} E_\varepsilon(U, B_k) \sim \pi |d_k| |\log \varepsilon|$$

(see Jerrard-Soner).



d_k vortices of degree 1

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