Vortex singularities in Ginzburg-Landau type problems

Radu Ignat*

Summary of the course. The purpose of this course is to analyse vortex singularities appearing in Ginzburg-Landau type problems. For that, we consider the following variational model:

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\,dx, \quad u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2,$$

where $\varepsilon > 0$ is a small parameter. We are interested in the asymptotic behaviour as $\varepsilon \to 0$ of critical points u_ε of E_ε that are solutions to the system of elliptic PDEs:

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1-|u_\varepsilon|^2) \quad \text{in} \quad \Omega.$$

As $\varepsilon \to 0$, it is expected that u_ε converges to a so-called $\mathbb{S}^1$-valued canonical harmonic map, whose prototype is the following complex function:

$$u_*(z) = e^{i\varphi_*}\left(\frac{z-a_1}{|z-a_1|}\right)^{d_1} \cdots \left(\frac{z-a_N}{|z-a_N|}\right)^{d_N}, \tag{1}$$

where $\varphi_* : \Omega \to \mathbb{R}$ is harmonic and $a_k \in \Omega$ are the vortex singularities of winding number $d_k \in \mathbb{Z}$. These vortices correspond to zeros of u_ε around which the functional E_ε concentrates and blows up at order $|\log\varepsilon|$ in the limit $\varepsilon \to 0$. Our aim is to present a variational approach in proving this concentration phenomenon of E_ε around vortices.

Organisation. I will start by introducing the problem: a quick physical motivation, the objects we focus on (vortices, jacobian, winding number...) and the main results we want to present (concentration of the jacobian of u_ε and of E_ε). To prove these results, I will review some basic facts of Functional Analysis, Calculus of Variations and Degree Theory, in particular, some properties of the jacobian, winding number, co-area formula, Γ-convergence etc. Then we will prove the main results.

Tentative schedule. Fridays at 10am-noon on May 14, May 21 and May 28, 2021.

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1 Summary of Lecture 1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded simply-connected domain. For $u : \Omega \to \mathbb{R}^2$, we consider the Ginzburg-Landau functional

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \, dx, \tag{2}$$

where $\varepsilon > 0$ is a small parameter. Note that $W(t) = \frac{1}{4\varepsilon^2}(1-t^2)^2$ is a nonnegative double-well potential leading to the limit target space $\mathbb{S}^1 = \{u \in \mathbb{R}^2 : W(|u|) = 0\}$. Thus, as $\varepsilon \to 0$, we expect that the limit configurations of the above model are $\mathbb{S}^1$-valued maps $u_* : \Omega \to \mathbb{S}^1$.

Motivation. This "toy" model arises in physics, in particular, in the Landau theories for phase transitions (e.g., superconductivity, Bose-Einstein condensates, liquid crystals, micromagnetics...) where u represents the order parameter. Typically, for superconductors of type II, $|u|^2$ represents the density of Cooper pairs of superconducting electrons. The state $|u| \sim 1$ is the so called superconducting phase (the Meissner state), while $|u| \sim 0$ is the normal phase (corresponding to a "normal" conductor). In the mixed state (the prototype of our model), u has zeros corresponding to the so-called vortices away from which $|u| \sim 1$ (see more details in [13]).

This model can also be seen as a regularisation of the harmonic map problem. Indeed, it is known that $\mathbb{S}^1$-valued harmonic maps u with nontrivial topology (e.g., the boundary data has nonzero winding number on $\partial\Omega$) have infinite Dirichlet energy. Thus, it is natural to seek energetically optimal maps by relaxing the constraint $|u| = 1$ and replacing it with a term that penalises deviations of u from unit length, then considering a suitable limit which presumably should have an energetically optimal placement of singularities (see more details in [2]).

Goal. We are interested in the asymptotic behaviour as $\varepsilon \to 0$ of critical points u_ε of E_ε that are solutions to the system of PDEs:

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \quad \text{in} \quad \Omega. \tag{3}$$

As $\varepsilon \to 0$, it is expected that u_ε converges to a $\mathbb{S}^1$-valued harmonic map u_*, i.e.,

$$-\Delta u_* = u_* |\nabla u_*|^2 \quad \text{in} \quad \Omega. \tag{4}$$

Moreover, u_* has in general point singularities called vortices that are limits of the "topological" zeros of u_ε. These vortices are detected by the jacobian jac (u_*) and they represent high energy concentration regions of E_ε with a cost of order $|\log \varepsilon|$ as $\varepsilon \to 0$. Our aim is to present a variational approach based on Γ-convergence in proving this concentration phenomenon of E_ε around vortices.

Asymptotic behaviour of minimisers. First, we study the minimisers u_ε of the energy E_ε for a given $\mathbb{S}^1$-valued boundary data $g : \partial\Omega \to \mathbb{S}^1$ that is smooth. Such minimisers u_ε of E_ε with $u_\varepsilon = g$ on $\partial\Omega$ exist (by the direct method in Calculus of Variations), they are smooth solutions to the PDE system (3) and their asymptotic behaviour (in particular, the nucleation of vortices) strongly depend on the winding number of g:

$$\deg(g) = \frac{1}{2\pi} \int_{\partial\Omega} g \wedge \partial_\tau g \, d\mathcal{H}^1 \in \mathbb{Z}, \tag{5}$$

where $\tau = \nu^\perp$ is the unit tangent vector field at $\partial\Omega$ orthogonal to the unit outer normal field ν and $a \wedge b = a_1 b_2 - a_2 b_1$ for $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$.

Case 1: $\deg(g) = 0$. In this case, g admits a smooth lifting $\varphi_0 : \partial\Omega \to \mathbb{R}$, i.e., $g = e^{i\varphi_0}$ on $\partial\Omega$, φ being unique up to an additive $2\pi\mathbb{Z}$ constant. For small $\varepsilon > 0$, one has uniqueness of minimisers u_ε of E_ε with $u_\varepsilon = g$ on $\partial\Omega$ (see [14]). The asymptotic behaviour of u_ε is given by the following result of Bethuel-Brezis-Hélein [1]: as $\varepsilon \to 0$,

$$u_\varepsilon \to u_* \quad \text{in } H^1 \cap C^m_{loc} \cap C^{1,\alpha}(\Omega)$$

for every $m \in \mathbb{N}$ and $\alpha \in (0,1)$ where $u_* : \Omega \to \mathbb{S}^1$ is the unique $\mathbb{S}^1$-valued harmonic map satisfying (4) with the boundary condition $u_* = g$ on $\partial\Omega$. More precisely, $u_* = e^{i\varphi_*}$ with $\varphi_* : \Omega \to \mathbb{R}$ satisfying $\Delta\varphi_* = 0$ in Ω and $\varphi_* = \varphi_0$ on $\partial\Omega$. As a consequence, u_ε does not have zeros for small ε since $|u_\varepsilon| \to |u_*| = 1$ uniformly in Ω as $\varepsilon \to 0$. Also,

$$E_\varepsilon(u_\varepsilon) = \frac{1}{2} \int_\Omega |\nabla u_*|^2 \, dx + o(1) \quad \text{as } \varepsilon \to 0.$$

Case 2: $\deg(g) \neq 0$. In this case, g does no longer have a smooth lifting, but BV liftings with jumps and we expect the nucleation of vortices inside the domain. At the limit $\varepsilon \to 0$, the prototype of a vortex vector field of winding number 1 at $a = 0$ in the unit disk B_1 is given by

$$u_*(x) = \frac{x}{|x|} = e^{i\theta}.$$

Note that u_* does no longer belong to H^1, but only in $W^{1,p}$ for $p < 2$ in B_1; so, u_* is a singular $\mathbb{S}^1$-valued harmonic map. At the level $\varepsilon > 0$, a minimiser u_ε converging at u_* in B_1 is expected to have the form

$$u_\varepsilon(x) = f_\varepsilon(|x|)e^{i\theta}$$

where the radial profile f_ε solves the following ODE:

$$\begin{cases} -f_\varepsilon'' - \frac{1}{r}f_\varepsilon' + \frac{1}{r^2}f_\varepsilon = \frac{1}{\varepsilon^2}f_\varepsilon(1-f_\varepsilon^2) \quad \text{for every} \quad r \in (0,1), \\ f_\varepsilon(0) = 0, f_\varepsilon(1) = 1 \end{cases}$$

(see [11]). This symmetric approximation u_ε has a topological zero at a (around which there is a circulation of the phase of 2π) that induces a large energy on disks of radius $r > 0$: $E_\varepsilon(u_\varepsilon, B_r) = \pi \log \frac{r}{\varepsilon} + O(1)$ as $\varepsilon \to 0$. The asymptotic behaviour of u_ε is given by the following result of Bethuel-Brezis-Hélein [2]:

Theorem 1.1 ([2]) *If $d = \deg(g) > 0$ and u_ε is a minimiser of E_ε with the boundary data g, then for a sequence $\varepsilon \to 0$,*

$$u_\varepsilon \to u_*(x) = e^{i\varphi_*(x)} \prod_{k=1}^{d} \frac{x-a_k}{|x-a_k|} \quad in \quad W^{1,1} \cap C^m_{loc}(\Omega \setminus \{a_k\}_k) \cap C^{1,\alpha}_{loc}(\bar{\Omega} \setminus \{a_k\}_k),$$

for every $m \in \mathbb{N}$ and $\alpha \in (0,1)$ where $a_1, \ldots, a_d \in \Omega$ are d distinct vortex points in Ω of winding number 1 and $\varphi_* : \Omega \to \mathbb{R}$ satisfies $\Delta\varphi_* = 0$ in Ω and $\varphi_* = \tilde{\varphi}_0$ on $\partial\Omega$ with[1]

$$e^{i\tilde{\varphi}_0} = g(x) \prod_{k=1}^{d} \overline{\frac{x-a_k}{|x-a_k|}} \quad on \quad \partial\Omega.$$

Moreover, $E_\varepsilon(u_\varepsilon) = \pi d |\log \varepsilon| + O(1)$ as $\varepsilon \to 0$.

The second order term in the expansion of $E_\varepsilon(u_\varepsilon)$ was also determined in [2] and contains the so-called renormalized energy corresponding to the interaction energy between the vortices. More precisely, as $\varepsilon \to 0$,

$$E_\varepsilon(u_\varepsilon) = \pi d|\log \varepsilon| + W(a_1, \ldots, a_d) + d\gamma + o(1)$$

where the renormalized energy is given by

$$W(a_1, \ldots, a_d) = \lim_{r \to 0} \Big(\int_{\Omega \setminus \cup_{k=1}^d B_r(a_k)} \frac{1}{2}|\nabla u_*|^2 \, dx - \pi d|\log \varepsilon| \Big)$$

while γ is a constant corresponding to the energy of a radially symmetric minimiser $u_\varepsilon = f_\varepsilon(|x|)e^{i\theta}$ in the unit disk B_1 with the boundary data $g(x) = x$ on ∂B_1, i.e.,

$$\gamma = \lim_{\varepsilon \to 0} \Big(E_\varepsilon(u_\varepsilon, B_1) - \pi|\log \varepsilon| \Big).$$

The renormalized energy W governs the optimal position of vortices, the interaction between them corresponds to a logarithmic repulsion, i.e., $W \sim -\pi \sum_{k \neq \ell} \log|a_k - a_\ell| + O(1)$ as $\min_{k \neq \ell} |a_k - a_\ell| \to 0$.

[1]For $x \in \mathbb{R} \sim \mathbb{C}$, we denote by $\bar{x}$ the complex conjugate of x.

2 Summary of Lecture 2. Jacobian and degree theory

Jacobian. The Jacobian determinant plays an important role in the Ginzburg-Landau theory as it detects the vortices of the order parameter. Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be smooth; then

$$\operatorname{jac}(u) = \det(\nabla u) = \partial_1 u \wedge \partial_2 u.$$

Note that $\operatorname{jac}(u)$ is well defined in the larger class $u \in H^1(\Omega, \mathbb{R}^2)$ since $\partial_1 u, \partial_2 u \in L^2$, so $\operatorname{jac}(u) \in L^1$. But how to define the Jacobian for a canonical harmonic map $u_* \notin H^1$? In fact, the notion of Jacobian $\operatorname{jac}(u)$ can be extended as a distribution when $u \in W^{1,1} \cap L^\infty(\Omega, \mathbb{R}^2)$:

$$\operatorname{jac}(u) = \frac{1}{2}\operatorname{curl}(u \wedge \nabla u)$$

where the current $j = (u \wedge \partial_1 u, u \wedge \partial_2 u) \in L^1$ as $u \in L^\infty$ and $\nabla u \in L^1$. Moreover, $\operatorname{jac}(u)$ belongs to $W^{-1,1}$ as dual of Lipschitz functions ζ vanishing at the boundary $\partial\Omega$, i.e.,

$$< \operatorname{jac}(u), \zeta > := -\frac{1}{2}\int_\Omega \nabla^\perp \zeta \cdot u \wedge \nabla u \, dx, \quad \zeta \in W^{1,\infty}(\Omega),\ \zeta = 0 \text{ on } \partial\Omega.$$

As $u_* \in W^{1,1}(\Omega, \mathbb{S}^1)$, then $\operatorname{jac}(u_*) \in W^{-1,1}$.

Examples. If $u \in H^1(\Omega, \mathbb{S}^1)$, then $\operatorname{jac}(u) = 0$. If $u_*(x) = \frac{x}{|x|} \in W^{1,1} \cap L^\infty$, then $\operatorname{jac}(u_*) = \pi\delta_0$ where δ_0 is the Dirac mass at 0. Moreover, if u_* is the canonical harmonic map in (1), then $\operatorname{jac}(u_*) = \pi \sum_{k=1}^N d_k \delta_{a_k}$.

The following characterisation of the Jacobian holds in the space $W^{1,1}(\Omega, \mathbb{S}^1)$ (see [5] for the space $BV(\Omega, \mathbb{S}^1)$):

Theorem 2.1 ([3]) *If $u_* \in W^{1,1}(\Omega, \mathbb{S}^1)$ and $\operatorname{jac}(u_*) \in \mathcal{M}(\Omega)$ is a finite measure in Ω, then there exist N distinct points $a_1, \ldots, a_N \in \Omega$ and $d_1, \ldots, d_N \in \mathbb{Z} \setminus \{0\}$ such that*

$$\operatorname{jac}(u_*) = \pi \sum_{k=1}^N d_k \delta_{a_k}.$$

In particular, $\|\operatorname{jac}(u_*)\|_{\mathcal{M}} = \pi \sum_{k=1}^N |d_k|$.

Main Theorem. The aim of lecture 3 will be to study the asymptotic behaviour of the energy E_ε for more general configurations u_ε that are not necessarily critical points of E_ε. The framework is given by the Γ-convergence of the functionals $\frac{1}{|\log\varepsilon|}E_\varepsilon$ in the strong $W^{1,1}$ topology. Recall that $E_\varepsilon(u) < \infty$ if and only if $u \in H^1(\Omega, \mathbb{R}^2)$. We extend $E_\varepsilon : W^{1,1}(\Omega, \mathbb{R}^2) \to [0, +\infty]$ by setting $E_\varepsilon(u) = +\infty$ if $u \in W^{1,1} \setminus H^1(\Omega, \mathbb{R}^2)$. The following result holds:

Theorem 2.2 ([10]) *Let $E_0 : W^{1,1}(\Omega, \mathbb{R}^2) \to [0, +\infty]$ be defined by*

$$E_0(u) = \begin{cases} \|\operatorname{jac}(u)\|_{\mathcal{M}} & \text{if } u \in W^{1,1}(\Omega, \mathbb{S}^1), \operatorname{jac}(u) \in \mathcal{M}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\frac{1}{|\log \varepsilon|} E_\varepsilon \rightharpoonup E_0$ in the Γ-convergence sense in the topology $W^{1,1}$, i.e.,

a) lower bound: if $u_\varepsilon \to u$ in $W^{1,1}$, then $\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} E_\varepsilon(u_\varepsilon) \geq E_0(u)$.

b) upper bound: if $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ with $\operatorname{jac}(u) \in \mathcal{M}(\Omega)$, then there exists $u_\varepsilon \to u$ in $W^{1,1}$ such that $\frac{1}{|\log \varepsilon|} E_\varepsilon(u_\varepsilon) \to E_0(u)$ as $\varepsilon \to 0$.

In the proof of this theorem, an important role is played by the notion of topological degree of a $H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$-map that we present in the following.

Degree theory. We focus on maps $g : \mathbb{S}^1 \to \mathbb{S}^1$ where we often embed $\mathbb{S}^1 \subset \mathbb{C}$.

Case 1. Smooth maps. If $g : \mathbb{S}^1 \to \mathbb{S}^1$ is C^1, then the topological degree (also called winding number or index) is given by

$$\deg(g) = \frac{1}{2\pi} \int_{\mathbb{S}^1} g \wedge g' \, d\mathcal{H}^1 = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \bar{g} g' \, d\mathcal{H}^1,$$

where $\bar{g}$ is the conjugate of g and g' is the angular derivative of g. If $A = (1, 0)$, then on $\mathbb{S}^1 \setminus \{A\}$, $g = e^{i\varphi}$ for a C^1 lifting $\varphi : \mathbb{S}^1 \setminus \{A\} \to \mathbb{R}$ so that

$$\deg(g) = \frac{\varphi(A+) - \varphi(A-)}{2\pi} \in \mathbb{Z} \tag{6}$$

because $g(A) = e^{i\varphi(A+)} = e^{i\varphi(A-)}$. Example: if $g(e^{i\theta}) = e^{id\theta}$ for some $d \in \mathbb{Z}$, then $\deg(g) = d$.

Proposition 2.3 *If $z \in \mathbb{S}^1$ is a regular value of g, i.e., $g^{-1}(\{z\}) = \{x_1, \ldots, x_n\}$ with $g'(x_k) \neq 0$ for $1 \leq k \leq n$, then*

$$\deg(g) = \sum_{k=1}^{n} sgn(g \wedge g'(x_k)).$$

Proposition 2.4 *If $u : B^2 \to \mathbb{R}^2$ is a C^2 extension of g inside the unit disk B_1, then*

$$\deg(g) = \frac{1}{\pi} \int_{B_1} \operatorname{jac}(u) \, dx.$$

Indeed, integration by parts yields

$$\frac{1}{\pi}\int_{B_1} \operatorname{jac}(u)\,dx = \frac{1}{2\pi}\int_{B_1} \operatorname{curl}(u\wedge\nabla u)\,dx = \frac{1}{2\pi}\int_{\partial B_1} u\wedge\partial_\theta u\,d\mathcal{H}^1 = \deg(g).$$

Case 2. Continuous maps. If $g:\mathbb{S}^1\to\mathbb{S}^1$ is C^0, then the winding number can be defined as in (6): indeed, letting $A=(1,0)$, then on $\mathbb{S}^1\setminus\{A\}$, $g=e^{i\varphi}$ for a unique C^0 lifting $\varphi:\mathbb{S}^1\setminus\{A\}\to\mathbb{R}$ (up to an additive $2\pi\mathbb{Z}$ constant) so that (6) has a meaning. It is easy to check that the definition is independent on the point $A\in\mathbb{S}^1$.

Another way to define it is to show the existence of a unique continuous extension to the space $(C^0(\mathbb{S}^1,\mathbb{S}^1),\|\cdot\|_{L^\infty})$ of the degree map defined in Case 1, i.e.,

$$Deg: g\in C^1(\mathbb{S}^1,\mathbb{S}^1)\mapsto \deg(g)\in\mathbb{Z}.$$

For that, we need the following:

Fact 1: If $f,g\in C^1(\mathbb{S}^1,\mathbb{S}^1)$ such that $\|f-g\|_{L^\infty}<\frac{1}{2}$, then $\deg(f)=\deg(g)$. In particular, the map Deg is continuous on $(C^1(\mathbb{S}^1,\mathbb{S}^1),\|\cdot\|_{L^\infty})$.

Fact 2: $C^1(\mathbb{S}^1,\mathbb{S}^1)$ is dense in $(C^0(\mathbb{S}^1,\mathbb{S}^1),\|\cdot\|_{L^\infty})$.

Conclusion: there exists a unique continuous extension of Deg to

$$g\in(C^0(\mathbb{S}^1,\mathbb{S}^1),\|\cdot\|_{L^\infty})\mapsto\deg(g)\in\mathbb{Z}.$$

In particular, the connected components of $(C^0(\mathbb{S}^1,\mathbb{S}^1),\|\cdot\|_{L^\infty})$ are given by the sets $\{g\in C^0(\mathbb{S}^1,\mathbb{S}^1)\,:\,\deg(g)=k\}_{k\in\mathbb{Z}}$.

Sketch of Fact 1: define the homotopy $H:[0,1]\times\mathbb{S}^1\to\mathbb{S}^1$ between f and g as

$$H(t,x)=\frac{tg(x)+(1-t)f(x)}{|tg(x)+(1-t)f(x)|},\quad t\in[0,1], x\in\mathbb{S}^1$$

and show that $H(t,\cdot)\in C^1(\mathbb{S}^1,\mathbb{S}^1)$ and the continuous map $t\in[0,1]\mapsto\deg(H(t,\cdot))\in\mathbb{Z}$ is constant.

Sketch of Fact 2: For $g\in C^0(\mathbb{S}^1,\mathbb{S}^1)$, identify $g(e^{i\theta})\sim g(\theta)$ as a 2π-periodic function in $\mathbb{R}$ and for small $\varepsilon>0$, consider a family of nonnegative mollifiers $\{\rho_\varepsilon\}$ and the convolution $\bar{g}_\varepsilon=g*\rho_\varepsilon$ is a smooth 2π-periodic function in $\mathbb{R}$. Identifying again $\bar{g}_\varepsilon(e^{i\theta})\sim\bar{g}_\varepsilon(\theta)$ $\bar{g}_\varepsilon\in C^1(\mathbb{S}^1,\mathbb{R}^2)$, $\bar{g}_\varepsilon\to g$ uniformly, in particular, $|\bar{g}_\varepsilon|\to 1$ uniformly as $\varepsilon\to 0$. Finally, for small $\varepsilon>0$, define $g_\varepsilon:=\frac{\bar{g}_\varepsilon}{|\bar{g}_\varepsilon|}\in C^1(\mathbb{S}^1,\mathbb{S}^1)$ and check that $g_\varepsilon\to g$ uniformly as $\varepsilon\to 0$.

Proposition 2.5 *If $m\in\mathbb{N}$ and $g\in C^m(\mathbb{S}^1,\mathbb{S}^1)$, then $\deg(g)=0$ if and only if $g=e^{i\varphi}$ for a lifting $\varphi\in C^m(\mathbb{S}^1,\mathbb{R})$.*

Case 3. $H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$-maps. Recall that $g \in H^{1/2}(\mathbb{S}^1,\mathbb{R}^2)$ if $g \in L^2$ and

$$\|g\|^2_{\dot H^{1/2}} = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|g(x)-g(y)|^2}{|x-y|^2}\,dxdy < \infty.$$

Equivalently, $g \in H^{1/2}(\mathbb{S}^1,\mathbb{R}^2)$ if and only if g is the trace of a map $u \in H^1(B_1,\mathbb{R}^2)$. Define $H^{1/2}(\mathbb{S}^1,\mathbb{S}^1) = \{g \in H^{1/2}(\mathbb{S}^1,\mathbb{R}^2) : |g| = 1 \text{ a.e. }\}$ with the norm $\|g\|_{H^{1/2}} = \|g\|_{L^2} + \|g\|_{\dot H^{1/2}}$.

To define a degree for $g \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$, we proceed as in Case 2, i.e., we show the existence of a unique continuous extension to the space $(H^{1/2}(\mathbb{S}^1,\mathbb{S}^1), \|\cdot\|_{H^{1/2}})$ of the degree map

$$Deg : g \in C^1(\mathbb{S}^1,\mathbb{S}^1) \mapsto \deg(g) \in \mathbb{Z}.$$

For that, we need the following:

<u>Fact 1</u>: There exists a constant $C>0$ such that

$$|\deg(f)-\deg(g)| \le C\|f-g\|_{H^{1/2}}\Big(\|f\|_{H^{1/2}} + \|g\|_{H^{1/2}}\Big) \quad \text{for every } f,g \in C^1(\mathbb{S}^1,\mathbb{S}^1).$$

<u>Fact 2</u>: $C^1(\mathbb{S}^1,\mathbb{S}^1)$ is dense in $(H^{1/2}(\mathbb{S}^1,\mathbb{S}^1), \|\cdot\|_{H^{1/2}})$.

3 Exercises

Exercise 1 Let $\Omega \subset \mathbb{R}^N$ be a connected open set and $u : \Omega \to \mathbb{S}^M$ with $N, M \ge 1$.

a) If $\Delta u = 0$, then prove that u is a constant.

b) If $u \in H^1(\Omega,\mathbb{S}^M)$ is a $\mathbb{S}^M$-valued harmonic map, i.e.,

$$\frac{d}{dt}\Big|_{t=0} \int_\Omega \left|\nabla\left(\frac{u+t\zeta}{|u+t\zeta|}\right)\right|^2 dx = 0 \quad \text{for every } \zeta \in C_c^\infty(\Omega;\mathbb{R}^{M+1}),$$

then $-\Delta u = u|\nabla u|^2$ in Ω.

c) Assume that $N=2$ and $M=1$.

c1) Let $u \in C^2(\Omega,\mathbb{S}^1)$ and $j = u \wedge \nabla u$ be the associated current. Prove that u is a $\mathbb{S}^1$-valued harmonic map if and only if $\operatorname{div} j = 0$ and $\operatorname{curl} j = 0$ in Ω.

c2) If $u \in W^{1,1}(\Omega,\mathbb{S}^1)$ is the canonical harmonic map

$$u(x) = e^{i\varphi}\prod_{k=1}^n \left(\frac{x-a_k}{|x-a_k|}\right)^{d_k},$$

for $a_1,\dots,a_n \in \Omega$, $d_1,\dots,d_n \in \mathbb{Z}$ and $\varphi \in W^{1,1}(\Omega,\mathbb{R})$, prove that the current $j = u \wedge \nabla u$ satisfies $\operatorname{div} j = 0$ and $\operatorname{curl} j = 2\pi\sum_{k=1}^n d_k\delta_{a_k}$ in Ω.

Exercise 2 Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded open set and $g : \partial\Omega \to \mathbb{R}^2$ be a smooth function.

a) Prove that for every $\varepsilon > 0$, there exists a minimiser u_ε of the energy E_ε defined in (2) over the space

$$H^1_g(\Omega, \mathbb{R}^2) = \{u \in H^1(\Omega, \mathbb{R}^2) \, : \, u = g \text{ on } \partial\Omega\}.$$

Moreover, u_ε satisfies the Euler-Lagrange equation (3).

b) Prove that every solution $u_\varepsilon \in H^1_g(\Omega, \mathbb{R}^2)$ to the PDE (3) is smooth in $\bar{\Omega}$.

c) If $|g| = 1$ on $\partial\Omega$, prove that every solution $u_\varepsilon \in H^1_g(\Omega, \mathbb{R}^2)$ to the PDE (3) satisfies $|u_\varepsilon| \leq 1$ in Ω. (Hint: start by proving that $\rho = 1 - |u_\varepsilon|^2$ satisfies $-\Delta\rho + \frac{2}{\varepsilon^2}|u_\varepsilon|^2\rho \geq 0$ in Ω and conclude by the maximum principle...)

d) Prove that if $\varepsilon > 0$ is large enough, then E_ε is convex over $H^1(\Omega, \mathbb{R}^2)$; as a consequence, there exists a unique solution $u_\varepsilon \in H^1_g(\Omega, \mathbb{R}^2)$ to the PDE (3).

Exercise 3 Let B_1 be the unit disk in $\mathbb{R}^2$, $d \geq 1$ be an integer and $g : \mathbb{S}^1 \to \mathbb{S}^1$ be given by $g(e^{i\theta}) = e^{id\theta}$ for every $\theta \in [0, 2\pi]$. Prove that for every $\varepsilon > 0$, there exists a solution $u_\varepsilon \in H^1_g(B_1, \mathbb{R}^2)$ of the form $u_\varepsilon(x) = f_\varepsilon(|x|)e^{id\theta}$ for every $x \in B_1$ to the PDE (3). Moreover, the radial profile f_ε solves the following ODE:[2]

$$\begin{cases} -f_\varepsilon'' - \frac{1}{r}f_\varepsilon' + \frac{d^2}{r^2}f_\varepsilon = \frac{1}{\varepsilon^2}f_\varepsilon(1 - f_\varepsilon^2) \quad \text{for every} \quad r \in (0,1), \\ f_\varepsilon(0) = 0, f_\varepsilon(1) = 1. \end{cases}$$

Exercise 4 Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded simply connected domain.

a) If $k \in \mathbb{N}$ and $u \in C^k(\Omega, \mathbb{S}^1)$ prove that there exists a lifting $\varphi \in C^k(\Omega, \mathbb{R})$ of u, i.e., $u = e^{i\varphi}$ in Ω and φ is unique up to an additive constant $2\pi\mathbb{Z}$.

b) If $p \geq 2$ and $u \in W^{1,p}(\Omega, \mathbb{S}^1)$ prove that there exists a lifting $\varphi \in W^{1,p}(\Omega, \mathbb{R})$ of u, i.e., $u = e^{i\varphi}$ in Ω and φ is unique up to an additive constant $2\pi\mathbb{Z}$.

(Hint: Start by proving that $\operatorname{curl}(u \wedge \nabla u) = 0$ and apply Poincaré lemma to obtain $\nabla\varphi = u \wedge \nabla u$...)

c) Using b), prove that for a boundary data $g = e^{i\varphi_0}$ with $\varphi_0 : \partial\Omega \to \mathbb{R}$ smooth, there exists a unique minimizing $\mathbb{S}^1$-valued harmonic map u_* of the problem

$$\min\{\int_\Omega |\nabla u|^2 \, dx \, : \, u \in H^1(\Omega, \mathbb{S}^1), u = g \text{ on } \partial\Omega\}.$$

[2]Such solution f_ε is unique, see e.g. [8].

Moreover, $u_* = e^{i\varphi_*} \in C^\infty(\bar{\Omega}, \mathbb{S}^1)$ where $\varphi_* : \Omega \to \mathbb{R}$ is the unique solution of $\Delta\varphi_* = 0$ in Ω and $\varphi_* = \varphi_0$ on $\partial\Omega$.

Exercise 5 Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $u \in C^1(\Omega, \mathbb{R}^2)$ and $j(u) = u \wedge \nabla u$ be the associated current to u.

a) if $u \neq 0$ in Ω, prove that

$$|\nabla u|^2 = \left|\frac{j(u)}{|u|}\right|^2 + \big|\nabla |u|\big|^2.$$

b) if $|u| = 1$ in Ω and $\rho \in C^1(\Omega, \mathbb{R})$, prove that

$$|\nabla u|^2 = |j(u)|^2, \quad j(\rho u) = \rho^2 j(u).$$

Deduce that

$$|\nabla(\rho u)|^2 = \rho^2 \, |j(u)|^2 + |\nabla \rho|^2 = \rho^2 \, |\nabla u|^2 + |\nabla \rho|^2.$$

c) if $\varphi \in C^1(\Omega, \mathbb{R})$, prove that

$$j(e^{i\varphi} u) = j(u) + |u|^2 \, \nabla\varphi.$$

Exercise 6 Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain.

a) If $u, v \in H^1(\Omega, \mathbb{R}^2)$ and $\zeta \in W^{1,\infty}(\Omega, \mathbb{R})$ with $\zeta = 0$ on $\partial\Omega$, prove that

$$\Big|\int_\Omega (\mathrm{jac}\,(u) - \mathrm{jac}\,(v))\zeta \, dx\Big| \leq \frac{1}{2}\|u - v\|_{L^2}\big(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\big)\|\nabla\zeta\|_{L^\infty}.$$

b) If $u \in H^1(\Omega, \mathbb{S}^1)$, then $\mathrm{jac}\,(u) = 0$.

c) If $d \in \mathbb{Z}$, $a \in \Omega$, $\varphi \in C^1(\Omega, \mathbb{R})$ and $u(x) = e^{i\varphi}\left(\frac{x-a}{|x-a|}\right)^d$, prove that $\mathrm{jac}\,(u) = \pi d \delta_a$.

References

[1] F. Bethuel, H. Brezis, F. Hélein, *Asymptotics for the minimization of a Ginzburg-Landau functional*, Calc. Var. Partial Differential Equations **1** (1993), 123-148.

[2] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg-Landau vortices*, Birkhäuser, Boston, 1994.

[3] H. Brezis, P. Mironescu, A. Ponce, *$W^{1,1}$-maps with values into* $\mathbb{S}^1$, Geometric analysis of PDE and several complex variables, 69-100, Contemp. Math. **368**, Amer. Math. Soc., Providence, RI, 2005.

[4] H. Brezis, L. Nirenberg, *Degree theory and BMO. I. Compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), 197–263.

[5] R. Ignat, *The space $BV(\mathbb{S}^2, \mathbb{S}^1)$: minimal connection and optimal lifting*, Ann. Inst. H. Poincaré, Anal. Non linéaire **22** (2005), 283-302.

[6] R. Ignat, R.L. Jerrard, *Renormalized energy between vortices in some Ginzburg-Landau models on 2-dimensional Riemannian manifolds*, Arch. Ration. Mech. Anal. **239** (2021), 1577–1666.

[7] R. Ignat, L. Nguyen, V. Slastikov, A. Zarnescu, *On the uniqueness of minimisers of Ginzburg-Landau functionals*, Ann. Sci. Éc. Norm. Supér. **53** (2020), 589–613.

[8] R. Ignat, L. Nguyen, V. Slastikov, A. Zarnescu, *Uniqueness results for an ODE related to a generalized Ginzburg-Landau model for liquid crystals*, SIAM Journal on Mathematical Analysis **46** (2014), 3390-3425.

[9] R.L. Jerrard, *Lower bounds for generalized Ginzburg-Landau functionals*, SIAM J. Math. Anal. **30** (1999), 721-746.

[10] R.L. Jerrard, H.M. Soner, *The Jacobian and the Ginzburg-Landau energy,* Calc. Var. PDE **14** (2002), 151-191.

[11] F. Pacard and T. Rivière, *Linear and nonlinear aspects of vortices*, Birkhäuser, Boston, 2000.

[12] E. Sandier, *Lower bounds for the energy of unit vector fields and applications* J. Funct. Anal. **152** (1998), 379-403.

[13] E. Sandier, S. Serfaty, *Vortices in the magnetic Ginzburg-Landau model*, Birkhäuser, 2007.

[14] D. Ye, F. Zhou, *Uniqueness of solutions of the Ginzburg-Landau problem*, Nonlinear Anal. **26** (1996), 603-612.