RENORMALIZED ENERGY BETWEEN VORTICES IN SOME GINZBURG-LANDAU MODELS ON 2-DIMENSIONAL RIEMANNIAN MANIFOLDS

R. Ignat and R.L. Jerrard
October 29, 2020

Abstract. We study a variational Ginzburg-Landau type model depending on a small parameter $\varepsilon > 0$ for (tangent) vector fields on a 2-dimensional Riemannian manifold S. As $\varepsilon \to 0$, these vector fields tend to have unit length so they generate singular points, called vortices, of a (non-zero) index if the genus $\mathfrak{g}$ of S is different than 1. Our first main result concerns the characterization of canonical harmonic unit vector fields with prescribed singular points and indices. The novelty of this classification involves flux integrals constrained to a particular *vorticity-dependent* lattice in the $2\mathfrak{g}$-dimensional space of harmonic 1-forms on S if $\mathfrak{g} \geq 1$. Our second main result determines the interaction energy (called renormalized energy) between vortex points as a Γ-limit (at the second order) as $\varepsilon \to 0$. The renormalized energy governing the optimal location of vortices depends on the Gauss curvature of S as well as on the quantized flux. The coupling between flux quantization constraints and vorticity, and its impact on the renormalized energy, are new phenomena in the theory of Ginzburg-Landau type models. We also extend this study to two other (extrinsic) models for embedded hypersurfaces $S \subset \mathbb{R}^3$, in particular, to a physical model for non-tangent maps to S coming from micromagnetics.

Contents

1. Introduction

We consider three related asymptotic variational problems similar to the Ginzburg-Landau model that are described by singularly perturbed functionals depending on a small parameter $\varepsilon > 0$. These functionals are defined for smooth vector fields on a 2-dimensional compact Riemannian manifold S (or otherwise, for embedded surfaces, we consider smooth maps whose non-tangential component is strongly penalized). As $\varepsilon \to 0$, we expect that these maps generate point singularities, called vortices, carrying a topological degree (or index). In every case, our goal is to characterize the limit of minimizers of these functionals as $\varepsilon \to 0$, or more generally, to prove a Γ-convergence result at second order that captures a "renormalized energy" between the vortex singularities and identifies a "canonical harmonic unit vector field" associated to these vortices.

We classify all harmonic unit vector fields with singularities at prescribed vortex points with prescribed indices (satisfying a certain constraint coming from the topology of S). The subtlety for surfaces of genus $\mathfrak{g} \geq 1$ is that a harmonic unit vector field depends not only on the prescribed vortex points with their topological degrees, but on some *flux integrals* constrained to belong to a particular vorticity-dependent lattice in the $2\mathfrak{g}$-dimensional space of harmonic 1-forms on S. The renormalized energy associated to a configuration of vortices depends on vortex interaction (mediated by the Green's and Robin's functions for the Laplacian on S), a term arising from the Gaussian curvature of S, and the flux integrals. The dependence on vortex position and degree of the flux constraints, and through them the renormalized energy, constitutes a new phenomenon in the theory of Ginzburg-Landau type models.

1.1. Three models.

We will always assume that the potential $F : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that there exists some $C > 0$ with

$$F(1) = 0, \qquad F(s^2) \geq C(1-s)^2, \text{ for all } s > 0. \tag{1}$$

Problem 1: Let (S, g) be a closed (i.e., compact, connected without boundary) oriented 2-dimensional Riemannian manifold of genus $\mathfrak{g}$. Consider (tangent) vector fields [1]

$$u : S \to TS, \quad \text{i.e., } u(x) \in T_xS \text{ for every } x \in S$$

[1] In the sequel, a vector field on S is always tangent at S (the standard definition in differential geometry).

where $TS = \cup_{x \in S} T_x S$ is the tangent bundle of S, and minimize the *intrinsic* energy

$$E_\varepsilon^{in}(u) = \int_S e_\varepsilon^{in}(u) \,\mathrm{vol}_g, \quad e_\varepsilon^{in}(u) := \frac{1}{2}|Du|_g^2 + \frac{1}{4\varepsilon^2} F(|u|_g^2). \tag{2}$$

Here, vol_g is the volume 2-form on S, $|v|_g$ is the length of a vector field v with respect to (w.r.t.) the metric g and

$$|Du|_g^2(x) := |D_{\tau_1} u|_g^2(x) + |D_{\tau_2} u|_g^2(x)$$

where $D_v u$ denotes covariant differentiation (with respect to the Levi-Civita connection) of u (in direction v) and $\{\tau_1, \tau_2\}$ is any orthonormal basis for $T_x S$.

Problem 2: Let (S, g) be a closed oriented 2-dimensional Riemannian manifold *isometrically embedded* in $(\mathbb{R}^3, \bar{g})$. To simplify the notation, we will still denote by g the metric $\bar{g}$ on $\mathbb{R}^3$, which in applications is typically the Euclidean metric. Consider sections m of the tangent bundle TS (i.e., $m(x) \in T_x S$ for a.e. $x \in S$), and minimize the *extrinsic* energy

$$E_\varepsilon^{ex}(m) = \int_S e_\varepsilon^{ex}(m) \,\mathrm{vol}_g, \quad e_\varepsilon^{ex}(m) := \frac{1}{2}|\bar{D}m|_g^2 + \frac{1}{4\varepsilon^2} F(|m|_g^2).$$

That is, $|\cdot|_g$ denotes the length in the metric g on $\mathbb{R}^3$ and

$$|\bar{D}m|_g^2 := |\bar{D}_{\tau_1} \bar{m}|_g^2 + |\bar{D}_{\tau_2} \bar{m}|_g^2,$$

where $\bar{m}$ is an extension of m to a neighborhood of S, $\{\tau_1(x), \tau_2(x)\}$ form a basis for $T_x S$, and $\bar{D}_v$ denotes covariant derivative (with respect to the Levi-Civita connection) in $(\mathbb{R}^3, g)$ in the v direction. As is well known, $|\bar{D}m|_g^2$ is independent of the choice of extension $\bar{m}$. The difference between $|\bar{D}m|_g^2$ in $e_\varepsilon^{ex}(m)$ and $|Dm|_g^2$ in $e_\varepsilon^{in}(m)$ consists in the normal component $|\bar{D}m \cdot N|_g^2$ of the full differential $\bar{D}m$ (the so called shape operator, see (23) and Lemma 10.2 below) where N is the Gauss map at S. Problem 2 is relevant to liquid crystals, as a relaxation of the model proposed in [31, 32] and studied (for the torus) in [39].

Problem 3: Let (S, g) be a closed oriented 2-dimensional Riemannian manifold *isometrically embedded* in $\mathbb{R}^3$ (that is endowed with the Euclidean metric). Consider maps $M : S \to \mathbb{R}^3$ with $|M| = 1$ *a.e.* (standing for the magnetization), and minimize the micromagnetic energy on S:

$$E_\varepsilon^{mm}(M) = \int_S e_\varepsilon^{mm}(M) \,\mathrm{vol}_g, \quad e_\varepsilon^{mm}(M) := \frac{1}{2}|\bar{D}M|^2 + \frac{1}{4\varepsilon^2} F\big(1 - (M \cdot N)^2\big).$$

Here $|\bar{D}M|^2 := |\tau_1 \cdot \bar{D}\bar{M}|^2 + |\tau_2 \cdot \bar{D}\bar{M}|^2$, where $|\cdot|$ denotes the Euclidean length of a vector in $\mathbb{R}^3$, $\bar{D}$ is the differential operator in $\mathbb{R}^3$, $\bar{M}$ is an extension of M to a neighborhood of S and $\{\tau_1(x), \tau_2(x)\}$ form an orthonormal basis for $T_x S$ and $N(x)$ is the Gauss map at S. As usual, $|\bar{D}M|^2$ is independent of the choice of extension $\bar{M}$. Note that if M is decomposed as

$$M = m + (M \cdot N)N,$$

where m is the projection of M on the tangent plane TS, then the energy $E_\varepsilon^{mm}(M)$ can be seen as a nonlinear perturbation of $E_\varepsilon^{ex}(m)$ in terms of the tangent component m with the potential $F(|m|^2)$ since $|m|^2 = 1 - (M \cdot N)^2$ (see Section 11). The above variational problem is a reduced model for thin ferromagnetic films for the potential $F(s^2) = 1 - s^2$ for $s \in [0, 1]$ (satisfying (1) with $C = 1$) (see Section 3.2).

Roughly speaking, Problem 1 can be reduced *locally* in the limit $\varepsilon \to 0$ to a linear problem through the notion of "lifting", i.e., the rotation of a unit vector field with respect to a fixed (locally smooth) unit vector field – see Lemma 10.4 for some details. In particular, a canonical harmonic unit vector field is locally determined by a lifting. Problems 2 and 3 are however fully nonlinear; to determine the renormalized energy in these two extrinsic problems, our strategy is to use canonical harmonic vector fields as Coulomb gauge similarly to the Ginzburg-Landau model with magnetic field. This strategy will enable a splitting of the extrinsic renormalized energy into the intrinsic renormalized energy and a nonlinear scalar variational problem (involving the shape operator) for the optimal lifting with respect to a canonical harmonic vector field.

1.2. **Vortices.** Let (S, g) be a closed oriented 2-dimensional Riemannian manifold of genus $\mathfrak{g}$ (not necessarily embedded in $\mathbb{R}^3$). We will identify vortices of a vector field u with small geodesic balls centered at some points around which u has a (non-zero) index. To be more precise, we introduce the Sobolev space (for $p \geq 1$)

$$\mathcal{X}^{1,p}(S) := \{\text{vector fields } u : S \to TS : \ |u|_g, |Du|_g \in L^p(S)\}.$$

We will also write $\mathcal{X}(S)$ to denote the space of smooth vector fields on S. Given $u \in \mathcal{X}^{1,p}(S) \cap L^q(S)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$, we define the current $j(u)$ as the following 1-form:

$$j(u) = (Du, iu)_g, \tag{3}$$

where $(\cdot, \cdot)_g$ is the scalar product on TS (more generally, the inner product associated to k-forms, $k = 0, 1, 2$) and $i : TS \to TS$ is an isometry of T_xS to itself for every $x \in S$ satisfying

$$i^2 w = -w, \qquad (iw, v)_g = -(w, iv)_g \ = \ \mathrm{vol}_g(w, v). \tag{4}$$

In particular, $j(u)$ is a well-defined 1-form in $L^1(S)$ if $u \in \mathcal{X}^{1,1}(S)$ with $|u|_g = 1$ almost everywhere in S. To introduce the notion of index, we assume that $\mathcal{O}$ is an open subset of S of Lipschitz boundary and $u \in \mathcal{X}^{1,2}(\mathcal{N})$ is a vector field in a neighborhood $\mathcal{N}$ of $\partial\mathcal{O}$ such that $|u|_g \geq \frac{1}{4}$ a.e. in $\mathcal{N}$; then the *index* (or topological degree) of u along $\partial\mathcal{O}$ is defined by

$$\deg(u; \partial\mathcal{O}) := \frac{1}{2\pi}\left(\int_{\partial\mathcal{O}} \frac{j(u)}{|u|_g^2} + \int_{\mathcal{O}} \kappa \ \mathrm{vol}_g\right), \tag{5}$$

where κ is the Gauss curvature on S and the curve $\partial\mathcal{O}$ has the orientation inherited in the usual way from $\mathcal{O}$ as oriented by the volume form, so that Stokes' Theorem holds with the standard sign conventions (see [15] Chapter 6.1). In particular, if u is smooth enough in $\mathcal{O}$ and has unit length on $\partial\mathcal{O}$, then one has

$$\deg(u; \partial\mathcal{O}) = \frac{1}{2\pi}\int_{\mathcal{O}} \omega(u)$$

where $\omega(u)$ is the *vorticity* (as a 2-form) associated to the vector field u:

$$\omega(u) := dj(u) + \kappa \, \mathrm{vol}_g, \tag{6}$$

where $dj(u)$ is the exterior derivative of $j(u)$ (for more details, see Lemma 6.3 below). Sometimes we will identify the index of u at a point $P \in S$ with the index of u along a sufficiently small curve around P. We show in Lemma 5.4 that every smooth vector field $u \in \mathcal{X}(\mathcal{O})$ (or more generally, $u \in \mathcal{X}^{1,2}(\mathcal{O})$) of unit length in $\mathcal{O}$ has $\deg(u; \partial\mathcal{O}) = 0$; moreover, a vortex with non-zero index will carry infinite energy in Problems 1, 2 and 3 as $\varepsilon \to 0$.

1.3. **Aim.** We will prove a Γ-convergence result (at the second order) for the three energy functionals introduced above, as $\varepsilon \to 0$. The genus $\mathfrak{g}$ and the Euler characteristic

$$\chi(S) = 2 - 2\mathfrak{g}$$

of S will play an important role. In particular, at the level of minimizers u_ε of E_ε^{in}, we show that as $\varepsilon \to 0$, u_ε converges weakly in $\mathcal{X}^{1,p}(S)$ for $p < 2$, see Theorem 12.1 (for a subsequence) to a canonical harmonic vector field u^* of unit length that is smooth [2] away from $n = |\chi(S)|$ distinct singular points $a_1, \dots, a_n$, each singular point a_k carrying the same index $d_k = \operatorname{sign} \chi(S)$ for $k = 1, \dots, n$ so that [3]

$$\sum_{k=1}^{n} d_k = \chi(S). \tag{7}$$

Moreover, the vorticity $\omega(u^*)$ detects the singular points $\{a_k\}_{k=1}^n$ of u^*:

$$\omega(u^*) = 2\pi \sum_{k=1}^{n} d_k \delta_{a_k} \qquad \text{in } S, \tag{8}$$

where δ_{a_k} is the Dirac measure (as a 2-form) at a_k. The expansion of the minimal intrinsic energy E_ε^{in} at the second order is given by

$$E_\varepsilon^{in}(u_\varepsilon) = n\pi \log \frac{1}{\varepsilon} + \lim_{r \to 0} \left(\int_{S \setminus \cup_{k=1}^n B_r(a_k)} \frac{1}{2} |Du^*|_g^2 \operatorname{vol}_g + n\pi \log r \right) + n\iota_F + o(1), \text{ as } \varepsilon \to 0,$$

where $\iota_F > 0$ is a constant depending only on the potential F and $B_r(a_k)$ is the geodesic ball centered at a_k of radius r, see again Theorem 12.1. The second term in the above right-hand side (RHS) is called the *renormalized energy* between the vortices $a_1, \dots, a_n$ and governs the optimal location of these singular points; in the Euclidean case, this notion was introduced by Bethuel-Brezis-Hélein in their seminal book [3]. In particular, if S is the unit sphere in $\mathbb{R}^3$ endowed with the standard metric g, then $n = 2$ and a_1 and a_2 are two diametrically opposed points on S. Our results will give an explicit description of this intrinsic renormalized energy using the Green's and Robin's functions for the Laplacian on S. The canonical harmonic vector fields are crucial in the extrinsic Problems 2 and 3, as they play the role of a Coulomb gauge that yields an explicit formula for the extrinsic renormalized energy involving the intrinsic one and a scalar minimization problem, see Section 2.2.

2. Main results

2.1. **Canonical harmonic vector fields of unit length.** Let (S, g) be a closed oriented 2-dimensional Riemannian manifold of genus $\mathfrak{g}$ (not necessarily embedded in $\mathbb{R}^3$). We will say that a canonical [4] harmonic vector field of unit length having distinct singular points

[2] In the case of a surface (S, g) with genus 1 (i.e., homeomorphic with the flat torus), then $n = 0$ and u^* is smooth in S.

[3] In fact, $\deg(u^*; \gamma) = d_k$ for every closed simple curve γ around a_k and lying near a_k.

[4] We use the term "canonical harmonic unit vector field" to emphasize the parallel to the canonical harmonic map introduced in [3], but for genus $\mathfrak{g} > 0$, specifying a canonical harmonic unit vector field requires not only the points a_k and degrees d_k but also some flux integrals, see (12) below. Thus, such a vector field can only claim to be canonical once this data is given. In addition, uniqueness holds only up to a global rotation (see Theorem 2.1).

$a_1, \dots, a_n \in S$ of index $d_1, \dots, d_n \in \mathbb{Z}$ for some $n \geq 1$, is a vector field $u^* \in \mathcal{X}^{1,1}(S)$ such that $|u^*|_g = 1$ in S, (8) holds, i.e.,

$$dj(u^*) = -\kappa \operatorname{vol}_g + 2\pi \sum_{k=1}^{n} d_k \delta_{a_k}$$

and

$$d^* j(u^*) = 0 \qquad \text{in } S. \tag{9}$$

Here, d^* is the adjoint of the exterior derivative d, i.e., $d^* j(u^*)$ is the unique 0-form on S such that

$$\int_S \big(d^* j(u^*), \zeta\big)_g \operatorname{vol}_g = \int_S \big(j(u^*), d\zeta\big)_g \operatorname{vol}_g \quad \text{for every smooth 0-form } \zeta,$$

where $(\cdot, \cdot)_g$ is the inner product associated to k-forms, $k = 0, 1, 2$. If u^* satisfies (8), then (6) combined with Gauss-Bonnet theorem imply that necessarily (7) holds.

Equation (9) asserts that u^* is a harmonic section of the unit tangent bundle; see Remark 2.3 below for details.

We will see that condition (7) is also sufficient. Indeed, if (7) holds, we will construct solutions of (8) and (9), as follows: if $\psi = \psi(a; d)$ is the unique 2-form on S solving

$$-\Delta\psi = -\kappa \operatorname{vol}_g + 2\pi \sum_{k=1}^{n} d_k \delta_{a_k} \qquad \text{in } S, \qquad\qquad \int_S \psi = 0, \tag{10}$$

with the sign convention that $-\Delta = dd^* + d^*d$, then the idea is to find u^* such that $j(u^*) - d^*\psi$ belongs to the space of harmonic 1-forms, i.e.,

$$Harm^1(S) = \{\text{integrable 1-forms } \eta \text{ on } S \;:\; d\eta = d^*\eta = 0 \text{ as distributions}\}. \tag{11}$$

The dimension of the space $Harm^1(S)$ is twice the genus (i.e., $2\mathfrak{g}$) of (S, g) and **we fix an orthonormal basis** $\eta_1, \dots, \eta_{2\mathfrak{g}}$ **of** $Harm^1(S)$ such that

$$\int_S (\eta_k, \eta_l)_g \operatorname{vol}_g = \delta_{kl} \quad \text{for } k, l = 1, \dots, 2\mathfrak{g}.$$

Therefore, our ansatz for $j(u^*)$ may be written

$$j(u^*) = d^*\psi + \sum_{k=1}^{2\mathfrak{g}} \Phi_k \eta_k \qquad \text{in } S \tag{12}$$

for some constant vector $\Phi = (\Phi_1, \dots, \Phi_{2\mathfrak{g}}) \in \mathbb{R}^{2\mathfrak{g}}$. We call these constants *flux integrals* as they can be recovered by

$$\Phi_k = \int_S (j(u^*), \eta_k)_g \operatorname{vol}_g, \qquad \text{for } k = 1, \dots, 2\mathfrak{g}.$$

These flux integrals play an essential role in our analysis. They depend nontrivially on (a, d); this phenomenon is new, as far as we know, in the study of of Ginzburg-Landau models, see Section 4 for more details. Note that (12) combined with (10) automatically yield (8) and (9). One important point is to characterize for which values of Φ the RHS of (12) arises as $j(u^*)$ for some vector field u^* of unit length in S. For that condition, we need to recall the following theorem of Federer-Fleming [17]: there exist $2\mathfrak{g}$ simple closed curves γ_ℓ on S, $\ell = 1, \dots, 2\mathfrak{g}$, such that for any closed Lipschitz curve γ on S, one can find integers $c_1 \dots, c_{2\mathfrak{g}}$ such that

$$\gamma \text{ is homologous to } \sum_{\ell=1}^{2\mathfrak{g}} c_\ell \gamma_\ell$$

i.e., there exists an integrable function $f : S \to \mathbb{Z}$ such that

$$\int_\gamma \zeta - \sum_{\ell=1}^{2\mathfrak{g}} c_\ell \int_{\gamma_\ell} \zeta \ = \ \int_S f\, d\zeta \quad \text{for all smooth 1-forms } \zeta$$

(see more details in Section 5.4). **We fix a choice of such curves** $\{\gamma_\ell\}_{\ell=1}^{2\mathfrak{g}}$. With these chosen curves $\{\gamma_\ell\}_{\ell=1}^{2\mathfrak{g}}$ and the harmonic 1-forms $\{\eta_k\}_{k=1}^{2\mathfrak{g}}$, we denote by

$$\alpha_{\ell k} := \int_{\gamma_\ell} \eta_k, \quad k, \ell = 1, \dots, 2\mathfrak{g}. \tag{13}$$

The matrix $\alpha = (\alpha_{\ell k})_{1 \le k, \ell \le 2\mathfrak{g}}$ is invertible[5] (see Lemma 5.2).

Theorem 2.1. *Let $n \ge 1$ and $d = (d_1, \dots, d_n) \in \mathbb{Z}^n$ satisfy* (7). *Then for every $a = (a_1, \dots, a_n) \in S^n$, there exist equivalence classes (mod 2π)*

$$\zeta_\ell = \zeta_\ell(a; d) \in \mathbb{R}/2\pi\mathbb{Z}, \qquad \ell = 1, \dots, 2\mathfrak{g}$$

such that if a vector field $u^* \in \mathcal{X}^{1,1}(S)$ of unit length solves (8) *and* (9)*, then $j(u^*)$ has the form* (12) *for constants $\Phi_1, \dots, \Phi_{2\mathfrak{g}}$ such that*

$$-\sum_{k=1}^{2\mathfrak{g}} \alpha_{\ell k} \Phi_k \in \zeta_\ell(a; d), \qquad \ell = 1, \dots, 2\mathfrak{g}, \tag{14}$$

where $(\alpha_{\ell k})$ are defined in (13). *Conversely, given any $\Phi_1, \dots, \Phi_{2\mathfrak{g}}$ satisfying* (14)*, there exists a vector field $u^* \in \mathcal{X}^{1,1}(S)$ of unit length solving* (8) *and* (9) *and such that $j(u^*)$ satisfies* (12). *In addition, the following hold:*

1) *$\zeta_\ell(\cdot; d)$ depends continuously on $a \in S^n$ for every $\ell = 1, \dots, 2\mathfrak{g}$. More generally, if*[6]

$$\mu_t := 2\pi \sum_{l=1}^{n_t} d_{l,t} \delta_{a_{l,t}} \to \mu_0 := 2\pi \sum_{l=1}^{n_0} d_{l,0} \delta_{a_{l,0}} \qquad \text{in } W^{-1,1} \qquad \text{as } t \downarrow 0, \tag{15}$$

$\{d_{l,t}\}_l$ are integers with (7) *and $\sum_{l=1}^{n_t} |d_{l,t}|$ is uniformly bounded in t, then $\zeta_\ell(a_t; d_t) \to \zeta_\ell(a_0; d_0)$ as $t \downarrow 0$.*

2) *any u^* solving* (8) *and* (9) *belongs to $\mathcal{X}^{1,p}(S)$ for all $1 \le p < 2$, and is smooth away from $\{a_k\}_{k=1}^n$.*

3) *If $u^*, \tilde{u}^*$ both satisfy* (12) *for the same $(a; d)$ and the same $\{\Phi_k\}_{k=1}^{2\mathfrak{g}}$, then $\tilde{u}^* = e^{i\beta} u^*$ for some $\beta \in \mathbb{R}$ where $e^{i\beta} = \cos\beta + i \sin\beta$ for the isometry i defined in* (4).

Remark 2.2. Throughout this paper, objects that we write as functions of $(a; d)$, such as $\psi(a; d), \zeta_\ell(a; d)$, and so on, in fact depend only on the measure $2\pi \sum_{l=1}^n d_l \delta_{a_l}$. As a result, one can always do the reduction of a set $(a; d)$ of points $a = (a_1, \dots, a_n) \in S^n$ (not necessarily distinct) and integers $d = (d_1, \dots, d_n)$ (that can be zero) satisfying (7) to a

[5] In fact, by changing the choice of curves and the basis in $Harm^1(S)$, the matrix α is multiplied by an invertible matrix (similar to the standard change of coordinates in vector spaces) due to the above definition of homologous curves where $\int_\gamma \eta = \sum_{\ell=1}^{2\mathfrak{g}} c_\ell \int_{\gamma_\ell} \eta$ for every harmonic 1-form η, see also Lemma 5.2.

[6] See Section 5.3 for the definition of $W^{-1,1}$.

set $(\tilde{a};\tilde{d})$ where the points a_k are distinct and $d_k \neq 0$; indeed, one can just put together all the identical a_k, sum their degrees d_k, relabel them and then cancel the a_k with zero degree d_k (of course, (7) is conserved). This is why we can always assume that the points (a_k) are distinct and that every d_k is nonzero.

The equivalence classes $\{\zeta_\ell(a;d)\}_{\ell=1}^{2\mathfrak{g}}$ are determined as follows. For every $\ell = 1,\dots,2\mathfrak{g}$, we let λ_ℓ be some smooth simple closed curve such that λ_ℓ is homologous to γ_ℓ (the curves fixed in (13)) and $\{a_k\}_{k=1}^n$ is disjoint from λ_ℓ; for example, λ_ℓ is either γ_ℓ or, if γ_ℓ intersects some a_k, a small perturbation thereof. We now define $\zeta_\ell(a;d)$ to be the element of $\mathbb{R}/2\pi\mathbb{Z}$ such that [7]

$$\zeta_\ell(a;d) := \int_{\lambda_\ell} (d^*\psi + A) \mod 2\pi, \quad \ell = 1,\dots,2\mathfrak{g}, \tag{16}$$

where $\psi = \psi(a;d)$ is the 2-form given by (10) and A is the connection 1-form associated to any moving frame defined in a neighborhood of λ_ℓ (see Section 5.2). The proof of Theorem 2.1 will show that $\zeta_\ell(a;d)$ is well-defined. In general, $\zeta_\ell(a;d) \neq 0 \mod 2\pi$ for $\ell = 1,\dots,2\mathfrak{g}$ as we will see in Example 6.7 in which it can be explicitly computed. We remark that in this example, S is the flat torus; it would be interesting to compute ζ_ℓ for other manifolds, including examples of genus $\mathfrak{g} \geq 2$, and in particular to determine whether in general ζ_ℓ depends nontrivially on (a,d).

Remark 2.3. For a smooth unit vector field u in an open set, we have in some coordinates

$$d^*j(u) = d^*[\sum_{k=1}^2 (iu, D_k u)_g dx^k] = -\star d \star [\sum_{k=1}^2 (iu, D_k u)_g dx^k] = -\star d[\sum_{k=1}^2 (iu, D_k u)_g \star dx^k]$$
$$= -\star \sum_{k,\ell=1}^2 [(iD_\ell u, D_k u)_g + (iu, D_\ell D_k u)_g] dx^\ell \wedge \star dx^k.$$

By considering exponential normal coordinates at a point $p \in S$ (see Section 9.1), one can check that the above expression reduces to

$$d^*j(u) = (iu, D^*Du)_g \quad \text{where} \quad D^*Du = -\frac{1}{\sqrt{g}} D_\ell(\sqrt{g} g^{\ell k} D_k u)$$

for $g := \det(g_{\ell k})$ and $g^{-1} = (g^{\ell k})$. Thus $d^*j(u) = 0$ if and only if D^*Du is parallel to u. As with standard computations for harmonic maps into spheres, this can be rewritten as $D^*Du = |Du|_g^2 u$. This is exactly the equation for a harmonic section of the unit tangent bundle, that is, the condition that a section be a critical point of the covariant Dirichlet energy in the unit tangent bundle. See [42], especially Proposition 1.1, for more. Note that for less smooth unit vector fields u, the equation $D^*Du = |Du|_g^2 u$ makes sense distributionally for $u \in \mathcal{X}^{1,2}$, whereas $d^*j(u) = 0$ requires only $u \in \mathcal{X}^{1,1}$.

The lattice $\mathcal{L}(a;d)$. Due to Theorem 2.1, we introduce the following set corresponding to n distinct points $a = (a_1,\dots,a_n) \in S^n$ and nonzero integers $d = (d_1,\dots,d_n) \in \mathbb{Z}^n$ satisfying (7):

$$\mathcal{L}(a;d) := \{\Phi = (\Phi_1,\dots,\Phi_{2\mathfrak{g}}) \in \mathbb{R}^{2\mathfrak{g}} : \sum_{k=1}^{2\mathfrak{g}} \alpha_{\ell k}\Phi_k + \zeta_\ell(a;d) \in 2\pi\mathbb{Z},\ \ell = 1,\dots,2\mathfrak{g}\}.$$

[7] By this we mean that $\zeta_\ell(a;d) = \{2\pi n + \int_{\lambda_\ell}(d^*\psi + A) : n \in \mathbb{Z}\}$. We will consistently abuse notation in a similar way.

It is a lattice (up to a translation). Indeed, if $\alpha = (\alpha_{\ell k})_{1\le \ell,k\le 2\mathfrak{g}}$ is the matrix defined in (13) with the inverse α^{-1}, then

$$\Phi \in \mathcal{L}(a;d) \Longleftrightarrow \Phi \in 2\pi\alpha^{-1}\mathbb{Z}^{2\mathfrak{g}} - \alpha^{-1}\zeta, \tag{17}$$

i.e., the lattice is determined by the columns of the matrix α^{-1} and it is shifted by the vector $\alpha^{-1}\zeta$ with $\zeta(a;d) = (\zeta_1,\dots,\zeta_{2\mathfrak{g}})$ being any element of the equivalence class defined by (16). Due to the relation on Φ, the above discussed change of curves $\{\gamma_k\}$ and basis of harmonics $\{\eta_k\}$ would be equivalent to a change of coordinates in the lattice $\mathcal{L}(a;d)$.

The continuity of ζ stated at Theorem 2.1 point 1) can be quantified as follows:

Lemma 2.4. *For every $K \in \mathbb{Z}_+$, there exists $C_K > 0$ such that for every two measures $\mu = 2\pi\sum_{k=1}^n d_k\delta_{a_k}$ and $\tilde\mu = 2\pi\sum_{k=1}^{\tilde n}\tilde d_k\delta_{\tilde a_k}$ with the distinct points $a = (a_k)_{k=1}^n$, $\tilde a = (\tilde a_k)_{k=1}^{\tilde n} \subset S$ and the nonzero integers $d = \{d_k\}_{k=1}^n$ and $\tilde d = \{\tilde d_k\}_{k=1}^{\tilde n}$ satisfying (7) and $\sum_{k=1}^n |d_k|$, $\sum_{\ell=1}^{\tilde n}|\tilde d_k| \le K$, then*

$$\operatorname{dist}_{\mathbb{R}^{2\mathfrak{g}}}\left(\mathcal{L}(a;d),\mathcal{L}(\tilde a;\tilde d)\right) \le C_K\|\mu-\tilde\mu\|_{W^{-1,1}(S)}. \tag{18}$$

Here $\operatorname{dist}_{\mathbb{R}^{2\mathfrak{g}}}(\mathcal{L},\tilde{\mathcal{L}}) = \inf_{\Phi\in\mathcal{L},\tilde\Phi\in\tilde{\mathcal{L}}}|\Phi-\tilde\Phi|$, which coincides with the Hausdorff distance, since $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are both translations of a fixed lattice $2\pi\alpha^{-1}\mathbb{Z}^{2\mathfrak{g}}$.

2.2. **Renormalized energy.**

The intrinsic Dirichlet energy. Let (S,g) be a closed oriented 2-dimensional Riemannian manifold of genus $\mathfrak{g}$ (not necessarily embedded in $\mathbb{R}^3$). For any $n \ge 1$, we consider n **distinct** points $a = (a_1,\dots,a_n) \in S^n$. Let $d = (d_1,\dots,d_n) \in \mathbb{Z}^n$ satisfying (7), $\{\zeta_\ell(a;d)\}_{\ell=1}^{2\mathfrak{g}}$ be given in Theorem 2.1 and $\Phi \in \mathbb{R}^{2\mathfrak{g}}$ be a constant vector inside the lattice $\mathcal{L}(a;d)$ defined in (17). We define the *renormalized energy* between the vortices a of indices d by

$$W(a,d,\Phi) := \lim_{r\to 0}\left(\int_{S\setminus\cup_{k=1}^n B_r(a_k)} \frac{1}{2}|Du^*|_g^2 \operatorname{vol}_g + \pi\log r\sum_{k=1}^n d_k^2\right), \tag{19}$$

where $u^* = u^*(a,d,\Phi)$ is the unique (up to a multiplicative complex number) canonical harmonic vector field given in Theorem 2.1 and $B_r(a_k)$ is the geodesic ball centered at a_k of radius r. (Our arguments will show that the above limit indeed exists, see (50)). As in the Euclidean case (see the pioneering work of Bethuel, Brezis and Hélein [3]), we can compute the renormalized energy by using the Green's function. For that, let $G(x,y)$ be the unique function on $S\times S$ such that

$$-\Delta_x(G(\cdot,y)\operatorname{vol}_g) = \delta_y - \frac{\operatorname{vol}_g}{\operatorname{Vol}_g(S)} \text{ distributionally in } S, \quad \int_S G(x,y)\operatorname{vol}_g(x) = 0,\ \forall y\in S,$$

with $\operatorname{Vol}_g(S) := \int_S \operatorname{vol}_g$. Then G may be represented in the form (see Chapter 4.2[8]) [2]

$$G(x,y) = G_0(x,y) + H(x,y), \qquad \text{with } H \in C^1(S\times S),$$

[8]More precisely, according to [2], page 109, eqn (17), one may define G_0 as above such that $H := G - G_0$ can be represented in the form

$$H(x,y) = \int_S \Delta_z G_0(x,z)G_0(z,y)\operatorname{vol}(z) + \text{ smoother terms},$$

(where here Δ_z denotes the pointwise Laplacian rather than the distributional Laplacian) and in addition $\|\Delta_z G_0\|_{L^\infty(S)} \le C$.

where G_0 is smooth away from the diagonal, with

$$G_0(x,y) = -\frac{1}{2\pi}\log(\mathrm{dist}_S(x,y))$$
$$\text{if the geodesic distance } \mathrm{dist}_S(x,y) < \frac{1}{2}(\text{injectivity radius of } S).$$

The 2-form $\psi = \psi(a;d)$ defined in (10) can be written as:

$$\psi = 2\pi \sum_{k=1}^{n} d_k G(\cdot, a_k)\,\mathrm{vol}_g \; + \psi_0\,\mathrm{vol}_g \qquad \text{in } S, \tag{20}$$

where $\psi_0 \in C^\infty(S)$ has zero average on S and solves

$$-\Delta\psi_0 = -\kappa + \bar{\kappa}, \qquad \text{for } \bar{\kappa} = \frac{1}{\mathrm{Vol}(S)}\int_S \kappa\,\mathrm{vol}_g = \frac{2\pi\chi(S)}{\mathrm{Vol}(S)}. \tag{21}$$

In other words, the 2-form $x \mapsto \psi(x) + d_k \log \mathrm{dist}_S(x, a_k)\,\mathrm{vol}_g$ is C^1 in a neighborhood of a_k for every $1 \le k \le n$. We have the following expression of the renormalized energy:

Proposition 2.5. *Given $n \ge 1$ distinct points $a_1, \ldots, a_n \in S$, integers $d_1, \ldots, d_n$ with* (7) *and $\Phi \in \mathcal{L}(a;d)$, then*

$$\begin{aligned} W(a,d,\Phi) = 4\pi^2 &\sum_{1\le l<k\le n} d_l d_k G(a_l, a_k) + 2\pi \sum_{k=1}^{n}\left[\pi d_k^2 H(a_k,a_k) + d_k\psi_0(a_k)\right] \\ &+ \frac{1}{2}|\Phi|^2 + \int_S \frac{|d\psi_0|_g^2}{2}\,vol_g\,, \end{aligned} \tag{22}$$

where ψ_0 is defined in (21).

Given $(a_k; d_k)$ for $k = 1, \ldots, n$, minimizing the renormalized energy leads one to choose $\Phi_* = \Phi_*(a;d) \in \mathcal{L}(a;d)$ such that $|\Phi_*|^2 = \min_{\Phi\in\mathcal{L}(a;d)} |\Phi|^2$. Then $u^*(a,d,\Phi_*)$ is arguably the *most* canonical harmonic unit vector field associated to the singularities $(a;d)$. Note however that Φ_* may fail to be unique, for some choices of $(a;d)$, as we illustrate in Example 6.7. A renormalized energy that depends only on the singularities is then given by $\min_{\Phi\in\mathcal{L}(a;d)} W(a,d,\Phi) = W(a,d,\Phi_*(a;d))$.

In the case of the unit sphere S in $\mathbb{R}^3$ endowed with the standard metric (in particular, ψ_0 vanishes in S), if $n = 2$ and $d_1 = d_2 = 1$, then the second term in the RHS of (22) is independent of a_k (as $x \mapsto H(x,x)$ is constant, see [40]). Moreover, $\mathfrak{g} = 0$, so $\mathcal{L}(a;d) = \emptyset$ for every $(a;d)$, and W depends only on $(a;d)$. Thus minimizing W is equivalent by minimizing the Green's function $G(a_1,a_2)$ over the set of pairs (a_1,a_2) in $S \times S$, namely, the minimizing pairs are diametrically opposed.

More generally, if $S = \mathbb{S}^2$ is endowed with a non-standard metric g, then Steiner [40] proves that $x \mapsto H(x,x) + \frac{1}{2\pi}\psi_0(x)$ is constant.[9] Therefore, an optimal pair (a_1,a_2) of vortices of degree $d_1 = d_2 = 1$ minimizes the following energy

$$(a_1,a_2) \in S\times S \mapsto 4\pi G(a_1,a_2) + \psi_0(a_1) + \psi_0(a_2).$$

In general this is a complicated expression, but it should be possible to find minima in special cases. For example, if S is an ellipsoid, then we expect the vortices a_1 and a_2 will be placed at the two poles of the major axis as they have maximal Gauss curvature (the maximum principle suggests that this will minimize ψ_0), and they maximize the distance

[9]The function $x \mapsto H(x,x)$ is called the Robin's mass on $\mathbb{S}^2$, see e.g. [40].

$\operatorname{dist}_S(a_1, a_2)$ (so minimize $G(a_1, a_2)$).

The extrinsic Dirichlet energy. In the case of an embedded surface $S \subset \mathbb{R}^3$, when dealing with the extrinsic Dirichlet energy in Problems 2 and 3, a second interaction energy between vortices a of degree d is important next to $W(a, d, \Phi)$. For that, we denote by $\mathcal{S} : TS \to TS$ the shape operator on S, that is,

$$\mathcal{S}(v) = -\bar{D}_v N, \quad \text{for every } v \in TS, \tag{23}$$

where N is the Gauss map on S. To determine the renormalized energy in the two extrinsic problems, our strategy is to use canonical harmonic vector fields as Coulomb gauge similarly to the Ginzburg-Landau model with magnetic field. Let $u^* = u^*(a, d, \Phi)$ be the unique (up to a multiplicative complex number) canonical harmonic vector field given in Theorem 2.1. We consider

$$\tilde{W}(a, d, \Phi) = \min_{\Theta: S \to \mathbb{R}} \frac{1}{2} \int_S |d\Theta|_g^2 + \left|\mathcal{S}(e^{i\Theta} u^*)\right|_g^2 \operatorname{vol}_g. \tag{24}$$

(Existence of a minimizer is standard, as we discuss in more detail later.) We will prove in Theorems 10.1 and 11.1 in Sections 10 and 11 that the renormalized energy associated to the extrinsic energy E_ε^{ex} (as well as the one associated to the energy E_ε^{mm} in Problem 3) is given by

$$W(a, d, \Phi) + \tilde{W}(a, d, \Phi).$$

This splitting highlights the role of the intrinsic problem in understanding the extrinsic one; more precisely, the canonical harmonic vector field which is a "minimizer" of the limit intrinsic problem (as $\varepsilon \to 0$) plays the role of the Coulomb gauge in the extrinsic problem leading to the above explicit formula for the renormalized energy in Problems 2 and 3.

Note that for the unit sphere S in $\mathbb{R}^3$ endowed with the standard metric, the shape operator satisfies $|\mathcal{S}(u)|_g = 1$ for any $x \in S$ and unit vector $u \in T_x S$, so that $\tilde{W}(a, d, \Phi) = 2\pi$ for all (a, d, Φ). Therefore, the total renormalized energy $W + \tilde{W}$ has the same minimizers as W.

2.3. **Γ-convergence.** Given the potential F in Section 1, we compute the intrinsic energy of the radial profile of a vortex of index 1 inside a ball of radius $R > 0$ with respect to the Euclidean structure on $\mathbb{R}^2$:

$$I_F^{in}(R, \varepsilon) := \min \left\{ \int_{B_R(0)} e_\varepsilon(v)\, dy \;:\; v : B_R(0) \to \mathbb{C}, v(y) = \frac{y}{R} \text{ for } |y| = R \right\} \tag{25}$$
$$\text{with} \quad e_\varepsilon(v) := \frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2} F(|v|^2).$$

The above minimum is indeed achieved[10] and $I_F^{in}(R, \varepsilon) = I_F^{in}(\lambda R, \lambda \varepsilon) = I_F^{in}(1, \frac{\varepsilon}{R}) =: I_F^{in}(\frac{\varepsilon}{R})$ for every $\lambda > 0$, and the following limit exists (see [3, Lemma III.1]):

$$\iota_F := \lim_{t \downarrow 0} (I_F^{in}(t) + \pi \log t). \tag{26}$$

The extrinsic energy of the radial profile of a vortex of index 1 in Problem 2 will also correspond to the one above. However, for Problem 3, due to the constraint of unit-length

[10] In fact, the minimizer is unique and symmetric [34, 30]. For other uniqueness results, see [21, 22].

on the magnetization M, the following expression comes out:

$$I_F^{mm}(R,\varepsilon) := \min\left\{\int_{B_R(0)} \tilde{e}_\varepsilon(v)\,dy \,:\, v: B_R(0)\to \mathbb{S}^2, v(y) = \frac{1}{R}(y,0) \text{ for } |y| = R\right\} \tag{27}$$

$$\text{with} \quad \tilde{e}_\varepsilon(v) := \frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2}F(1-v_3^2) \quad \text{where } v = (v_1, v_2, v_3).$$

Again, the above minimum is indeed achieved for every fixed $R, \varepsilon > 0$ and writing $I_F^{mm}(R,\varepsilon) =: I_F^{mm}(\frac{\varepsilon}{R})$, we obtain the following quantity (see (114))

$$\tilde{\iota}_F := \lim_{t\downarrow 0}(I_F^{mm}(t) + \pi \log t). \tag{28}$$

We state our main result for Problem 1 in a closed oriented 2-dimensional Riemannian manifold of genus $\mathfrak{g}$:

Theorem 2.6. *The following* Γ*-convergence result holds.*

1) *(Compactness) Let $(u_\varepsilon)_{\varepsilon\downarrow 0}$ be a family of vector fields in $\mathcal{X}^{1,2}(S)$ satisfying $E_\varepsilon^{in}(u_\varepsilon) \leq T\pi|\log\varepsilon| + C$ for some integer $T \geq 0$ and a constant $C > 0$. We denote by*
$$\Phi(u_\varepsilon) := \left(\int_S (j(u_\varepsilon), \eta_1)_g\, vol_g, \ldots, \int_S (j(u_\varepsilon), \eta_{2\mathfrak{g}})_g\, vol_g\right) \in \mathbb{R}^{2\mathfrak{g}},$$
where $\{\eta_k\}_{k=1}^{2\mathfrak{g}}$ are fixed in (12). Then there exists a sequence $\varepsilon \downarrow 0$ such that [11]
$$\omega(u_\varepsilon) \longrightarrow 2\pi \sum_{k=1}^n d_k \delta_{a_k} \quad \text{in } W^{-1,1}, \quad \text{as } \varepsilon \to 0, \tag{29}$$
where $\{a_k\}_{k=1}^n$ are distinct points in S and $\{d_k\}_{k=1}^n$ are nonzero integers satisfying (7) and $\sum_{k=1}^n |d_k| \leq T$. Moreover, if $\sum_{k=1}^n |d_k| = T$, then $n = T$ and $|d_k| = 1$ for every $k = 1, \ldots, n$; in this case, for a further subsequence, there exists $\Phi \in \mathcal{L}(a;d)$ such that $\Phi(u_\varepsilon) \to \Phi$.

2) *(Γ-liminf inequality) Assume that the vector fields $u_\varepsilon \in \mathcal{X}^{1,2}(S)$ satisfy (29) for n distinct points $\{a_k\}_{k=1}^n \in S^n$ and $|d_k| = 1$, $k = 1, \ldots n$ that satisfy (7) and $\Phi \in \mathcal{L}(a;d)$. Then*
$$\liminf_{\varepsilon\to 0}\left[E_\varepsilon^{in}(u_\varepsilon) - n\pi|\log\varepsilon|\right] \;\geq\; W(a,d,\Phi) + n\iota_F.$$

3) *(Γ-limsup inequality) For every n distinct points $a_1, \ldots, a_n \in S$ and $d_1, \ldots, d_n \in \{\pm 1\}$ satisfying (7) and every $\Phi \in \mathcal{L}(a;d)$ there exists a sequence of vector fields u_ε on S such that $|u_\varepsilon|_g \leq 1$ in S, (29) holds and*
$$E_\varepsilon^{in}(u_\varepsilon) - n\pi|\log\varepsilon| \longrightarrow W(a,d,\Phi) + n\iota_F \quad \text{as } \varepsilon \to 0.$$

In fact, in the case $|d_k| = 1$, we will prove a sharper lower bound than the one stated in point (2) above, see Proposition 9.1 below. In the general case of arbitrary degrees $d_k \in \mathbb{Z}\setminus\{0\}$ satisfying (7), we only prove a lower bound at the first order, implicit in the fact that $\sum_{k=1}^n |d_k| \leq T$; see also Corollary 8.3.

If $T = 0$, the theorem implies that $n = 0$. In this case, then, there are no limiting vortices, so necessarily $\mathfrak{g} = 1$ (i.e., S is diffeomorphic to the 1-torus). Also, $\mathcal{L}(a,d)$ is a fixed lattice $\mathcal{L}$. See also Remark 12.2 point 2) below. By (22), the renormalized energy in this case is exactly $\frac{1}{2}|\Phi|^2 + \frac{1}{2}\int_S |d\psi_0|^2 \text{ vol}_g$. It is not clear whether $\Phi = 0$ belongs to the lattice $\mathcal{L}$ if the torus is not flat.

[11] In fact, we prove in Proposition 8.1 below that (29) holds in $W^{-1,p}$ for every $p \in [1,2)$.

The situation in points 2) and 3) above (i.e., all vortices have degree ± 1) is typical when the vector fields u_ε are minimizers of E_ε^{in} (or energetically close to minimizing configurations). For more details, see Theorem 12.1.

For Problem 2 where the surface S is isometrically embedded in $\mathbb{R}^3$, one has the similar result by replacing the interaction energy between vortices with:

$$W(a,d,\Phi) + \tilde{W}(a,d,\Phi)$$

see Theorem 10.1. While for Problem 3, the difference with respect to the result of Problem 2 consists in replacing ι_F by $\tilde{\iota}_F$ (see Theorem 11.1); so, up to this constant, there is no change of the vortex location when minimizing the interaction energy in Problem 3 w.r.t. Problem 2.

This theorem is the generalization of the Γ-convergence result for E_ε^{in} in the Euclidean case (see [12, 25, 38, 1]) and it is based on topological methods for energy concentration (vortex ball construction, vorticity estimates etc.) as introduced in [24, 37]. A part of our results were announced in [20].

Outline of the article. In Section 3, we give a motivation for our models coming from micromagnetics and geometry, while in Section 4, we present some challenges and novelties of our results with respect to other Ginzburg-Landau type models. Before giving the proofs of our results, we present in Section 5 some notation and background on differential forms, Sobolev spaces on manifolds and some useful computations involving the current. In Section 6, we prove the characterization of canonical harmonic vector fields in Theorem 2.1 as well as the stability estimate for the lattice $\mathcal{L}(a;d)$ in Lemma 2.4; we also give Example 6.7 for the non-triviality of the lattice $\mathcal{L}(a;d)$ in the case of the flat torus $\mathbb{R}^2/\mathbb{Z}^2$. In Section 7 we prove the formula of the renormalized energy in Proposition 2.5. In Section 8, we prove the compactness result for the vorticity measure in Theorem 2.6 point 1); as a consequence, we deduce the Γ-limit at the first order of the intrinsic energy E_ε^{in}. The lower / upper bound in Theorem 2.6 are proved in Section 9; in particular, we show an improved lower bound of the intrinsic energy E_ε^{in} in Proposition 9.1. In Sections 10 and 11, we prove the Γ-convergence result at the second order for the extrinsic energy E_ε^{ex} and micromagnetic energy E_ε^{mm} (see Theorems 10.1 and 11.1). Finally, in Section 12, we characterize the asymptotic behavior of minimizers of our three energy functionals. In Appendix, we give the so-called "ball construction" adapted to a surface S which is a key tool in proving the lower bound of our functionals.

3. Motivation

3.1. **Geometry and topology.** One of the first theorems one encounters in topology states that there does not exist any continuous nonvanishing vector field on any closed oriented surface S of genus $\mathfrak{g} \neq 1$. A unit vector field on such a surface must therefore have singularities. If the surface has a Riemannian metric, one might hope to use the metric structure to seek an energetically optimal unit vector field, which presumably should have an energetically optimal placement of singularities. This line of thought leads to the problem of minimizing the covariant Dirichlet energy

$$\int_S \frac{1}{2}|Du|_g^2 \,\mathrm{vol}_g \tag{30}$$

among all unit vector fields on S. However, it follows from results in [39] (an extension to the Sobolev space $W^{1,2}$ of the "Hairy Ball Theorem", see also related results in [9]) that when $\mathfrak{g} \neq 1$, there does not exist any unit vector field on S of finite energy. It is then reasonable (by analogy with standard considerations in the analysis of the Ginzburg-Landau functional) to seek energetically optimal vector fields by relaxing the constraint $|u|_g = 1$ and replacing it with a term that penalizes deviations of u from unit length, then considering a suitable limit. This leads to Problem 1, or to Problem 2 if one is interested in the extrinsic Dirichlet energy on an embedded surface. One may thus interpret our results about these problems as describing an optimal placement of singularities, as sought above.

In the case of genus 1, a number of results about minimization of the extrinsic Dirichlet energy, in the space of unit tangent vector fields, are proved in [39], motivated by models of liquid crystals [31, 32].

3.2. **Micromagnetics.** One of the motivation of our study comes from micromagnetics. Micromagnetics is a variational principle describing the behavior of small ferromagnetic bodies considered here of cylindrical shape $\Omega = \Omega' \times (0, t)$ where Ω' is the cross section of the sample of diameter ℓ and t is the thickness of the cylinder (see Figure 1). A

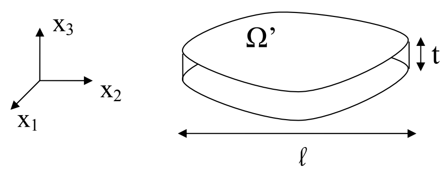

FIGURE 1. A ferromagnetic sample.

ferromagnetic material is described by a $\mathbb{S}^2$-valued map

$$m : \Omega \to \mathbb{S}^2,$$

called magnetization, corresponding to the stable states of the energy functional (written here in the absence of anisotropy and external magnetic field):

$$E^{3D}(m) = \eta^2 \int_\Omega |\nabla m|^2 \, dx + \int_{\mathbb{R}^3} |\nabla U|^2 \, dx. \tag{31}$$

The first term, called exchange energy, penalizes the variations of m according to the material constant $\eta > 0$ (the exchange length) that is of the order of nanometers. The second term of E^{3D} is the stray field energy that favors flux closure; more precisely, the stray field potential $U : \mathbb{R}^3 \to \mathbb{R}$ is determined by the static Maxwell equation

$$\Delta U = \nabla \cdot \big(m \mathbf{1}_\Omega\big) \quad \text{in} \quad \mathbb{R}^3, \tag{32}$$

$$\text{i.e.,} \quad \int_{\mathbb{R}^3} \nabla U \cdot \nabla \zeta \, dx = \int_\Omega m \cdot \nabla \zeta \, dx, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^3).$$

In other words, the stray field ∇U is the Helmholtz projection of $m\mathbf{1}_\Omega$ onto the L^2-gradient fields and

$$\int_{\mathbb{R}^3} |\nabla U|^2 \, dx = \|\nabla \cdot \big(m \mathbf{1}_\Omega\big)\|^2_{\dot{W}^{-1,2}(\mathbb{R}^3)}.$$

Thin film regime of very small ferromagnets. Assume the following asymptotic regime[12]:

$$h := \frac{t}{\ell} \to 0 \quad \text{and} \quad \varepsilon := \frac{\eta}{\ell} = \text{constant}$$

[12] A thin film regime is characterized by a small aspect ratio h; the ferromagnetic samples considered here are very small because ℓ has the order of nanometers as η.

for some fixed parameter $\varepsilon > 0$. Set $x = (x', x_3)$, $x' = (x_1, x_2)$ where $'$ stands only in this section for an in-plane quantity. In order to study the asymptotic behavior as $h \to 0$, we rescale the variables: $y' := x'/\ell$ (so, $\omega' := \Omega'/\ell$ is of diameter 1), $y_3 := x_3/t$, $m_h(y) := m(x)$ and

$$E_h(m_h) := \frac{1}{\eta^2 t} E^{3D}(m), \quad m_h : \omega = \omega' \times (0,1) \to \mathbb{S}^2,$$

where the diameter of ω' equals 1. In this context, Gioia-James [18] proved the following Γ-convergence result in strong L^2-topology:

$$E_h \xrightarrow{\Gamma} E_0$$

where the Γ-limit functional E_0 is given by

$$E_0(M) = \int_\omega \left\{ |\nabla M|^2 + \frac{1}{\varepsilon^2} M_3^2 \right\} dy = \int_{\omega'} \left\{ |\nabla' M|^2 + \frac{1}{\varepsilon^2} M_3^2 \right\} dy'$$

for a limit magnetization $M = (M', M_3) : \omega \to \mathbb{S}^2$ that is invariant in y_3-direction, i.e., $\partial_{y_3} M = 0$ in ω, $\nabla' = (\partial_1, \partial_2)$, so that one can write

$$M = M(y') \in W^{1,2}(\omega', \mathbb{S}^2), \quad y' = (y_1, y_2) \in \omega'.$$

The hint is the following: since the exchange energy term in $E_h(m_h)$ of m_h is given by

$$m_h \mapsto \int_\omega \left(|\nabla' m_h|^2 + \frac{1}{h^2} |\partial_{y_3} m_h|^2 \right) dy,$$

it is clear that configurations m_h of uniformly bounded energy (i.e., $E_h(m_h) \leq C$) tend to converge strongly in L^2 to a limit M depending only on y'-variables. The more delicate issue consists in understanding the scaling of the stray field energy term. For that, we assume for simplicity that m_h is invariant in y_3-direction (i.e., $m(x') = m_h(x'/\ell)$ for $x' \in \Omega'$). Then the Maxwell equation (32) turns into:

$$\Delta U = \nabla' \cdot m' \, \mathcal{H}^3 \llcorner \Omega - m \cdot \nu \, \mathcal{H}^2 \llcorner \partial\Omega \quad \text{in} \quad \mathbb{R}^3,$$

where ν is the unit outer normal vector on $\partial\Omega$ and $\mathcal{H}^k$ is the Hausdorff measure of dimension k. This equation is a transmission problem that can be solved explicitly using the Fourier transform $\mathcal{F}(\cdot)$ in the in-plane variables x' and the computation yields (see e.g. [19]):

$$\int_{\mathbb{R}^3} |\nabla U|^2 \, dx = t \int_{\mathbb{R}^2} \tilde{f}\left(\frac{t}{2}|\xi'|\right) \left|\mathcal{F}(m_3 \mathbf{1}_{\Omega'})\right|^2 d\xi' + t \int_{\mathbb{R}^2} f\left(\frac{t}{2}|\xi'|\right) \left|\frac{\xi'}{|\xi'|} \cdot \mathcal{F}(m' \mathbf{1}_{\Omega'})\right|^2 d\xi',$$

where

$$\tilde{f}(s) = \frac{1 - e^{-2s}}{2s} \quad \text{and} \quad f(s) = 1 - \tilde{f}(s) \text{ if } s \geq 0.$$

To conclude, one formally approximates $\tilde{f}(s) \approx 1$ and $f(s) \approx s$ if $s = o(1)$ so that[13]

$$\frac{1}{\eta^2 t} \int_{\mathbb{R}^3} |\nabla U|^2 \, dx \approx \frac{1}{\eta^2} \int_{\Omega'} m_3^2 \, dx' + O\left(\frac{h}{\varepsilon^2}\right) \approx \frac{1}{\varepsilon^2} \int_{\omega'} m_{h,3}^2 \, dy',$$

as $h \to 0$.

Very small magnetic shells. The situation of curved ferromagnetic samples was considered by Carbou [10]. The context is the following: let $S \subset \mathbb{R}^3$ be a surface isometrically

[13] A different regime is studied in [23].

embedded in $\mathbb{R}^3$ of diameter $\ell = 1$ and N be the Gauss map at S. A curved magnetic shell is considered occupying the domain

$$\Omega := \left\{ x' + sN(x') \,:\, s \in (0,t),\ x' \in S \right\}.$$

Then Carbou [10] proved the corresponding Γ-convergence result as in Gioia-James [18] where the Γ-limit is given by

$$M \in H^1(S;\mathbb{S}^2) \mapsto \int_S |\bar{D}M|^2 + \frac{1}{\varepsilon^2}(M\cdot N)^2 \, d\mathcal{H}^2$$

where $\bar{D}M$ is the extrinsic differential of M and $M \cdot N$ is the normal component of M on the surface S. In the context of energy E_ε^{mm}, denoting $|m|^2 = 1 - (M\cdot N)^2$ we have $F(|m|^2) = 1 - |m|^2 = (M\cdot N)^2 \geq (1-|m|)^2$, so (1) is satisfied for $C = 1$.

3.3. **Ginzburg-Landau model with magnetic field on a 2-dimensional manifold.**

Similar to the above asymptotic analysis, a Γ-convergence study was also carried out for a Ginzburg-Landau model with magnetic field on thin shells, see [14, 13]. The Γ-limit energy functional arising in the limit as the shell thickness tends to 0 is given for a complex-valued function ϕ defined on a 2-dimensional manifold S as follows:

$$\phi \in H^1(S;\mathbb{C}) \mapsto \int_S |d\phi - i\phi\mathcal{A}|_g^2 + \frac{1}{\varepsilon^2}(1-|\phi|^2)^2 \ \mathrm{vol}_g, \tag{33}$$

where $\mathcal{A}$ is a 1-form on S obtained as the tangential restriction of a 1-form that generates the applied magnetic field in the ambient space. Among other results, [14, 13] investigate changes in the behaviour of minimizers of (33) as $\mathcal{A}$ varies (for simply-connected surfaces S, often assumed to be surfaces of revolution), including the critical magnetic field at which vortices become energetically favorable. In our setting, if one chooses a canonical harmonic unit vector field u^* (for some vortex points a with degrees d satisfying (7) and flux integrals $\Phi \in \mathcal{L}(a,d)$), then the connection 1-form $A = -j(u^*)$ associated to the frame $\{u^*, iu^*\}$ (see Section 5.2 below) plays the role of a (tangent) magnetic field carried by $\mathcal{A}$ and satisfies $d^*A = -d^*j(u^*) = 0$ (by (9)). In other words, the canonical harmonic vector field plays the role of the Coulomb gauge. Indeed, associating a vector field $u := \phi u^*$ to any $\phi \in H^1(S;\mathbb{C})$, through the formula (35) below, it yields $|Du|_g = |d\phi - i\phi\mathcal{A}|_g$ and $|u|_g = |\phi|$ which gives a formal link [14] between (33) with our (intrinsic) Problem 1.

4. Challenges

One first main result, Theorem 2.1, contains a classification of all harmonic unit vector fields in $\mathcal{X}^{1,1}(S)$ with singularities at prescribed points. This classification is surprisingly subtle on manifolds of genus ≥ 1. Indeed, Theorem 2.1 shows that if a harmonic vector field u^* with singularities of degree d_k at points $a_k \in S$ for $k = 1,\dots,n$ exists, then the harmonic part of the associated current $j(u^*)$ — that is, the projection of $j(u^*)$ onto the space of harmonic 1-forms — belongs to a particular lattice $\mathcal{L}(a;d)$ in the $2\mathfrak{g}$-dimensional space of harmonic 1-forms, and conversely, every element of $\mathcal{L}(a;d)$ can be realized as $j(u^*)$ for some canonical harmonic unit vector field u^*. (The degrees must also satisfy the natural topological constraint $\sum_{k=1}^n d_k = \chi(S)$; this is clear and unsurprising.) Every $\mathcal{L}(a;d)$ is a translation of a lattice that depends only on geometry of S, but we show that

[14] However, $A = -j(u^*) \notin L^2$ whenever u^* has singular points (in particular, whenever $\mathfrak{g} \neq 1$), and then the space of finite-energy ϕ contains functions not in H^1, making it inconvenient to work with the representation (33).

the translation may depend nontrivially on $(a;d)$ in a concrete example (see Example 6.7), and we believe this to be the case in general. Although flux quantization constraints appear in many Ginzburg-Landau models on non-simply connected Euclidean domains (see for example [28, 36]), the dependence (encoded in $\mathcal{L}(a;d)$) of the constraints on the vortex locations and the geometry of S seems to be a new phenomenon.

The lattice $\mathcal{L}(a;d)$ reappears and gives rise to novel issues in the proof of our main results. There we must control energy coming from the harmonic part of the current $j(u_\varepsilon)$ for a sequence u_ε of vector fields; this requires a detailed understanding of the way in which the distribution of vorticity in (approximately) unit vector fields imposes vorticity-dependent (approximate) constraints on the harmonic part of the associated currents.

These points do not appear in earlier work on related problems. Besides the papers [14, 13] mentioned above[15], this includes papers of Orlandi [33] and Qing [35] that describe the asymptotic behaviour of minimizers of a Ginzburg-Landau energy for a section of a complex line bundle over a Riemannian manifold[16]. This minimization problem involves finding not only an optimal unit-length section u of the bundle (corresponding in our setting to a tangent unit vector field), but also an optimal connection on the bundle. By contrast, we insist on working with the Levi-Civita connection, natural in our setting. A consequence of the freedom to choose an optimal connection is that the vorticity-dependent constraints described by the lattice $\mathcal{L}(a,d)$ do not arise in [33, 35], either in the description of optimal maps or the characterization of energy asymptotics.

A distinct and important technical issue arises from the need to isolate the energetic contribution of the vortex cores, reflected in the constants ι_F and $\tilde{\iota}_F$ arising in Theorems 2.6, 10.1, and 11.1. As usual, these terms are captured by sharp energy estimates carried out near the vortex cores. The new feature is that, in order to approximate the metric g well by the Euclidean metric – this is necessary to correctly resolve ι_F and $\tilde{\iota}_F$ – we must carry out these estimates on geodesic balls that contain the vortices and *whose radii vanish as ε tends to* 0. This requirement forces us to rely on refined quantitative control of the vorticity throughout our analysis.

Very closely related is the recent work of Canevari and Segatti [8], characterizing the asymptotics of a spatially-discretized covariant Dirichlet energy (30) on a surface, in the limit as the discretization scale tends to zero. These authors prove results quite parallel to ours, but their main focus is on the discrete-to-continuum limit, and the renormalized energy that they find (see [8], equations (18), (20)) is described in a way that leaves its dependence on $(a;d)$ very implicit and does not resolve the issues appearing in our Theorem 2.1 and elsewhere in this paper.

5. Notation and background

Let (S,g) be a closed oriented 2-dimensional Riemannian manifold of genus $\mathfrak{g}$, not necessarily embedded in $\mathbb{R}^3$. We will write $\chi(S)$ and $\mathfrak{g}$ to denote the Euler characteristic and the genus of S that are related by $\chi(S)=2-2\mathfrak{g}$. We write D to denote the Levi-Civita connection on (S,g). We will write $\mathrm{dist}_S(p,q)$ to denote the geodesic distance between $p\in S$ and $q\in S$:

$$\mathrm{dist}_S(p,q) := \inf\left\{\int_0^1 |\gamma'(s)|_g ds \ : \ \gamma:[0,1]\to S \ \text{Lipschitz}, \quad \gamma(0)=p,\gamma(1)=q\right\}.$$

[15] The papers [14, 13] study only the case of genus $\mathfrak{g}=0$ where the flux quantization related with the lattice $\mathcal{L}$ does not occur.

[16] For a non-variational approach to existence results in this setting, see for example [11].

We will write $B_r(x)$ (or $B(x,r)$) to denote the open geodesic ball

$$B_r(p) := \{q \in S : \mathrm{dist}_S(p,q) < r\}.$$

and $\bar{B}_r(x)$ is the closure of this ball. Given points $a_1, \ldots, a_n \in S$ and $\sigma > 0$, we also write $a = (a_1, \ldots, a_n)$ and

$$S_\sigma(a) := S \setminus \cup_{k=1}^n \bar{B}_\sigma(a_k)$$

and

$$\rho_a := \min_{k \neq \ell} \mathrm{dist}_S(a_k, a_\ell).$$

We will also write simply S_σ, when it is clear which points $(a_1, \ldots, a_n)$ we have in mind. We write $\mathbf{1}_{S_\sigma}$ for the characteristic function of S_σ.

5.1. **Differential forms.** If η, ζ are k-forms, $k = 0, 1, 2$, we will write $(\eta, \zeta)_g$ to denote the inner product induced by the metric g, and the length $|\eta|_g := (\eta, \eta)_g^{1/2}$. We will always fix a global volume 2-form, denoted vol_g, associated to the metric for which we define the isometry

$$i : TS \to TS$$

by (4). The Hodge-star operator, mapping k-forms to $2-k$ forms, is defined by requiring that

$$\eta \wedge \star\zeta = (\eta, \zeta)_g \,\mathrm{vol}_g \qquad \text{for all } k\text{-forms } \eta, \zeta.$$

It is well-known, and straightforward to check, that $\star\star = (-1)^{k(2-k)}$ for a two-dimensional surface S. Also, for dimension 2, we define the adjoint of the exterior derivative d by $d^* := - \star d \star$ on S. Then it follows that

$$\int_S (d\eta, \zeta)_g \,\mathrm{vol}_g = \int_S (\eta, d^*\zeta)_g \,\mathrm{vol}_g \qquad \text{for a } k\text{-form } \zeta \text{ and a } k-1\text{-form } \eta,\ k = 1, 2.$$

If we instead integrate over a subset of S of the form $S \setminus \mathcal{O}$, then this identity becomes

$$\int_{S\setminus\mathcal{O}} (d\eta, \zeta)_g \,\mathrm{vol}_g - \int_{S\setminus\mathcal{O}} (\eta, d^*\zeta)_g \,\mathrm{vol}_g = -\int_{\partial\mathcal{O}} \eta \wedge \star\zeta \tag{34}$$

where we consider $\partial\mathcal{O}$ to have the orientation inherited from $\mathcal{O}$ (rather than $S \setminus \mathcal{O}$, hence the minus sign on the right-hand side). If η is a 0-form then we will omit the wedge on the right-hand side of (34).

For $p \in S$, we will write δ_p to denote the (measure-valued) 2-form such that

$$\int_S f\delta_p = f(p) \qquad \text{for every continuous } f : S \to \mathbb{R}$$

If A is a 1-form on S and $v \in T_xS$, then $A(v)$ denotes the number obtained via the action of the 1-covector $A_x \in T_x^*S$ on $v \in T_xS$. If v is a vector field, then $A(v)$ denotes the function whose value at x is $A(v(x))$.

5.2. **The connection 1-form.** A *moving frame* on an open subset $\mathcal{O} \subset S$ will mean a pair of smooth, properly oriented, orthonormal vector fields $\tau_k \in \mathcal{X}(\mathcal{O})$ for $k = 1, 2$, i.e.,

$$(\tau_k, \tau_\ell)_g = \delta_{k\ell} \qquad\qquad \mathrm{vol}_g(\tau_1, \tau_2) = 1$$

everywhere in $\mathcal{O}$. Note that if τ is any smooth unit vector field on $\mathcal{O}$, then $\{\tau_1, \tau_2\} = \{\tau, i\tau\}$ provides a moving frame, and if $\{\tau_1, \tau_2\}$ is any moving frame, then $\tau_2 = i\tau_1$. In general a moving frame exists only locally on S.

On an open subset $\mathcal{O} \subset S$, we will define the *connection 1-form* A associated to a moving frame $\{\tau_1, \tau_2\}$ by

$$A(v) = (D_v\tau_2, \tau_1)_g = -(D_v\tau_1, \tau_2)_g, \qquad v \in \mathcal{X}(\mathcal{O}).$$

Since $0 = v(\tau_k, \tau_k)_g = 2(D_v\tau_k, \tau_k)_g$ for $k = 1, 2$, it follows that $D_v\tau_1 = -A(v)\tau_2$ and $D_v\tau_2 = A(v)\tau_1$. Note that if $\{\tau_1, \tau_2\}$ is a moving frame on $\mathcal{O} \subset S$, then $A = -j(\tau_1)$ on $\mathcal{O}$ where $j(\tau_1)$ is the 1-form defined in (3). In complex notation, this fact and the Leibniz rule imply that for any smooth complex-valued function ϕ on $\mathcal{O}$,

$$D_v(\phi\tau_1) = (d\phi(v) - iA(v)\phi)\tau_1. \tag{35}$$

The definition of A is clearly independent of any coordinate system on $\mathcal{O}$ (since our definition does not refer to any coordinates) but depends on the choice of a frame. However, it is a standard fact that dA is independent of the frame. In particular, we have the identity

$$dA = \kappa \ \mathrm{vol}_g \tag{36}$$

where κ is the Gaussian curvature of S. (See do Carmo [15], Proposition 2 on page 92; our 1-form A is written as $-\omega_{12}$ in do Carmo's notation, see [15] p. 94.) In fact, this may be taken as the definition of Gaussian curvature. We recall several attributes of the Gaussian curvature. First, the Gauss-Bonnet Theorem states that

$$\int_S \kappa \,\mathrm{vol}_g = 2\pi\chi(S)$$

where $\chi(S)$ is the Euler characteristic. (For a proof, with the definition of the Euler characteristic, consult for example [15], section 6.1.) Another classical fact that we will use is the *Bertrand-Diguet-Puiseux Theorem*, which says that

$$\kappa(P) = \lim_{r\searrow 0} 3\frac{2\pi r - \mathcal{H}^1(\partial B_r(P))}{\pi r^3} = \lim_{r\searrow 0} 12\frac{\pi r^2 - \mathrm{Vol}_g(B_r(P))}{\pi r^4}.$$

5.3. **Sobolev spaces.** For $q \in [1, \infty]$, we define $L^q(S;\mathbb{R})$ the space of q-integrable functions w.r.t. the volume form vol_g and the Sobolev spaces

$$W^{1,q}(S;\mathbb{R}) = \{f \in L^q(S;\mathbb{R}) \, : \, \|f\|_{W^{1,q}} := \max\{\|f\|_{L^q}, \|df\|_{L^q}\} < \infty\}.$$

If μ is a 2-form (possibly measure-valued) then we write for $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ that $W^{-1,p}$ is the dual of the Sobolev space $W^{1,q}$, i.e.,

$$\|\mu\|_{W^{-1,p}} := \sup\left\{\int_S f\mu \ : \ f \in W^{1,q}(S;\mathbb{R}),\, \|f\|_{W^{1,q}} \le 1\right\}.$$

We also recall the Hodge decomposition. The following version will suffice for us: if ζ is any square-integrable 1-form on S, then there exist a 0-form ξ, 2-form β, and $\eta \in Harm^1(S)$ (see (11)) such that

$$\zeta = d\xi + d^*\beta + \eta. \tag{37}$$

Moreover, this decomposition is unique. By integrating by parts one easily sees that for any 0-form ξ, 2-form β, and $\eta \in Harm^1(S)$, one has

$$\int_S (d\xi, d^*\beta)_g \,\mathrm{vol}_g \ = \ \int_S (d\xi, \eta)_g \,\mathrm{vol}_g \ = \ \int_S (d^*\beta, \eta)_g \,\mathrm{vol}_g \ = \ 0.$$

and it follows that if (37) holds, then

$$\|d\xi\|_{L^2}^2 + \|d^*\beta\|_{L^2}^2 + \|\eta\|_{L^2}^2 = \|\zeta\|_{L^2}^2.$$

We have the following density result in $\mathcal{X}^{1,p}(S)$ (which is standard, see e.g. [29, Section 6]):

Lemma 5.1. *For any open set $\mathcal{O} \subset S$, if $u \in \mathcal{X}^{1,p}(\mathcal{O})$ for $p \geq 1$, then for any open set $\mathcal{O}' \subset\subset \mathcal{O}$ compactly supported in $\mathcal{O}$, there exists a family of smooth vector fields $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)} \subset \mathcal{X}(\mathcal{O}')$ that converges to u in $\mathcal{X}^{1,p}(\mathcal{O}')$. If $|u|_g \leq 1$ in $\mathcal{O}$, then one can arrange that $|u_\varepsilon|_g \leq 1$ in $\mathcal{O}'$ for $\varepsilon < \varepsilon_0$. Moreover, if $p \geq 2$ and $|u|_g = 1$ in $\mathcal{O}$, then one can arrange that $|u_\varepsilon|_g = 1$ everywhere in $\mathcal{O}'$.*

Proof. Let $u \in \mathcal{X}^{1,p}(\mathcal{O})$. One considers a standard radial mollifier $\rho \in C^\infty(\mathbb{R}^2)$ such that $0 \leq \rho \leq 1$, ρ has support in the unit ball and $\int_{\mathbb{R}^2} \rho(z)\,dz = 1$. For $x \in S$, we consider the exponential map $\exp_x : T_xS \to S$ and for $\varepsilon \in (0, \varepsilon_0)$ (with ε_0 be the injectivity radius of S), let

$$\rho_{\varepsilon,x}(y) = \frac{1}{\varepsilon^2}\rho\left(\frac{\exp_x^{-1}(y)}{\varepsilon}\right) \quad \text{in a neighborhood of } x$$

where we identified T_xS with $\mathbb{R}^2$; we also consider the renormalized mollifiers

$$\tilde{\rho}_{\varepsilon,x}(y) = \frac{\rho_{\varepsilon,x}(y)}{\int_S \rho_{\varepsilon,x}\ \mathrm{vol}_g}.$$

Now for $x \in \mathcal{O}$ such that $\mathrm{dist}_S(x, \partial\mathcal{O}) > \varepsilon$, we define

$$u_\varepsilon(x) = \int_{\mathcal{O}} \tilde{\rho}_{\varepsilon,x}(y)\tau_{y,x}u(y)\ \mathrm{vol}_g(y) \in T_xS,$$

where $\tau_{y,x} : T_yS \to T_xS$ is the parallel transport along the shortest geodesic from y to x. Then for any $\mathcal{O}' \subset\subset \mathcal{O}$, there exists ε_0 such that $u_\varepsilon \in \mathcal{X}(\mathcal{O}')$ for $0 < \varepsilon < \varepsilon_0$, and $u_\varepsilon \to u$ in $\mathcal{X}^{1,p}(\mathcal{O}')$ (see [29] for more details). Moreover, the following Poincaré-Wirtinger inequality holds:

$$\int_{B_\varepsilon(x)} |u_\varepsilon(x) - \tau_{y,x}u(y)|_g\ \mathrm{vol}_g(y) \leq c\varepsilon \int_{B_\varepsilon(x)} |Du|_g\ \mathrm{vol}_g,$$

for some universal constant $c > 0$. Also, note that $|u|_g \leq 1$ in S implies that $|u_\varepsilon|_g \leq 1$ in $\mathcal{O}'$.

Assume now that $|u|_g = 1$ in $\mathcal{O}$ and that $p \geq 2$. As $|\tau_{y,x}u(y)|_g = |u(y)|_g = 1$ a.e. in $\mathcal{O}$, we deduce:

$$\begin{aligned}\sup_{x\in\mathcal{O}'}\Big|1 - |u_\varepsilon(x)|_g\Big| &\leq C \sup_{x\in\mathcal{O}'} \frac{1}{\varepsilon^2}\int_{B_\varepsilon(x)} |u_\varepsilon(x) - \tau_{y,x}u(y)|_g\ \mathrm{vol}_g(y)\\ &\leq C \sup_{x\in\mathcal{O}'} \frac{1}{\varepsilon}\int_{B_\varepsilon(x)} |Du|_g\ \mathrm{vol}_g \leq C \sup_{x\in\mathcal{O}'} \|Du\|_{L^2(B_\varepsilon(x))} \to 0 \text{ as } \varepsilon \to 0,\end{aligned}$$

where we used the equiintegrability of $|Du|_g^2$ on $\mathcal{O}$. Therefore, $|u_\varepsilon|_g \to 1$ uniformly in $\mathcal{O}'$ as $\varepsilon \to 0$ so that the smooth vector fields $\tilde{u}_\varepsilon = u_\varepsilon/|u_\varepsilon|_g$ are of unit length and converge to u in $\mathcal{X}^{1,p}(\mathcal{O}')$. □

5.4. **A little homology.** Suppose that $\lambda_1, \dots, \lambda_\ell$ are closed Lipschitz curves on S, by which we mean that λ_k is a Lipschitz continuous function $[0, 1] \to S$ such that $\lambda_k(0) = \lambda_k(1)$ for every k. Given integers $c_1, \dots, c_\ell$, we say that

$$\sum_{k=1}^{\ell} c_k\lambda_k \text{ is homologous to } 0$$

if there exists an integrable function $f : S \to \mathbb{Z}$ such that

$$\sum_k c_k \int_{\lambda_k} \phi \ = \ \int_S f\, d\phi \qquad \text{for all smooth 1-forms } \phi.$$

Here and below we use the notation

$$\int_\lambda \phi := \int_0^1 (\lambda)^*\phi = \int_0^1 \sum_{k=1}^2 \phi_k(\lambda(s))\, \frac{d}{ds}\lambda^k(s)\, ds \ \text{ in local coordinates.}$$

In the last expression, with respect to a system $x = (x_1, x_2) : V \to \mathbb{R}^2$ of local coordinates on an open subset $V \subset S$, we write $\phi = \phi_1(x)dx^1 + \phi_2(x)dx^2$, and $\lambda^k(s) = x^k \circ \lambda(s)$, $k = 1, 2$.

We also say that λ is homologous to $\sum_k c_k\lambda_k$ if $\lambda - \sum c_k\lambda_k$ is homologous to 0.

The next lemma summarizes some standard facts:

Lemma 5.2. *If S is a compact Riemannian manifold of genus $\mathfrak{g}$, then there exist simple*[17] *closed curves γ_k, for $k = 1, \ldots, 2\mathfrak{g}$, such that if γ is any closed Lipschitz curve, then there exist integers $c_1 \ldots, c_{2\mathfrak{g}}$ such that*

$$\gamma \text{ is homologous to } \sum_{k=1}^{2\mathfrak{g}} c_k\gamma_k.$$

Moreover, these curves $\{\gamma_k\}_{k=1}^{2\mathfrak{g}}$ have the property that for $\eta \in Harm^1(S)$ defined in (11), the following equivalences take place:

$$\begin{aligned} \eta = 0 \qquad &\Longleftrightarrow \quad \int_\gamma \eta = 0 \qquad \text{for every closed Lipschitz curve } \gamma \\ &\Longleftrightarrow \quad \int_{\gamma_k} \eta = 0 \qquad \text{for } k = 1, \ldots, 2\mathfrak{g}. \end{aligned} \tag{38}$$

In particular, the matrix $\alpha = (\alpha_{\ell k})$ defined in (13) is invertible.

In fact, the existence of curves γ_k with the stated properties is a consequence of the classification of surfaces and de Rham's Theorem. To see that the matrix α is invertible, consider a vector $b \in \mathbb{R}^{2\mathfrak{g}}$ such that $\alpha b = 0$. By (13), it means that $\int_{\gamma_\ell} \sum_{k=1}^{2\mathfrak{g}} b_k\eta_k = 0$ for every $\ell = 1, \ldots, 2\mathfrak{g}$. Then (38) yields $\sum_{k=1}^{2\mathfrak{g}} b_k\eta_k = 0$; as $\{\eta_k\}_k$ is a basis of $Harm^1(S)$, one has $b = 0$. Thus the nullspace of α is trivial.

5.5. **Some useful calculations.** In this section we record some straightforward facts that we will use repeatedly. Let u be a smooth vector field in an open set $\mathcal{O} \subset S$. First, note that wherever $u \neq 0$, for every smooth unit vector field v we have

$$D_v u = (D_v u, \frac{iu}{|u|_g})_g \frac{iu}{|u|_g} + (D_v u, \frac{u}{|u|_g})_g \frac{u}{|u|_g} = \frac{j(u)(v)}{|u|_g}\frac{iu}{|u|_g} + v(|u|_g)\frac{u}{|u|_g},$$

It follows that

$$|Du|_g^2 = \left|\frac{j(u)}{|u|_g}\right|_g^2 + |d|u|_g|^2. \tag{39}$$

In particular, if u is of unit length (i.e., $|u|_g = 1$) and ρ is a smooth scalar function, then

$$|Du|_g^2 = |j(u)|_g^2, \quad j(\rho u) = \rho^2 j(u),$$

and thus

$$|D(\rho u)|_g^2 = \rho^2\, |j(u)|_g^2 + |d\rho|_g^2 = \rho^2\, |Du|_g^2 + |d\rho|_g^2.$$

[17] That is, non self-intersecting.

Writing in complex variable $e^{i\Theta} = \cos\Theta + i\sin\Theta$ for a smooth scalar function Θ where i is the isometry (4), then

$$j(e^{i\Theta}u) = j(u) + |u|_g^2 \, d\Theta.$$

The above properties generalize to suitable Sobolev spaces by a standard density argument (see Lemma 5.1).

Lemma 5.3. *Let $\mathcal{O}$ be an open set in S. Then $j : \mathcal{X}^{1,2}(\mathcal{O}) \to L^p(\mathcal{O})$ is a continuous map for every $p \in [1,2)$ and $|dj(u)|_g \le |Du|_g^2$ a.e. in $\mathcal{O}$ for every $u \in \mathcal{X}^{1,2}(\mathcal{O})$. As a consequence, the map $u \in \mathcal{X}^{1,2}(\mathcal{O}) \mapsto dj(u)$ is continuous as a map with values into the set of 2-forms endowed with the $W^{-1,p}$-norm for every $p \in [1,2)$. Moreover, if $u \in \mathcal{X}^{1,2}(\mathcal{O})$, then $\frac{j(u)}{|u|_g} = (Du, \frac{iu}{|u|_g})_g$ is well defined and belongs to L^2.*

Proof. If $u, v \in \mathcal{X}^{1,2}(\mathcal{O})$ and $p \in [1,2)$, then the Hölder inequality implies

$$\begin{aligned}\int_{\mathcal{O}} |j(u) - j(v)|_g^p \,\mathrm{vol}_g &\le C\bigg(\int_{\mathcal{O}} |(D(u-v), iu)_g|^p \,\mathrm{vol}_g + \int_{\mathcal{O}} |\big(Dv, i(u-v)\big)_g|^p \,\mathrm{vol}_g\bigg)\\ &\le C\bigg(\|D(u-v)\|_{L^2}^p \|u\|_{L^q}^p + \|Dv\|_{L^2}^p \|u-v\|_{L^q}^p\bigg)\\ &\le C\|D(u-v)\|_{L^2}^p (\|Du\|_{L^2}^p + \|Dv\|_{L^2}^p),\end{aligned}$$

where $q = p/(2-p)$ and we used the Sobolev embedding $\mathcal{X}^{1,2} \subset L^q$. Therefore, $j : \mathcal{X}^{1,2}(\mathcal{O}) \to L^p(\mathcal{O})$ is a continuous map. As $d : L^p \to W^{-1,p}$ is continuous, we deduce that $u \mapsto dj(u)$ is continuous as map with values into the set of 2-forms endowed with the $W^{-1,p}$-norm for every $p \in [1,2)$.

We now prove that $|dj(u)|_g \le |Du|_g^2$ a.e. in $\mathcal{O}$. Assume for the moment that u is smooth in $\mathcal{O}$. Fix some $x \in \mathcal{O}$, and choose (properly oriented) coordinates near x such that the coordinate vector fields $\partial_{x_1}, \partial_{x_2}$ are orthonormal *at* x. In these coordinates, $j(u) = \sum_{k=1,2} (D_k u, iu)_g dx^k$ and thus, by the Schwartz lemma,

$$dj(u) = \sum_{k,\ell=1,2} (D_k u, iD_\ell u)_g dx^\ell \wedge dx^k = 2(iD_1 u, D_2 u)_g dx^1 \wedge dx^2. \tag{40}$$

Thus at x,

$$|dj(u)|_g = 2|(iD_1 u, D_2 u)_g| \le |D_1 u|_g^2 + |D_2 u|_g^2 = |Du|_g^2,$$

where we have used several times the choice of coordinates, which implies that dx^1, dx^2, are orthonormal at x, in particular that $dx^1 \wedge dx^2 = \mathrm{vol}_g$. In the general case, by a standard density argument (via Lemma 5.1), one deduces that the above inequality holds a.e. in $\mathcal{O}$ for every $u \in \mathcal{X}^{1,2}(\mathcal{O})$. The last part of the statement follows from (39). □

As a consequence, we have the following:

Lemma 5.4. *Assume that $\mathcal{O}$ is an open subset of S and that $u \in \mathcal{X}^{1,2}(\mathcal{O})$ satisfies $|u|_g = 1$. Then*

$$dj(u) = -\kappa \, vol_g \qquad in\ \mathcal{O}.$$

In particular, we have $\omega(u) = 0$ in $\mathcal{O}$.

Proof. If u is smooth in $\mathcal{O}$, then we define $\tau_1 = u$ and $\tau_2 = i\tau_1$ in $\mathcal{O}$, and the definitions imply that the connection 1-form associated to this choice of orthonormal frame is exactly $A = -j(u)$. So the conclusion follows immediately as $dA = \kappa \,\mathrm{vol}_g$. For general $u \in \mathcal{X}^{1,2}(\mathcal{O})$ of unit length, one argues by density (see Lemma 5.1) and the continuity properties of $j(\cdot)$ in Lemma 5.3. □

6. The canonical harmonic vector field. Proof of Theorem 2.1

In this section we consider (S, g) to be a 2-dimensional closed oriented Riemannian manifold (not assumed to be embedded in any Euclidean space). We will need the following

Lemma 6.1. *If B is any nonempty open subset of S, then there exists a moving frame $\{\tau_1, \tau_2\}$ on $S \setminus B$.*

Proof. A standard construction (see for example [15] pages 103-4) yields a smooth vector field that does not vanish outside some finite set (the vertices of a triangulation of S). After pushing forward via a diffeomorphism of S that maps every point of this finite set into B, we get a vector field v such that $|v|_g > 0$ outside B. Then we obtain a moving frame on $S \setminus B$ by setting $\tau = v/|v|_g$ and $\{\tau_1, \tau_2\} = \{\tau, i\tau\}$. □

Lemma 6.2. *Let γ be any closed Lipschitz curve on S. If $\{\tau_1, \tau_2\}$ and $\{\tilde{\tau}_1, \tilde{\tau}_2\}$ are moving frames defined in a neighborhood of γ, and A and $\tilde{A}$ are the associated connection 1-forms, then*

$$\int_\gamma A = \int_\gamma \tilde{A} \qquad \mod 2\pi.$$

Proof. In the domain where they are both defined, there exists a smooth $\mathbb{C}$-valued function ϕ such that $\tilde{\tau}_1 = \phi\tau_1$, since $\{\tau_1(x), \tau_2(x)\} = \{\tau_1(x), i\tau_1(x)\}$ form a basis for T_xS. It then follows that $|\phi| = 1$ everywhere and that $\tilde{\tau}_2 = \phi\tau_2$ as well. If we write $\phi = \phi_1 + i\phi_2$, the definition of the connection 1-form together with Section 5.5 imply that $A - \tilde{A} = \phi_1 d\phi_2 - \phi_2 d\phi_1 =: (d\phi, i\phi)$.

Next, it is convenient to abuse notation and write γ to denote both the curve in S and a Lipschitz function $\gamma : [0, 1] \to S$, with $\gamma(0) = \gamma(1)$, that parametrizes the given curve, with the correct orientation. We will also write $\varphi = \phi \circ \gamma : [0, 1] \to \mathbb{S}^1 \subset \mathbb{C}$. Clearly φ is Lipschitz, so we can find a Lipschitz function $f : [0, 1] \to \mathbb{R}$ such that $\varphi(s) = e^{if(s)}$ for $s \in [0, 1]$. Then one readily checks that

$$\int_\gamma (A - \tilde{A}) = \int_\gamma (d\phi, i\phi) = \int_0^1 (\varphi'(s), i\varphi(s))ds = \int_0^1 f'(s)ds = f(1) - f(0) \in 2\pi\mathbb{Z}$$

since $\varphi(0) = \varphi(1)$. □

As a consequence, we deduce that the index (or topological degree) defined in (5) is an integer number:

Lemma 6.3. *Let $\mathcal{O}$ be a simply connected open subset of S of nonempty Lipschitz boundary and $u \in \mathcal{X}^{1,2}(\mathcal{N})$ is a vector field in a neighborhood $\mathcal{N}$ of $\partial\mathcal{O}$ such that $|u|_g \geq \frac{1}{2}$ a.e. in $\mathcal{N}$; then the* index *of u along $\partial\mathcal{O}$ defined in* (5) *is well defined and it is an integer.*

Proof. We start by explaining why the definition (5) makes sense for $u \in \mathcal{X}^{1,2}(\mathcal{N})$. In fact, if $\{\tau, i\tau\}$ is a moving frame in $\mathcal{N} \cup \mathcal{O}$ (which exists due to Lemma 6.1 as by our assumption $S \setminus \mathcal{O}$ has nonempty interior) and $\tilde{u} = u/|u|_g$, then $\tilde{u} = \phi\tau$ for some $\phi \in H^1(\mathcal{N}, \mathbb{S}^1)$. Denoting by A the connection 1-form associated to the frame, by (35), we have that $D\tilde{u} = (d\phi - iA\phi)\tau$ so that $j(u)/|u|_g^2 = j(\tilde{u}) = (d\phi, i\phi) - A$ where $(d\phi, i\phi) = \phi_1 d\phi_2 - \phi_2 d\phi_1$ is the current associated to the unit-length complex function ϕ belonging to $H^{1/2}(\partial\mathcal{O}, \mathbb{S}^1)$ by the trace theorem. Therefore, since $dA = \kappa \operatorname{vol}_g$, the Stokes theorem implies that (5) writes

$$2\pi \deg(u; \partial\mathcal{O}) = \int_{\partial\mathcal{O}} (d\phi, i\phi) - \int_{\partial\mathcal{O}} A + \int_{\mathcal{O}} \kappa \operatorname{vol}_g = \int_{\partial\mathcal{O}} (d\phi, i\phi)$$

where the meaning of the last term is given by the duality $(H^{-1/2}(\partial\mathcal{O}), H^{1/2}(\partial\mathcal{O}))$. Moreover, it is known (see [5, 7]) that this number is a multiple of 2π as long as $\phi \in H^{1/2}(\partial\mathcal{O}, \mathbb{S}^1)$. □

The following lemma is a main point in the proof of Theorem 2.1.

Lemma 6.4. *Let u be a smooth unit vector field defined on an open set $\mathcal{O} \subset S$. If γ is any smooth closed curve in $\mathcal{O}$, and if A is the connection 1-form associated to any moving frame defined in a neighborhood of γ, then*

$$\int_\gamma (j(u) + A) \in 2\pi\mathbb{Z}. \tag{41}$$

Conversely, if j is a smooth 1-form in an open set $\mathcal{O} \subset S$ such that

$$\int_\gamma (j + A) \in 2\pi\mathbb{Z} \tag{42}$$

for any curve γ and connection 1-form A as above, then there exists a smooth unit vector field u in the open set $\mathcal{O}$, such that $j(u) = j$.

Proof. The first part is a direct consequence of Lemma 6.2 as $\{u, iu\}$ is a moving frame around γ to which the connection 1-form $\tilde{A}$ is associated so that $j(u) = -\tilde{A}$. However, we give in the following a different proof that is needed for the last part of the statement. Let u be a smooth unit vector field on $\mathcal{O} \subset S$. For simplicity we write $j_u := j(u)$.

Step 1. An ODE argument. Fix some smooth curve $\gamma : [0,1] \to \mathcal{O}$ with $\gamma(0) = \gamma(1)$ and for $s \in [0,1]$, let $U(s) := u(\gamma(s)) \in T_{\gamma(s)}S$. Then for $s \in [0,1]$, we have

$$\begin{aligned} D_{\gamma'(s)}U(s) &= (D_{\gamma'(s)}U(s), U(s))_g\, U(s) + (D_{\gamma'(s)}U(s), iU(s))_g\, iU(s) \\ &= j_u(\gamma'(s))\, iU(s)\,, \end{aligned} \tag{43}$$

since $0 = \frac{d}{ds}|U(s)|_g^2 = 2(D_{\gamma'}U(s), U(s))_g$. (We remind the reader of our convention that if j_u is a 1-form and $v \in T_xS$, then $j_u(v)$ denotes $j_u|_x(v)$.) Now let $\{\tau_1, \tau_2\} = \{\tau, i\tau\}$ be any moving frame defined in a neighborhood of γ, and let A be the connection 1-form associated to it. Writing $U(s)$ in terms of the frame, we have

$$U(s) = \phi(s)\tau(s) = (\phi_1(s) + i\phi_2(s))\,\tau(s)$$

where $\tau(s) := \tau(\gamma(s))$ and $\phi_j(s) = (U(s), \tau_j(s))_g$, $j = 1, 2$. Using (35) to rewrite the ODE (43) in terms of ϕ, we obtain

$$\phi'(s) = (j_u + A)(\gamma'(s))\ i\phi(s).$$

We solve to find that

$$U(s) = \phi(s)\tau(s) = \phi(0)\exp\left[i\int_0^s (j_u + A)(\gamma'(t))\,dt\right]\tau(s), \tag{44}$$

for $0 < s \leq 1$. Since $\gamma(0) = \gamma(1)$, however, it must be the case that $U(0) = U(1)$, and thus

$$\int_0^1 (j_u + A)(\gamma'(t))\,dt = \int_\gamma (j_u + A) = 0 \mod 2\pi.$$

This proves (41).

Step 2. Strategy. To establish the converse, we now assume that j satisfies (42) on an open set $\mathcal{O}$. We may assume that $\mathcal{O}$ is connected, as otherwise we may follow the procedure described below on every connected component. Now fix some $x \in \mathcal{O}$ and $v \in T_xS$ such that $|v|_g = 1$. Given any other $y \in \mathcal{O}$, we define $u(y)$ by the following procedure: Fix a smooth curve $\gamma : [0,1] \to \mathcal{O}$ such that $\gamma(0) = x, \gamma(1) = y$. If u exists, then $u(\gamma(s))$ must satisfy the ODE (43) found above. Motivated by this, we let $U(s) \in T_{\gamma(s)}S$ be the solution of (43) with initial data as below:

$$D_{\gamma'(s)}U(s) = j(\gamma'(s))\ iU(s), \qquad U(0) = v. \tag{45}$$

We hope to define

$$u(y) := U(1).$$

Step 3. Independence of the connecting path. We must verify that the above definition makes sense (in particular, is independent of the choice of path connecting x to y). For this, it suffices to show that for any piecewise smooth curve $\gamma : [0,1] \to \mathcal{O}$ such that $\gamma(0) = \gamma(1)$,

$$\text{if } U \text{ solves (45) along } \gamma, \quad \text{then } U(0) = U(1).$$

Indeed, if γ_1 and γ_2 are two such curves joining x to y, then

$$\gamma(s) := \begin{cases} \gamma_1(1-2s) & \text{if } 0 \le s \le \frac{1}{2} \\ \gamma_2(2s-1) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

is a piecewise smooth curve beginning and ending at y and passing through x when $s = 1/2$. If we consider the solution of (45) such that $U(\frac{1}{2}) = v \in T_xS$, then $U(0) - U(1)$ characterizes the difference between the vectors obtained by transporting v from x to y, using the ODE (45), along γ_1 and γ_2.

Now, exactly as above, by writing (45) in terms of a moving frame $\{\tau_1, \tau_2\} = \{\tau, i\tau\}$ and solving the resulting equation, we find that (44) holds, and thus that $U(0) = U(1)$ if and only if (42) is satisfied. Thus the above procedure gives a well-defined vector field u on $\mathcal{O}$, which is clearly a unit vector field in view of (44).

Step 4. Smoothness of u and $j_u = j$.

As j is smooth and generating u via (45), by regularity of ODEs w.r.t. change of parameters and initial data, we deduce that u is smooth in $\mathcal{O}$. It remains to check that $j(u) = j$. Again we will write j_u instead of $j(u)$. Given any $y \in \mathcal{O}$ and $v \in T_yS$, fix a smooth curve $\gamma : [0,1] \to \mathcal{O}$ such that

$$\gamma(0) = x, \qquad \gamma(1) = y, \qquad \gamma'(1) = v.$$

Let $U(s) \in T_{\gamma(s)}S$ solve the ODE (45). By construction, $U(s) = u(\gamma(s))$ for all s. Then at the point y (corresponding to $s = 1$) we have

$$j_u(v) \overset{(3)}{=} (D_v u, iu)_g = (D_{\gamma'}U, iU)_g \overset{(45)}{=} j(\gamma') = j(v).$$

Since v was arbitrary, it follows that $j(u) = j_u = j$, completing the proof. □

Before proving Theorem 2.1, we need the following result:

Lemma 6.5. *Assume $a_1, \dots, a_n$ are n distinct points in S, $d_1, \dots, d_n \in \mathbb{Z}$ such that (7) is satisfied and let ψ be the zero average 2-form solving (10). Let λ_ℓ, $\ell = 1, \dots, J$ be closed Lipschitz curves in S, all disjoint from the set $\cup_{k=1}^n \{a_k\}$ appearing in (10), and such that $\sum_{\ell=1}^J \tilde{d}_\ell \lambda_\ell$ is homologous to 0, for some integers $\tilde{d}_1, \dots, \tilde{d}_J$. Finally, let $\{\tau_1, \tau_2\}$ be a moving frame defined in a neighborhood of $\cup_{\ell=1}^J \lambda_\ell$, and let A be the connection 1-form associated to it. Then*

$$\sum_{\ell=1}^J \tilde{d}_\ell \int_{\lambda_\ell} (d^*\psi + A) = 0 \mod 2\pi. \tag{46}$$

Remark 6.6. The proof shows that the conclusion of the Lemma still holds if

$$-\Delta\psi + \kappa \operatorname{vol}_g = \omega$$

where ω is a 2-form supported in a union $\cup_{k=1}^n B_k$ of disjoint balls such that $\int_{B_k} \omega = 2\pi d_k$ for every $1 \le k \le n$, (7) holds and the curves $\{\lambda_\ell\}_{1\le \ell \le J}$ are disjoint from $\cup_{k=1}^n \bar{B}_k$.

Proof. The assumption that $\sum \tilde{d}_\ell \lambda_\ell$ is homologous to 0 means that there exists an integrable function $f : S \to \mathbb{Z}$ such that

$$\sum_{k=1}^J \tilde{d}_\ell \int_{\lambda_\ell} \phi = \int_S f\, d\phi \qquad \text{for every smooth 1-form } \phi. \tag{47}$$

It follows from the above that $df = 0$ in the sense of distributions away from the curves λ_ℓ (a closed set of measure zero), and thus f is locally constant away from this set $\cup_{\ell=1}^J \lambda_\ell$. Since we can add a constant to f without changing the integral in (47), we can therefore assume that $f = 0$ on an open set B. After shrinking B if necessary, we may assume that its closure does not intersect $\cup_{\ell=1}^J \lambda_\ell$. Then, according to Lemma 6.1, there exists a moving frame $\{\tau_1, \tau_2\}$ defined on a neighborhood of the support of f. Let A denote the associated connection 1-form. In view of Lemma 6.2, it suffices to prove (46) for this choice of A. We wish to substitute $\phi = d^*\psi + A$ in (47) (but ψ is not smooth on S) to find that

$$\sum_{\ell=1}^J \tilde{d}_\ell \int_{\lambda_\ell} (d^*\psi + A) = \int_S f\, d(d^*\psi + A) \in 2\pi\mathbb{Z}\,,$$

since f is integer-valued and $d(d^*\psi + A) = -\Delta\psi + \kappa \operatorname{vol}_g = 2\pi \sum_{k=1}^n d_k \delta_{a_k}$, according to (10). To justify this, we approximate ψ by smooth functions proceeding as follows. First, it is a standard fact that if $\int f d\phi = 0$ for all smooth 1-forms with support in an open set U, then f is constant[18] in U. It thus follows from (47) that f is locally constant away from $\cup \lambda_\ell$, and in particular in a neighborhood of each a_k. For $0 < \sigma < \frac{1}{2} \min_{j\ne k} \operatorname{dist}_S(a_j, a_k)$, let Q_σ be a smooth function supported in $\cup_{k=1}^n B(a_k, \sigma)$, with $d_k Q_\sigma \ge 0$ inside $B(a_k,\sigma)$, and such that $\int_{B(a_k,\sigma)} Q_\sigma \operatorname{vol}_g = d_k$ for every k and σ, and let ψ_σ solve

$$-\Delta\psi_\sigma = -\kappa \operatorname{vol}_g + 2\pi Q_\sigma \operatorname{vol}_g\ .$$

Then (47) implies that for every $\sigma > 0$,

$$\sum_{\ell=1}^J \tilde{d}_\ell \int_{\lambda_\ell} (d^*\psi_\sigma + A) = 2\pi \int_S f Q_\sigma \operatorname{vol}_g.$$

The last integral belongs to $2\pi\mathbb{Z}$ for every $\sigma < \min_k \operatorname{dist}_S(a_k, \cup_\ell \lambda_\ell)$, and standard theory (for example, properties of the Green's function recalled in Section 2.2) implies that $d^*\psi_\sigma \to d^*\psi$ as $\sigma \searrow 0$, locally uniformly away from $\{a_k\}$. Thus we deduce (46) by taking the limit $\sigma \searrow 0$. □

We can now give the main result of this section:

Proof of Theorem 2.1. Let $\psi = \psi(a; d)$ solve (10), for fixed $d_1, \dots, d_n \in \mathbb{Z}$ and distinct $a_1, \dots, a_n \in S$ such that (7) holds. Let $j^* = j^*(a, d, \Phi)$ be defined by (12), that is, $j^* = d^*\psi + \sum_{k=1}^{2\mathfrak{g}} \Phi_k \eta_k$.

[18] Modifying f on a null set, if necessary, we assume that $f(x) = \lim_{r\to 0} \fint_{B(r,x)} f(y) \operatorname{vol}_g$ wherever this limit exists.

Step 1.Definition of $\zeta_k(a;d)$ and its consequences. We recall the definition of $\zeta_k(a;d)$. For every $k=1,\dots,2\mathfrak{g}$, we let λ_k be a smooth curve that is homologous to γ_k (the curves fixed in Lemma 5.2) and disjoint from $\{a_l\}_{l=1}^n$. We now define $\zeta_k(a;d)\in\mathbb{R}/2\pi\mathbb{Z}$ by (16), i.e.,

$$\zeta_k(a;d) := \int_{\lambda_k} (d^*\psi + A) \mod 2\pi, \quad k=1,\dots,2\mathfrak{g},$$

where A is the connection 1-form associated to any moving frame defined in a neighborhood of λ_k. It follows from Lemmas 6.2 and 6.5 that the above integral is independent, modulo $2\pi\mathbb{Z}$, of the choice of moving frame and of the curve λ_k homologous to γ_k, and hence that ζ_k is well-defined as an element of $\mathbb{R}/2\pi\mathbb{Z}$.

With this choice of ζ_k, we deduce from (12) that

$$\int_{\lambda_k} (j^*+A) = \int_{\lambda_k} \big(d^*\psi + A + \sum_{\ell=1}^{2\mathfrak{g}} \Phi_\ell \eta_\ell\big) = \zeta_k + \sum_\ell \alpha_{k\ell}\Phi_\ell.$$

where $\{\alpha_{k\ell}\}$ were defined in (13), noting that for λ_k homologous to γ_k, one has $\int_{\lambda_k}\eta_\ell = \int_{\gamma_k}\eta_\ell = \alpha_{k\ell}$, by the definition of homologous and because $d\eta_k=0$. Also, it follows from Lemma 5.2 that any γ is homologous to a linear combination of $\gamma_1,\dots,\gamma_{2\mathfrak{g}}$, and hence to a linear combination of $\lambda_1,\dots,\lambda_{2\mathfrak{g}}$, say $\sum_{k=1}^{2\mathfrak{g}} \tilde d_k \lambda_k$. Then Lemma 6.5 implies that

$$\int_\gamma (j^*+A) = \sum_{k=1}^{2\mathfrak{g}} \tilde d_k \int_{\lambda_k} (j^*+A) = \sum_{k=1}^{2\mathfrak{g}} \tilde d_k \left(\zeta_k + \sum_\ell \alpha_{k\ell}\Phi_\ell\right).$$

It follows that for $\zeta_k(a;d)$ as defined above, $j^*(a,d,\Phi)$ satisfies

$$\int_\gamma (j^*+A) = 0 \mod 2\pi \quad \text{for every Lipschitz path } \gamma \text{ in } S\setminus\cup_{l=1}^n\{a_l\}$$

(48)

$$\iff \qquad \sum_{\ell=1}^{2\mathfrak{g}} \alpha_{k\ell}\Phi_\ell + \zeta_k = 0 \mod 2\pi \text{ for all } 1\le k\le 2\mathfrak{g},$$

Step 2. First implication. Assume that u^* is a unit vector field satisfying (8) and (9). These conditions and the equation (10) for ψ imply that $j(u^*)-d^*\psi$ is a harmonic 1-form, and it follows that $j(u^*) =: j^*(a,d,\Phi)$ for certain constants Φ_k. Then by combining (41) and (48), we conclude that $\sum_\ell \alpha_{k\ell}\Phi_\ell + \zeta_k = 0 \mod 2\pi$ for every k, which is (14).

Step 3. Converse implication. Fix constants (Φ_k) satisfying (14). By combining (48) and the sufficiency assertion from Lemma 6.4, we conclude that there exists a smooth unit vector field u^* in $\mathcal{O} := S\setminus\cup_{l=1}^n\{a_l\}$ satisfying $j(u^*)=j^*$ so that (12) is fulfilled.

Step 4. Continuity of ζ_k, $k=1,\dots,2\mathfrak{g}$. To prove the continuity of ζ_k, consider a sequence μ_t as in (15), and let $\nu_t := \mu_t - \mu_0$ with $t>0$ small. Then (15) and basic properties of the $W^{-1,1}$ norm (see e.g. [6, Theorem 5.1]) imply that ν_t can be written in the form

$$\nu_t = \sum_{l=1}^{K_t} 2\pi(\delta_{p_{l,t}} - \delta_{q_{l,t}}), \qquad \text{with } \sum_l \mathrm{dist}_S(p_{l,t},q_{l,t}) \to 0 \quad \text{as } t\to 0$$

and $\{K_t\}_{t\to 0}$ is uniformly bounded. For sufficiently small $r>0$, whenever $t>0$ is small enough, we can find Lipschitz paths $\lambda_{k,t}$, for $k=1,\dots,2\mathfrak{g}$, such that

$$\mathrm{dist}_S(\lambda_{k,t},\{p_{l,t},q_{l,t}\}_l)\ge r, \qquad \lambda_{k,t} \text{ is homologous to } \gamma_k, \qquad \text{and} \qquad \mathcal{H}^1(\lambda_{k,t})\le C$$

for all k (where γ_k are the curves fixed in Lemma 5.2). In fact, the curves $\lambda_{k,t}$ can be taken to be γ_k, modified whenever they pass through $B(p_{l,t},2r)$ or $B(q_{l,t},2r)$, by replacing that portion of the path with an arc of the circle $\partial B(p_{l,t},2r)$ or $\partial B(q_{l,t},2r)$. By (20), we write for $t>0$ small:

$$\psi_t = 2\pi \sum_{l=1}^{K_t} \Big[\big(G(\cdot,p_{l,t}) - G(\cdot,q_{l,t})\big)\,\mathrm{vol}_g \Big],$$

so that for every $k=1,\dots,2\mathfrak{g}$, the definition (16) of ζ_k implies that

$$2\pi \sum_{l=1}^{K_t} \int_{\lambda_{l,t}} d^* \Big[\big(G(\cdot,p_{l,t}) - G(\cdot,q_{l,t})\big)\,\mathrm{vol}_g \Big] = \zeta_k(a_t,d_t) - \zeta_k(a_0,d_0) \mod 2\pi.$$

But facts about the Green's function summarized in Section 2.2 imply that $d^*\Big[\big(G(\cdot,p) - G(\cdot,q)\big)\,\mathrm{vol}_g\Big] \to 0$ as $\mathrm{dist}_S(p,q)\to 0$, uniformly in the set $\{(x,p,q) : \mathrm{dist}_S(x,\{p,q\}) \geq r\}$. Hence the sum of integrals on the left-hand side above tends to 0 as $t\to 0$, which is what we needed to prove.

Step 5. Uniqueness (modulo a global rotation) of u^*. Assume that u^* and $\tilde{u}^*$ are two solutions of (8) and (9) such that $j(u^*) = j(\tilde{u}^*)$. Fixing x and $v = u^*(x)$ as at the start of the construction of u^* (in (45)). Since both v and $\tilde{v} := \tilde{u}^*(x)$ are unit vectors, there exists some α such that $\tilde{v} = e^{i\alpha}v$. Then by inspection we see that if γ is any Lipschitz curve avoiding the points $\cup_{k=1}^n\{a_k\}$, then $\tilde{U}(s) = e^{i\alpha}U(s)$ solves (45) with initial data $\tilde{U}(0) = \tilde{v}$. It follows that $\tilde{u}^*(y) = e^{i\alpha}u(y)$ for every $y \notin \cup_{k=1}^n\{a_k\}$. Thus $\tilde{u}^* = e^{i\alpha}u^*$ a.e. in S.

Step 6. Regularity. Standard estimates, such as those recalled in Section 2.2 for example, imply that Green functions belong to $W^{1,p}$ for all $p<2$ and smooth away from $\cup_k\{a_k\}$ which by (20) it leads to ψ being in the same Sobolev space and smooth away from $\cup_k\{a_k\}$. Moreover, (12) in combination with $|Du^*|_g = |j(u^*)|_g$ (by (43)) yields $u \in \mathcal{X}^{1,p}(S)$ for all $p<2$. As $j(u^*)$ is smooth away from $\cup_k\{a_k\}$, then Lemma 6.4 through the construction (43) yield u^* is smooth away from $\cup_k\{a_k\}$. □

We also prove the estimate (18):

Proof of Lemma 2.4. First note from (17) that there exists some $C = C(\alpha)$ such that

$$\mathrm{dist}_{\mathbb{R}^{2\mathfrak{g}}}\big(\mathcal{L}(a;d),\mathcal{L}(\tilde{a};\tilde{d})\big) \leq C \qquad \text{for all } (a;d),(\tilde{a},\tilde{d}).$$

It therefore suffices to prove (18) under the assumption that $\|\mu-\tilde{\mu}\|_{W^{-1,1}} \leq 1$. As in Step 4 in the proof of Theorem 2.1, we can rewrite $\mu - \tilde{\mu} = 2\pi\sum_{l=1}^{\hat{n}}(\delta_{p_l} - \delta_{q_l})$ for the dipoles $\{p_l,q_l\}_{l=1}^{\hat{n}} \subset a\cup\tilde{a}$ with $\hat{n}\leq 2K$. It follows from our specific choice of the $W^{-1,1}$ norm (see Section 5.3) and the fact that $\|\mu-\tilde{\mu}\|_{W^{-1,1}}\leq 1$ that the $W^{-1,1}$ norm of $\mu-\tilde{\mu}$ represents the minimal connection (see e.g. [6, Theorem 5.1])

$$\|\mu-\tilde{\mu}\|_{W^{-1,1}} = 2\pi \min_{\sigma\in\mathcal{E}_{\hat{n}}} \sum_{l=1}^{\hat{n}} \mathrm{dist}_S(p_l,q_{\sigma(l)}),$$

where $\mathcal{E}_{\hat{n}}$ is the set of permutations of $\hat{n}$ elements. After relabelling, we can assume that an optimal permutation σ is the identity. For sufficiently small $r>0$, we can find Lipschitz paths λ_k homologous to γ_k (where γ_k are the curves fixed in Lemma 5.2) and of uniformly bounded length, for $k = 1, \dots, 2\mathfrak{g}$, such that $\mathrm{dist}_S(\lambda_k, \{p_l, q_l\}_l) \geq r$ for all k. If we denote by $\psi(a; d)$ and $\tilde{\psi}(\tilde{a}, \tilde{d})$ the solutions defined in (10) associated to μ and $\tilde{\mu}$, we have by (20):

$$\hat{\psi} := \psi - \tilde{\psi} = 2\pi \sum_{l=1}^{\hat{n}} \Big[\big(G(\cdot, p_l) - G(\cdot, q_l)\big) \mathrm{vol}_g \Big].$$

As $\big|d^*\big[\big(G(x,p) - G(x,q)\big)\,\mathrm{vol}_g\big]\big| \leq C_r \,\mathrm{dist}_S(p,q)$ for $\mathrm{dist}_S(x, \{p,q\}) \geq r$, we deduce by (16) that

$$|\zeta_k(a;d) - \zeta_k(\tilde{a}, \tilde{d})| \leq \int_{\gamma_k} \big|d^*\hat{\psi}\big| \leq 2\pi \sum_{l=1}^{\hat{n}} \int_{\gamma_k} \big|d^*\big[\big(G(x,p_l) - G(x,q_l)\big)\,\mathrm{vol}_g\big]\big| \leq C_r \|\mu - \tilde{\mu}\|_{W^{-1,1}}.$$

The conclusion is now straightforward. □

Example 6.7. Let S be the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ with the standard (x,y) coordinates and the standard metric $ds^2 = dx^2 + dy^2$. We will often identify S with the unit square with periodic boundary conditions. Here the genus $\mathfrak{g} = 1$, and the 1-forms η_k (fixed as an orthonormal basis in (11)) may be taken to be

$$\eta_1 = dx, \qquad \eta_2 = dy.$$

In addition, we may take the curves from Section 5.4 to be

$$\gamma_1(s) = (s, 0), \qquad \gamma_2(s) = (0, s), \qquad \text{for } 0 \leq s \leq 1.$$

We let $\{\tau_1, \tau_2\}$ denote the standard coordinate vector fields, yielding a global moving frame for which the connection 1-form A is identically 0 and $\kappa = 0$.

Fix some $(a; d) \in S^n \times \mathbb{Z}^n$ such that (7) holds and let ψ solve (10), i.e., $-\Delta\psi = 2\pi \sum_{k=1}^n d_k \delta_{a_k}$ in S. We will identify each a_k with the point $a_k := (a_k^x, a_k^y) \in [0,1)^2$ and we write

$$I(y) := \int_{\lambda_1^y} d^*\psi, \qquad \text{for } \lambda_1^y(s) = (s, y), \quad 0 \leq s \leq 1.$$

For every $y \in [0,1)$, note that λ_1^y is homologous to the curve γ_1. According to the definition (16), if $y \notin \{a_k^y\}_{k=1}^n$, then $\zeta_1(a;d)$ is the equivalence class in $\mathbb{R}/2\pi\mathbb{Z}$ containing $I(y)$. We may assume by a translation that $0 \notin \{a_k^y\}_{k=1}^n$. Then by Stokes Theorem and the equation (10) for ψ,

$$I(0) - I(y) = \int_{[0,1]\times(0,y)} dd^*\psi = 2\pi \int_{[0,1]\times(0,y)} \sum_{k=1}^n d_k \delta_{a_k} = 2\pi \sum_{k=1}^n d_k \mathbf{1}_{(a_k^y, 1)}(y) \tag{49}$$

for $a.e.$ $y \in (0,1)$. On the other hand, the 2-form ψ may be written $\psi = \tilde{\psi}(x,y) dx \wedge dy$ for some function $\tilde{\psi} : S \to \mathbb{R}$, which we may identify with a $\mathbb{Z}^2$ periodic function on $\mathbb{R}^2$. Then $d^*\psi = -\partial_x \tilde{\psi} dy + \partial_y \tilde{\psi} dx$, so that

$$\int_0^1 I(y)\, dy = \int_0^1 \left(\int_0^1 \partial_y \tilde{\psi}(s,y) ds \right) dy = 0$$

by the periodicity of $\tilde{\psi}$. We can thus integrate (49) and simplify (using the fact that $\sum d_k = 0$) to find that

$$I(0) = \int_0^1 I(0)\, dy = 2\pi \sum_{k=1}^n d_k (1 - a_k^y) = -2\pi \sum_{k=1}^n d_k a_k^y.$$

This determines $\zeta_1(a;d)$. A nearly identical computation shows that $\zeta_2(a;d)$ is the equivalence class in $\mathbb{R}/2\pi\mathbb{Z}$ containing $2\pi \sum_{k=1}^n d_k a_k^x$.

In this example, we may also use (17), (13) and the above computations to find that

$$\mathcal{L}(a;d) = \left\{ \left(2\pi(m_1 + \sum_{k=1}^n d_k a_k^y), 2\pi(m_2 - \sum_{k=1}^n d_k a_k^x) \right) : (m_1, m_2) \in \mathbb{Z}^2 \right\}.$$

In particular, if either $\sum_{k=1}^n d_k a_k^y = \frac{1}{2}$ or $\sum_{k=1}^n d_k a_k^x = \frac{1}{2}$ mod 1, then there is not a *unique* element Φ of $\mathcal{L}(a;d)$ of minimal Euclidean norm.

7. The intrinsic renormalized energy. Proof of Proposition 2.5

In this section, we prove the characterization of the intrinsic renormalized energy in Proposition 2.5.

Proof of Proposition 2.5. Let $r > 0$ be small satisfying

$$\sqrt{r} \le \rho_a := \min_{k \neq l} \operatorname{dist}_S(a_k, a_l)$$

and recall the notation $S_r := S \setminus \cup_{k=1}^n B_r(a_k)$. The fact that u^* is a unit vector field implies that $|Du^*|_g^2 = |j(u^*)|_g^2$. Then the form (12) of $j(u^*)$ implies that

$$\frac{1}{2}\int_{S_r} |j(u^*)|_g^2 \operatorname{vol}_g = \frac{1}{2}\int_{S_r} \left(|d^*\psi|_g^2 + 2\sum_{k=1}^{2\mathfrak{g}} \Phi_k (d^*\psi, \eta_k)_g + \sum_{l,k=1}^{2\mathfrak{g}} \Phi_l \Phi_k (\eta_l, \eta_k)_g \right) \operatorname{vol}_g.$$

Step 1. Computing the integrals depending on Φ. As $\{\eta_k\}_{k=1}^{2\mathfrak{g}}$ are smooth forming an orthonormal basis of (11), we compute

$$\int_{S_r} \sum_{l,k} \Phi_l \Phi_k (\eta_l, \eta_k)_g \ \operatorname{vol}_g = \int_S \sum_{l,k} \Phi_l \Phi_k (\eta_l, \eta_k)_g \ \operatorname{vol}_g + O(|\Phi|^2 r^2) = |\Phi|^2 + O(|\Phi|^2 r^2).$$

Similarly, integrating by parts,

$$\begin{aligned}\int_{S_r} \sum_l \Phi_l (d^*\psi, \eta_l)_g \ \operatorname{vol}_g &= \int_S \sum_l \Phi_l (\psi, \underbrace{d\eta_l}_{=0})_g \ \operatorname{vol}_g + O(|\Phi|\, r) \\ &= O(|\Phi|^2 r^{3/2} + r^{1/2})\end{aligned}$$

where we used (20) and the properties on Green's function, which imply that

$$\int_{B_r} |d^*\psi| \ \operatorname{vol}_g \sim \int_0^r \frac{1}{s} s ds = O(r).$$

Step 2. Computing $\int_{S_r} |d^*\psi|_g^2$ *vol*$_g$. We rewrite (20) as follows:

$$\psi = (\psi_0 + \psi_1) \operatorname{vol}_g, \qquad \psi_1 := \sum_{k=1}^n 2\pi d_k G(\cdot, a_k).$$

(Observe that we have taken ψ_0, ψ_1 to be functions, whereas ψ is a 2-form.) Then $|d^*\psi|_g^2 = |\star d \star \psi|_g^2 = |d \star \psi|_g^2 = |d(\psi_0 + \psi_1)|_g^2$. Since ψ_0 is smooth and $\psi_1 \in W^{1,p}$ for $p < 2$, it follows that

$$\int_{S_r} |d^*\psi|_g^2 \operatorname{vol}_g = \int_{S_r} |d\psi_1|_g^2 \operatorname{vol}_g + \int_{S_r} \left(2(d\psi_1, d\psi_0)_g + |d\psi_0|_g^2 \right) \operatorname{vol}_g.$$

Step 2a. Computing $\int_{S_r} |d\psi_1|_g^2 \, vol_g$. We use Stokes Theorem (see (34)) to write

$$\int_{S_r} |d\psi_1|_g^2 \,\mathrm{vol}_g = \int_{S_r} (d^* d\psi_1, \psi_1)_g \,\mathrm{vol}_g - \sum_{k=1}^n \int_{\partial B(a_k,r)} \psi_1 \star d\psi_1 \ .$$

Since ψ_1 has mean 0 and $d^* d\psi_1$ is constant, equal with $-\bar\kappa$ (see (21)) away from $\{a_k\}$, it follows [19]

$$\int_{S_r} (d^* d\psi_1, \psi_1)_g \,\mathrm{vol}_g = -\bar\kappa \int_{S_r} \psi_1 \,\mathrm{vol}_g = \bar\kappa \int_{\cup_k B_r(a_k)} \psi_1 \ \mathrm{vol}_g = O(r^2(|\log r| + 1))$$

where we used the Green functions properties recalled in Section 2.2 and the fact that the distance between the points a_k is larger than $\sqrt{r}$, i.e.,

$$\int_{B_r(a_k)} G(\cdot, a_k) \ \mathrm{vol}_g \le C \int_0^r |\log s| \, s ds = O(r^2(|\log r| + 1)),$$

$$\int_{B_r(a_k)} G(\cdot, a_l) \ \mathrm{vol}_g \le C |\log \mathrm{dist}_S(a_k, a_l)| \mathrm{Vol}(B_r(a_k)),$$

We now fix $k \in \{1 \dots, n\}$, and we write

$$R_k(x) := \psi_1(x) + d_k \log \mathrm{dist}_S(x, a_k) = 2\pi d_k H(x, a_k) + \sum_{l \ne k} 2\pi d_l G(x, a_l)$$

to denote the regular part of ψ_1 near a_k. Since $H \in C^1(S \times S)$ and $\mathrm{dist}_S(a_l, a_k) \ge \sqrt{r}$ for every $l \ne k$, it is clear that R_k is Lipschitz in $B_r(a_k)$, with Lipschitz constant bounded by $Cr^{-1/2}$. In addition, $|d\psi_1|_g \le C/r$ on $\partial B(a_k, r)$, so

$$\begin{aligned} \int_{\partial B(a_k,r)} \psi_1(\star d\psi_1) &= \int_{\partial B(a_k,r)} (R_k - d_k \log r)(\star d\psi_1) \\ &= \big(R_k(a_k) - d_k \log r + O(\sqrt{r})\big) \int_{\partial B(a_k,r)} \star d\psi_1 \end{aligned}$$

and (recalling that $\eta = \star\eta \ \mathrm{vol}_g$ for any 2-form η)

$$\begin{aligned} \int_{\partial B(a_k,r)} \star d\psi_1 &= \int_{B(a_k,r)} d \star d\psi_1 = \int_{B(a_k,r)} \underbrace{\star d \star}_{=-d^*} d\psi_1 \,\mathrm{vol}_g = \int_{B(a_k,r)} \Delta\psi_1 \,\mathrm{vol}_g \\ &= -2\pi d_k + \bar\kappa \mathrm{Vol}(B(a_k, r)) = -2\pi d_k - O(r^2). \end{aligned}$$

Combining the above, we find that

$$\int_{S_r} |d\psi_1|^2 \,\mathrm{vol}_g = -\sum 2\pi d_k^2 \log r + \sum_k 4\pi^2 d_k^2 H(a_k, a_k) + 8\pi^2 \sum_{1 \le l < k \le n} d_k d_l G(a_k, a_l) + O(\sqrt{r}).$$

Step 2b. Computing $\int_{S_r} (d\psi_1, d\psi_0)_g \, vol_g$. Since ψ_0 is smooth in S and $\psi_1 \in W^{1,p}(S)$ for $p < 2$, Hölder's inequality leads to

$$\int_{S_r} (d\psi_1, d\psi_0)_g \ \mathrm{vol}_g = \int_S (d\psi_1, d\psi_0)_g \ \mathrm{vol}_g + \|d\psi_1\|_{L^{4/3}} O(r^{1/2}) = \int_S (d\psi_1, d\psi_0)_g \,\mathrm{vol}_g + O(\sqrt{r}).$$

As ψ_0 has mean 0, Stokes theorem and the equation satisfied by ψ_1 imply that

$$\int_S (d\psi_0, d\psi_1)_g \,\mathrm{vol}_g = \int_S (\psi_0, d^* d\psi_1)_g \,\mathrm{vol}_g = \int_S (\psi_0, -\Delta\psi_1)_g \,\mathrm{vol}_g = \sum_k 2\pi d_k \psi_0(a_k).$$

[19] Recall that $\Delta(\psi_1 \,\mathrm{vol}_g) = (\Delta\psi_1) \,\mathrm{vol}_g$.

Step 3. Conclusion. As a consequence of the above computation, we obtain the following: there exists $r_0(S) > 0$ such that if $r \in (0, r_0)$ satisfies

$$\sqrt{r} \leq \min_{k \neq l} \operatorname{dist}_S(a_k, a_l)$$

then any 1-form $j^* = j^*(a, d, \Phi)$ satisfying (12) (with $\psi = \psi(a; d)$ given by (10) and $\{\Phi\}_{k=1}^{2\mathfrak{g}}$ not necessarily in $\mathcal{L}(a; d)$) we have that

(50)

$$\frac{1}{2} \int_{S_r} |j^*(a, d, \Phi)|^2 \operatorname{vol}_g = -\pi \log r \sum_{k=1}^{n} d_k^2 + 4\pi^2 \sum_{1 \leq l < k \leq n} d_l d_k G(a_l, a_k)$$
$$+ 2\pi \sum_{k=1}^{n} \left[\pi d_k^2 H(a_k, a_k) + d_k \psi_0(a_k) \right] + \frac{1}{2} |\Phi|^2 + \int_S \frac{|d\psi_0|_g^2}{2} \operatorname{vol}_g + O(\sqrt{r}) + O(|\Phi|^2 r^{3/2}).$$

Moreover, the constants above depend only on S and $\sum_{k=1}^n |d_k|$. We conclude that the limit in the definition (19) of $W(a, d, \Phi)$ exists and the desired formula (22) holds true. □

8. Compactness

The result of this section will be crucial in proving point 1 of our main result in Theorem 2.6. It is stated as precise estimates for the vorticity and the flux integrals in terms of the intrinsic energy, but immediately implies parallel results for the other energies (in view of (96) and (116), see below).

Proposition 8.1. *For every $p \in [1, 2)$ and $T, C > 0$, every integer $n > T - 1$ and every $0 < q < 1 - \frac{T}{n+1}$, there exist $\varepsilon_0 \in (0, \frac{1}{2})$, $C_p > 0$ such that the following holds true: if $0 < \varepsilon < \varepsilon_0$ and $u \in \mathcal{X}^{1,2}(S)$ with*

$$\frac{1}{2} \int_S |Du|_g^2 + \frac{1}{2\varepsilon^2} F(|u|_g^2) \ vol_g \leq T\pi |\log \varepsilon| + C, \tag{51}$$

then there exist K distinct points $a_1, \ldots, a_K \in S$ and nonzero integers $d_1, \ldots, d_K \in \mathbb{Z}$ such that (7) holds, $\sum_{k=1}^K |d_k| \leq n$ (so, $K \leq n$) and

$$\|\omega(u) - 2\pi \sum_{k=1}^{K} d_k \delta_{a_k}\|_{W^{-1,p}} \leq C_p (n+1) T |\log \varepsilon| \varepsilon^{q(\frac{2}{p} - 1)}. \tag{52}$$

Moreover, if we define

$$\Phi(u) = (\Phi_1(u), \ldots, \Phi_{2\mathfrak{g}}(u)) := \left(\int_S (j(u), \eta_1)_g \ vol_g, \ldots, \int_S (j(u), \eta_{2\mathfrak{g}})_g \ vol_g \right), \tag{53}$$

for the orthonormal basis $\{\eta_k\}_{k=1}^{2\mathfrak{g}}$ fixed in (11), then

$$\operatorname{dist}_{\mathbb{R}^{2\mathfrak{g}}}(\Phi(u), \mathcal{L}(a; d)) \leq C_q \varepsilon^q \ , \tag{54}$$

where $\mathcal{L}(a; d)$ is the set defined in Section 2.2 for $a = (a_1, \ldots, a_K)$ and $d = (d_1, \ldots, d_K)$.

In the above Proposition, n can be 0 (if $T \in (0, 1)$), in which case, $K = 0$. Our proof will rely on the following result:

Proposition 8.2. *For every $T, C > 0$, every integer $n > T-1$ and every $0 < q < 1-\frac{T}{n+1}$, there exist $\varepsilon_0, r_0, c > 0$ such that the following holds true: if $\varepsilon \in (0, \varepsilon_0)$, $\sigma \in [\varepsilon^q, r_0]$ and $u \in \mathcal{X}(S)$ is a smooth vector field with (51), then there exists a collection of pairwise disjoint balls $\mathcal{B}^\sigma = \{B_{l,\sigma}\}_{l=1}^{K_\sigma}$ of centers $a_{l,\sigma} \in S$ and radius $r_{l,\sigma} > 0$ such that*

$$\{x \in S \,:\, |u(x)|_g \leq \frac{1}{2}\} \subset \cup_{l=1}^{K_\sigma} B_{l,\sigma}, \tag{55}$$

$$\sum_{l=1}^{K_\sigma} |d_{l,\sigma}| \ \leq n\,, \qquad\qquad \textit{where } d_{l,\sigma} := \deg(u; \partial B_{l,\sigma})\ . \tag{56}$$

$$\sum_{l=1}^{K_\sigma} r_{l,\sigma} \ \leq (n+1)\sigma, \tag{57}$$

$$\int_{B_{l,\sigma}} e_\varepsilon^{in}(u)\, vol_g \geq |d_{l,\sigma}|(\pi \log\frac{\sigma}{\varepsilon} - c), \quad l = 1, \dots, K_\sigma. \tag{58}$$

If $n = 0$ above, then K_σ is not necessarily 0 (as balls of degree zero may appear). Proposition 8.2 is proved by a rather standard vortex balls argument, as introduced in [24, 37] for the Ginzburg-Landau energy in flat 2-dimensional domains. We present some details in Appendix A. With Proposition 8.2 available, the proof of the basic compactness assertion (52) follows classical arguments, which we recall for the convenience of the reader. The main new point is the estimate (54) of the flux integrals.

Proof of Proposition 8.1. In what follows, $c > 0$ is a constant that can change from line to line and that can depend on all parameters appearing the hypotheses of the proposition.

Step 1. Reduction to smooth bounded vector fields. We consider $h : \mathbb{R}^+ \to \mathbb{R}^+$ to be the Lipschitz cut-off function $h(s) = 1$ if $s \leq 1$ and $h(s) = 1/s$ if $s > 1$ and

$$\hat{u} := h(|u|_g)u.$$

First, we want to show that we can replace u by $\hat{u}$ in the statement of Proposition 8.1. Indeed, $\hat{u} \in \mathcal{X}^{1,2}(S)$ and since $|D\hat{u}|_g \leq |Du|_g$ (see Section 5.5) and $F(|\hat{u}|_g) \leq F(|u|_g)$ (because $F(1) = 0$), we get that

$$\int_{\mathcal{O}} e_\varepsilon^{in}(\hat{u})\, \mathrm{vol}_g \leq \int_{\mathcal{O}} e_\varepsilon^{in}(u)\, \mathrm{vol}_g, \text{ for every } \mathcal{O} \subset S,$$

so the bound (51) is conserved for $\hat{u}$. Moreover, by Section 5.5,

$$|j(\hat{u}) - j(u)|_g = |h^2(|u|_g) - 1|\, |j(u)|_g \ \leq |u|_g\, |h^2(|u|_g) - 1|\, |Du|_g.$$

Moreover, the definition of h and (1) imply that

$$|u|_g\, |h^2(|u|_g) - 1| \ \leq 2\, |1 - |u|_g| \leq c\sqrt{F(|u|_g^2)}.$$

It follows that

$$\|j(\hat{u}) - j(u)\|_{L^1(S)} \leq c\varepsilon E_\varepsilon^{in}(u)\ .$$

In particular, by (53), we have for any $k \in \{1, \dots, 2\mathfrak{g}\}$ that

$$\Phi_k(u) = \int_S (j(\hat{u}), \eta_k)_g\, \mathrm{vol}_g \ + O(\varepsilon E_\varepsilon^{in}(u)) \ = \ \Phi_k(\hat{u}) + O(\varepsilon |\log \varepsilon|).$$

Moreover, for any $\varphi \in W^{1,\infty}(S)$, we have

$$\left| \int_S \varphi\, [\omega(u) - \omega(\hat{u})] \right| = \left| \int_S \varphi\, d\, [j(u) - j(\hat{u})] \right| \leq c \|d\varphi\|_{L^\infty} \varepsilon E_\varepsilon^{in}(u);$$

this yields $\|\omega(\hat{u})-\omega(u)\|_{W^{-1,1}(S)}\leq c\varepsilon|\log\varepsilon|$. To estimate $\|\omega(\hat{u})-\omega(u)\|_{W^{-1,p}(S)}$ for $1<p<2$, we use Lemma 5.3:

$$\|\omega(u)-\omega(\hat{u})\|_{L^1}\leq\int_S|dj(u)|_g+|dj(\hat{u})|_g\,\mathrm{vol}_g\leq\int_S|Du|_g^2+|D\hat{u}|_g^2\,\mathrm{vol}_g\leq c|\log\varepsilon|$$

and then the interpolation inequality:

$$\|\omega(\hat{u})-\omega(u)\|_{W^{-1,p}}\leq C\|\omega(\hat{u})-\omega(u)\|_{W^{-1,1}}^{\frac{2}{p}-1}\|\omega(\hat{u})-\omega(u)\|_{L^1}^{2-\frac{2}{p}}=O(\varepsilon^{\frac{2}{p}-1}|\log\varepsilon|).$$

Therefore, it is enough to prove the statement for $\hat{u}$ instead of u. Furthermore, due to the density result in Lemma 5.1 and the continuity results in Theorem 2.1 point 1) and Lemma 5.3, **we can assume that u is a smooth vector field in S with $|u|_g\leq 1$.** (The cutting-off procedure $|\hat{u}|_g\leq 1$ is needed in order that the potential term in the energy E_ε^{in} passes to the limit, as F could increase very fast at infinity.)

Step 2. An approximation $\tilde{u}$ of u. Let $\tilde{h}:\mathbb{R}^+\to\mathbb{R}^+$ be a smooth function such that

$$\tilde{h}(s)=1\text{ for }0\leq s\leq\frac{1}{4},\qquad\tilde{h}(s)=1/s\text{ for }s\geq\frac{1}{2},\qquad s\mapsto s\tilde{h}(s)\text{ is nondecreasing}$$

and define the smooth vector field

$$\tilde{u}=\tilde{h}(|u|_g)u. \tag{59}$$

The advantage of working with $\tilde{u}$ is that $|\tilde{u}|_g=1$ if $|u|_g\geq\frac{1}{2}$. Then Section 5.5 implies $j(\tilde{u})=\tilde{h}^2(|u|_g)j(u)$ and $|D\tilde{u}|_g\leq c|Du|_g$ in S since by (39), we have

$$|D\tilde{u}|_g^2\leq\big[\tilde{h}^2(|u|_g)+\big(\frac{d}{ds}(s\tilde{h}(s))\big)^2\big|_{s=|u|_g}\big]|Du|_g^2.$$

By the computations in Step 1, we deduce

$$\|j(\tilde{u})-j(u)\|_{L^1(S)}\leq c\varepsilon E_\varepsilon^{in}(u)\ .$$

Step 3. Proof of (52). For any $\varphi\in W^{1,\infty}(S)$, it follows from Step 2 that

$$\left|\int_S\varphi\,[\omega(u)-\omega(\tilde{u})]\right|=\left|\int_S\varphi\,d\,[j(u)-j(\tilde{u})]\right|\leq c\|d\varphi\|_{L^\infty}\varepsilon E_\varepsilon^{in}(u).$$

With the notations of Proposition 8.2 applied for the smooth vector field u, we claim that for φ as above and $\varepsilon^q\leq\sigma\leq r_0$,

$$\left|\int_S\varphi\big[\omega(\tilde{u})-2\pi\sum_{l=1}^{K_\sigma}d_{l,\sigma}\delta_{a_{l,\sigma}}\big]\right|\leq c\|d\varphi\|_{L^\infty}\Big(\sum_{l=1}^{K_\sigma}r_{l,\sigma}\Big)E_\varepsilon^{in}(u).$$

Indeed, it follows from Proposition 8.2 (see (55)) and (59) that $|\tilde{u}|_g=1$ outside the balls $\cup_{l=1}^{K_\sigma}B_{l,\sigma}$ so that Lemma 5.4 implies $\omega(\tilde{u})=0$ outside $\cup_{l=1}^{K_\sigma}B_{l,\sigma}$. So

$$\int_S\varphi\omega(\tilde{u})=\sum_{l=1}^{K_\sigma}\int_{B_{l,\sigma}}\varphi\,\omega(\tilde{u}).$$

For each $1\leq l\leq K_\sigma$, we have that

$$\begin{aligned}\int_{B_{l,\sigma}}\varphi\,\omega(\tilde{u})&=\varphi(a_{l,\sigma})\int_{B_{l,\sigma}}\omega(\tilde{u})+\int_{B_{l,\sigma}}(\varphi(x)-\varphi(a_{l,\sigma}))\omega(\tilde{u})\\&=\varphi(a_{l,\sigma})\underbrace{\left(\int_{\partial B_{l,\sigma}}j(\tilde{u})+\int_{B_{l,\sigma}}\kappa\,\mathrm{vol}_g\right)}_{=2\pi d_{l,\sigma}}+\int_{B_{l,\sigma}}(\varphi(x)-\varphi(a_{l,\sigma}))\omega(\tilde{u}),\end{aligned}$$

where we used (5), (6) and the fact that $|\tilde{u}|_g = 1$ on $\partial B_{l,\sigma}$ by (55). In particular, for $\varphi = 1$ in S, one has that

$$2\pi \sum_{l=1}^{K_\sigma} d_{l,\sigma} = \int_S \omega(\tilde{u}) = \int_S \kappa \operatorname{vol}_g = 2\pi\chi(S),$$

i.e., (7) holds for the integers $\{d_{l,\sigma}\}_l$. To estimate the last term in the above RHS, note that clearly

$$|\varphi(x) - \varphi(a_{l,\sigma})| \leq \|d\varphi\|_{L^\infty} r_{l,\sigma} \quad \text{ for } x \in B_{l,\sigma}.$$

Moreover, Lemma 5.3 and the definition of ω imply that $|\omega(\tilde{u})|_g \leq |D\tilde{u}|_g^2 + |\kappa|$ in S, and as a consequence,

$$\int_{B_{l,\sigma}} |\omega(\tilde{u})|_g \operatorname{vol}_g \leq \int_S |D\tilde{u}|_g^2 \operatorname{vol}_g + c \leq c \int_S |Du|_g^2 \operatorname{vol}_g + c \leq c(E_\varepsilon^{in}(u) + 1).$$

We may assume that $\varepsilon_0 < \frac{1}{2}$, and then we can absorb the additive constant in the multiplicative constant. By combining these estimates with (57), we see that for any smooth φ,

$$\left| \int_S \varphi \Big[\omega(u) - 2\pi \sum_{l=1}^{K_\sigma} d_{l,\sigma} \delta_{a_{l,\sigma}} \Big] \right| \leq c(n+1)T|\log\varepsilon|\sigma\|d\varphi\|_{L^\infty}.$$

Setting $\sigma = \varepsilon^q$, this is the case $p = 1$ of (52), noting that all the points $\{a_{l,\sigma}\}_{l=1}^{K_\sigma}$ are disjoint (as they belong to pairwise disjoint balls), (7) holds and $\sum_{l=1}^{K_\sigma} |d_{l,\sigma}| \leq n$ by (56). For $1 < p < 2$, we complete the proof of (52) using (again) the interpolation inequality $\|\mu\|_{W^{-1,p}} \leq C\|\mu\|_{W^{-1,1}}^{\frac{2}{p}-1} \|\mu\|_{L^1}^{2-\frac{2}{p}}$, where L^1 norm is understood to mean the total variation if μ is a measure, together with the fact that

$$\|\omega(u) - 2\pi \sum_{l=1}^{K_\sigma} d_{l,\sigma}\delta_{a_{l,\sigma}}\|_{L^1} \leq \int_S (|Du|_g^2 + |\kappa|) \operatorname{vol}_g + 2\pi n \leq cnT\pi|\log\varepsilon|,$$

provided that $\varepsilon < 1/2$. This follows easily from (6), (56) and Lemma 5.3. Also, (52) holds for $\omega(\tilde{u})$ (as the interpolation argument works for $\omega(\tilde{u})$ exactly as for $\omega(u)$). Discarding the points $a_{l,\sigma}$ with zero degree $d_{l,\sigma} = 0$, one may assume that in (52) all the integers d_k are nonzero.

Step 4. Proof of (54). For any $k \in \{1, \ldots, 2\mathfrak{g}\}$, it follows from Step 2 that

$$\Phi_k(u) = \int_S (j(\tilde{u}), \eta_k)_g \operatorname{vol}_g + O(\varepsilon E_\varepsilon^{in}(u)) = \Phi_k(\tilde{u}) + O(\varepsilon E_\varepsilon^{in}(u)). \tag{60}$$

Since $h^2(|u|_g)|u|_g \leq c$, we have $|j(\tilde{u})|_g \leq c|Du|_g \in L^2$. As $\tilde{u}$ is smooth in S, the Hodge decomposition (37) implies

$$j(\tilde{u}) = d\xi + d^*\tilde{\psi} + \sum_{k=1}^{2\mathfrak{g}} \tilde{\Phi}_k \eta_k, \qquad \tilde{\Phi}_k := \Phi_k(\tilde{u}) \tag{61}$$

for some smooth function ξ and 2-form $\tilde{\psi}$. Taking the exterior derivative of this, we find that

$$-\Delta\tilde{\psi} + \kappa \operatorname{vol}_g = \omega(\tilde{u}) \quad \text{in} \quad S,$$

and the RHS is supported in $\cup_{l=1}^{K_\sigma} B_{l,\sigma}$ (see Step 2). As in Step 4 of the proof of Theorem 2.1, for some $r > 0$ (small but fixed, independent of ε) and every small enough $\varepsilon > 0$ (and hence also $\sigma = \varepsilon^q$), we fix Lipschitz paths λ_k, for $k = 1, \dots, 2\mathfrak{g}$ such that

$$\lambda_k \cap \left(\cup_{l=1}^{K_\sigma} B_{l,\sigma}\right) = \emptyset, \qquad \mathrm{dist}_S(\lambda_k, \cup_{l=1}^{K_\sigma} B_{l,\sigma}) \geq r \text{ if } d_{l,\sigma} \neq 0,$$

and

$$\lambda_k \text{ is homologous to } \gamma_k, \qquad \text{and} \qquad \mathcal{H}^1(\lambda_k) \leq c$$

for all $k \in \{1, \dots, 2\mathfrak{g}\}$ (recall that $\{\gamma_k\}$ are the curves fixed in Lemma 5.2). The point is that, as in Theorem 2.1, we obtain λ_k by starting with γ_k and modifying it as necessary, first to make it disjoint from all $B_{l,\sigma}$, increasing the arclength by at most $2\pi(n+1)\sigma$ due to (57); and next to arrange that it is always a distance at least r from every ball with nonzero degree $d_{l,\sigma}$. Since the number of such balls is at most n, due to (56) this can be done in such a way that the arclength increases by a controlled amount, for example $2\pi n r$. If r and σ are small enough, these modifications preserve the homology class. We next define

$$\tilde{\zeta}_k := \int_{\lambda_k} (d^*\tilde{\psi} + A) \ \in \mathbb{R}/2\pi\mathbb{Z}$$

where A is the connection 1-form associated to any moving frame defined in a neighborhood of λ_k. It follows from (61) and Lemma 6.4 (as $|\tilde{u}|_g = 1$ outside $\cup_{l=1}^{K_\sigma} B_{l,\sigma}$) that for $k = 1, \dots, 2\mathfrak{g}$

$$\tilde{\zeta}_k + \sum_{\ell=1}^{2\mathfrak{g}} \alpha_{k\ell}\tilde{\Phi}_\ell \ = \ \int_{\lambda_k} (d^*\psi + A + \sum_{\ell=1}^{2\mathfrak{g}} \tilde{\Phi}_\ell \eta_\ell) \ = \int_{\lambda_k} j(\tilde{u}) + A - \underbrace{\int_{\lambda_k} d\xi}_{=0} = \ 0 \quad \mathrm{mod}\ 2\pi,$$

where $(\alpha_{k\ell})$ were defined in (13). Let us write $(\alpha^{k\ell})$ to denote the inverse of $(\alpha_{k\ell})$. Denoting $d^\sigma = (d_{1,\sigma}, \dots, d_{K_\sigma,\sigma})$, as $d^\sigma = \{d_{l,\sigma}\}_{l=1}^{K_\sigma}$ satisfy (7), we may consider the unique solution $\psi = \psi(a^\sigma; d^\sigma)$ of (10) of zero mean on S, i.e.,

$$-\Delta\psi = -\kappa \,\mathrm{vol}_g + 2\pi \sum_{l=1}^{K_\sigma} d_{l,\sigma}\delta_{a_{l,\sigma}} \quad \text{in} \quad S.$$

Considering $\zeta_\ell(a; d)$ given by (16) with $a = (a_{1,\sigma}, \dots, a_{K_\sigma,\sigma})$ and $d = (d_{1,\sigma}, \dots, d_{K_\sigma,\sigma})$, we deduce that

$$\zeta_\ell(a; d) + \sum_{k=1}^{2\mathfrak{g}} \alpha_{\ell k} \Bigg[\underbrace{\tilde{\Phi}_k + \sum_{m=1}^{2\mathfrak{g}} \alpha^{km}(\tilde{\zeta}_m - \zeta_m(a; d))}_{=:\hat{\Phi}_k}\Bigg] = 0 \quad \mathrm{mod}\ 2\pi \qquad \text{for } \ell = 1, \dots, 2\mathfrak{g},$$

which implies that the vector in square brackets $\{\hat{\Phi}_k\}_{k=1}^{2\mathfrak{g}}$ belongs to $\mathcal{L}(a; d)$. Hence, in view of (60), we have that

$$\mathrm{dist}_{\mathbb{R}^{2\mathfrak{g}}}(\Phi(u), \mathcal{L}(a; d)) \leq c\varepsilon E_\varepsilon^{in}(u) + c \sup_\ell |\tilde{\zeta}_\ell - \zeta_\ell(a; d)|. \tag{62}$$

To estimate $\tilde{\zeta}_\ell - \zeta_\ell(a; d)$, we investigate the equation

$$-\Delta(\tilde{\psi} - \psi) = \omega(\tilde{u}) - 2\pi \sum_{l=1}^{K_\sigma} d_{l,\sigma}\delta_{a_{l,\sigma}}.$$

Thus, for any $p \in (1, 2)$, we see from Step 2 and elliptic regularity that

$$\|\tilde{\psi} - \psi\|_{W^{1,p}} \leq C_p(n+1)T|\log\varepsilon|\varepsilon^{q(\frac{2}{p}-1)}.$$

Also, $\tilde{\psi} - \psi$ is harmonic away from $\cup_{k=1}^{K_\sigma} B_{k,\sigma}$, so we further deduce from standard elliptic theory that for $r > 0$ fixed above,

$$\|\tilde{\psi} - \psi\|_{C^1(\{x\in S\,:\,\mathrm{dist}_S(x,\cup B_{k,\sigma})>r\})} \le C_{p,r}(n+1)T|\log\varepsilon|\varepsilon^{q(\frac{2}{p}-1)}.$$

In particular this estimate holds on λ_ℓ for every $\ell = 1, \dots, 2\mathfrak{g}$. Thus, as a direct consequence of the definitions of $\tilde{\zeta}_\ell$ and $\zeta_\ell(a;d)$, we obtain for a fixed small $r > 0$:

$$\left|\tilde{\zeta}_\ell - \zeta_\ell(a;d)\right| = \left|\int_{\lambda_\ell} d^*(\tilde{\psi} - \psi)\right| \le C_p(n+1)T|\log\varepsilon|\varepsilon^{q(\frac{2}{p}-1)} .$$

For any $\tilde{q} \in (0, 1 - \frac{T}{n+1})$, one chooses some $q \in (\tilde{q}, 1 - \frac{T}{n+1})$ and $p \in (1,2)$ close to 1 so that $(n+1)T|\log\varepsilon|\varepsilon^{q(\frac{2}{p}-1)} \le \varepsilon^{\tilde{q}}$ for some $\varepsilon \le \varepsilon_{\tilde{q}}$ and the above inequality and (62) together yield (54) for $\tilde{q}$. □

As a direct consequence, we have partially the point 1 in Theorem 2.6, together with a lower bound (at the first order) of the intrinsic energy:

Corollary 8.3. *Let $(u_\varepsilon)_{\varepsilon\downarrow 0}$ be a sequence of vector fields in $\mathcal{X}^{1,2}(S)$ satisfying* (51) *for some fixed $T, C > 0$. Then there exists a subsequence for which the vorticities $\omega(u_\varepsilon)$ converge in $W^{-1,p}(S)$, for all $1 \le p < 2$, to a limit of the form $2\pi\sum_{k=1}^K d_k\delta_{a_k}$ for K distinct points $a_1, \dots a_K \in S$ and nonzero $d_1, \dots, d_K \in \mathbb{Z}$ with* (7) *and $\sum_{k=1}^K |d_k| \le T$ (so, $K \le T$). Moreover,*

$$\liminf_{\varepsilon\to 0} \frac{1}{\pi|\log\varepsilon|} E_\varepsilon^{in}(u_\varepsilon) \ge \sum_{k=1}^K |d_k|.$$

Proof. Fix the integer n satisfying $n + 1 > T \ge n$ and $q \in (0, 1 - \frac{T}{n+1})$. By Step 1 in the proof of Proposition 8.1, we may assume that u_ε are smooth vector fields with $|u_\varepsilon|_g \le 1$ in S. Furthermore, for each $\varepsilon > 0$, as in the proof of Proposition 8.1, we consider $\sigma = \varepsilon^q$ and the set of pairwise disjoint balls $\cup_{l=1}^{K_\varepsilon} B_{l,\varepsilon}$ of center $\{a_{l,\varepsilon}\}_l$ associated to u_ε such that $d_{l,\varepsilon}$ is the degree of u_ε on $\partial B_{l,\varepsilon}$ satisfying (7). Moreover, $\sum_l |d_{l,\varepsilon}| \le n$ which entails that for a subsequence $\varepsilon \downarrow 0$, there exist points $a_1, \dots, a_K \in S$ (not necessarily distinct) and $d_1, \dots, d_K \in \mathbb{Z}$ such that the measures $\mu_\varepsilon := 2\pi\sum_{l=1}^{K_\varepsilon} d_{l,\varepsilon}\delta_{a_{l,\varepsilon}}$ converge to

$$\mu := 2\pi\sum_{l=1}^K d_l\delta_{a_l}$$

as measures, and thus, in $W^{-1,p}$ for any $p \in [1,2)$ (as $W^{1,\tilde{p}}(S)$ embeds in the space of continuous functions, for the conjugate real $\tilde{p} = \frac{p}{p-1} > 2$). Relabeling the indices, we may assume that $a = (a_1, \dots, a_K)$ are distinct and that $d_k \ne 0$ for $k = 1, \dots, K$. Obviously, (7) holds (as $\mu_\varepsilon(S)$ is preserved by the convergence, i.e, equal to $2\pi\chi(S)$), as well as the upper bound of the total variation of those measures is conserved leading to

$$\sum_{l=1}^K |d_l| = \frac{|\mu|(S)}{2\pi} \le \liminf_{\varepsilon\to 0} \frac{|\mu_\varepsilon|(S)}{2\pi} = \liminf_{\varepsilon\to 0}\sum_{l=1}^{K_\varepsilon} |d_{l,\varepsilon}| \le n \le T. \tag{63}$$

By (52), we conclude that $\omega(u_\varepsilon) \to \mu$ in any $W^{-1,p}$ for $p \in [1,2)$ as $\varepsilon \to 0$. Finally, the lower bound of the energy is obtain by (58) for $\sigma = \varepsilon^q$:

$$\liminf_{\varepsilon\to 0} \frac{1}{\pi|\log\varepsilon|} E_\varepsilon^{in}(u_\varepsilon) \geq \liminf_{\varepsilon\to 0} \frac{1}{\pi|\log\varepsilon|} \sum_{k=1}^{K_\varepsilon} \int_{B_{l,\varepsilon}} e_\varepsilon^{in}(u_\varepsilon)\,\mathrm{vol}_g$$

$$\geq \liminf_{\varepsilon\to 0} \sum_{1\leq l\leq K_\varepsilon,\, d_{l,\varepsilon}\neq 0} (1-q)|d_{l,\varepsilon}| \geq (1-q)\sum_{l=1}^{K} |d_l| = (1-q)\frac{|\mu|(S)}{2\pi},$$

where we used (63). As μ is the limit of $\omega(u_\varepsilon)$ (so independent of q), passing to the limit $q \to 0$, the conclusion is straightforward. □

Remark 8.4. At this stage, we cannot conclude that the sequence $\{\Phi(u_\varepsilon)\}_{\varepsilon\downarrow 0}$ is bounded as large oscillations might arise a-priori in the current $j(u_\varepsilon)$. To handle this difficulty, we need to insure that the excess of energy away from vortices is of order $O(1)$ (see Proposition 9.1).

9. Renormalized energy as a Γ-limit in the intrinsic case. Proof of Theorem 2.6

In this section, we focus on the situation where all vortices have degree ± 1 and the excess of energy away from vortices is of order $O(1)$. We will prove that the flux integrals converge and that we have a stronger lower bound (than the one stated in Theorem 2.6, point 2). This is typically the situation when the vector fields u_ε are minimizers of E_ε^{in} (or energetically close to minimizing configurations). The following Proposition together with Corollary 8.3 lead to the final conclusion of Theorem 2.6.

Proposition 9.1. *1) Let $(u_\varepsilon)_{\varepsilon\in(0,1)}$ be a family of vector fields in $\mathcal{X}^{1,2}(S)$ satisfying*

$$E_\varepsilon^{in}(u_\varepsilon) \leq n\pi|\log\varepsilon| + C \qquad \text{for every } \varepsilon \tag{64}$$

for some integer $n > 0$, and assume that there exist $n_0 (\leq n)$ distinct points $a_1, \dots, a_{n_0} \in S$, and nonzero integers $d_1, \dots, d_{n_0}$ satisfying (7) such that

$$\omega(u_\varepsilon) \overset{W^{-1,1}}{\longrightarrow} 2\pi \sum_{k=1}^{n_0} d_k \delta_{a_k} \qquad \text{with} \qquad \sum_{k=1}^{n_0} |d_k| = n. \tag{65}$$

Then $n_0 = n$ and $|d_k| = 1$ for every k, and there exists $\Phi \in \mathcal{L}(a; d)$ such that, after passing to a further subsequence if necessary,

$$\Phi(u_\varepsilon) \to \Phi, \qquad \Phi(u_\varepsilon) \text{ defined in (53).} \tag{66}$$

Moreover, for every $\sigma > 0$,

$$\liminf_{\varepsilon\to 0} \left[E_\varepsilon^{in}(u_\varepsilon) - n(\pi|\log\varepsilon| + \iota_F)\right] \geq W(a, d, \Phi) + \liminf_{\varepsilon\to 0} \int_{S\setminus\cup_{k=1}^n B_\sigma(a_k)} \left[\frac{1}{2}\Big|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j(u^*)\Big|_g^2 + e_\varepsilon^{in}(|u_\varepsilon|_g)\right] vol_g \tag{67}$$

for $u^* = u^*(a, d, \Phi)$, $a = (a_1, \dots, a_n)$ and $d = (d_1, \dots, d_n)$.
2) Conversely, for every distinct $a_1, \dots, a_n$ and $d_1, \dots, d_n \in \{\pm 1\}$ satisfying (7), and every $\Phi \in \mathcal{L}(a; d)$ there exist sequences of smooth vector fields u_ε such that $|u_\varepsilon|_g \leq 1$ in S, (65) and (66) hold and

$$E_\varepsilon^{in}(u_\varepsilon) - n(\pi|\log\varepsilon| + \iota_F) \to W(a, d, \Phi) \quad \text{as } \varepsilon \to 0. \tag{68}$$

9.1. **Useful coordinates.** It will be useful to carry out certain computations in exponential normal coordinates near certain points (typically, one of the points $P \in S$ about which $\omega(u_\varepsilon)$ concentrates). These are defined by the map $y \in \mathbb{R}^2 \mapsto \exp_P(y_1\tau_{1,P} + y_2\tau_{2,P}) =: \Psi(y)$, where $\{\tau_{1,P}, \tau_{2,P}\}$ is an orthonormal basis for T_PS. This map is a diffeomorphism when restricted to a suitable neighborhood of the origin in $\mathbb{R}^2$. In this neighborhood,

$$x = \Psi(y) \ \Rightarrow \ \mathrm{dist}_S(P, x) = |y|, \qquad \text{so } B_r(P) \cong \{y \in \mathbb{R}^2 : |y| < r\}.$$

Here $|y|$ denotes the Euclidean norm of $y \in \mathbb{R}^2$. We will write $g_{lk}(y) := (\partial_l\Psi(y), \partial_k\Psi(y))_g$ to denote the components of the metric tensor in this coordinate system (where we identify $\partial_l\Psi(y)$ with an element of $T_{\Psi(y)}S$ in the natural way). It is then a standard fact that

$$g_{lk}(y) := \delta_{lk} + O(|y|^2), \qquad \text{and hence} \quad g(y) := \det(g_{lk}(y)) = 1 + O(|y|^2). \tag{69}$$

Furthermore, we can also find a moving frame $\{\tau_1, \tau_2\}$ near P such that the connection 1-form A satisfies

$$|A(\Psi(y))|_g = O(|y|). \tag{70}$$

Indeed, (70) can be achieved by starting with an arbitrary moving frame $\{\tau_1, \tau_2\}$ near P, and replacing it by $\{e^{i\phi}\tau_1, e^{i\phi}\tau_2\}$ for a suitable function ϕ.

For any point $P \in S$ there is a $\sigma > 0$ such that both normal coordinates and the above moving frame are defined in $B_\sigma(P) \subset S$. Thus, given a vector field u, in this neighborhood we can define $v = v_1 + iv_2 : B_\sigma(0) \subset \mathbb{R}^2 \to \mathbb{C}$ by requiring that

$$u(\Psi(y)) = v(y)\tilde\tau_1(y) = v_1(y)\tilde\tau_1(y) + v_2(y)\tilde\tau_2(y), \qquad \tilde\tau_k(y) = \tau_k(\Psi(y)),\ k = 1, 2. \tag{71}$$

We will write $|v| = (v, v)^{1/2} := (v_1^2 + v_2^2)^{1/2}$, so that $|v(y)| = |u(\Psi(y))|_g$. We will also write the energy density $e_\varepsilon(v)$ and the current $j(v)$ to denote the Euclidean quantities

$$e_\varepsilon(v) := \frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2}F(|v|^2), \qquad j(v) := \sum_{k=1}^{2}(iv, \partial_{y_k}v)dy_k,$$

where here all norms and inner products $(\cdot, \cdot)$ are understood with respect to the Euclidean structure on $\mathbb{R}^2$, with respect to which the 1-forms $\{dy_1, dy_2\}$ are orthonormal. It is then routine to check that

$$\begin{aligned} e_\varepsilon^{in}(u)(\Psi(y)) &= [1 + O(|y|^2)]e_\varepsilon(v)(y) + O(|v|^2(y)) \\ \Psi^* j(u) &= j(v) + O(|y|)|v|^2 \\ \Psi^* \mathrm{vol}_g &= [1 + O(|y|^2)]dy \end{aligned} \tag{72}$$

where $dy = dy_1 \wedge dy_2$ denotes the Euclidean area element. Thus for example

$$\int_{B_\sigma(P)} e_\varepsilon^{in}(u)\ \mathrm{vol}_g = (1 + O(\sigma^2)) \int_{\{y \in \mathbb{R}^2 : |y| < \sigma\}} e_\varepsilon(v) + O(|v|^2)\ dy\ . \tag{73}$$

9.2. **Upper bound.** Given F satisfying (1), we recall the notations (25) for the intrinsic energy of the radial vortex profile $I_F^{in}(R, \varepsilon)$ as well as the limit ι_F defined in (26). The coordinate system described above will allow us to reduce energy estimates on small balls to classical facts about the Ginzburg-Landau energy in the Euclidean setting. We first use this reduction to prove the upper bound part of Proposition 9.1.

Proof of Proposition 9.1, point 2). Recall that we constructed a canonical harmonic unit vector field $u^* = u^*(x; a, d, \Phi)$ in Theorem 2.1. We will construct an appropriate vector field $U_\varepsilon = U_\varepsilon(a, d, \Phi)$ for the upper bound in Proposition 9.1, point 2) as follows: first, we choose

$$U_\varepsilon := u^* \text{ in } S_{\sqrt{\varepsilon}} := S \setminus \cup_{k=1}^n B(a_k, \sqrt{\varepsilon}).$$

In order to define U_ε inside the balls $B(a_k, \sqrt{\varepsilon})$, we need to prove that u^* has the appropriate behavior at the boundary $\partial B(a_k, \sqrt{\varepsilon})$ which is done in the next step.

Step 1. Estimating u^* on $\partial B(a_k, \sqrt{\varepsilon})$. Writing $j^* := j(u^*)$, by (12), (20), and properties of the Green's function G (see Section 2.2), we have in a neighborhood of the vortices a_k:

$$\begin{aligned} j^*(x) &= d^*[2\pi d_k G(x, a_k)\,\mathrm{vol}_g + \text{smooth terms}] \\ &= d^*[-d_k \log(\mathrm{dist}_S(x, a_k))\,\mathrm{vol}_g + C^1 \text{ terms}] \\ &= \star d[d_k \log(\mathrm{dist}_S(x, a_k))] + C^0 \text{ terms}. \end{aligned} \tag{74}$$

Let $v^* : B(0, \sqrt{\varepsilon}) \to \mathbb{S}^1$ be the representation of u^* in exponential normal coordinates near a_k given by (71). Since within these coordinates $\Psi^* j(u^*) = j(v^*) - \Psi^* A$, near a_k, we deduce that

$$j(v^*) = d_k d\theta + C^0 \text{ terms} \quad \text{in } B(0, \sqrt{\varepsilon}),$$

where $d\theta$ is the angular 1-form $d\theta := \frac{1}{|y|^2}(y_1 dy_2 - y_2 dy_1)$. In particular, we have that

$$v^* = e^{i(d_k\theta + \eta)} \text{ on } \partial B(0, \sqrt{\varepsilon}), \tag{75}$$

for a C^1 function $\eta : \partial B(0, \sqrt{\varepsilon}) \to \mathbb{R}$ that we write $\eta = \eta(\theta)$ with the angular derivative $|\partial_\theta \eta| \le C\sqrt{\varepsilon}$. Moreover, as $|d_k| = 1$, it follows that

$$|j(v^*)|^2(y) = \frac{1}{|y|^2} + O(\frac{1}{|y|}) \quad \text{in } B(0, \sqrt{\varepsilon}).$$

Step 2. Defining U_ε inside the ball $B(a_k, \sqrt{\varepsilon})$. We define $V_\varepsilon := v^* = e^{i(d_k\theta + \eta)}$ on $\partial B(0, \sqrt{\varepsilon})$. Setting $\bar{\eta}$ to be the mean of η over $\partial B(0, \sqrt{\varepsilon})$, we define V_ε inside the annulus $B(0, \sqrt{\varepsilon}) \setminus B(0, \frac{\sqrt{\varepsilon}}{2})$ by linear interpolation in the lifting as follows:

$$V_\varepsilon(re^{i\theta}) = e^{i[d_k\theta + \bar{\eta} + 2(\frac{r}{\sqrt{\varepsilon}} - \frac{1}{2})(\eta - \bar{\eta})]} \quad \text{for } r \in (\frac{\sqrt{\varepsilon}}{2}, \sqrt{\varepsilon}).$$

Finally, as $|d_k| = 1$, we define V_ε inside the ball $B(0, \frac{\sqrt{\varepsilon}}{2})$ as being a minimizer of $I_F^{in}(\frac{\sqrt{\varepsilon}}{2}, \varepsilon)$ if $d_k = 1$ (or its complex conjugate if $d_k = -1$) up to a rotation of angle $\bar{\eta}$. The minimizing property of V_ε implies that $|V_\varepsilon| \le 1$ everywhere (by cutting off at 1). Through the normal coordinates (71), we define U_ε to be the corresponding vector field to V_ε inside the ball $B(0, \sqrt{\varepsilon})$. Note that by construction $U_\varepsilon \in \mathcal{X}^{1,2}(S)$ (in fact, it is Lipschitz since every minimizer in $I_F^{in}(\frac{\sqrt{\varepsilon}}{2}, \varepsilon)$ is Lipschitz) and $|U_\varepsilon|_g \le 1$ in S.

Step 3. Estimating the energy of U_ε and $j(U_\varepsilon)$ inside the ball $B(a_k, \sqrt{\varepsilon})$. First, by definition of V_ε inside the ball $B(0, \frac{\sqrt{\varepsilon}}{2})$, we obtain via (26)

$$\int_{B(0, \frac{\sqrt{\varepsilon}}{2})} e_\varepsilon(V_\varepsilon)\, dy = \pi \log \frac{\sqrt{\varepsilon}}{2\varepsilon} + \iota_F + o(1).$$

Second, inside the annulus $B(0, \sqrt{\varepsilon}) \setminus B(0, \frac{\sqrt{\varepsilon}}{2})$, since $|d_k| = 1$, we have

$$\begin{aligned} \int_{B(0,\sqrt{\varepsilon})\setminus B(0,\frac{\sqrt{\varepsilon}}{2})} \frac{1}{2}|\nabla V_\varepsilon|^2\, dy &= \int_{\frac{\sqrt{\varepsilon}}{2}}^{\sqrt{\varepsilon}} \int_0^{2\pi} \frac{1}{2r}\Big|d_k + 2(\frac{r}{\sqrt{\varepsilon}} - \frac{1}{2})\partial_\theta \eta\Big|^2 + \frac{2r}{\varepsilon}|\eta - \bar{\eta}|^2\, d\theta dr \\ &\le \pi \int_{\frac{\sqrt{\varepsilon}}{2}}^{\sqrt{\varepsilon}} \frac{1}{r}(1 + O(\sqrt{\varepsilon}))\, dr + \int_0^{2\pi} |\partial_\theta \eta|^2\, d\theta = \pi \log 2 + o(1) \end{aligned} \tag{76}$$

where we used the Poincaré inequality and $|\partial_\theta \eta| \le C\sqrt{\varepsilon}$. Finally, by (69) and (72), we compute

$$\int_{B(a_k,\sqrt{\varepsilon})} e_\varepsilon^{in}(U_\varepsilon)\ \mathrm{vol}_g = \int_{\{y\in\mathbb{R}^2:|y|<\sqrt{\varepsilon}\}} [(1+O(\varepsilon))e_\varepsilon(V_\varepsilon)+O(1)]\sqrt{g(y)}dy$$
$$\le \pi\log\frac{\sqrt{\varepsilon}}{\varepsilon} + \iota_F + o(1) \quad \text{as } \varepsilon\to 0.$$

To estimate the current $j(U_\varepsilon)$, note that since $|V_\varepsilon|\le 1$ everywhere,

$$|j(V_\varepsilon)|^2 \le |V_\varepsilon|^2|\nabla V_\varepsilon|^2 \le |\nabla V_\varepsilon|^2 \le 2e_\varepsilon(V_\varepsilon).$$

Thus for every $p\in[1,2)$, we have by Hölder's inequality

$$\int_{B(0,\sqrt{\varepsilon})} |j(V_\varepsilon)|^p\,dy \le \int_{B(0,\sqrt{\varepsilon})} 2^{p/2}e_\varepsilon(V_\varepsilon)^{p/2}\,dy \le |B(0,\sqrt{\varepsilon})|^{1-\frac{p}{2}}\left(\int_{B(0,\sqrt{\varepsilon})} 2e_\varepsilon(V_\varepsilon)\,dy\right)^{p/2} \to 0.$$

Combined with the equality $\Psi^* j(U_\varepsilon) = j(V_\varepsilon) - |V_\varepsilon|^2\Psi^* A$ near a_k, as $|V_\varepsilon|\le 1$ everywhere, we conclude that $\int_{B(a_k,\sqrt{\varepsilon})}|j(U_\varepsilon)|^p \to 0$ for every $p\in[1,2)$ as $\varepsilon\to 0$.

Step 4. Conclusion. Using the definition of $W(a,d,\Phi)$ in Section 2.2 and Step 3, we compute

$$E_\varepsilon^{in}(U_\varepsilon) = \int_{S_{\sqrt{\varepsilon}}} \frac{1}{2}|Du^*|_g^2\ \mathrm{vol}_g + \sum_{k=1}^n \int_{B(a_k,\sqrt{\varepsilon})} e_\varepsilon^{in}(U_\varepsilon)\ \mathrm{vol}_g$$
$$= W(a,d,\Phi) + n\pi\log\frac{1}{\sqrt{\varepsilon}} + o(1) + \pi n\log\frac{\sqrt{\varepsilon}}{\varepsilon} + n\iota_F + o(1)$$
$$= W(a,d,\Phi) + n(\pi\log\frac{1}{\varepsilon}+\iota_F) + o(1) \quad \text{as } \varepsilon\to 0.$$

As $U_\varepsilon := u^*$ in $S_{\sqrt{\varepsilon}}$, by Steps 1 and 3, we deduce that

$$j(U_\varepsilon) - j(u^*) \to 0 \quad \text{strongly in} \quad L^p(S) \tag{77}$$

for $p\in[1,2)$ which entails $dj(U_\varepsilon)\to dj(u^*)$ strongly in $W^{-1,p}(S)$ for $p\in[1,2)$, in particular (65) and (66) hold where $\Phi\in\mathcal{L}(a;d)$ was given in the hypothesis as the flux integrals associated to u^* by (53). As U_ε is not smooth, using the smoothness argument in Step 1 of the proof of Proposition 8.1, U_ε can be replaced by a smooth vector field u_ε with the desired properties.

□

9.3. **Lower bound. Throughout most of this section, we assume that $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is a sequence of *smooth* vector fields with $|u_\varepsilon|_g\le 1$ in S satisfying the hypotheses (64) and (65) of Proposition 9.1 point 1)** (the smoothness assumption follows by the argument in Step 1 of the proof of Proposition 8.1). We will drop the assumed bound on $|u_\varepsilon|_g$ only in Step 6 in the proof of Proposition 9.1 point 1), where we explain how to get the lower bound in the general case. All constants appearing in our estimates may depend on S, n, and the constant C in (64). Our first lemma allows us to approximate the vorticity $\omega(u_\varepsilon)$ by a sum of point masses that are well-separated, relative to the scale of the approximation.

Lemma 9.2. *There exists $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0)$, we can find $r_\varepsilon\in(\varepsilon^{\frac{1}{2(n+1)}},\varepsilon^\beta)$ for some $\beta=\beta(n)>0$ and $K=K(\varepsilon)\in\mathbb{Z}_+$ distinct points $a^\varepsilon=(a_{1,\varepsilon},\dots,a_{K,\varepsilon})$ in S and* *nonzero integers* $d^\varepsilon = (d_{1,\varepsilon}, \ldots, d_{K,\varepsilon})$ *with* (7) *such that* $\sum_{k=1}^K |d_{k,\varepsilon}| \le n$ *(so,* $K \le n$*) and* (78)

$$\|\omega(u_\varepsilon) - 2\pi \sum_{k=1}^K d_{k,\varepsilon}\delta_{a_{k,\varepsilon}}\|_{W^{-1,1}} \le r_\varepsilon^2, \qquad \mathrm{dist}_S(a_{k,\varepsilon}, a_{l,\varepsilon}) \ge \sqrt{r_\varepsilon} \text{ for all } 1 \le k < l \le K.$$

In addition, there exists $\Phi^\varepsilon \in \mathcal{L}(a^\varepsilon, d^\varepsilon)$ *such that* $|\Phi(u_\varepsilon) - \Phi^\varepsilon| \le C\sqrt{r_\varepsilon}$*.*

Proof. Let $0 < q < \frac{1}{n+1}$ and $\sigma_1 = \varepsilon^{q/2}$ Apply now Proposition 8.1 for $T = n$, $p = 1$ and q, and consider the $K(\le n)$ distinct points $a_{k,\varepsilon}$ and nonzero integers $d_{k,\varepsilon}$ provided by it $(1 \le k \le K)$. We know by (54) that $\mathrm{dist}_{\mathbb{R}^{2\mathfrak{g}}}(\Phi(u_\varepsilon), \mathcal{L}(a^\varepsilon, d^\varepsilon)) \le \sigma_1^2$. Set the associated measure $\mu_1(a^\varepsilon, d^\varepsilon) = 2\pi\sum_k d_{k,\varepsilon}\delta_{a_{k,\varepsilon}}$. If $\mathrm{dist}_S(a_{k,\varepsilon}, a_{l,\varepsilon}) \ge \sqrt{\sigma_1}$ for all $k \ne \ell$, then this collection satisfies (78) with $r_\varepsilon = \sigma_1$, for small enough ε. If not, define a new collection of points as follows: consider some pair $a_{l,\varepsilon} \ne a_{\ell,\varepsilon}$ such that $\mathrm{dist}_S(a_{l,\varepsilon}, a_{\ell,\varepsilon}) < \sqrt{\sigma_1}$. Remove this pair from $\{a_{k,\varepsilon}\}$ and replace them by a point P with the associated degree $d = d_{l,\varepsilon} + d_{\ell,\varepsilon}$ such that $\mathrm{dist}_S(P, a_{l,\varepsilon}) < \frac{1}{2}\sqrt{\sigma_1}$ and $\mathrm{dist}_S(P, a_{\ell,\varepsilon}) < \frac{1}{2}\sqrt{\sigma_1}$. The total sum of absolute values of the new degrees could decrease, so it stays $\le n$. Note that

$$\|(d_{l,\varepsilon}\delta_{a_{l,\varepsilon}} + d_{\ell,\varepsilon}\delta_{a_{\ell,\varepsilon}}) - (d_{l,\varepsilon} + d_{\ell,\varepsilon})\delta_P\|_{W^{-1,1}} \le (|d_{l,\varepsilon}| + |d_{\ell,\varepsilon}|)\frac{\sqrt{\sigma_1}}{2} \le n\frac{\sqrt{\sigma_1}}{2}. \tag{79}$$

Continue in this fashion until a new collection is reached (still denoted $\{a_{k,\varepsilon}\}$ and where the points of zero degree are suppressed) such that $\mathrm{dist}_S(a_{l,\varepsilon}, a_{\ell,\varepsilon}) \ge \sqrt{\sigma_1}$ for all distinct $a_{l,\varepsilon} \ne a_{\ell,\varepsilon}$. This takes at most $K - 1 \le n - 1$ of the above steps because at each step the number of points decreases. It follows from (79) that

$$\|\omega(u_\varepsilon) - 2\pi \sum_{k=1}^K d_{k,\varepsilon}\delta_{a_{k,\varepsilon}}\|_{W^{-1,1}} \le \sigma_1^2 + n(n-1)\frac{\sqrt{\sigma_1}}{2} \le n^2\sqrt{\sigma_1} =: \sigma_2^2.$$

Denoting μ_2 the measure associated to this new collection of points $a_{k,\varepsilon}$ and degrees $d_{k,\varepsilon}$, we note that $\|\mu_1 - \mu_2\|_{W^{-1,1}} \le \|\mu_1 - \omega(u_\varepsilon)\|_{W^{-1,1}} + \|\omega(u_\varepsilon) - \mu_2\|_{W^{-1,1}} \le (n^2+1)\sqrt{\sigma_1}$. If, for this collection, $\mathrm{dist}_S(a_{\ell,\varepsilon}, a_{l,\varepsilon}) \ge \sqrt{\sigma_2}$ for all $l \ne \ell$, then again we are finished. If not, we continue in the same fashion. Within (at most) $n-1$ iterations of this procedure, we obtain a collection of points satisfying (78) for some $r_\varepsilon \le C(n)\varepsilon^\beta$ for some (large) $C(n)$ and (small) positive β. By decreasing $\beta > 0$ we may suppose that $C(n) = 1$. Moreover, if $\tilde{a}^\varepsilon$ is the final collection of points with the nonzero degrees $\tilde{d}^\varepsilon$, denoting $\tilde{\mu}(\tilde{a}^\varepsilon, \tilde{d}^\varepsilon)$ the associated measure, we have that $\|\mu_1 - \tilde{\mu}\|_{W^{-1,1}} \le C\sqrt{r_\varepsilon}$. Now we use (18) and (54) to conclude that

$$\mathrm{dist}_{\mathbb{R}^{2\mathfrak{g}}}(\Phi(u_\varepsilon), \mathcal{L}(\tilde{a}^\varepsilon, \tilde{d}^\varepsilon)) \le \mathrm{dist}_{\mathbb{R}^{2\mathfrak{g}}}(\Phi(u_\varepsilon), \mathcal{L}(a^\varepsilon, d^\varepsilon)) + \mathrm{dist}_{\mathbb{R}^{2\mathfrak{g}}}(\mathcal{L}(a^\varepsilon, d^\varepsilon), \mathcal{L}(\tilde{a}^\varepsilon, \tilde{d}^\varepsilon)) \le C\sqrt{r_\varepsilon}.$$

□

Our next lemma provides a good lower energy bound away from the vortices. This will be used several times in the proof of the compactness and lower bound assertions of Proposition 9.1.

Lemma 9.3. *Using the notations in Lemma 9.2, let* $a^\varepsilon = (a_{1,\varepsilon}, \ldots, a_{K,\varepsilon})$, $d^\varepsilon = (d_{1,\varepsilon}, \ldots, d_{K,\varepsilon})$ *satisfy* (78) *for some* $r_\varepsilon \in (\varepsilon^{\frac{1}{2(n+1)}}, \varepsilon^\beta)$ *for some* $\beta = \beta(n) > 0$ *and* $\Phi^\varepsilon = (\Phi_{k,\varepsilon})_{k=1}^{2\mathfrak{g}} \in \mathcal{L}(a^\varepsilon, d^\varepsilon)$ *such that* $|\Phi(u_\varepsilon) - \Phi^\varepsilon| \le C\sqrt{r_\varepsilon}$*. Let* $u^*(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon)$ *be a canonical harmonic vector field given in Theorem 2.1 with the associated current* $j_\varepsilon^* := j(u^*(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon))$*. Then for* *all sufficiently small* $\varepsilon > 0$*,*

$$\int_{S_{r_\varepsilon}} e_\varepsilon^{in}(u_\varepsilon)\, vol_g \geq \pi(\sum_{k=1}^K d_{k,\varepsilon}^2)\log\frac{1}{r_\varepsilon} + W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon)$$

$$+\int_{S_{r_\varepsilon}} \Big(\frac{1}{2}\left|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*\right|_g^2 + e_\varepsilon^{in}(|u_\varepsilon|_g)\Big)\, vol_g - O(r_\varepsilon^{1/3}) - O(r_\varepsilon^{1/2}|\Phi^\varepsilon|^2) \tag{80}$$

for $S_{r_\varepsilon} := S \setminus \cup_{k=1}^K B_{r_\varepsilon}(a_{k,\varepsilon})$*.*

Proof. The proof uses some arguments from [27], Theorem 2. First, we use Section 5.5 and elementary algebra to find that

$$e_\varepsilon^{in}(u_\varepsilon) = \frac{1}{2}|j_\varepsilon^*|_g^2 + \frac{1}{2}\left|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*\right|_g^2 + (j_\varepsilon^*, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*)_g + e_\varepsilon^{in}(|u_\varepsilon|_g) \quad \text{in } S_{r_\varepsilon}. \tag{81}$$

In addition, by (50), we have that

$$\frac{1}{2}\int_{S_{r_\varepsilon}} |j_\varepsilon^*|_g^2\, \mathrm{vol}_g = \pi(\sum_k d_{k,\varepsilon}^2)\log\frac{1}{r_\varepsilon} + W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) + O(\sqrt{r_\varepsilon}) + O(r_\varepsilon^{3/2}|\Phi^\varepsilon|^2) \quad \text{as } \varepsilon \to 0\ .$$

After combining these, we find that to prove (80), it suffices to prove that

$$\left|\int_{S_{r_\varepsilon}} (j_\varepsilon^*, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*)_g\, \mathrm{vol}_g\right| = O({r_\varepsilon}^{1/3}) + O(r_\varepsilon^{1/2}|\Phi^\varepsilon|^2) \quad \text{as } \varepsilon \to 0. \tag{82}$$

Toward this end, we let $\psi_\varepsilon := \psi(a^\varepsilon, d^\varepsilon)$ be the solution of (10) and we start by using (12) to write

$$\int_{S_{r_\varepsilon}} (j_\varepsilon^*, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*)_g\, \mathrm{vol}_g = \int_{S_{r_\varepsilon}} (d^*\psi_\varepsilon, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*)_g\, \mathrm{vol}_g + \sum_{k=1}^{2\mathfrak{g}} \Phi_{k,\varepsilon}\int_{S_{r_\varepsilon}} (\eta_k, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*)_g\, \mathrm{vol}_g$$

$$=: L_0 + \sum_{k=1}^{2\mathfrak{g}} \Phi_{k,\varepsilon} L_k.$$

Step 1. Estimate of L_k *for* $k = 1, \dots, 2\mathfrak{g}$. We decompose

$$L_k = \int_S (\eta_k, j(u_\varepsilon) - j_\varepsilon^*)_g\ \mathrm{vol}_g + \int_S (\eta_k, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g})_g (1 - |u_\varepsilon|_g)\, \mathrm{vol}_g - \int_{S\setminus S_{r_\varepsilon}} (\eta_k, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*)_g\, \mathrm{vol}_g.$$

We estimate the terms on the right-hand side. First, as Φ^ε are the flux integrals associated to $u^*(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon)$, we deduce

$$|\int_S (\eta_k, j(u_\varepsilon) - j_\varepsilon^*)_g\, \mathrm{vol}_g| = |\Phi_k(u_\varepsilon) - \Phi_{k,\varepsilon}| = O(\sqrt{r_\varepsilon}).$$

Next, since by (39), $\left|\frac{|j(u_\varepsilon)|_g}{|u_\varepsilon|_g}(1 - |u_\varepsilon|_g)\right| \leq \big|1 - |u_\varepsilon|_g\big||Du_\varepsilon|_g \leq C\varepsilon e_\varepsilon^{in}(u_\varepsilon)$, it is clear that

$$\left|\int_S (\eta_k, \frac{j(u_\varepsilon)}{|u_\varepsilon|})_g (1 - |u_\varepsilon|_g)\ \mathrm{vol}_g\right| \leq C\|\eta_k\|_{L^\infty}\varepsilon|\log\varepsilon|.$$

We split the remaining term into two pieces. By Cauchy-Schwarz,

$$\left|\int_{\cup_{\ell=1}^{K_\varepsilon} B_{r_\varepsilon}(a_{\ell,\varepsilon})} (\eta_k, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g})_g\, \mathrm{vol}_g\right| \leq \left(\int_{\cup_{\ell=1}^{K_\varepsilon} B_{r_\varepsilon}(a_{\ell,\varepsilon})} |\eta_k|^2\, \mathrm{vol}_g \int_{\cup_{\ell=1}^{K_\varepsilon} B_{r_\varepsilon}(a_{\ell,\varepsilon})} |Du_\varepsilon|_g^2\, \mathrm{vol}_g\right)^{1/2}$$

$$= O(r_\varepsilon|\log\varepsilon|^{1/2}) = O({r_\varepsilon}^{1/2}).$$

Next, from (12), (20) and properties of the Green's function in Section 2.2 (in particular that $\|d^*G(\cdot,a_{k,\varepsilon})\|_{L^1(B_{r_\varepsilon}(a_{\ell,\varepsilon}))} = O(r_\varepsilon)$ for every k and ℓ), one readily checks that

$$\left|\int_{\cup_{\ell=1}^{K_\varepsilon} B_{r_\varepsilon}(a_{\ell,\varepsilon})} (\eta_k, j^*_\varepsilon)_g \operatorname{vol}_g\right| \le C\|\eta_k\|_{L^\infty}(r_\varepsilon + |\Phi^\varepsilon| r_\varepsilon^2 \|\eta_k\|_{L^\infty}).$$

By combining the above, we conclude that

$$|\Phi_{k,\varepsilon} L_k| = O\bigg((\sqrt{r_\varepsilon} + |\Phi^\varepsilon| r_\varepsilon^2)|\Phi^\varepsilon|\bigg) = O(\sqrt{r_\varepsilon}) + O(\sqrt{r_\varepsilon}|\Phi^\varepsilon|^2) \qquad \text{for every } k = 1,\dots,2\mathfrak{g}.$$

Step 2. Estimate of L_0. Next, with $\psi_\varepsilon := \psi(a^\varepsilon, d^\varepsilon)$ the 2-form solving (10), we define

$$\tilde\psi_\varepsilon(x) := \begin{cases} \psi_\varepsilon(x) & \text{in } S_{r_\varepsilon} \\ \psi_\varepsilon(x) + d_k(\log \operatorname{dist}_S(x, a_{\ell,\varepsilon}) - \log r_\varepsilon)\operatorname{vol}_g & \text{in } B_{r_\varepsilon}(a_{\ell,\varepsilon}),\ \ell = 1,\dots,K_\varepsilon. \end{cases}$$

Since $\operatorname{dist}_S(a_{l,\varepsilon}, a_{\ell,\varepsilon}) \ge \sqrt{r_\varepsilon}$, it follows from (20) and properties of the Green's function from Section 2.2 that $\tilde\psi_\varepsilon$ is Lipschitz continuous in S and C^1 in $\cup_{\ell=1}^{K_\varepsilon} B_{r_\varepsilon}(a_{\ell,\varepsilon})$, with Lipschitz constant bounded by $C/\sqrt{r_\varepsilon}$ in S. Thus we can write

$$d^*\tilde\psi_\varepsilon = \mathbf{1}_{S_{r_\varepsilon}} d^*\psi_\varepsilon + \xi_\varepsilon, \qquad \text{with} \quad \|d^*\tilde\psi_\varepsilon\|_{L^\infty(S)}, \|\xi_\varepsilon\|_{L^\infty(S)} \le C/\sqrt{r_\varepsilon}$$

where ξ_ε is a 1-form supported in $\cup_{\ell=1}^{K_\varepsilon} B_{r_\varepsilon}(a_{\ell,\varepsilon})$. With this notation we have

$$L_0 = \int_S (d^*\tilde\psi_\varepsilon, j(u_\varepsilon) - j^*_\varepsilon)_g \operatorname{vol}_g + \int_S (d^*\tilde\psi_\varepsilon, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g})_g (1 - |u_\varepsilon|_g)\operatorname{vol}_g - \int_{S\setminus S_{r_\varepsilon}} (\xi_\varepsilon, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j^*_\varepsilon)_g \operatorname{vol}_g.$$

We consider the terms on the right-hand side. First, writing $u^*_\varepsilon := u^*(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon)$, by the Stokes theorem and the definition of the Hodge star operator, we have

$$\int_S (d^*\tilde\psi_\varepsilon, j(u_\varepsilon) - j^*_\varepsilon)_g \operatorname{vol}_g = \int_S \star\tilde\psi_\varepsilon\,(\omega(u_\varepsilon) - \omega(u^*_\varepsilon)) = \int_S \star\tilde\psi_\varepsilon\left(\omega(u_\varepsilon) - 2\pi\sum_{\ell=1}^{K_\varepsilon} d_{\ell,\varepsilon}\delta_{a_{\ell,\varepsilon}}\right)$$

and from this, together with (78), we conclude that

$$\left|\int_S (d^*\tilde\psi_\varepsilon, j(u_\varepsilon) - j^*_\varepsilon)_g \operatorname{vol}_g\right| \le \|\star\tilde\psi_\varepsilon\|_{W^{1,\infty}} \|\omega(u_\varepsilon) - 2\pi\sum_{\ell=1}^{K_\varepsilon} d_{\ell,\varepsilon}\delta_{a_{\ell,\varepsilon}}\|_{W^{-1,1}} \le C{r_\varepsilon}^{3/2}.$$

The other terms in the decomposition of L_0 are estimated exactly like their counterparts in Step 1 above, using the estimates $\|d^*\tilde\psi_\varepsilon\|_\infty$, $\|\xi_\varepsilon\|_\infty \le C r_\varepsilon^{-1/2}$. This leads to

$$|L_0| \le C(\ {r_\varepsilon}^{3/2} + \frac{\varepsilon}{r_\varepsilon}|\log\varepsilon| + r_\varepsilon^{1/2}|\log\varepsilon|^{1/2} + {r_\varepsilon}^{1/2} + {r_\varepsilon}^{3/2}|\Phi^\varepsilon|) = O({r_\varepsilon}^{1/3} + r_\varepsilon^2|\Phi^\varepsilon|^2)$$

as $\varepsilon \to 0$. □

Our next lemma provides a rather crude estimate of the energy near the "vortex cores".

Lemma 9.4. *Using the notations in Lemma 9.2, let* $a^\varepsilon = (a_{1,\varepsilon},\dots,a_{K,\varepsilon})$, $d^\varepsilon = (d_{1,\varepsilon},\dots,d_{K,\varepsilon})$ *satisfy* (78) *for some* $r_\varepsilon \in (\varepsilon^{\frac{1}{2(n+1)}}, \varepsilon^\beta)$ *for some* $\beta = \beta(n) > 0$. *Then for all sufficiently small* $\varepsilon > 0$,

$$\int_{B_{r_\varepsilon}(a_{k,\varepsilon})} e^{in}_\varepsilon(u_\varepsilon)\, vol_g \ge |d_{k,\varepsilon}|(\pi\log\frac{r_\varepsilon}{\varepsilon} - C), \quad k = 1,\dots,K. \tag{83}$$

Proof. Consider the collection of balls $\{B_{\ell,\sigma}\}_\ell$ provided by applying Proposition 8.2 to u_ε, with $T = n$ and $\sigma = \frac{1}{4n+4} r_\varepsilon$. In view of (58), it suffices to show that for every $1 \le k \le K$,

$$\sum_{\ell : B_{\ell,\sigma} \subset B_{r_\varepsilon}(a_{k,\varepsilon})} |d_{\ell,\sigma}| \ge |d_{k,\varepsilon}|. \tag{84}$$

To do this, we fix some $k \in \{1, \dots, K\}$, and we define the set

$$\mathcal{T}_k := \{r \in (0, r_\varepsilon) : \partial B_r(a_{k,\varepsilon}) \cap (\cup_\ell B_{\ell,\sigma}) = \emptyset\}.$$

It follows from (57) that $|\mathcal{T}_k| \ge \frac{1}{2} r_\varepsilon$. Now define a Lipschitz function $\varphi : S \to \mathbb{R}$ by

$$\varphi(x) := f(R_k(x)), \qquad \text{for} \quad f(r) := |\mathcal{T}_k| - \int_0^r \mathbf{1}_{\mathcal{T}_k}(s)ds, \qquad R_k(x) := \mathrm{dist}_S(x, a_{k,\varepsilon}).$$

It is clear that $\varphi(a_{k,\varepsilon}) = |\mathcal{T}_k|$, $\|\varphi\|_{W^{1,\infty}} \le 2$ and the support of φ is inside $B_{r_\varepsilon}(a_{k,\varepsilon})$, so

$$|\int_S \varphi\, \omega(u_\varepsilon) - 2\pi d_{k,\varepsilon} \varphi(a_{k,\varepsilon})| \le \|\varphi\|_{W^{1,\infty}} \|\omega(u_\varepsilon) - 2\pi \sum_{l=1}^K d_{l,\varepsilon} \delta_{a_{l,\varepsilon}}\|_{W^{-1,1}} \le 2 r_\varepsilon^2,$$

because by (78), $a_{l,\varepsilon} \notin B_{r_\varepsilon}(a_{k,\varepsilon})$ if $l \ne k$. Next, we define $\tilde{u}_\varepsilon$ as in (59), so that by Step 3 in the proof of Proposition 8.1:

$$\int_S \varphi\, \omega(u_\varepsilon) = \int_S \varphi\, \omega(\tilde{u}_\varepsilon) + O(\varepsilon |\log \varepsilon|).$$

We fix a moving frame $\{\tau_1, \tau_2\}$ defined in $B_{r_\varepsilon}(a_{k,\varepsilon})$ and let A be the connection 1-form associated to it. Then by (36) we may write $\omega(\tilde{u}_\varepsilon) = d(j(\tilde{u}_\varepsilon) + A)$ in $B_{r_\varepsilon}(a_{k,\varepsilon})$. It follows

$$\int_S \varphi\, \omega(\tilde{u}_\varepsilon) = \underbrace{\int_S d\Big(\varphi(j(\tilde{u}_\varepsilon) + A)\Big)}_{=0} - \int_S d\varphi \wedge (j(\tilde{u}_\varepsilon) + A) = - \int_S f'(R_k) dR_k \wedge (j(\tilde{u}_\varepsilon) + A).$$

Recalling that the integrands are supported in $\mathrm{supp}(f) \subset B_{r_\varepsilon}(a_{k,\varepsilon})$, the coarea formula and the definition of f imply that

$$\begin{aligned} -\int_{B_{r_\varepsilon}(a_{k,\varepsilon})} f'(R_k) dR_k \wedge (j(\tilde{u}_\varepsilon) + A) &= \int_{r \in \mathcal{T}_k} \int_{\partial B_r(a_{k,\varepsilon})} (j(\tilde{u}_\varepsilon) + A)\, dr \\ &= 2\pi \int_{r \in \mathcal{T}_k} \deg(\tilde{u}_\varepsilon, \partial B_r(a_{k,\varepsilon}))\, dr, \end{aligned}$$

where we used (5) and $|\tilde{u}_\varepsilon|_g = 1$ on $\partial B_r(a_{k,\varepsilon})$ for every $r \in \mathcal{T}_k$. Combining these, we find that

$$\left| 2\pi d_{k,\varepsilon} |\mathcal{T}_k| - 2\pi \int_{r \in \mathcal{T}_k} \deg(\tilde{u}_\varepsilon, \partial B_r(a_{k,\varepsilon}))\, dr \right| \le C r_\varepsilon^2$$

for small $\varepsilon > 0$. As $|\mathcal{T}_k| \ge \frac{r_\varepsilon}{2}$, it follows that if ε is small enough, then $|\deg(\tilde{u}_\varepsilon, \partial B_r(a_{k,\varepsilon}))| \ge |d_{k,\varepsilon}|$ for a large set of $r \in \mathcal{T}_k$. Choose one of these $r \in \mathcal{T}_k$. Since

$$\deg(\tilde{u}_\varepsilon, \partial B_r(a_{k,\varepsilon})) = \sum_{\ell \,:\, B_{\ell,\sigma} \subset B_r(a_{k,\varepsilon})} d_{\ell,\sigma},$$

this implies (84). □

We now present the proof of Proposition 9.1. The bulk of the proof is devoted to a sharp lower bound near the vortices, which uses preliminary estimates provided by Lemmas 9.3 and 9.4 to refine the conclusion of Lemma 9.4.

Proof of Proposition 9.1 point 1). By Step 1 in the proof of Proposition 8.1, we may assume that u_ε are smooth vector fields on S with $|u_\varepsilon|_g \le 1$ everywhere (as the cutting u_ε by $\hat{u}_\varepsilon$ and then regularizing as in Lemma 5.1, the new vector field satisfies the hypotheses of the Proposition but has less energy). We will explain in Step 6 below how to get the result for general vector fields without the constraint on the length of u_ε. Next to the distinct points $a = (a_1, \dots, a_{n_0}) \in S$ and nonzero integers $d = (d_1, \dots, d_{n_0})$ (given in the hypothesis of Proposition 9.1), using Lemma 9.2, we find $K = K_\varepsilon$ distinct points $a^\varepsilon = (a_{1,\varepsilon}, \dots, a_{K,\varepsilon})$ and nonzero integers $d^\varepsilon = (d_{1,\varepsilon}, \dots, d_{K,\varepsilon})$ satisfying (78) for some $r_\varepsilon \in (\varepsilon^{\frac{1}{2(n+1)}}, \varepsilon^\beta)$ and for some $\beta = \beta(n) > 0$, $\sum_k |d_{k,\varepsilon}| \le n$ and $\Phi^\varepsilon = (\Phi_{k,\varepsilon})_{k=1}^{2\mathfrak{g}} \in \mathcal{L}(a^\varepsilon, d^\varepsilon)$ such that $|\Phi(u_\varepsilon) - \Phi^\varepsilon| \le C\sqrt{r_\varepsilon}$. Let $j^*_\varepsilon := j(u^*(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon))$ as defined in (12).

Step 1. We prove that $\sum_{k=1}^K |d_{k,\varepsilon}| = n$, and we control $\operatorname{dist}_S(a_{k,\varepsilon}, a_\ell)$ and the signs of $d_{k,\varepsilon}$ for small $\varepsilon > 0$, i.e., every $a_{k,\varepsilon}$ is close to some a_ℓ with $\operatorname{sign} d_{k,\varepsilon} = \operatorname{sign} d_\ell$. Moreover, $W(a^\varepsilon, d^\varepsilon, \cdot)$ is coercive in Φ^ε and $W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \to \infty$ if a limit degree satisfies $|d_\ell| > 1$.
First, Lemma 9.2 and (65) imply that

$$\Big\| \sum_{k=1}^{K_\varepsilon} d_{k,\varepsilon} \delta_{a_{k,\varepsilon}} - \sum_{\ell=1}^{n_0} d_\ell \delta_{a_\ell} \Big\|_{W^{-1,1}} := s_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0. \tag{85}$$

For small ε and for $\ell = 1, \dots, n_0$, we consider the Lipschitz function

$$f_{\varepsilon,\ell}(x) := \operatorname{sign}(d_\ell)[2s_\varepsilon - \operatorname{dist}_S(x, a_\ell)]^+, \quad x \in S,$$

where $[\cdots]^+ = \max\{[\cdots], 0\}$. We also define, for $\ell = 1, \dots, n_0$,

$$I_\ell^\varepsilon := \{k \in \{1, \dots, K\} : \operatorname{dist}_S(a_{k,\varepsilon}, a_\ell) \le 2s_\varepsilon\},$$
$$I_\ell^{\varepsilon,+} := \{k \in I_\ell^\varepsilon : \operatorname{sign}(d_{k,\varepsilon}) = \operatorname{sign}(d_\ell)\}.$$

We henceforth assume that ε is small enough that $s_\varepsilon < 1$ and the closed balls $\{\bar{B}_{2s_\varepsilon}(a_\ell)\}$ are disjoint, and hence $\{I_\ell^\varepsilon\}_\ell$ are pairwise disjoint. It follows from our convention for defining Sobolev norms (see Section 5.3) that $\|f_{\varepsilon,\ell}\|_{W^{1,\infty}} = 1$. Thus

$$\int f_{\varepsilon,\ell} \Big(\sum_{l=1}^{n_0} d_l \delta_{a_l} - \sum_{k=1}^{K_\varepsilon} d_{k,\varepsilon} \delta_{a_{k,\varepsilon}} \Big) \le \|f_{\varepsilon,\ell}\|_{W^{1,\infty}} \Big\| \sum_{k=1}^{K_\varepsilon} d_{k,\varepsilon} \delta_{a_{k,\varepsilon}} - \sum_{l=1}^{n_0} d_l \delta_{a_l} \Big\|_{W^{-1,1}} = s_\varepsilon.$$

However, the definition and the smallness condition on s_ε imply that

$$\begin{aligned} \int f_{\varepsilon,\ell} \Big(\sum_{l=1}^{n_0} d_l \delta_{a_l} - \sum_{k=1}^{K_\varepsilon} d_{k,\varepsilon} \delta_{a_{k,\varepsilon}} \Big) &= 2s_\varepsilon |d_\ell| - \operatorname{sign}(d_\ell) \sum_{k \in I_\ell^\varepsilon} d_{k,\varepsilon} (2s_\varepsilon - \operatorname{dist}_S(a_{k,\varepsilon}, a_\ell)) \\ &\ge 2s_\varepsilon |d_\ell| - 2s_\varepsilon \sum_{k \in I_\ell^{\varepsilon,+}} |d_{k,\varepsilon}|. \end{aligned}$$

We combine these facts and divide by $2s_\varepsilon$ to find that $|d_\ell| - \sum_{k \in I_\ell^{\varepsilon,+}} |d_{k,\varepsilon}| \le \frac{1}{2}$. Since both terms on the left are integers, it follows that

$$|d_\ell| \le \sum_{k \in I_\ell^{\varepsilon,+}} |d_{k,\varepsilon}| \quad \text{for } \ell = 1, \dots, n_0. \tag{86}$$

Summing over ℓ and using the disjointness of $\{I_\ell^\varepsilon\}_\ell$, we obtain

$$n = \sum_{\ell=1}^{n_0} |d_\ell| \le \sum_\ell \sum_{k \in I_\ell^{\varepsilon,+}} |d_{k,\varepsilon}| \le \sum_\ell \sum_{k \in I_\ell^\varepsilon} |d_{k,\varepsilon}| \le \sum_{k=1}^{K_\varepsilon} |d_{k,\varepsilon}| \le n.$$

It follows that in fact for small ε,

$$\sum_{k=1}^{K}|d_{k,\varepsilon}|=n,\qquad \operatorname{sign}d_{k,\varepsilon}=\operatorname{sign}d_\ell \ \text{ for all } k\in I_\ell^\varepsilon,\qquad \sum_{k\in I_\ell^\varepsilon}d_{k,\varepsilon}=d_\ell \tag{87}$$

where the last equality holds due to (86) and the first two equalities in (87). In particular, one has that $a_{k,\varepsilon}\to a_\ell$ for every $k\in I_\ell^\varepsilon$; as for all $1\le \ell\le n_0$ the indices $\{d_{k,\varepsilon}\,:\,k\in I_\ell^\varepsilon\}$ have the same sign, the explicit formula (22) for W implies that $W(a^\varepsilon,d^\varepsilon,\cdot)$ is coercive in Φ_ε:

$$\begin{aligned} &W(a^\varepsilon,d^\varepsilon,\Phi^\varepsilon)\ge -C(a,d)+|\Phi_\varepsilon|^2 &&\text{for all } \varepsilon\in(0,\varepsilon_0),\\ &W(a^\varepsilon,d^\varepsilon,\Phi^\varepsilon)\to+\infty &&\text{as } \varepsilon\to 0, \text{ if } |d_\ell|>1 \text{ for any } \ell. \end{aligned}$$

The point is that in the sum $\sum_{1\le l<k\le n}d_{l,\varepsilon}d_{k,\varepsilon}G(a_{k,\varepsilon},a_{l,\varepsilon})$ in formula (22) for W, Step 1 implies that $\operatorname{dist}_S(a_{k,\varepsilon},a_{l,\varepsilon})$ is bounded away from 0, for small ε, for all pairs k,l such that $d_{k,\varepsilon}d_{l,\varepsilon}<0$. Thus contributions from this sum are bounded below, and all other terms are manifestly also bounded from below as $\operatorname{dist}_S(a_{k,\varepsilon},a_{l,\varepsilon})$ is bounded. Moreover, if $|d_\ell|>1$ for any ℓ, then multiple points $a_{k,\varepsilon}$ with the same sign $d_{k,\varepsilon}$ must converge to the same a_ℓ, causing the sum $\sum_{1\le l<k\le n}d_{l,\varepsilon}d_{k,\varepsilon}G(a_{k,\varepsilon},a_{l,\varepsilon})$ to diverge.

Step 2. We prove that $|d_\ell|=|d_{k,\varepsilon}|=1$ *for all* $1\le k\le K$ *and* $1\le\ell\le n_0$ *(so,* $K=n=n_0$*),* $\operatorname{dist}_S(a_{k,\varepsilon},a_{\ell,\varepsilon})\ge C>0$ *for every* $k\ne\ell$ *and* $\{\Phi_\varepsilon\}$ *converge (for a subsequence) as* $\varepsilon\to 0$. Indeed, by combining the energy estimates away from the vortex cores and inside the vortex cores as shown in Lemmas 9.3 and 9.4, we find for all sufficiently small $\varepsilon>0$:

$$\begin{aligned}\int_S e_\varepsilon^{in}(u_\varepsilon)\operatorname{vol}_g \ge{}& \pi\sum_{k=1}^{K}|d_{k,\varepsilon}|\log\frac{1}{\varepsilon}+\pi\sum_{k=1}^{K}(d_{k,\varepsilon}^2-|d_{k,\varepsilon}|)\log\frac{1}{r_\varepsilon}+W(a^\varepsilon,d^\varepsilon,\Phi^\varepsilon)\\ &+\int_{S_{r_\varepsilon}}\frac12\Big|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g}-j_\varepsilon^*\Big|_g^2+e_\varepsilon^{in}(|u_\varepsilon|_g)\operatorname{vol}_g\ -C\,,\end{aligned}\tag{88}$$

where $S_{r_\varepsilon}=S\setminus\cup_{k=1}^K B_{r_\varepsilon}(a_{k,\varepsilon})$. Then the upper bound (64) and (87) imply that

$$\pi\sum_{k=1}^{K}(d_{k,\varepsilon}^2-|d_{k,\varepsilon}|)\log\frac{1}{r_\varepsilon}+W(a^\varepsilon,d^\varepsilon,\Phi^\varepsilon)\le C.$$

Combined with the coercivity of W proved in Step 1, it follows that

$$|d_{k,\varepsilon}|=1 \text{ for all } k, \text{ (so, } K=n),\qquad |d_\ell|=1\ \text{ for all } \ell,\qquad |\Phi^\varepsilon|^2\le C$$

for all sufficiently small ε. Also, Step 1 implies $I_\ell^{\varepsilon,+}=I_\ell^\varepsilon$ containing only one point $a_{\ell,\varepsilon}$ that converges to a_ℓ as $\varepsilon\to0$, thus, yielding $\operatorname{dist}_S(a_{k,\varepsilon},a_{\ell,\varepsilon})\ge C>0$ for every $k\ne\ell$ for small $\varepsilon>0$. In particular, (66) holds (that is, Φ^ε converges) after possibly passing to a subsequence, and (88) implies that

$$\int_{S_{r_\varepsilon}}\frac12\left|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g}-j_\varepsilon^*\right|_g^2+e_\varepsilon^{in}(|u_\varepsilon|_g)\operatorname{vol}_g\ \le C.$$

It follows from the coarea formula that

$$\int_{r_\varepsilon}^{\sqrt{r_\varepsilon}}\sum_{k=1}^{n}\int_{\partial B_t(a_{k,\varepsilon})}\left(\frac12\left|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g}-j_\varepsilon^*\right|_g^2+e_\varepsilon^{in}(|u_\varepsilon|_g)\right)d\mathcal{H}^1dt\le C$$

and hence, since $r_\varepsilon \le \varepsilon^\beta$ for some positive β, that there exists $t_\varepsilon \in (r_\varepsilon, \sqrt{r_\varepsilon})$ such that

$$\sum_{k=1}^{n} t_\varepsilon \int_{\partial B_{t_\varepsilon}(a_{k,\varepsilon})} \left(\frac{1}{2} \left| \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^* \right|_g^2 + e_\varepsilon^{in}(|u_\varepsilon|_g) \right) d\mathcal{H}^1 \le C|\log\varepsilon|^{-1}. \tag{89}$$

Step 3. Passage in normal coordinates. We now fix some $k \in \{1, \ldots, n\}$. We assume for concreteness, and to simplify the notation, that $d_{k,\varepsilon} = +1$. We aim to rewrite the integral around $\partial B(t_\varepsilon, a_{k,\varepsilon})$ in exponential normal coordinates near $a_{k,\varepsilon}$, using a moving frame such that (70) holds (see (71) and the discussion in Section 9.1 for notation). Since we have arranged that $|u_\varepsilon|_g \le 1$ everywhere, it follows from (69), (72) and (74) that on $\partial B_{t_\varepsilon}(a_{k,\varepsilon})$,

$$\begin{aligned} \left| \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^* \right|_g^2 (\Psi(y)) &= [1 + O(t_\varepsilon^2)] \left| \frac{j(v_\varepsilon)}{|v_\varepsilon|}(y) - d\theta + O(1) \right|^2 \\ &= [1 + O(t_\varepsilon^{1/2})] \left| \frac{j(v_\varepsilon)}{|v_\varepsilon|}(y) - d\theta \right|^2 + O(t_\varepsilon^{-1/2}). \end{aligned} \tag{90}$$

where $d\theta := \frac{1}{|y|^2}(y_1 dy_2 - y_2 dy_1)$ and we used the Young inequality $|z_1 + z_2|^2 \le (1 + t_\varepsilon^{1/2})|z_1|^2 + (1 + t_\varepsilon^{-1/2})|z_2|^2$. Combining this with (89) and again using (72), we obtain

$$t_\varepsilon \int_{\{y \in \mathbb{R}^2 : |y| = t_\varepsilon\}} \frac{1}{2} \left| \frac{j(v_\varepsilon)}{|v_\varepsilon|}(y) - d\theta \right|^2 + e_\varepsilon(|v_\varepsilon|)(y) \; d\mathcal{H}^1(y) \le C|\log\varepsilon|^{-1}. \tag{91}$$

Step 4. We will show that

$$\int_{\{y \in \mathbb{R}^2 : |y| < t_\varepsilon\}} e_\varepsilon(v_\varepsilon) \; dy \ge \pi \log \frac{t_\varepsilon}{\varepsilon} + \iota_F + o(1) \quad \textit{as } \varepsilon \to 0.$$

To do this, it is convenient to define

$$\beta_\varepsilon(v; r) := r \int_{\partial B_r} \left(\left| \frac{j(v)}{|v|} - d\theta \right|^2 + e_\varepsilon(|v|) \right) d\mathcal{H}^1(y), \tag{92}$$

for $v \in H^1(O; \mathbb{C})$, where O is a neighborhood of the origin in $\mathbb{R}^2$ containing the disk $B_r = \{y \in \mathbb{R}^2 : |y| < r\}$. We further define for small $\delta > 0$:

$$I^\delta(\varepsilon, r) := \inf \left\{ \int_{B_r} e_\varepsilon(v) \;:\; v \in H^1(B_r; \mathbb{C}),\; \beta_\varepsilon(v, r) \le \delta \right\}.$$

By a change of variables one finds that for $v^r(x) := v(rx)$,

$$\beta_\varepsilon(v, r) = \beta_{\varepsilon/r}(v^r, 1), \qquad \text{and thus} \qquad I^\delta(\varepsilon, r) = I^\delta(\varepsilon/r, 1).$$

Note that $\beta_\varepsilon(v, 1) = 0$ implies that $|v| = 1$ and $j(v) = d\theta$ on ∂B_1 (i.e., $v = e^{i(\theta + \bar\eta)}$ for some constant $\bar\eta$) so that by (26):

$$\lim_{\varepsilon \to 0} \left(I^0(\varepsilon, 1) - \pi \log \frac{1}{\varepsilon} \right) = \iota_F$$

(see [3, Lemma III.1]). We also claim that for small $\delta > 0$:

$$I^\delta(\varepsilon, 1) \ge \pi \log \frac{1}{\varepsilon} + \iota_F - C\delta + o(1) \quad \text{as } \varepsilon \to 0. \tag{93}$$

This follows from the fact that if $v \in H^1(B_1;\mathbb{C})$ and $\beta_\varepsilon(v;1) \le \delta$, then v admits an extension to a function $\tilde v \in H^1(B_2;\mathbb{C})$ such that $\beta_\varepsilon(\tilde v, 2) = 0$ and

$$\int_{B_2\setminus B_1} e_\varepsilon(\tilde v) \le \pi \log 2 + C\delta.$$

Indeed, this may be done by writing $v(y) = \rho(y)e^{i(\theta+\eta(y))}$ for $|y| = 1$ with $\rho, \eta \in H^1(\partial B_1)$ with $|1-\rho| = O(\varepsilon)$ pointwise on ∂B_1 and $\|\partial_\tau \eta\|^2_{L^2(\partial B_1)} = O(\delta)$ (due to the assumption $\beta_\varepsilon(v,1) \le \delta$). Then for $1 \le r \le 2$, we set $\tilde v(ry) := [1 + (2-r)(\rho(y)-1)] \exp i[\theta + \bar\eta + (2-r)(\eta(y) - \bar\eta)]$, where $\bar\eta$ is the mean of η over ∂B_1. Thus for small $\varepsilon > 0$:

$$I^0(\varepsilon, 2) \le \int_{B_2} e_\varepsilon(\tilde v)\, dy \le \int_{B_1} e_\varepsilon(v)\, dy + \pi \log 2 + C\delta.$$

It follows that $I^\delta(\varepsilon, 1) \ge I^0(\varepsilon, 2) - \pi \log 2 - C\delta$, which implies (93). Then by combining (93) and (91), we conclude Step 4.

Step 5. Lower bound (67). By Step 4 combined with (73), we deduce that

$$\int_{B_{t_\varepsilon}(a_{k,\varepsilon})} e^{in}_\varepsilon(u_\varepsilon)\, \mathrm{vol}_g \ge \pi \log \frac{t_\varepsilon}{\varepsilon} + \iota_F + o(1) \qquad \text{as } \varepsilon \to 0, \text{ for every } 1 \le k \le n.$$

As $t_\varepsilon \in (r_\varepsilon, \sqrt{r_\varepsilon})$ and $\mathrm{dist}_S(a_{k,\varepsilon}, a_{\ell,\varepsilon}) = O(1) \ge t_\varepsilon$ for $k \ne \ell$, we see that (78) holds true for t_ε so that we can apply Lemma 9.3 for t_ε yielding:

$$\int_{S_{t_\varepsilon}} e^{in}_\varepsilon(u_\varepsilon)\, \mathrm{vol}_g \ge n\pi \log \frac{1}{t_\varepsilon} + W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon)$$
$$+ \int_{S_{t_\varepsilon}} \frac{1}{2} \left| \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j^*_\varepsilon \right|^2_g + e^{in}_\varepsilon(|u_\varepsilon|_g)\, \mathrm{vol}_g \;- o(1)$$

as $\varepsilon \to 0$. By adding these inequalities and noting that for every fixed $\sigma > 0$, $j^*_\varepsilon \to j^* = j(u^*(a,d,\Phi))$ uniformly on $S \setminus \cup_{k=1}^n B_\sigma(a_k)$ and $W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \to W(a,d,\Phi)$ as $\varepsilon \to 0$, we complete the proof of (67).

Step 6. Conclusion. We finally consider the general case, without the assumption $|u_\varepsilon|_g \le 1$. Due to the cutting u_ε by $\hat u_\varepsilon$ (see Step 1 of the proof of Proposition 8.1), as $e^{in}_\varepsilon(|\hat u_\varepsilon|_g) = 0$ a.e. in $\{|u_\varepsilon|_g \ge 1\}$, it remains to check for a fixed $\sigma > 0$:

$$E^{in}_\varepsilon(u_\varepsilon) - E^{in}_\varepsilon(\hat u_\varepsilon)$$
$$\ge \int_{\{x\in S_\sigma\,:\,|u_\varepsilon|_g(x)>1\}} \frac{1}{2}\left|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j^*\right|^2_g + e^{in}_\varepsilon(|u_\varepsilon|_g) - \frac{1}{2}\left|j(\hat u_\varepsilon) - j^*\right|^2_g \;\mathrm{vol}_g + o(1), \text{ as } \varepsilon \to 0,$$

where we denoted by $S_\sigma = S \setminus \cup_{k=1}^n B_\sigma(a_k)$ and $j^* = j(u^*(a,d,\Phi))$. Using (81) and $j(u_\varepsilon) = |u_\varepsilon|^2_g j(\hat u_\varepsilon)$ in $\{|u_\varepsilon|_g > 1\}$, the above inequality will follow from

$$\int_{\{x\in S_\sigma\,:\,|u_\varepsilon|_g(x)>1\}} (|u_\varepsilon|_g - 1)\,(j(\hat u_\varepsilon), j^*)_g\; \mathrm{vol}_g = o(1) \text{ as } \varepsilon \to 0.$$

To prove this, we use $|j(\hat u_\varepsilon)|_g = |D\hat u_\varepsilon|_g \le |Du_\varepsilon|_g$ in $\{|u_\varepsilon|_g > 1\}$ so that we obtain by (1) and the Cauchy-Schwarz inequality:

$$\Big|\int_{\{x\in S_\sigma\,:\,|u_\varepsilon|_g(x)>1\}} (|u_\varepsilon|_g - 1)\,(j(\hat u_\varepsilon), j^*)_g\; \mathrm{vol}_g\Big|$$
$$\le \|j^*\|_{L^\infty(S_\sigma)} \Big(\int_S F(|u_\varepsilon|^2_g)\,\mathrm{vol}_g\Big)^{1/2} \Big(\int_S |Du_\varepsilon|^2_g\,\mathrm{vol}_g\Big)^{1/2} = O\Big(\frac{\varepsilon|\log\varepsilon|}{\sigma}\Big) = o(1).$$

□

Proof of Theorem 2.6. It is a direct consequence of Corollary 8.3 and Proposition 9.1. □

10. Γ-limit in the extrinsic case

In this section we prove the counterpart of Proposition 9.1 for the extrinsic energy E_ε^{ex} in Problem 2. Here, the surface S is isometrically embedded in $\mathbb{R}^3$.

Theorem 10.1. *The following Γ-convergence result holds.*

1) *(Compactness) Let $(m_\varepsilon)_{\varepsilon\downarrow 0}$ be a family of sections of $\mathcal{X}^{1,2}(S)$ satisfying $E_\varepsilon^{ex}(m_\varepsilon) \leq T\pi|\log\varepsilon| + C$ for some integer $T > 0$ and a constant $C > 0$. Then there exists a sequence $\varepsilon \downarrow 0$ such that for every $p \in [1, 2)$,*

$$\omega(m_\varepsilon) \longrightarrow 2\pi \sum_{k=1}^{n} d_k \delta_{a_k} \quad \text{in } W^{-1,p}, \quad \text{as } \varepsilon \to 0, \tag{94}$$

where $\{a_k\}_{k=1}^n$ are distinct points in S and $\{d_k\}_{k=1}^n$ are nonzero integers satisfying (7) and $\sum_{k=1}^n |d_k| \leq T$. Moreover, if $\sum_{k=1}^n |d_k| = T$, then $n = T$ and $|d_k| = 1$ for every $k = 1, \dots, n$; in this case, for a further subsequence, there exists $\Phi \in \mathcal{L}(a; d)$ such that $\Phi(m_\varepsilon)$ defined in (53) converges to Φ as $\varepsilon \to 0$.

2) *(Γ-liminf inequality) Assume that the sections $m_\varepsilon \in \mathcal{X}^{1,2}(S)$ satisfy (94) for $p = 1$ and n distinct points $\{a_k\}_{k=1}^n \in S^n$ and $|d_k| = 1$, $k = 1, \dots n$ such that (7) holds true and $\Phi(m_\varepsilon) \to \Phi \in \mathcal{L}(a; d)$. Then*

$$\liminf_{\varepsilon\to 0} \left[E_\varepsilon^{ex}(m_\varepsilon) - n(\pi|\log\varepsilon| + \iota_F)\right] \;\geq\; W(a, d, \Phi) + \tilde{W}(a, d, \Phi) \tag{95}$$

for $u^* = u^*(a, d, \Phi)$, $a = (a_1, \dots, a_n)$, $d = (d_1, \dots, d_n)$ and $\tilde{W}(a, d, \Phi)$ defined in (24).

3) *(Γ-limsup inequality) For every n distinct points $a_1, \dots, a_n \in S$ and $d_1, \dots, d_n \in \{\pm 1\}$ satisfying (7) and every $\Phi \in \mathcal{L}(a; d)$ there exists a sequence of smooth sections $m_\varepsilon : S \to TS$ such that $|m_\varepsilon|_g \leq 1$ in S, (94) holds, $\Phi(m_\varepsilon) \to \Phi$ and*

$$E_\varepsilon^{ex}(m_\varepsilon) - n\pi|\log\varepsilon| \longrightarrow W(a, d, \Phi) + \tilde{W}(a, d, \Phi) + n\iota_F \quad \text{as } \varepsilon \to 0.$$

10.1. **Compactness.** Let us start by computing the extrinsic Dirichlet energy of a section m:

Lemma 10.2. *If $m : S \to TS$ is a section of $\mathcal{X}^{1,2}$ then*

$$|\bar{D}m|_g^2 = |Dm|_g^2 + |\mathcal{S}(m)|_g^2 \quad \text{a.e. in } S,$$

where $\mathcal{S} : TS \to TS$ is the shape operator defined in (23).

Proof. Let $\{\tau_1, \tau_2 = i\tau_1\}$ be a local moving frame on S, i.e.,

$$\tau_\ell \cdot \tau_k := (\tau_\ell, \tau_k)_g = \delta_{\ell k}, \quad \ell, k = 1, 2.$$

We write

$$m = \sum_{k=1}^{2} m^k \tau_k, \quad D_\ell m := D_{\tau_\ell} m, \quad \bar{D}_\ell m := \tau_\ell \cdot \bar{D}m, \; \ell = 1, 2.$$

Denoting N the Gauss map at S, we decompose the extrinsic differential as follows:

$$\bar{D}m = Dm + (\bar{D}m \cdot N) \otimes N, \quad \text{i.e.,} \quad \bar{D}_\ell m = D_\ell m + (\bar{D}_\ell m \cdot N)N, \quad \ell = 1, 2.$$

Therefore,

$$|\bar{D}m|_g^2 = \sum_{\ell=1}^{2} |\bar{D}_\ell m|_g^2 = \sum_{\ell=1}^{2} \Big(|D_\ell m|_g^2 + (\bar{D}_\ell m \cdot N)^2 \Big). \tag{96}$$

Recall the definition of the shape operator (23) (in particular, $\mathcal{S}(\tau_\ell) = -\bar{D}_\ell N$ for $\ell = 1,2$). It is a standard fact that $\mathcal{S}$ is a symmetric operator corresponding to the second fundamental form H of S; in other words, we have in the frame $\{\tau_1, \tau_2\}$ that [20]

$$H_{\ell k} = \tau_k \cdot \mathcal{S}(\tau_\ell) = \tau_\ell \cdot \mathcal{S}(\tau_k) = H_{k\ell}.$$

Therefore, as $m \cdot N = 0$ on S, we compute for every $\ell = 1, 2$:

$$\bar{D}_\ell m \cdot N = -m \cdot \bar{D}_\ell N = \sum_{k=1}^{2} m^k \tau_k \cdot \mathcal{S}(\tau_\ell) = \sum_{k=1}^{2} m^k \tau_\ell \cdot \mathcal{S}(\tau_k) = \tau_\ell \cdot \mathcal{S}(m)$$

so that

$$\sum_{\ell=1}^{2} (\bar{D}_\ell m \cdot N)^2 = |\mathcal{S}(m)|_g^2 = \sum_{1\le l,k\le 2} m^l m^k H_{lk}^2, \quad \text{where } H_{lk}^2 = \sum_{\ell=1}^{2} H_{\ell l} H_{\ell k}.$$

□

Proof of Theorem 10.1 point 1). By Lemma 10.2, we see that $E_\varepsilon^{in}(m_\varepsilon) \le E_\varepsilon^{ex}(m_\varepsilon)$, so that we can apply Corollary 8.3 and Theorem 2.6 point 1) to reach the conclusion. Note that the lower bound of $E_\varepsilon^{in}(m_\varepsilon)$ in Corollary 8.3 holds also true for $E_\varepsilon^{ex}(m_\varepsilon)$. □

10.2. **Upper bound.** In the following, we adapt the construction from the proof of Proposition 9.1, point 2) to the case of Problem 2.

Proof of Theorem 10.1, point 3). Let $u^* = u^*(a, d, \Phi)$ be a canonical harmonic map and Θ be a minimizer in (24) (such a minimizer exists by the direct method in calculus of variations). Then Θ satisfies the associated Euler-Lagrange equation to (24):

$$-\Delta\Theta + \frac{1}{2}\big(\cos(2\Theta)(\mathcal{S}(u^*), \mathcal{S}(iu^*))_g + \sin(2\Theta)(|\mathcal{S}(iu^*)|_g^2 - |\mathcal{S}(u^*)|_g^2)\big) = 0 \quad \text{in} \quad S. \tag{97}$$

Therefore, $\Delta\Theta \in L^\infty$ so $\Theta \in C^1(S)$. Let $U_\varepsilon := U_\varepsilon(a, d, \Phi)$ be the vector field constructed for the upper bound in Proposition 9.1. We set

$$m_\varepsilon := e^{i\Theta} U_\varepsilon \quad \text{in} \quad S. \tag{98}$$

By Lemma 10.2, we have that $|\bar{D}m_\varepsilon|_g^2 = |Dm_\varepsilon|_g^2 + |\mathcal{S}(m_\varepsilon)|_g^2$. We compute the intrinsic part as follows:

$$D_\ell m_\varepsilon = e^{i\Theta} D_\ell U_\varepsilon + i e^{i\Theta} \partial_\ell \Theta U_\varepsilon$$

yielding

$$|Dm_\varepsilon|_g^2 = |DU_\varepsilon|_g^2 + |U_\varepsilon|_g^2 |d\Theta|_g^2 + 2\sum_\ell \partial_\ell\Theta \big(D_\ell U_\varepsilon, iU_\varepsilon\big)_g$$

$$\text{so that} \qquad |\bar{D}m_\varepsilon|_g^2 = |DU_\varepsilon|_g^2 + |U_\varepsilon|_g^2 |d\Theta|_g^2 + |\mathcal{S}(m_\varepsilon)|_g^2 + 2\big(j(U_\varepsilon), d\Theta\big)_g.$$

[20] The symmetry of H follows from $H_{\ell k} = -\tau_k \cdot \bar{D}_\ell N = N \cdot (\bar{D}_\ell \tau_\beta - D_\ell \tau_\beta)$ as $\tau_k \cdot N = 0$ and $\bar{D}_\ell \tau_\beta - D_\ell \tau_\beta = \bar{D}_\beta \tau_\ell - D_\beta \tau_\ell + \underbrace{[\bar{\tau}_\ell, \bar{\tau}_\beta] - [\tau_\ell, \tau_\beta]}_{=0}$ where $[\cdot,\cdot]$ represents the commutator in $\mathbb{R}^3$ for the metric g.

Recall that $|U_\varepsilon|_g \le 1$ in S and $U_\varepsilon = u^*$ in $S_{\sqrt{\varepsilon}} = S \setminus \cup_k B_{\sqrt{\varepsilon}}(a_k)$. Since $|m_\varepsilon|_g = |U_\varepsilon|_g$, we deduce by (24):

$$E_\varepsilon^{ex}(m_\varepsilon) \le \int_S e_\varepsilon^{in}(U_\varepsilon)\,\mathrm{vol}_g + \tilde{W}(a,d,\Phi) + \int_S \big(j(U_\varepsilon), d\Theta\big)_g\,\mathrm{vol}_g + \frac{1}{2}\int_{S\setminus S_{\sqrt{\varepsilon}}} |\mathcal{S}(m_\varepsilon)|_g^2\,\mathrm{vol}_g.$$

The desired upper bound follows by the upper bound of $E_\varepsilon^{in}(U_\varepsilon)$ in Proposition 9.1 point 2), by noting that as $\varepsilon \to 0$:

$$\int_{S\setminus S_{\sqrt{\varepsilon}}} |\mathcal{S}(m_\varepsilon)|_g^2\,\mathrm{vol}_g = o(1)$$

(as $|m_\varepsilon|_g \le 1$) and

$$\begin{aligned}\left|\int_S \big(j(U_\varepsilon), d\Theta\big)_g\,\mathrm{vol}_g\right| &\le \left|\int_S \big(j(u^*), d\Theta\big)_g\,\mathrm{vol}_g\right| + \left|\int_{S\setminus S_{\sqrt{\varepsilon}}} \big(j(U_\varepsilon) - j(u^*), d\Theta\big)_g\,\mathrm{vol}_g\right| \\ &\le \left|\int_S \big(\underbrace{d^* j(u^*)}_{=0}, \Theta\big)_g\,\mathrm{vol}_g\right| + \|d\Theta\|_{L^\infty}\int_{S\setminus S_{\sqrt{\varepsilon}}} |j(U_\varepsilon) - j(u^*)|_g\,\mathrm{vol}_g \\ &= o(1)\end{aligned}$$

because $j(U_\varepsilon) - j(u^*) \to 0$ strongly in $L^p(S)$ for every $p \in [1,2)$ (see (77)). It remains to prove the convergence of the vorticity $\omega(m_\varepsilon)$ and of the flux integrals $\Phi(m_\varepsilon)$ as $\varepsilon \to 0$. For that, we use $j(m_\varepsilon) = d\Theta|U_\varepsilon|_g^2 + j(U_\varepsilon)$; since $d\Theta(1 - |U_\varepsilon|_g^2) \to 0$ in $L^p(S)$ for every $p \in [1,2)$ (as $d\Theta \in L^\infty$ and $|U_\varepsilon|_g = 1$ in $S_{\sqrt{\varepsilon}}$), we deduce by (77):

$$dj(m_\varepsilon) = d\Big(j(U_\varepsilon) + d\Theta - d\Theta(1 - |U_\varepsilon|_g^2)\Big) \to dj(u^*)$$

in $W^{-1,p}(S)$ for $\varepsilon \to 0$ yielding (94). Also, for every harmonic 1-form η, integration by parts and (77) yield

$$\begin{aligned}\int_S (j(m_\varepsilon), \eta)_g\,\mathrm{vol}_g &= \int_S (j(U_\varepsilon), \eta)_g + (d\Theta, \eta)_g\,\mathrm{vol}_g - \int_{S\setminus S_{\sqrt{\varepsilon}}} (1 - |U_\varepsilon|_g^2)(d\Theta, \eta)_g\,\mathrm{vol}_g \\ &= \int_S (j(u^*), \eta)_g\,\mathrm{vol}_g + o(1)\end{aligned}$$

because $d\Theta \in L^\infty$; in particular, $\Phi(m_\varepsilon) \to \Phi(u^*)$ for $\varepsilon \to 0$. The smoothing argument follows as in Step 1 in the proof of Proposition 8.1. □

10.3. **Lower bound.** Following the proof in the intrinsic case in Proposition 9.1, point 1), we start with a sharp lower bound away from the vortices, parallel to Lemma 9.3 above. Let u_ε satisfy the assumption in Theorem 10.1 point 2). As $E_\varepsilon^{ex}(m_\varepsilon) \ge E_\varepsilon^{in}(m_\varepsilon)$, by the proof of Proposition 9.1, point 1), there exist $\beta = \beta(n) > 0$ and

- n distinct points $a^\varepsilon = (a_{1,\varepsilon}, \ldots, a_{n,\varepsilon})$ and integers $d^\varepsilon = (d_{1,\varepsilon}, \ldots, d_{n,\varepsilon})$ such that (78) is satisfied for the vorticity $\omega(m_\varepsilon)$ for some $r_\varepsilon \in (\varepsilon^{\frac{1}{2(n+1)}}, \varepsilon^\beta)$, and with $|d_{k,\varepsilon}| = 1$ for all k;
- $\Phi^\varepsilon = (\Phi_{k,\varepsilon})_{k=1}^{2\mathfrak{g}} \in \mathcal{L}(a^\varepsilon, d^\varepsilon)$ such that $|\Phi(m_\varepsilon) - \Phi^\varepsilon| \le C\sqrt{r_\varepsilon}$.

Moreover, $\mathrm{dist}_S(a_{k,\varepsilon}, a_{\ell,\varepsilon}) \geq C > 0$ for every $k \neq \ell$, $a_{k,\varepsilon} \to a_k$ as $\varepsilon \to 0$ and $|\Phi^\varepsilon| \leq C$ (because $\Phi(m_\varepsilon) \to \Phi$ by hypothesis). In addition, $d_{k,\varepsilon} \to d_k$ as $\varepsilon \to 0$, and thus in fact $d_{k,\varepsilon} = d_k$ for all k, when ε is small enough. Finally, from Lemma 9.4 we have

$$\int_{B_{r_\varepsilon}(a_{k,\varepsilon})} e_\varepsilon^{ex}(m_\varepsilon)\,\mathrm{vol}_g \geq \int_{B_{r_\varepsilon}(a_{k,\varepsilon})} e_\varepsilon^{in}(m_\varepsilon)\,\mathrm{vol}_g \geq (\pi \log \frac{r_\varepsilon}{\varepsilon} - C), \quad k = 1, \dots, n. \tag{99}$$

Also, we may assume that

$$E_\varepsilon^{ex}(m_\varepsilon) \leq n(\pi|\log \varepsilon| + C) \tag{100}$$

for a constant $C = C(a, d, \Phi) > 0$, since otherwise (95) is obvious.

Lemma 10.3. *Under the above hypotheses,*

$$\int_{S_{r_\varepsilon}} e_\varepsilon^{ex}(m_\varepsilon)\, vol_g \geq \pi n \log \frac{1}{r_\varepsilon} + (W + \tilde{W})(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) - o(1) \quad as\ \varepsilon \to 0, \tag{101}$$

for $S_{r_\varepsilon} := S \setminus \cup_k B_{r_\varepsilon}(a_{k,\varepsilon})$.

Proof. For the proof it is useful to define[21] a functional $\mathcal{I}_\varepsilon[\,\cdot\,; a, d, \Phi]$ on $H^1(S;\mathbb{C})$:

$$\mathcal{I}_\varepsilon[w; a, d, \Phi] = \int_S \frac{1}{2}|dw|_g^2 + \frac{1}{2}|\mathcal{S}(wu^*(a, d, \Phi))|_g^2 + \frac{1}{4\varepsilon^2}F(|w|_g^2)\ \mathrm{vol}_g. \tag{102}$$

We will also write $u_\varepsilon^* := u^*(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon)$ and $j_\varepsilon^* = j(u_\varepsilon^*)$.

Step 1. It follows from (81) and (82) that

$$\int_{S_{r_\varepsilon}} e_\varepsilon^{in}(m_\varepsilon)\,\mathrm{vol}_g = \int_{S_{r_\varepsilon}} \frac{1}{2}|j_\varepsilon^*|_g^2 + \frac{1}{2}\left|\frac{j(m_\varepsilon)}{|m_\varepsilon|_g} - j_\varepsilon^*\right|_g^2 + e_\varepsilon^{in}(|m_\varepsilon|_g)\,\mathrm{vol}_g + o(1). \tag{103}$$

For every $x \in S \setminus \{a^\varepsilon\}$, since $\{u_\varepsilon^*(x), iu_\varepsilon^*(x)\}$ is a basis for T_xS, there is a $w_\varepsilon(x) \in \mathbb{C}$ such that

$$m_\varepsilon = w_\varepsilon u_\varepsilon^*.$$

If $m_\varepsilon \in \mathcal{X}^{1,2}(S)$, then it is clear that the function $w_\varepsilon : S \to \mathbb{C}$ defined in this way belongs to H^1_{loc} away from $\{a^\varepsilon\}$, and Section 5.5 shows that

$$d|w_\varepsilon| = d|m_\varepsilon|_g, \qquad \frac{(iw_\varepsilon, dw_\varepsilon)}{|w_\varepsilon|} = \frac{j(m_\varepsilon)}{|m_\varepsilon|_g} - |m_\varepsilon|_g j_\varepsilon^*, \qquad |dw_\varepsilon|_g^2 = |d|w_\varepsilon||_g^2 + |\frac{(iw_\varepsilon, dw_\varepsilon)}{|w_\varepsilon|}|_g^2,$$

where here $(\,\cdot\,,\,\cdot\,)$ denotes the real inner product on $\mathbb{C}$, defined by $(v, w) := \frac{1}{2}(v\bar{w} + w\bar{v})$. From Lemma 10.2 we also know that $e_\varepsilon^{ex}(m_\varepsilon) = e_\varepsilon^{in}(m_\varepsilon) + \frac{1}{2}|\mathcal{S}(m_\varepsilon)|_g^2$. It follows that

$$\begin{aligned}\int_{S_{r_\varepsilon}} e_\varepsilon^{ex}(m_\varepsilon)\,\mathrm{vol}_g = \int_{S_{r_\varepsilon}} \frac{1}{2}|j_\varepsilon^*|_g^2\ \mathrm{vol}_g + \int_{S_{r_\varepsilon}} \frac{1}{2}|dw_\varepsilon|_g^2 + \frac{1}{2}|\mathcal{S}(w_\varepsilon u_\varepsilon^*)|_g^2 + \frac{1}{4\varepsilon^2}F(|w_\varepsilon|^2)\ \mathrm{vol}_g \\ + \frac{1}{2}\int_{S_{r_\varepsilon}} \left|\frac{j(m_\varepsilon)}{|m_\varepsilon|_g} - j_\varepsilon^*\right|_g^2 - \left|\frac{j(m_\varepsilon)}{|m_\varepsilon|_g} - |m_\varepsilon|_g j_\varepsilon^*\right|_g^2 \mathrm{vol}_g + o(1).\end{aligned}$$

Clearly

$$\left|\frac{j(m_\varepsilon)}{|m_\varepsilon|_g} - j_\varepsilon^*\right|_g^2 - \left|\frac{j(m_\varepsilon)}{|m_\varepsilon|_g} - |m_\varepsilon|_g j_\varepsilon^*\right|_g^2 = 2\frac{j(m_\varepsilon)}{|m_\varepsilon|_g} \cdot j_\varepsilon^*(|m_\varepsilon|_g - 1) + |j_\varepsilon^*|^2(1 - |m_\varepsilon|_g^2).$$

[21] Recall from Theorem 2.1 that $u^*(a, d, \Phi)$ is unique only up to a rotation. For purposes of this definition, we assume that a representative u^* has been (arbitrarily) fixed. In the end we are only interested in $\inf \mathcal{I}_\varepsilon$, and this is independent of the chosen rotation. We will therefore feel free to adjust the rotations as needed.

Since j^*_ε is smooth away from $\{a^\varepsilon\}$ and blows up like $\mathrm{dist}_S(\cdot, a_{k,\varepsilon})^{-1}$ near each $a_{k,\varepsilon}$, see (74), it is clear that $|j^*_\varepsilon|_g \le C r_\varepsilon^{-1} \le C\varepsilon^{-\frac{1}{2(n+1)}}$ on S_{r_ε}. Recalling that $\frac{|j(m_\varepsilon)|_g}{|m_\varepsilon|_g} \le |Dm_\varepsilon|_g$, straightforward estimates then show that

(104)
$$\left|\int_{S_{r_\varepsilon}} \left|\frac{j(m_\varepsilon)}{|m_\varepsilon|_g} - j^*_\varepsilon\right|_g^2 - \left|\frac{j(m_\varepsilon)}{|m_\varepsilon|_g} - |m_\varepsilon|_g j^*_\varepsilon\right|_g^2 \mathrm{vol}_g\right| \le C\varepsilon r_\varepsilon^{-1} E^{in}_\varepsilon(m_\varepsilon) + C r_\varepsilon^{-2}\varepsilon\sqrt{E^{in}_\varepsilon(m_\varepsilon)} = o(1)$$

as $\varepsilon \to 0$. We infer that

(105)
$$\int_{S_{r_\varepsilon}} e^{ex}_\varepsilon(m_\varepsilon)\,\mathrm{vol}_g = \int_{S_{r_\varepsilon}} \frac{1}{2}|j^*_\varepsilon|_g^2 \ \mathrm{vol}_g + \int_{S_{r_\varepsilon}} \frac{1}{2}|dw_\varepsilon|_g^2 + \frac{1}{2}|\mathcal{S}(w_\varepsilon u^*_\varepsilon)|_g^2 + \frac{1}{4\varepsilon^2}F(|w_\varepsilon|^2) \ \mathrm{vol}_g + o(1).$$

Step 2. It follows from (99) and (103) that

$$\int_{S_{r_\varepsilon}} \frac{1}{2}|dw_\varepsilon|_g^2 + \frac{1}{2}|\mathcal{S}(w_\varepsilon u^*_\varepsilon)|_g^2 + \frac{1}{4\varepsilon^2}F(|w_\varepsilon|^2) \ \mathrm{vol}_g$$
$$\le E^{ex}_\varepsilon(m_\varepsilon) - \int_{S_{r_\varepsilon}} \frac{1}{2}|j^*_\varepsilon|_g^2 \ \mathrm{vol}_g - n\pi\log\frac{r_\varepsilon}{\varepsilon} + C.$$

We recall also that, in view of (39), the fact that u^*_ε is a *unit* vector field implies that $|j^*_\varepsilon|_g = |Du^*_\varepsilon|_g$, and hence (from the definition of the intrinsic renormalized energy) that

$$\int_{S_{r_\varepsilon}} \frac{1}{2}|j^*_\varepsilon|_g^2 \ \mathrm{vol}_g = W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) + n\pi\log\frac{1}{r_\varepsilon} + o(1). \tag{106}$$

Combining the above estimates with (100), we deduce that

$$\int_{S_{r_\varepsilon}} \frac{1}{2}|dw_\varepsilon|_g^2 + \frac{1}{2}|\mathcal{S}(w_\varepsilon u^*_\varepsilon)|_g^2 + \frac{1}{4\varepsilon^2}F(|w_\varepsilon|^2) \ \mathrm{vol}_g \le C \tag{107}$$

for some constant independent of ε.

Step 3. We next claim that exists $\tilde{w}_\varepsilon \in H^1(S;\mathbb{C})$ such that $\tilde{w}_\varepsilon = w_\varepsilon$ on $S_{\sqrt{r_\varepsilon}}$ and

$$\mathcal{I}_\varepsilon[\tilde{w}_\varepsilon; a^\varepsilon, d^\varepsilon, \Phi^\varepsilon] \le \int_{S_{r_\varepsilon}} \frac{1}{2}|dw_\varepsilon|_g^2 + \frac{1}{2}|\mathcal{S}(w_\varepsilon u^*_\varepsilon)|_g^2 + \frac{1}{4\varepsilon^2}F(|w_\varepsilon|^2) \ \mathrm{vol}_g + o(1) \tag{108}$$

as $\varepsilon \to 0$, where $\mathcal{I}_\varepsilon$ was defined at (102).

First, fix some $k \in \{1, \dots, n\}$ and consider exponential normal coordinates Ψ at $a_{k,\varepsilon}$, mapping a Euclidean ball $\{y \in \mathbb{R}^2 : |y| < \sigma\}$ onto the geodesic ball $B_\sigma(a_{k,\varepsilon})$ in S, see Section 9.1. For $y \in \mathbb{R}^2$ such that $r_\varepsilon \le |y| \le \sqrt{r_\varepsilon}$, let $v_\varepsilon(y) := w_\varepsilon(\Psi(y))$. We may then rewrite the energy of w_ε in this annulus in terms of v_ε. Using (69) to approximate the metric g by the Euclidean metric, we find that

$$\int_{B_{\sqrt{r_\varepsilon}}(a_{k,\varepsilon})\setminus B_{r_\varepsilon}(a_{k,\varepsilon})} \frac{1}{2}|dw_\varepsilon|_g^2 + \frac{1}{4\varepsilon^2}F(|w_\varepsilon|^2)\,\mathrm{vol}_g = (1 + O(r_\varepsilon)) \int_{\{y\in\mathbb{R}^2 : r_\varepsilon < |y| < \sqrt{r_\varepsilon}\}} e_\varepsilon(v_\varepsilon)(y)dy$$

where $e_\varepsilon(v_\varepsilon)$ denotes the Euclidean Ginzburg-Landau energy and dy is the Euclidean area element. By arguing as in Step 3 of the proof of Proposition 9.1, see (89), we may find some $\tilde{t}_\varepsilon \in (r_\varepsilon, \sqrt{r_\varepsilon})$ such that

$$\tilde{t}_\varepsilon \int_{\{y\in\mathbb{R}^2 : |y| = \tilde{t}_\varepsilon\}} e_\varepsilon(v_\varepsilon)(y)d\mathcal{H}^1(y) \le C|\log\varepsilon|^{-1}. \tag{109}$$

In particular, writing ∂_τ for the tangential derivative, it follows from Cauchy-Schwarz that

$$\int_{\{y\in\mathbb{R}^2:|y|=\tilde{t}_\varepsilon\}} |\partial_\tau v_\varepsilon| d\mathcal{H}^1 \;\le\; C|\log\varepsilon|^{-1/2}.$$

Hence the total variation of v_ε on $\{y\in\mathbb{R}^2 : |y| = \tilde{t}_\varepsilon\}$ is bounded by $C|\log\varepsilon|^{-1/2}$. Since $\tilde{t}_\varepsilon \ge \varepsilon^{1/2(n+1)}$ and $\tilde{t}_\varepsilon \int_{\{|y|=\tilde{t}_\varepsilon\}} F(|v_\varepsilon|^2)\, d\mathcal{H}^1 \le C\varepsilon^2|\log\varepsilon|^{-1}$, it follows that there is some constant v_0 of unit modulus such that $|v_\varepsilon(y) - v_0| \le C|\log\varepsilon|^{-1/2}$ whenever $|y| = \tilde{t}_\varepsilon$. We may thus write

$$v_\varepsilon(y) = \rho(y)e^{i\eta(y)} \qquad \text{for } |y| = \tilde{t}_\varepsilon,$$

where η is H^1 and real-valued. In particular, $\rho \le 2$ on $\{|y| = \tilde{t}_\varepsilon\}$. Let $\bar{\eta}$ denote the mean of η on $\{y\in\mathbb{R}^2 : |y| = \tilde{t}_\varepsilon\}$, and define a complex-valued function $\tilde{v}_\varepsilon$ on $\{|y| < \tilde{t}_\varepsilon\}$ by

$$\tilde{v}_\varepsilon(sy) := [1 + s(\rho(y) - 1)]\exp[i(\bar{\eta} + s(\eta(y) - \bar{\eta}))] \qquad \text{for } |y| = \tilde{t}_\varepsilon, 0 \le s \le 1,$$

Then one can check from (109) that

$$\int_{\{y\in\mathbb{R}^2:|y|<\tilde{t}_\varepsilon\}} e_\varepsilon(\tilde{v}_\varepsilon)(y)dy \le C|\log\varepsilon|^{-1}. \tag{110}$$

We next define $\tilde{w}_\varepsilon(x) = \tilde{v}_\varepsilon(\Psi^{-1}(x))$ in $B_{\tilde{t}_\varepsilon}(a_{k,\varepsilon}) \subset S$. We remark that since the area of $B_{\tilde{t}_\varepsilon}(a_{k,\varepsilon})$ is bounded by $C\tilde{t}_\varepsilon^2$ and $|\tilde{w}_\varepsilon|_g \le 2$ in $B_{\tilde{t}_\varepsilon}(a_{k,\varepsilon})$, it is clear that

$$\int_{B_{\tilde{t}_\varepsilon}(a_{k,\varepsilon})} |\mathcal{S}(\tilde{w}_\varepsilon u_\varepsilon^*)|_g^2 \operatorname{vol}_g \le C\int_{B_{\tilde{t}_\varepsilon}(a_{k,\varepsilon})} |\tilde{w}_\varepsilon|^2 \operatorname{vol}_g = o(1) \qquad \text{as } \varepsilon\to 0.$$

Now we proceed in the same fashion for every $k\in\{1,\dots,n\}$, and we set $\tilde{w}_\varepsilon = w_\varepsilon$ on $S_{\tilde{t}_\varepsilon}$ (in particular, $\tilde{w}_\varepsilon = w_\varepsilon$ on $S_{\sqrt{r_\varepsilon}}$). This yields a function $\tilde{w}_\varepsilon \in H^1(S;\mathbb{C})$. Again using (69), which implies that the difference between the metric g (appearing in e_ε^{ex}) and the Euclidean metric (appearing in e_ε) is negligible in small balls (in particular, inside $B_{\tilde{t}_\varepsilon}(a_{k,\varepsilon})$) for our choice of coordinates, we readily verify that $\tilde{w}_\varepsilon$ satisfies (108), proving the claim.

Step 4. We introduce[22] the functional $\mathcal{I}_0[\,\cdot\,; a, d, \Phi]$ for $v\in H^1(S;\mathbb{S}^1)$:

$$\mathcal{I}_0[v; a, d, \Phi] = \int_S \frac{1}{2}|dv|_g^2 + \frac{1}{2}|\mathcal{S}(vu^*(a, d, \Phi))|_g^2 \;\operatorname{vol}_g.$$

The definition (24) then implies that $\tilde{W}(a, d, \Phi) = \inf_{\Theta\in H^1(S;\mathbb{R})} \mathcal{I}_0[e^{i\Theta}; a, d, \Phi]$. In view of (105), (106), and (108), to conclude to our desired estimate (101), it suffices to prove that

$$\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon[\tilde{w}_\varepsilon; a^\varepsilon, d^\varepsilon, \Phi^\varepsilon] \ge \tilde{W}(a, d, \Phi) \tag{111}$$

and

$$\limsup_{\varepsilon\to 0} \tilde{W}(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \le \tilde{W}(a, d, \Phi). \tag{112}$$

Step 4'. We prove the second assertion (112). Toward this goal, we claim that after possible ε-dependent rotations of u_ε^*, we have

$$u_\varepsilon^* \to u^* \quad \text{a.e. in } S. \tag{113}$$

[22]The footnote 21 applies here as well.

First note that it is clear from the definition (10) of $\psi(a; d)$ and (20) that

$$j_\varepsilon^* := d^*\psi(a_\varepsilon, d_\varepsilon) + \sum_{k=1}^{2\mathfrak{g}} \Phi_{k,\varepsilon}\eta_k \to d^*\psi(a, d) + \sum_{k=1}^{2\mathfrak{g}} \Phi_k \eta_k = j^*$$

in C^1_{loc} away from $a_1, \ldots, a_n$ and globally in $L^q(S)$ for every $q \in [1, 2)$. Next, fix $x \in S \setminus \{a_k\}_k$ and a unit vector $v \in T_xS$. We may assume that $u_\varepsilon^*(x) = v$ for every ε. Now consider $y \in S \setminus \{a_k\}_k$ and a smooth curve $\gamma : [0, 1] \to S \setminus \{a_k\}_k$ such that $\gamma(0) = x, \gamma(1) = y$. For ε sufficiently small, the image of γ is bounded away from $\{a_{k,\varepsilon}\}_k$. When this holds, we define $U_\varepsilon(s) := u_\varepsilon^*(\gamma(s))$, and similarly $U(s) = u^*(\gamma(s))$. In (44), we have derived an explicit formula that gives $U_\varepsilon(s)$ in terms of $v \in T_xS$ and j_ε^* (or $U(s)$ in terms of v and j^*), and with the convergence of j_ε^* to j^*, this formula immediately implies that

$$u_\varepsilon^*(y) = U_\varepsilon(1) \to U(1) = u^*(y) \text{ as } \varepsilon \to 0.$$

Since y was an arbitrary point in $S \setminus \{a_k\}_k$, this proves the claim (113). [23]

Now the direct method leads to the existence of $\Theta_0 \in H^1(S; \mathbb{R})$ minimizing $\mathcal{I}_0[e^{i(\cdot)}; a, d, \Phi]$. The continuity of the shape operator and the convergence $u_\varepsilon^* \to u^*$ *a.e.* imply that $|\mathcal{S}(e^{i\Theta_0}u_\varepsilon^*)|_g^2 \to |\mathcal{S}(e^{i\Theta_0}u^*)|_g^2$ almost everywhere and hence in L^p for every $p < \infty$. It follows that

$$\limsup_\varepsilon \tilde{W}(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \le \lim_\varepsilon \mathcal{I}_0[e^{i\Theta_0}; a^\varepsilon, d^\varepsilon, \Phi^\varepsilon] = \mathcal{I}_0[e^{i\Theta_0}; a, d, \Phi] = \tilde{W}(a, d, \Phi),$$

proving (112).

Step 4". We prove (111). First note from (108), (107) that $\|\tilde{w}_\varepsilon\|_{H^1}^2 \le 2\mathcal{I}_\varepsilon[\tilde{w}_\varepsilon; a^\varepsilon, d^\varepsilon, \Phi^\varepsilon] \le C$. We may thus assume, after passing to a subsequence, that $\tilde{w}_\varepsilon \rightharpoonup w_0$ weakly in $H^1(S; \mathbb{C})$ and thus a.e. in S and strongly in L^p for every $p < \infty$. By Fatou's lemma,

$$\int_S F(|w_0|^2) \text{ vol}_g \le \liminf_{\varepsilon\to 0} \int_S F(|\tilde{w}_\varepsilon|^2) \text{ vol}_g \le \liminf_{\varepsilon\to 0} 4\varepsilon^2 \mathcal{I}_\varepsilon[\tilde{w}_\varepsilon; a^\varepsilon, d^\varepsilon, \Phi^\varepsilon] = 0,$$

so we deduce that $|w_0| = 1$ a.e. Standard weak lower semicontinuity arguments together with (113) and the continuity of the shape operator imply that

$$\mathcal{I}_0[w_0; a, d, \Phi] \le \liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon[\tilde{w}_\varepsilon; a^\varepsilon, d^\varepsilon, \Phi^\varepsilon].$$

To complete the proof of (111), it thus suffices to show that w_0 admits a lifting, that is, that there exists some $\Theta_0 \in H^1(S; \mathbb{R})$ such that $w_0 = e^{i\Theta_0}$. Note that this is a delicate issue as S is not simply connected while standard results (see e.g., [4]) requires this topological contraint on S. We will show in Lemma 10.4 that w_0 has indeed an H^1 lifting provided that w_0 satisfies the constraint $\Phi(w_0u^*) = \Phi(u^*) = \Phi$. Toward this end, as in Step 1, we note that $j(w_0u^*) = (iw_0, dw_0) + j^*$. Thus for $k \in \{1, \ldots, 2\mathfrak{g}\}$, we have

$$\Phi_k(w_0u^*) = \int_S ((iw_0, dw_0) + j^*, \eta_k)_g \text{ vol}_g = \int_S ((iw_0, dw_0), \eta_k)_g \text{ vol}_g + \Phi_k.$$

Similarly,

$$\Phi_k(\tilde{w}_\varepsilon u_\varepsilon^*) = \int_S ((i\tilde{w}_\varepsilon, d\tilde{w}_\varepsilon), \eta_k)_g \text{ vol}_g + \int_S (|\tilde{w}_\varepsilon|^2 - 1)j_\varepsilon^*, \eta_k)_g \text{ vol}_g + \Phi_{k,\varepsilon}.$$

[23] This argument proves the following continuity result (in addition to Theorem 2.1): if $a_\varepsilon \to a$ and $\Phi_\varepsilon \in \mathcal{L}(a_\varepsilon, d) \to \Phi \in \mathcal{L}(a, d)$, then up to rotations, $u_\varepsilon^*(a_\varepsilon, d, \Phi_\varepsilon) \to u^*(a, d, \Phi)$ almost everywhere and in L^p for all $p < \infty$.

From these and the convergence $\tilde{w}_\varepsilon \rightharpoonup w_0$ weakly in $H^1(S;\mathbb{C})$, $\tilde{w}_\varepsilon \to w_0$ strongly in $L^2(S)$ and $L^6(S)$ (in particular, $|\tilde{w}_\varepsilon|^2 \to 1$ in $L^3(S)$), $j^*_\varepsilon \to j^*$ in $L^q(S)$ for $q = \frac{3}{2} < 2$, and recalling that $\Phi^\varepsilon \to \Phi$, one can verify that $\Phi(w_0 u^*) = \lim_{\varepsilon\to 0} \Phi(\tilde{w}_\varepsilon u^*_\varepsilon)$.

Next, recall that by construction in Step 3, $\tilde{w}_\varepsilon u^*_\varepsilon = w_\varepsilon u^*_\varepsilon = m_\varepsilon$ in $S_{\tilde{t}_\varepsilon}$ and $|\tilde{w}_\varepsilon| \le 2$ in $\mathcal{O} := \cup_{k=1}^N B_{\tilde{t}_\varepsilon}(a_{k,\varepsilon})$. Therefore, $|j(\tilde{w}_\varepsilon u^*_\varepsilon)|_g \le 4|j^*_\varepsilon|_g + 2|d\tilde{w}_\varepsilon|_g$ in $\mathcal{O}$. Thus

$$
\begin{aligned}
&|\Phi_k(\tilde{w}_\varepsilon u^*_\varepsilon) - \Phi_k(m_\varepsilon)| \le C \int_{\mathcal{O}} |j(\tilde{w}_\varepsilon u^*_\varepsilon)|_g + |j(m_\varepsilon)|_g \operatorname{vol}_g \\
&\le C\bigg(\int_{\mathcal{O}} |j^*_\varepsilon|_g + |d\tilde{w}_\varepsilon|_g \operatorname{vol}_g + \int_{\mathcal{O}\cap\{|m_\varepsilon|_g\le 2\}} |Dm_\varepsilon|_g \operatorname{vol}_g + \int_{\mathcal{O}\cap\{|m_\varepsilon|_g\ge 2\}} (|m_\varepsilon|_g - 1)|Dm_\varepsilon|_g \operatorname{vol}_g \bigg) \\
&\le C\bigg(\tilde{t}_\varepsilon^{1-1/q} \|j^*_\varepsilon\|_{L^q(S)} + \tilde{t}_\varepsilon^{1/2} \|d\tilde{w}_\varepsilon\|_{L^2(\mathcal{O})} + \tilde{t}_\varepsilon^{1/2} E^{in}_\varepsilon(m_\varepsilon)^{1/2} + \varepsilon E^{in}_\varepsilon(m_\varepsilon) \bigg) \to 0,
\end{aligned}
$$

as $\varepsilon \to 0$, where we used Hölder's inequality, that (j^*_ε) is uniformly bounded in $L^q(S)$ for $q = 3/2$, (110), (100) and (1). Since $\Phi(m_\varepsilon) \to \Phi$ by assumption, we deduce that $\Phi(w_0 u^*) = \Phi(u^*)$ as claimed.

Now Lemma 10.4 below implies that w admits a lifting, completing the proof of (111) and hence of Lemma 10.3. □

Lemma 10.4. *Assume that $w \in H^1(S;\mathbb{S}^1)$ and that $\Phi(wu^*) = \Phi(u^*)$ for some canonical harmonic unit vector field $u^*(a,d,\Phi)$. Then there exists $\Theta \in H^1(S;\mathbb{R})$ such that $w = e^{i\Theta}$.*

Proof. It follows from [4] that any $w \in H^1(S;\mathbb{S}^1)$ can *locally* be written in the form $w = e^{i\theta}$. It follows that, again locally, $j(w)$ has the form $j(w) = d\theta$. Thus $dj(w) = 0$. As a result, $\int_S (j(w), d^*\beta)_g \operatorname{vol}_g = 0$ for all 2-forms β in $H^1(S)$. This implies that the Hodge decomposition (37) for $j(w)$ takes the form

$$
j(w) = (iw, dw) = d\Theta + \eta \qquad \text{where } \Theta \in H^1(S;\mathbb{R}) \text{ and } \eta \text{ is a harmonic 1-form.}
$$

Next, by hypothesis, we have

$$
\begin{aligned}
\Phi_k(u^*) = \Phi_k(wu^*) &= \int_S (j(wu^*), \eta_k)_g \operatorname{vol}_g \\
&= \int_S (j(w) + j^*, \eta_k)_g \operatorname{vol}_g = \int_S (j(w), \eta_k)_g \operatorname{vol}_g + \Phi_k(u^*).
\end{aligned}
$$

yielding $\int_S (j(w), \eta_k)_g \operatorname{vol}_g = 0$ for every $k = 1, \dots, 2\mathfrak{g}$. Thus the harmonic part of $j(w)$ in the decomposition (37) vanishes, and $j(w) = d\Theta$ for some $\Theta \in H^1(S;\mathbb{R})$, or equivalently, $j(we^{-i\Theta}) = 0$. Writing $v = we^{-i\Theta} \in H^1(S;\mathbb{S}^1)$, we deduce that

$$
dv = (dv, \frac{iv}{|v|})\frac{iv}{|v|} + (dv, \frac{v}{|v|})\frac{v}{|v|} = j(v)\frac{iv}{|v|} + d|v|\,\frac{v}{|v|} = 0.
$$

It follows that v is constant, from which we conclude that $w = e^{i(\Theta+\alpha)}$ for some $\alpha \in \mathbb{R}$. □

Remark 10.5. Note that

$$
\tilde{W}(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \to \tilde{W}(a,d,\Phi) \quad \text{if } a^\varepsilon \to a,\ d^\varepsilon \to d \text{ and } \Phi^\varepsilon \in \mathcal{L}(a^\varepsilon, d^\varepsilon) \to \Phi \in \mathcal{L}(a,d).
$$

In fact it is a consequence of (112) and $\liminf_{\varepsilon\to 0} \tilde{W}(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \ge \tilde{W}(a,d,\Phi)$ which follows the argument in Step 4" above. Indeed, if we denote by Θ_ε a minimizer of $\tilde{W}(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon)$ (such a minimizer exists as a consequence of the direct method in calculus of variations), we have that $\|d\Theta_\varepsilon\|^2_{L^2(S)} \le 2\tilde{W}(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \le \int_S |\mathcal{S}(e^{i\pi} u^*(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon))|^2_g \operatorname{vol}_g \le C$ (because $\mathcal{S}$ is bounded over the set of unit vector fields). Therefore, up to an additive constant, the Poincaré-Wirtinger inequality implies that (Θ_ε) is uniformly bounded in $H^1(S)$. Therefore, for a subsequence, there exists a limit $\Theta \in H^1(S)$ such that $\Theta_\varepsilon \rightharpoonup \Theta$ weakly in $H^1(S)$

and a.e. in S. As $u^*(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \to u^*(a, d, \Phi)$ a.e. in S (by footnote 23, recall that $d^\varepsilon = d$ for small ε), standard weak lower semicontinuity arguments, the continuity of the shape operator and Fatou's lemma imply

$$\liminf_{\varepsilon \to 0} \tilde{W}(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \geq \frac{1}{2} \int_S |d\Theta|_g^2 + |\mathcal{S}(e^{i\Theta} u^*(a, d, \Phi))|_g^2 \operatorname{vol}_g \geq \tilde{W}(a, d, \Phi).$$

Proof of Theorem 10.1, point 2). We may assume the hypothesis on $(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon)$ made at the beginning of Section 10.3. Then we argue as in the proof of Proposition 9.1 point 1). As $e_\varepsilon^{ex}(m_\varepsilon) \geq e_\varepsilon^{in}(m_\varepsilon)$ in S (by Lemma 10.2, by Step 3 in the proof of Proposition 9.1 point 1)), there exists $t_\varepsilon \in (r_\varepsilon, \sqrt{r}_\varepsilon)$ such that

$$\int_{B_{t_\varepsilon}(a_{k,\varepsilon})} e_\varepsilon^{ex}(m_\varepsilon) \operatorname{vol}_g \geq \int_{B_{t_\varepsilon}(a_{k,\varepsilon})} e_\varepsilon^{in}(m_\varepsilon) \operatorname{vol}_g \geq \pi \log \frac{t_\varepsilon}{\varepsilon} + \iota_F + o(1)$$

as $\varepsilon \to 0$, for every $1 \leq k \leq n$. As $t_\varepsilon \in (r_\varepsilon, \sqrt{r_\varepsilon})$ and $\operatorname{dist}_S(a_{k,\varepsilon}, a_{\ell,\varepsilon}) = O(1) \geq t_\varepsilon$ for $k \neq \ell$, then (78) holds true for t_ε so that we can apply Lemma 10.3 for t_ε yielding:

$$\int_{S_{t_\varepsilon}} e_\varepsilon^{ex}(m_\varepsilon) \operatorname{vol}_g \geq n\pi \log \frac{1}{t_\varepsilon} + (W + \tilde{W})(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) - o(1)$$

as $\varepsilon \to 0$. As $W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \to W(a, d, \Phi)$ (by Proposition 2.5, as $\Phi_\varepsilon \to \Phi$ and $d_\varepsilon = d$ for all small ε) and $\tilde{W}(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \to \tilde{W}(a, d, \Phi)$ (by Remark 10.5) in the limit $\varepsilon \to 0$, we reach the desired lower bound for $E_\varepsilon^{ex}(m_\varepsilon)$. □

11. Γ-limit for the micromagnetic energy

Before stating the main result for Problem 3, let us show that the quantity $\tilde{\iota}_F$ in (28) is well defined, i.e., the limit in (28) exists. For that, it is enough to prove the nondecreasing behaviour of $t \mapsto I_F^{mm}(t) + \pi \log t$ that follows as for ι_F (see [3, Lemma III.1]): for $0 < t_1 \leq t_2 \leq 1$, we want $I_F^{mm}(t_1) \leq \pi \log \frac{t_2}{t_1} + I_F^{mm}(t_2)$. Indeed, if v_2 is the minimizer of $I_F^{mm}(\frac{1}{t_2}, 1)$, then setting $v_1 = v_2$ in $B_{1/t_2}(0)$ and $v_1 = \frac{x}{|x|}$ in $B_{1/t_1}(0) \setminus B_{1/t_2}(0)$ we have

(114)
$$I_F^{mm}(t_1) = I_F^{mm}(\frac{1}{t_1}, 1) \leq \int_{B_{1/t_1}(0)} \tilde{e}_1(v_1)\, dy = I_F^{mm}(\frac{1}{t_2}, 1) + \pi \log \frac{t_2}{t_1} = I_F^{mm}(t_2) + \pi \log \frac{t_2}{t_1}.$$

In this section we prove the counterpart of Theorem 10.1 for the micromagnetic energy E_ε^{mm} in Problem 3 where the surface S is isometrically embedded in $\mathbb{R}^3$ endowed with the Euclidian metric g. For $M : S \to \mathbb{S}^2$, we always use the decomposition

$$M = m + M_\perp N, \quad m = \Pi(M) \tag{115}$$

where N is the Gauss map on S, $M_\perp = M \cdot N$ is the normal component of M and m is the projection Π of M on the tangent plane TS.

Theorem 11.1. *The following Γ-convergence result holds.*

1) *(Compactness) Let $(M_\varepsilon)_{\varepsilon \downarrow 0}$ be a family in $H^1(S; \mathbb{S}^2)$ satisfying*

$$E_\varepsilon^{mm}(M_\varepsilon) \leq T\pi |\log \varepsilon| + C$$

for some integer $T > 0$ and a constant $C > 0$. Then there exists a sequence $\varepsilon \downarrow 0$ such that for every $p \in [1, 2)$, the vorticity $\omega(m_\varepsilon)$ of the projection $m_\varepsilon = \Pi(M_\varepsilon)$ satisfies (94) for n distinct points $\{a_k\}_{k=1}^n$ and nonzero integers $\{d_k\}_{k=1}^n$ satisfying (7) and $\sum_{k=1}^n |d_k| \leq T$. Moreover, if $\sum_{k=1}^n |d_k| = T$, then $n = T$ and $|d_k| = 1$ for *every $k = 1, \dots, n$; in this case, for a further subsequence, there exists $\Phi \in \mathcal{L}(a; d)$ such that $\Phi(m_\varepsilon)$ defined in (53) converges to Φ as $\varepsilon \to 0$.*

2) *(Γ-liminf inequality) Assume that the projections $m_\varepsilon = \Pi(M_\varepsilon) \in \mathcal{X}^{1,2}(S)$ of some family $M_\varepsilon : S \to \mathbb{S}^2$ satisfy (94) for $p = 1$ and n distinct points $\{a_k\}_{k=1}^n \in S^n$ and $|d_k| = 1$, $k = 1, \dots n$ such that (7) holds true and $\Phi(m_\varepsilon) \to \Phi \in \mathcal{L}(a; d)$. Then*

$$\liminf_{\varepsilon\to 0} [E_\varepsilon^{mm}(M_\varepsilon) - n\pi|\log\varepsilon|] \geq W(a, d, \Phi) + \tilde{W}(a, d, \Phi) + n\tilde{\iota}_F + \liminf_{\varepsilon\to 0} \int_{S\setminus\cup_{k=1}^n B_\sigma(a_k)} |dM_{\perp,\varepsilon}|_g^2 \, vol_g, \ \forall \sigma > 0,$$

for $u^* = u^*(a, d, \Phi)$, $a = (a_1, \dots, a_n)$, $d = (d_1, \dots, d_n)$, $\tilde{W}(a, d, \Phi)$ defined in (24), $\tilde{\iota}_F$ is defined in (28) and $M_{\perp,\varepsilon}$ is the normal component of M_ε.

3) *(Γ-limsup inequality) For every n distinct points $a_1, \dots, a_n \in S$ and $d_1, \dots, d_n \in \{\pm 1\}$ satisfying (7) and every $\Phi \in \mathcal{L}(a; d)$ there exists a sequence of smooth maps $M_\varepsilon : S \to \mathbb{S}^2$ such that (94) holds for the projections $m_\varepsilon = \Pi(M_\varepsilon)$ (see (115)), $M_{\perp,\varepsilon} \to 0$ in $H^1_{loc}(S \setminus \{a_k\}_k)$, $\Phi(m_\varepsilon) \to \Phi$ and*

$$E_\varepsilon^{mm}(M_\varepsilon) - n\pi|\log\varepsilon| \longrightarrow W(a, d, \Phi) + \tilde{W}(a, d, \Phi) + n\tilde{\iota}_F \quad \text{as } \varepsilon \to 0.$$

Remark 11.2. The term $\liminf_{\varepsilon\to 0} \int_{S\setminus\cup_{k=1}^n B_\sigma(a_k)} |dM_{\perp,\varepsilon}|_g^2 \,\mathrm{vol}_g$ in the above Γ-liminf inequality will be used to show that if M_ε minimizes E_ε^{mm}, then $M_{\perp,\varepsilon} \to 0$ in $H^1_{loc}(S\setminus\{a_k\}_k)$, where $\{a_k\}_k$ are limiting vortex locations.

Proof. We divide the proof in several steps.

Step 1. A basic computation. Let $M : S \to \mathbb{S}^2$ such that $E_\varepsilon^{mm}(M) \leq C|\log\varepsilon|$. By (115), we start by computing the extrinsic differential of M:

$$\begin{aligned} \bar{D}M &= \bar{D}m + \bar{D}(M_\perp N) \\ &= Dm + (\bar{D}m \cdot N) \otimes N + M_\perp \bar{D}N + dM_\perp \otimes N \\ &= [Dm + M_\perp \bar{D}N] + [(\bar{D}m \cdot N) + dM_\perp] \otimes N. \end{aligned}$$

In other words, in terms of partial derivatives, we have for $\ell = 1, 2$:

$$\bar{D}_\ell M = [D_\ell m + M_\perp \bar{D}_\ell N] + [(\bar{D}_\ell m \cdot N) + \partial_\ell M_\perp] N.$$

This entails the following extrinsic Dirichlet energy density:

$$|\bar{D}M|_g^2 = \underbrace{|Dm + M_\perp \bar{D}N|_g^2}_{=:I} + \underbrace{|(\bar{D}m \cdot N) + dM_\perp|_g^2}_{=:II}.$$

Writing

$$E_\varepsilon^{mm}(M) = \int_S \frac{1}{2}(I + II) + \frac{1}{4\varepsilon^2} F(|m|_g^2) \,\mathrm{vol}_g \leq C|\log\varepsilon|,$$

we deduce by Young's inequality that

$$\begin{aligned} 2C|\log\varepsilon| \geq \int_S I \,\mathrm{vol}_g &= \int_S |Dm|_g^2 + M_\perp^2 |\bar{D}N|_g^2 + 2M_\perp (Dm, \bar{D}N)_g \,\mathrm{vol}_g \\ &\geq \int_S \frac{1}{2}|Dm|_g^2 - 3\|\bar{D}N\|_{L^\infty}^2 M_\perp^2 \,\mathrm{vol}_g \\ &\geq \int_S \frac{1}{2}|Dm|_g^2 \,\mathrm{vol}_g - O(\varepsilon^2|\log\varepsilon|). \end{aligned}$$

Therefore,

$$\int_S I \text{ vol}_g = \int_S |Dm|_g^2 + \underbrace{M_\perp^2|\bar{D}N|_g^2}_{=O(\varepsilon^2|\log\varepsilon|)} + 2\underbrace{M_\perp(Dm,\bar{D}N)_g}_{=O(\varepsilon|\log\varepsilon|)} \text{ vol}_g = \int_S |Dm|_g^2 \text{ vol}_g + O(\varepsilon|\log\varepsilon|).$$

The second term II is treated as follows:

$$\begin{aligned}\int_S II \text{ vol}_g &= \int_S |\bar{D}m\cdot N|_g^2 + |dM_\perp|_g^2 + 2(\bar{D}m\cdot N, dM_\perp)_g \text{ vol}_g \\ &= \int_S |\mathcal{S}(m)|_g^2 + |dM_\perp|_g^2 \text{ vol}_g + O(\varepsilon|\log\varepsilon|)\end{aligned}$$

because $\bar{D}m\cdot N = -m\cdot\bar{D}N$ so $|\bar{D}m\cdot N|_g^2 = |\mathcal{S}(m)|_g^2$ and integration by parts yields

$$\int_S (\bar{D}m\cdot N, dM_\perp)_g \text{ vol}_g = -\int_S \left(d^*(m\cdot\bar{D}N), M_\perp\right)_g \text{ vol}_g = O(\varepsilon|\log\varepsilon|).$$

Therefore, we obtain:

(116)
$$\int_S |\bar{D}M|_g^2 \text{ vol}_g = \int_S \left(|\bar{D}m|_g^2 + |dM_\perp|_g^2\right) \text{ vol}_g + O(\varepsilon|\log\varepsilon|) \geq \int_S |Dm|_g^2 \text{ vol}_g + O(\varepsilon|\log\varepsilon|).$$

Step 2. Compactness. Let M_ε satisfy the assumptions at Theorem 11.1 point 1). By Step 1, we deduce that $E_\varepsilon^{in}(m_\varepsilon) \leq T\pi|\log\varepsilon| + C$ where $m_\varepsilon = \Pi(M_\varepsilon)$ is the projection of M_ε on TS (recall that the potential F in E_ε^{mm} satisfies (1)). Thus m_ε satisfies the hypotheses of Theorem 2.6 point 1), and the desired conclusion follows from facts established there (in particular, Proposition 8.1).

Step 3. Upper bound. The difference with respect to Problem 2 is the following: within the notation in Step 1, as $|M| = 1$, one has that $|M_\perp| = \sqrt{1-|m|_g^2}$. By (116), the only term that changes in the renormalized energy for Problem 2 comes from $|dM_\perp|_g^2 = |d\sqrt{1-|m|^2}|_g^2$ that influences the energy of the radial profile of a vortex by a constant (therefore, ι_F in Problem 2 will be replaced by $\tilde{\iota}_F$). Let $u^* = u^*(a,d,\Phi)$ be a canonical harmonic map and Θ be a minimizer in (24). As Θ satisfies the associated Euler-Lagrange equation (97), it yields $\Delta\Theta \in L^\infty$ so $\Theta \in C^1(S)$. Let $U_\varepsilon := U_\varepsilon(a,d,\Phi)$ be the vector field constructed in the proof of Proposition 9.1 point 2). We have to modify U_ε in the balls $B_{\sqrt{\varepsilon}/2}(a_k)$ according to the micromagnetic radial profile of a vortex given in I_F^{mm}. For that, we recall that U_ε is denoted in exponential normal coordinates by V_ε around every vortex a_k of degree $d_k \in \{\pm 1\}$ and we have that $V_\varepsilon = e^{id_k\theta}$ on $\partial B_{\sqrt{\varepsilon}/2}(0)$ (up to a rotation). We define $\tilde{V}_\varepsilon : B(0,\frac{\sqrt{\varepsilon}}{2}) \to \mathbb{S}^2$ as being a minimizer of $I_F^{mm}(\frac{\sqrt{\varepsilon}}{2},\varepsilon)$ if $d_k = 1$ (or its complex conjugate if $d_k = -1$) up to a rotation. We set $\tilde{V}_\varepsilon = V_\varepsilon$ outside these balls of radius $\frac{\sqrt{\varepsilon}}{2}$. Denoting by $\tilde{U}_\varepsilon$ the tangential component of the corresponding map to $\tilde{V}_\varepsilon$ on S, we set $m_\varepsilon = e^{i\Theta}\tilde{U}_\varepsilon$ and $M_\varepsilon = m_\varepsilon + M_{\perp,\varepsilon}N$ where $M_{\perp,\varepsilon} = 0$ outside the balls $B_{\sqrt{\varepsilon}/2}(a_k)$ and $M_{\perp,\varepsilon}$ is the vertical component of $\tilde{V}_\varepsilon$ inside $B_{\sqrt{\varepsilon}/2}(a_k)$. By the proof of Proposition 9.1 point 2) and the above choice of M_ε inside the balls $B_{\sqrt{\varepsilon}/2}(a_k)$, we deduce that

$$\int_S |Dm_\varepsilon|_g^2 \text{ vol}_g \leq C|\log\varepsilon|$$

so that (116) implies

$$\int_S |\bar{D}M_\varepsilon|_g^2 \text{ vol}_g = \int_S \big(|\bar{D}m_\varepsilon|_g^2 + |dM_{\perp,\varepsilon}|_g^2\big) \text{ vol}_g + O(\varepsilon|\log\varepsilon|)$$
$$= \int_S (|D\tilde{U}_\varepsilon|_g^2 + \underbrace{|\tilde{U}_\varepsilon|_g^2|d\Theta|_g^2 + |\mathcal{S}(m_\varepsilon)|_g^2 + 2\big(j(\tilde{U}_\varepsilon), d\Theta\big)_g}_{III} + |dM_{\perp,\varepsilon}|_g^2) \text{ vol}_g + O(\varepsilon|\log\varepsilon|).$$

Estimating III. Recall that $|\tilde{U}_\varepsilon|_g^2 \le 1$ in S and $\tilde{U}_\varepsilon = u^*$ in $S_{\sqrt{\varepsilon}}$ so that

$$\int_S III \text{ vol}_g = \int_S |\tilde{U}_\varepsilon|_g^2|d\Theta|_g^2 + |\mathcal{S}(m_\varepsilon)|_g^2 + 2\big(j(\tilde{U}_\varepsilon), d\Theta\big)_g \text{ vol}_g$$
$$\le \int_S |d\Theta|_g^2 + |\mathcal{S}(e^{i\Theta}u^*)|_g^2 + 2\big(j(\tilde{U}_\varepsilon), d\Theta\big)_g \text{ vol}_g + \int_{S\setminus S_{\sqrt{\varepsilon}}} |\mathcal{S}(m_\varepsilon)|_g^2 \text{ vol}_g$$
$$\le 2\tilde{W}(a, d, \Phi) + o(1)$$

because

$$\int_{S\setminus S_{\sqrt{\varepsilon}}} |\mathcal{S}(m_\varepsilon)|_g^2 \text{ vol}_g = o(1)$$

(as $|m_\varepsilon|_g \le 1$) and

$$\left|\int_S \big(j(\tilde{U}_\varepsilon), d\Theta\big)_g \text{ vol}_g\right| \le \left|\int_S \big(j(u^*), d\Theta\big)_g \text{ vol}_g\right| + \left|\int_{S\setminus S_{\sqrt{\varepsilon}}} \big(j(\tilde{U}_\varepsilon) - j(u^*), d\Theta\big)_g \text{ vol}_g\right|$$
$$\le \left|\int_S \big(\underbrace{d^*j(u^*)}_{=0}, \Theta\big)_g \text{ vol}_g\right| + \|d\Theta\|_{L^\infty} \int_{S\setminus S_{\sqrt{\varepsilon}}} |j(\tilde{U}_\varepsilon) - j(u^*)|_g \text{ vol}_g$$
$$= o(1)$$

because $\|j(\tilde{U}_\varepsilon) - j(u^*)\|_{L^1(S\setminus S_{\sqrt{\varepsilon}})} \le \|j(\tilde{U}_\varepsilon))\|_{L^1(S\setminus S_{\sqrt{\varepsilon}})} + \|j(u^*)\|_{L^1(S\setminus S_{\sqrt{\varepsilon}})} \to 0$ (from Hölder's inequality in the small balls of radius $\sqrt{\varepsilon}$, using control over $\|D\tilde{U}_\varepsilon\|_{L^2}$ coming from the energy, and estimating $\|Du^*\|_{L^p}$ for $p < 2$, as in Steps 1 and 3 of the proof of Proposition 9.1).

Estimating the integral of $\frac{1}{2}(|D\tilde{U}_\varepsilon|_g^2 + |dM_{\perp,\varepsilon}|_g^2) + \frac{1}{4\varepsilon^2}F(|m_\varepsilon|_g^2)$ *on* S. First, by definition of $\tilde{V}_\varepsilon$ inside the ball $B(0, \frac{\sqrt{\varepsilon}}{2})$ and $\tilde{\iota}_F$, we obtain by (27)

$$\int_{B(0,\frac{\sqrt{\varepsilon}}{2})} \tilde{e}_\varepsilon(\tilde{V}_\varepsilon)\, dy = \pi \log \frac{\sqrt{\varepsilon}}{2\varepsilon} + \tilde{\iota}_F + o(1).$$

Recall that inside the annulus $B(0, \sqrt{\varepsilon}) \setminus B(0, \frac{\sqrt{\varepsilon}}{2})$, we have by (76)

$$\int_{B(0,\sqrt{\varepsilon})\setminus B(0,\frac{\sqrt{\varepsilon}}{2})} \frac{1}{2}|\nabla\tilde{V}_\varepsilon|^2\, dy = \pi\log 2 + o(1)$$

Thus, by (69) and (72) as $M_{\perp,\varepsilon} = 0$ outside $B(a_k, \sqrt{\varepsilon}/2)$:

$$\int_{B(a_k,\sqrt{\varepsilon})} \frac{1}{2}(|D\tilde{U}_\varepsilon|_g^2 + |dM_{\perp,\varepsilon}|_g^2) + \frac{1}{4\varepsilon^2}F(|m_\varepsilon|_g^2) \text{ vol}_g$$
$$= \int_{\{y\in\mathbb{R}^2 : |y| < \sqrt{\varepsilon}\}} \Big[(1 + O(\varepsilon))\tilde{e}_\varepsilon(\tilde{V}_\varepsilon) + O(1)\Big]\sqrt{g(y)}dy$$
$$= \pi\log\frac{\sqrt{\varepsilon}}{\varepsilon} + \tilde{\iota}_F + o(1) \quad \text{as } \varepsilon \to 0.$$

Finally, by definition of $W(a,d,\Phi)$, we have

$$\int_{S_{\sqrt{\varepsilon}}} \frac{1}{2}|D\tilde{U}_\varepsilon|_g^2 \,\mathrm{vol}_g = \int_{S_{\sqrt{\varepsilon}}} \frac{1}{2}|Du^*|_g^2 \,\mathrm{vol}_g = W(a,d,\Phi) + n\pi \log \frac{1}{\sqrt{\varepsilon}} + o(1).$$

Summing up, the desired upper bound follows.

The convergence $j(m_\varepsilon) \to j(u^*)$ in $L^p(S)$ for every $p \in [1,2)$ follows as in the proof of Theorem 10.1 point 3) because the change made above for $\tilde{U}_\varepsilon$ (instead of U_ε) in the small balls $B(a_k, \sqrt{\varepsilon})$ does not affect the convergence of the current due to $\|j(m_\varepsilon)\|_{L^p(B(a_k,\sqrt{\varepsilon}))} \to 0$ for every $p \in [1,2)$ (coming from the blow up of $\|j(m_\varepsilon)\|_{L^2(B(a_k,\sqrt{\varepsilon}))}$ as $|\log \varepsilon|$ and the Hölder inequality in the ball $B(a_k, \sqrt{\varepsilon})$). This entails also the convergence of the vorticities $\omega(m_\varepsilon)$ in (94), as well as $\Phi(m_\varepsilon) \to \Phi = \Phi(u^*)$.

Step 4. Lower bound. Let M_ε satisfy the assumptions at Theorem 11.1 point 2). Furthermore, we may assume that $E_\varepsilon^{mm}(M_\varepsilon) \le n\pi|\log\varepsilon| + c$ for some $c > 0$ (otherwise the lower bound is trivial). By (116), we deduce that

$$E_\varepsilon^{mm}(M_\varepsilon) = E_\varepsilon^{ex}(m_\varepsilon) + \int_S |dM_{\perp,\varepsilon}|^2 \,\mathrm{vol}_g + O(\varepsilon|\log\varepsilon|) \tag{117}$$

where $m_\varepsilon = \Pi(M_\varepsilon)$ is the projection of M_ε on TS and F satisfies (1). The rest of the proof follows the argument in the proof of Theorem 10.1, point 2). With the same notation, the only change here concerns the estimate inside the small balls $B_{t_\varepsilon}(a_{k,\varepsilon})$. As in Step 4 in the proof of Proposition 9.1 point 1), one uses the entire micromagnetic energy density and (27) to conclude using (69) and (72)

$$\begin{aligned}\int_{B_{t_\varepsilon}(a_{k,\varepsilon})} e_\varepsilon^{ex}(M_\varepsilon) + |dM_{\perp,\varepsilon}|^2 \,\mathrm{vol}_g &\ge \int_{\{y\in\mathbb{R}^2 : |y|<t_\varepsilon\}} \left[(1+O(\varepsilon))\tilde{e}_\varepsilon(v_\varepsilon) + O(1)\right] \sqrt{g(y)}dy \\ &\ge \pi \log \frac{t_\varepsilon}{\varepsilon} + \tilde{\iota}_F + o(1)\end{aligned}$$

as $\varepsilon \to 0$, for every $1 \le k \le n$ where $v_\varepsilon : B(0,t_\varepsilon) \to \mathbb{S}^2$ is the representation in normal coordinates of M_ε, with its in-plane component corresponding to m_ε and its vertical component to $M_{\perp,\varepsilon}$ (see (115)). Also note the extra term in $|dM_{\perp,\varepsilon}|^2$ in (117) that is used only inside the small balls, therefore, it leads to the extra term in the desired lower bound outside the fixed balls $B_\sigma(a_k)$ around the limit vortices. □

12. Minimizers of the considered functionals

In this section, we study the asymptotic behaviour of minimizers of our three functionals as $\varepsilon \to 0$.

The intrinsic case.

Theorem 12.1. *For $\varepsilon > 0$, let u_ε be a minimizer of E_ε^{in} over the set $\mathcal{X}^{1,2}(S)$. Then there exists a sequence $\varepsilon \downarrow 0$ such that for every $p \in [1,2)$,*

$$\begin{cases} \omega(u_\varepsilon) \longrightarrow 2\pi \sum_{k=1}^n d_k^* \delta_{a_k^*} \quad \text{in } W^{-1,p}(S), & \\ u_\varepsilon \rightharpoonup u^* \quad \text{weakly in } \mathcal{X}^{1,p}(S), & \text{as } \varepsilon \to 0, \\ \Phi(u_\varepsilon) \to \Phi^* & \end{cases}$$

where $n = |\chi(S)|$, $\{a_k^*\}_{k=1}^n$ are distinct points in S, $d_k^* = \mathrm{sign}(\chi(S))$, $\Phi^* \in \mathcal{L}(a^*, d^*)$ such that (a^*, d^*, ϕ^*) is a minimizer of the renormalized energy (for the above d^*)

$$\{W(a, d^*, \Phi) \,:\, a = (a_1, \ldots, a_n) \in S^n \text{ distinct points}, \Phi \in \mathcal{L}(a, d^*)\}$$

and u^* is a canonical harmonic vector field associated to (a^*, d^*, ϕ^*). Moreover, we have the following second order energy expansion:

$$E_\varepsilon^{in}(u_\varepsilon) = n\pi \log \frac{1}{\varepsilon} + W(a^*, d^*, \Phi^*) + n\iota_F + o(1), \text{ as } \varepsilon \to 0.$$

Remark 12.2. 1) We will also prove that $j(u_\varepsilon) \to j(u^*)$ in $L^p(S)$ for every $p < 2$, and that

$$\int_{S_\sigma} e_\varepsilon^{in}(u_\varepsilon) \operatorname{vol}_g \leq C_\sigma, \quad \int_{S_\sigma} \left[\frac{1}{2}\Big|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j(u^*)\Big|_g^2 + e_\varepsilon^{in}(|u_\varepsilon|_g)\right] \operatorname{vol}_g \to 0, \tag{118}$$

for every $\sigma > 0$, where $S_\sigma := S \setminus \cup_{k=1}^n B_\sigma(a_k^*)$. It follows that $|u_\varepsilon|_g \to 1$ in $H^1(S_\sigma)$ and $u_\varepsilon \rightharpoonup u^*$ weakly in $\mathcal{X}^{1,2}(S_\sigma)$.

2) In the case $\chi(S) = 0$, as smooth unit length vector fields do exist over S, the minimal energy E_ε^{in} is uniformly bounded as $\varepsilon \to 0$. Therefore, any sequence of minimizers u_ε of E_ε^{in} has a subsequence strongly convergent in $\mathcal{X}^{1,2}(S)$ to a smooth canonical harmonic vector field $u^* \in \mathcal{X}(S)$. As $|u_\varepsilon|_g \leq 1$ in S (by the standard cutting off argument at 1 for a minimizer u_ε), it entails $j(u_\varepsilon) \to j(u^*)$ in L^2 and therefore, $\omega(u_\varepsilon) \to 0$ in $H^{-1} \cap L^1(S)$ (by (40)). This case was treated in [39].

3) When $\mathfrak{g} = 0$ (that is, when S is a topological sphere) then Φ is not present, and the renormalized energy and vector field u^* both simplify significantly.

Proof of Theorem 12.1. Let $n = |\chi(S)|$. We assume $n > 0$ (the case $n = 0$ is treated in Remark 12.2 point 1)). Fix $d_k = \operatorname{sign}(\chi(S))$ for $1 \leq k \leq n$. Denoting $d = (d_1, \ldots, d_n)$, by Proposition 2.5, the direct method in the calculus of variation implies the existence of a minimizer of

$$\{W(a, d, \Phi) \,:\, a = (a_1, \ldots, a_n) \in S^n \text{ distinct points}, \Phi \in \mathcal{L}(a, d)\}.$$

Fix (a, d, Φ) such a minimizer. First, by the upper bound in Theorem 2.6 point 3) applied to the triple (a, d, Φ), we deduce that every minimizer u_ε of E_ε^{in} has to satisfy

$$E_\varepsilon^{in}(u_\varepsilon) \leq \pi n |\log \varepsilon| + W(a, d, \Phi) + n\iota_F + o(1). \tag{119}$$

By the compactness result in Theorem 2.6 point 1) (see also Corollary 8.3), we have for a subsequence that $\omega(u_\varepsilon) \longrightarrow 2\pi \sum_{k=1}^K d_k^* \delta_{a_k^*}$ in $W^{-1,p}(S)$ for some distinct points a_k^* and nonzero integers d_k^* satisfying $\sum_k d_k^* = \chi(S)$ and $\sum_k |d_k^*| \leq n = |\chi(S)|$. It entails that $K = n$ and $d_k^* = d_k = \operatorname{sign}(\chi(S))$. In this case, Theorem 2.6 point 1) gives us the existence of $\Phi^* \in \mathcal{L}(a^*, d^*)$ such that for a subsequence $\Phi(u_\varepsilon) \to \Phi^*$. Applying Theorem 2.6 point 2), we deduce that $E_\varepsilon^{in}(u_\varepsilon) \geq \pi n |\log \varepsilon| + W(a^*, d^*, \Phi^*) + n\iota_F + o(1)$. Then (119) leads as $\varepsilon \to 0$ to $W(a^*, d^*, \Phi^*) \leq W(a, d, \Phi)$. In other words, (a^*, d^*, Φ^*) is a minimizer of the intrinsic renormalized energy $W(\cdot, d^*, \cdot)$. Moreover, using the stronger lower bound in (67), we obtain the second estimate in (118).

It remains to prove the convergence of u_ε to a canonical harmonic vector field[24] associated to (a^*, d^*, Φ^*). Let u^* denote one such vector field. By Lemma 9.2 and Step 2 in the proof of Proposition 9.1, there exist n distinct points $a^\varepsilon = (a_{k,\varepsilon})_{1 \leq k \leq n}$ such that $\operatorname{dist}_S(a_{k,\varepsilon}, a_{\ell,\varepsilon}) \geq C_0 > 0$ for every $k \neq \ell$, $a_{k,\varepsilon} \to a_k^*$ and

$$\|\omega(u_\varepsilon) - 2\pi \operatorname{sign}(\chi(S)) \sum_{k=1}^n \delta_{a_{k,\varepsilon}}\|_{W^{-1,1}} = o(1) \text{ as } \varepsilon \to 0.$$

[24] which we recall is unique only up to a global rotation; in fact, $j(u^*) = j^*(a^*, d^*, \Phi^*)$ is genuinely unique as defined in (10), (12).

Then, by Proposition 8.2 and Lemma 9.4, we deduce that for every small $\varepsilon > 0$ and for every $r \in (\varepsilon^\beta, r_0(C_0))$ (with $\beta = \beta(n) > 0$),

$$\int_{B_r(a_{k,\varepsilon})} e_\varepsilon^{in}(u_\varepsilon)\,\mathrm{vol}_g \geq \pi \log \frac{r}{\varepsilon} - C, \quad k = 1, \ldots, n. \tag{120}$$

By the upper bound (119), it yields

$$\int_{S \setminus \cup_k B_r(a_{k,\varepsilon})} e_\varepsilon^{in}(u_\varepsilon)\,\mathrm{vol}_g \leq n\pi |\log r| + C$$

for every small $\varepsilon > 0$ and for every $r \in (\varepsilon^\beta, r_0(C_0))$. It follows by Lemma 12.3 below that $(u_\varepsilon)_{\varepsilon \to 0}$ is bounded in $\mathcal{X}^{1,p}(S)$, therefore, for a subsequence, $u_\varepsilon \rightharpoonup u_*$ in $\mathcal{X}^{1,p}(S)$ for every $p \in [1, 2)$ for a unit length vector field u_*. As $a_{k,\varepsilon} \to a_k^*$, by (120), we also deduce the first estimate in (118).

Now we aim to prove that u_* is a canonical harmonic unit vector field, *i.e.* that $|u_*|_g = 1$ (which is obvious) and $j(u_*) = j(u^*)$. By the Sobolev embedding, we have that $u_\varepsilon \to u_*$ strongly in $L^q(S)$ for every $q < \infty$, and we know that $Du_\varepsilon \rightharpoonup Du_*$ in L^p for every $p \in [1, 2)$. Together these imply that $j(u_\varepsilon) \rightharpoonup j(u_*)$ in $L^p(S)$ for every $p \in [1, 2)$. On the other hand, we claim that $j(u_\varepsilon) \to j(u^*)$ in $L^p(S)$ for every $p \in [1, 2)$. To prove this, fix $p \in [1, 2)$ and note that

$$\begin{aligned}\Big\|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j(u_\varepsilon)\Big\|_{L^p(S)} &\leq \big\||Du_\varepsilon|_g(1 - |u_\varepsilon|_g)\big\|_{L^p(S)} \\ &\leq \|Du_\varepsilon\|_{L^2(S)} \|1 - |u_\varepsilon|_g\|_{L^2(S)}^{\frac{2-p}{p}} \|D(1 - |u_\varepsilon|_g)\|_{L^2(S)}^{2-\frac{2}{p}}\end{aligned}$$

by Hölder and a Gagliardo-Nirenberg inequality. Since $(1 - |u_\varepsilon|_g)^2 \leq \frac{1}{C} F(|u_\varepsilon|_g^2)$ (by (1)) and $|D(1 - |u_\varepsilon|_g)|_g \leq |Du_\varepsilon|_g$, all terms on the right-hand side can be controlled by $E_\varepsilon^{in}(u_\varepsilon)$, leading to

$$\Big\|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j(u_\varepsilon)\Big\|_{L^p(S)} \leq C |\log \varepsilon|\, \varepsilon^{\frac{2-p}{p}}.$$

Also, (118) and Hölder's inequality readily imply that $\|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j(u^*)\|_{L^p(S_\sigma)} \to 0$ for every $\sigma > 0$, and hence that $\|j(u_\varepsilon) - j(u^*)\|_{L^p(S_\sigma)} \to 0$. Finally, for $q \in (p, 2)$, since $j(u^*) \in L^q(S)$ and $\{j(u_\varepsilon)\}$ is uniformly bounded in $L^q(S)$, Hölder's inequality implies that

$$\|j(u_\varepsilon) - j(u^*)\|_{L^p(S \setminus S_\sigma)} \leq \|1\|_{L^{\frac{qp}{q-p}}(S \setminus S_\sigma)} \|j(u_\varepsilon) - j(u^*)\|_{L^q(S \setminus S_\sigma)} \leq C \sigma^{\frac{2(q-p)}{pq}}.$$

As σ can be chosen arbitrarily small, summing up, we obtain that $j(u_\varepsilon) \to j(u^*)$ in $L^p(S)$ as claimed. (This completes the proof of Remark 12.2). Since $j(u_*)$ and $j(u^*)$ are both limits of $j(u_\varepsilon)$, we conclude that $j(u_*) = j(u^*)$ and hence that u_* is a canonical harmonic unit vector field. □

We use the following estimate reminiscent from the work of Struwe [41] in a ball $B_r \subset S$ of radius $r \in (0, r_0)$ (thus, r_0 is at most the injectivity radius of S) and we have

$$\left|\mathrm{Vol}_g(B_r) - \pi r^2\right| \leq c_1 r^4, \qquad \left|\mathcal{H}^1(\partial B_r) - 2\pi r\right| \leq c_1 r^3 \tag{121}$$

which is a consequence of Bertrand-Diguet-Puisseux Theorem (see Section 5.2) and the compactness of S.

Lemma 12.3. *Let r_0 be the injectivity radius of S, $\beta, c > 0$ and $\rho_0, R \in (0, r_0)$ with $\rho_0 < R$. For every $\varepsilon \in (0, \frac{1}{2})$, let $f_\varepsilon : B_R \to \mathbb{R}$ be a function on a ball $B_R \subset S$ such that*

$$\|f_\varepsilon\|_{L^2(B_R)}^2 \leq c(1 + |\log \varepsilon|)$$

and for every $\varepsilon^\beta \le \rho \le \rho_0$,

$$\|f_\varepsilon\|^2_{L^2(B_R\setminus B_\rho)} \le c(1+|\log\rho|).$$

Then for $1 \le p < 2$ *we have*

$$\|f_\varepsilon\|_{L^p(B_R)} \le C,$$

where $C > 0$ *is independent of* ε.

Proof. Let $1 \le p < 2$ and $\rho_j = 2^{-j}\rho_0$ for $0 \le j \le j_\beta$ with $j_\beta = \lfloor \log_2 \frac{\rho_0}{\varepsilon^\beta} \rfloor$ where $\lfloor s \rfloor$ is the integer part of a real s. Using Hölder's inequality, we have

$$\begin{aligned}
\int_{B_{\rho_0}\setminus B_{\rho_{j_\beta}}} |f_\varepsilon|^p \operatorname{vol}_g &= \sum_{j=0}^{j_\beta-1} \int_{B_{\rho_j}\setminus B_{\rho_{j+1}}} (|f_\varepsilon|^2)^{\frac{p}{2}} \operatorname{vol}_g \\
&\le \sum_{j=0}^{j_\beta-1} \left(\int_{B_{\rho_j}\setminus B_{\rho_{j+1}}} |f_\varepsilon|^2 \operatorname{vol}_g \right)^{\frac{p}{2}} \operatorname{Vol}_g(B_{\rho_j}\setminus B_{\rho_{j+1}})^{1-\frac{p}{2}} \\
&\le C(p,S) \sum_{j=0}^{j_\beta-1} \left(\int_{B_R\setminus B_{\rho_{j+1}}} |f_\varepsilon|^2 \operatorname{vol}_g \right)^{\frac{p}{2}} (2^{-j}\rho_0)^{2-p} \\
&\le C(p,S,\rho_0) \sum_{j=0}^{\infty} (1 + j\log 2 - \log\rho_0)^{\frac{p}{2}} 2^{-(2-p)j}.
\end{aligned}$$

Since

$$\lim_{j\to\infty} \frac{(1+(j+1)\log 2 - \log\rho_0)^{\frac{p}{2}} 2^{-(2-p)(j+1)}}{(1+j\log 2-\log\rho_0)^{\frac{p}{2}} 2^{-(2-p)j}} = 2^{-(2-p)} < 1,$$

the above sum converges so $\|f_\varepsilon\|_{L^p(B_{\rho_0}\setminus B_{\rho_{j_\beta}})} \le C$. Also, the hypothesis combined with Hölder's inequality yield $\|f_\varepsilon\|_{L^p(B_R\setminus B_{\rho_0})} \le C$ as well as

$$\|f_\varepsilon\|_{L^p(B_{\rho_{j_\beta}})} \le \|f_\varepsilon\|_{L^2(B_R)} \operatorname{Vol}_g(B_{\rho_{j_\beta}})^{\frac{1}{p}-\frac{1}{2}} = O(|\log\varepsilon|\varepsilon^{\beta(\frac{2}{p}-1)}) = o(1).$$

□

The extrinsic case.

If S is a surface isometrically embedded in $\mathbb{R}^3$, then Theorem 12.1 holds true also in the extrinsic case for minimizing sections m_ε of E_ε^{ex} with the natural change of having the limit triplet (a^*, d^*, ϕ^*) to be a minimizer of the extrinsic (instead of the intrinsic) renormalized energy

$$\{(W+\tilde{W})(a, d^*, \Phi) \,:\, a = (a_1, \dots, a_n) \in S^n \text{ distinct points}, \Phi \in \mathcal{L}(a, d^*)\}.$$

Moreover, we have the following formula for the minimal extrinsic energy:

$$E_\varepsilon^{ex}(m_\varepsilon) = n\pi \log\frac{1}{\varepsilon} + (W+\tilde{W})(a^*, d^*, \Phi^*) + n\iota_F + o(1), \tag{122}$$

as $\varepsilon \to 0$. Finally, after passing to a subsequence if necessary,

$$m_\varepsilon \rightharpoonup e^{i\Theta^*} u^* \text{ in } \mathcal{X}^{1,p}(S),$$

where u^* is some fixed canonical harmonic map $u^*(a^*, d^*, \phi^*)$ and Θ^* is a minimizer of $\Theta \mapsto \frac{1}{2}\int_S |d\Theta|_g^2 + |\mathcal{S}(e^{i\Theta}u^*)|_g^2 \operatorname{vol}_g$.

We sketch the proof of the above claim. The energy expansion (122) and the convergence

$$\omega(m_\varepsilon) \to 2\pi \sum_{k=1}^n d_k^* \delta_{a_k^*}, \qquad \Phi(m_\varepsilon) \to \Phi^*$$

are proved exactly as in the intrinsic case, using Theorem 10.1 in place of Theorem 2.6. Similarly, exactly the same arguments as for the intrinsic case prove that there exists some $m_* \in \mathcal{X}^{1,p}(S)$ for $m_\varepsilon \rightharpoonup m_*$ in $\mathcal{X}^{1,p}(S)$, for all $p \in [1,2)$.

It remains to prove that $m_* = e^{i\Theta^*} u^*(a^*, d^*, \Phi^*)$. We will deduce this conclusion from estimates carried out during the proof of Theorem 10.1. We start by recalling from the proof of point 2) Theorem 10.1 that for every sufficiently small $\varepsilon > 0$, there exists $t_\varepsilon \in (\varepsilon^{\frac{1}{2(n+1)}}, \varepsilon^{\beta/2})$, n distinct points $a^\varepsilon = (a_{k,\varepsilon})_{1\le k\le n} \in S^n$, $d^\varepsilon \in \{\pm 1\}^n$, and $\{\Phi_{k,\varepsilon}\}_{k=1}^{2\mathfrak{g}} \in \mathcal{L}(a^\varepsilon, d^\varepsilon)$ such that

$$a^\varepsilon \to a^*, \qquad d^\varepsilon \to d^* \ \ (\text{so } d^\varepsilon = d^* \text{ for all small } \varepsilon), \qquad \Phi^\varepsilon \to \Phi^*$$

and

$$\int_{B_{t_\varepsilon}(a_{k,\varepsilon})} e_\varepsilon^{ex}(m_\varepsilon)\,\mathrm{vol}_g \ge \pi \log \frac{t_\varepsilon}{\varepsilon} + \iota_F + o(1) \quad \text{ for every } 1 \le k \le n,$$

$$\int_{S_{t_\varepsilon}} e_\varepsilon^{ex}(m_\varepsilon)\,\mathrm{vol}_g \ge n\pi \log \frac{1}{t_\varepsilon} + (W + \tilde{W})(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) - o(1)$$

as $\varepsilon \to 0$. Let u_ε^* be a canonical harmonic map $u^*(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon)$ so that u_ε^* (up to a rotation as in (113)) satisfies $u_\varepsilon^* \to u^*$ in $L^p(S)$, $p < \infty$, as $\varepsilon \to 0$. We may then define $w_\varepsilon : S_{t_\varepsilon} \to \mathbb{C}$ by requiring that

$$m_\varepsilon = w_\varepsilon u_\varepsilon^* \qquad \text{in } S_{t_\varepsilon}.$$

Then (105), established during the proof of Lemma 10.3, can be applied in S_{t_ε} (instead of S_{r_ε}) implies that

$$\int_{S_{t_\varepsilon}} e_\varepsilon^{ex}(m_\varepsilon)\,\mathrm{vol}_g = W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) + n\pi \log \frac{1}{t_\varepsilon} + \int_{S_{t_\varepsilon}} \frac{1}{2}|dw_\varepsilon|_g^2 + \frac{1}{2}|\mathcal{S}(w_\varepsilon u_\varepsilon^*)|_g^2 + \frac{1}{4\varepsilon^2} F(|w_\varepsilon|^2)\ \mathrm{vol}_g + o(1).$$

Combining these and (122) and recalling that $W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \to W(a^*, d^*, \Phi^*)$ (by Proposition 2.5) and that $\tilde{W}(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) \to \tilde{W}(a^*, d^*, \Phi^*)$ (by Remark 10.5), we obtain

$$\int_{S_{t_\varepsilon}} \frac{1}{2}|dw_\varepsilon|_g^2 + \frac{1}{2}|\mathcal{S}(w_\varepsilon u_\varepsilon^*)|_g^2 + \frac{1}{4\varepsilon^2} F(|w_\varepsilon|^2)\ \mathrm{vol}_g \to \tilde{W}(a^*, d^*, \Phi^*)$$

as $\varepsilon \to 0$. Next, following the arguments in Step 3 of the proof of Lemma 10.3, we may construct a function $\tilde{w}_\varepsilon : S \to \mathbb{C}$ such that $\tilde{w}_\varepsilon = w_\varepsilon$ in $S_{\sqrt{r_\varepsilon}}$ and

$$\limsup_{\varepsilon\to 0} \int_S \frac{1}{2}|d\tilde{w}_\varepsilon|_g^2 + \frac{1}{2}|\mathcal{S}(\tilde{w}_\varepsilon u_\varepsilon^*)|_g^2 + \frac{1}{4\varepsilon^2} F(|\tilde{w}_\varepsilon|^2)\ \mathrm{vol}_g \le \tilde{W}(a^*, d^*, \Phi^*).$$

In particular $\{\tilde{w}_\varepsilon\}$ is uniformly bounded in $H^1(S)$. We pass to a subsequence that converges weakly in H^1, and hence strongly in L^p for every $p < \infty$, to a limit w_*. As in Step 4" of the proof of Lemma 10.3, one checks that $w_* \in H^1(S; \mathbb{S}^1)$ satisfying the compatibility condition $\Phi(w_* u^*) = \Phi(u^*)$ so that Lemma 10.4 yields $w_* = e^{i\Theta_*}$ for some $\Theta_* \in H^1(S; \mathbb{R})$.

Standard lower semicontinuity arguments imply

$$\begin{aligned}\tilde{W}(a^*, d^*, \Phi^*) &\le \int_S \frac{1}{2}|d\Theta_*|_g^2 + \frac{1}{2}|\mathcal{S}(e^{i\Theta_*}u^*)|_g^2 \ \mathrm{vol}_g \\ &= \int_S \frac{1}{2}|dw_*|_g^2 + \frac{1}{2}|\mathcal{S}(w_* u^*)|_g^2 \ \mathrm{vol}_g \\ &\le \liminf_{\varepsilon\to 0} \int_S \frac{1}{2}|d\tilde{w}_\varepsilon|_g^2 + \frac{1}{2}|\mathcal{S}(\tilde{w}_\varepsilon u_\varepsilon^*)|_g^2 + \frac{1}{4\varepsilon^2}F(|\tilde{w}_\varepsilon|^2) \ \mathrm{vol}_g \\ &\le \tilde{W}(a^*, d^*, \Phi^*).\end{aligned}$$

That is, Θ_* attains the minimum in the definition of $\tilde{W}$ given by $\tilde{W}(a^*, d^*, \Phi^*)$. Finally, the construction implies that for *a.e.* $x \in S$, $m_* = \lim_\varepsilon m_\varepsilon = \lim_\varepsilon \tilde{w}_\varepsilon u_\varepsilon^* = e^{i\Theta_*}u^*$, completing the proof.

The micromagnetic case.

Let S is a surface isometrically embedded in $\mathbb{R}^3$ endowed with the Euclidian metric. If $M_\varepsilon : S \to \mathbb{S}^2$ is a minimizer of E_ε^{mm}, then the projections $m_\varepsilon = \Pi(M_\varepsilon)$ on TS satisfy (for a subsequence) the convergences in Theorem 12.1 as $\varepsilon \to 0$ where the limit triplet (a^*, d^*, ϕ^*) is a minimizer of the extrinsic renormalized energy $W + \tilde{W}$. The second order expansion of the minimal micromagnetic energy has the form (122) with the natural change of ι_F by $\tilde{\iota}_F$. Moreover, the normal components $M_{\perp,\varepsilon}$ of M_ε satisfy $M_{\perp,\varepsilon} \to 0$ in $H^1(S \setminus \cup_{k=1}^n B_\sigma(a_k))$ as $\varepsilon \to 0$.

Appendix A. Ball construction. Proof of Proposition 8.2

In this section we present the vortex ball construction leading to Proposition 8.2. We start with several lemmas in which we verify, largely by adapting classical proofs to our setting, that basic ingredients needed for the vortex ball argument on the Euclidean plane remain available in the present setting. Once these ingredients are available, we follow classical arguments. Since the arguments are rather standard, our exposition is terse in places.

We first fix positive constants $c_1(S), r_0(S)$ such that $\partial B_r(x)$ is a homeomorphic to a circle for every $x \in S$, and $0 < r < r_0$ (thus, r_0 is at most the injectivity radius of S) and we recall (121). In several places later in our argument, we will impose additional smallness conditions on r_0.

In view of Lemma 5.1, in proving lower energy bounds we may restrict our attention to smooth vector fields.

Lemma A.1. *Given $u \in \mathcal{X}(S)$, let $\rho := |u|_g$. Assume that $\varepsilon < r < r_0(S)$. Then there exist positive constants c_2, c_3, c_4 such that the following hold. First,*

$$\frac{1}{2}\int_{\partial B_r(x)} |d\rho|_g^2 + \frac{1}{2\varepsilon^2}F(\rho^2)\, d\mathcal{H}^1 \ge \frac{c_2}{\varepsilon}\|1-\rho\|_{L^\infty}^2 \tag{123}$$

where $|d\rho|_g^2(x) := (d\rho(\tau_1))^2 + (d\rho(\tau_2))^2$ for any orthonormal basis $\{\tau_1, \tau_2\}$ of T_xS. Second,

$$\int_{\partial B_r(x)} e_\varepsilon^{in}(u) d\mathcal{H}^1 \ge \lambda_\varepsilon\left(\frac{r}{|d|}\right) \quad \text{for } d = \begin{cases} \deg(u; \partial B_r(x)) & \text{if } \rho \ge \frac{1}{2} \text{ on } \partial B_r(x) \\ \text{any positive integer} & \text{if not}\end{cases} \tag{124}$$

where

$$\lambda_\varepsilon(r) := \min_{0<s\le 1}\left[\frac{c_2}{4\varepsilon}(1-s)^2 + s^2\frac{\pi}{r}(1-c_3r^2)\right] \ge \frac{\pi(1-c_3r^2)}{r+c_4\varepsilon} \tag{125}$$

is a nonincreasing function, and we use the convention that for $r>0$, $\lambda_\varepsilon(r/0)=\lambda_\varepsilon(+\infty)=0$.

Proof. At $y\in\partial B_r(x)$, let $\tau\in T_yS$ denote the unit tangent to $\partial B_r(x)$, oriented in the standard way, and let $'$ denote differentiation with respect to τ. Further define $\zeta:=(1-\rho)^2$. Then by (1)

$$|\zeta'|+\frac{1}{\varepsilon}|\zeta|=2|\rho-1||\rho'|+\frac{1}{\varepsilon}\zeta\le\varepsilon|\rho'|^2+\frac{2}{\varepsilon}(1-\rho)^2\le C\varepsilon\left(|d\rho|_g^2+\frac{1}{2\varepsilon^2}F(\rho^2)\right).$$

Then (123) follows from a (suitably scaled) Sobolev embedding $W^{1,1}\hookrightarrow L^\infty$ on $\partial B_r(x)$, taking into account the fact that $\mathcal{H}^1(\partial B_r(x))\ge\varepsilon$.

Next, if $\rho\ge\frac12$ on $\partial B_r(x)$, then we can define $v=u/\rho$ and $d:=\deg(u;\partial B_r(x))$. Since $|v|_g=1$, we have $|Dv|_g=|j(v)|_g$, and thus

$$\int_{\partial B_r(x)}|Dv|_g^2\,d\mathcal{H}^1=\int_{\partial B_r(x)}|j(v)|_g^2\,d\mathcal{H}^1\ \ge\ \frac{1}{\mathcal{H}^1(\partial B_r(x))}\Big|\int_{\partial B_r(x)}j(v)\Big|_g^2$$
$$\overset{(5)}{=}\frac{1}{\mathcal{H}^1(\partial B_r(x))}\left(2\pi d-\int_{B_r(x)}\kappa\ \mathrm{vol}_g\right)^2.$$

Since S is compact and smooth, it follows from this and (121) that

$$\int_{\partial B_r(x)}\frac12|Dv|_g^2d\mathcal{H}^1\ge\frac{\pi d^2}{r}(1-c_3r^2)\,. \tag{126}$$

Finally, if we write $s:=\min_{\partial B_r(x)}(\rho\wedge1)>0$, then $|Du|_g^2=|d\rho|_g{}^2+\rho^2|Dv|_g{}^2\ge|d\rho|_g^2+s^2|Dv|_g^2$. Then one may deduce (124) from (123) and (126), after first taking r_0 small enough so that $c_3r_0^2\le1/2$, which yields $|d|(1-c_3r^2)\ge1-c_3\frac{r^2}{d^2}$ for every $|d|\ge1$. Then (124) follows directly in the case $\rho\ge\frac12$ on $\partial B_r(x)$, if $d\ne0$, whereas if $d=0$ it is immediate. If $\min_{\partial B_r(x)}\rho<\frac12$, then $\|1-\rho\|_{L^\infty}>\frac12$, and thus

$$\int_{\partial B_r(x)}e_\varepsilon^{in}(u)d\mathcal{H}^1\ge\frac12\int_{\partial B_r(x)}|d\rho|_g^2+\frac{1}{2\varepsilon^2}F(\rho^2)\,d\mathcal{H}^1\overset{(123)}{\ge}\frac{c_2}{4\varepsilon}$$
$$\ge\min_{0<s\le1}\left[\frac{c_2}{4\varepsilon}(1-s)^2+s^2\frac{\pi}{r}(1-c_3r^2)\right]=\lambda_\varepsilon(r)\ge\lambda_\varepsilon(\frac{r}{|d|})$$

for any d. (If $\rho=0$ somewhere on $\partial B_r(x)$, then the definition $v=u/\rho$ may not make sense, but the proof of (124) relies only on (123) and makes no mention of v.) □

We also need:

Lemma A.2. *Assume that u is a smooth vector field on S and that for some $0<r<r_0(S)$ and $x\in S$,*

$$\rho:=|u|_g\ge\frac12\ \text{on}\ \partial B_r(x),\qquad\deg(u;\partial B_r(x))=d\ne0.$$

Then if r_0 is sufficiently small,

$$\int_{B_r(x)}|Du|_g^2\,vol_g\ge\frac{\pi}{4}|d|\,.$$

Proof. First, let $O:=\{y\in B_r(x):\rho(y)<t\}$, where t is a regular value of $\rho(\cdot)$ such that $\frac18<t<\frac14$. Then O is an open set with smooth boundary, compactly contained in $B_r(x)$, and

$$\begin{aligned} d = \deg(u; \partial B_r(x)) &= \frac{1}{2\pi}\left(\int_{\partial B_r(x)} j(\frac{u}{|u|_g}) + \int_{B_r(x)} \kappa \operatorname{vol}_g\right) \\ &= \frac{1}{2\pi}\left(\int_{\partial O} \frac{j(u)}{t^2} + \int_O \kappa \operatorname{vol}_g\right) \end{aligned}$$

where the final equality follows from Lemma 5.4 and Stokes' Theorem, as well as the fact that $|u|_g = t$ on ∂O. Thus if r_0 is sufficiently small (depending on $\|\kappa\|_\infty$) then

$$2\pi t^2\left(|d| - \frac{1}{2}\right) \le \Big|\int_{\partial O} j(u)\Big|_g = \Big|\int_O dj(u)\Big|_g \le \int_O |dj(u)|_g \le \int_O |Du|_g^2,$$

where the last inequality follows from Lemma 5.3, see (40). □

Finally we recall

Lemma A.3. *Assume that u_ε is a smooth vector field satisfying* (51). *Then there exists $\varepsilon_0 > 0$ such that whenever $0 < \varepsilon < \varepsilon_0$, there exists a collection $\tilde{\mathcal{B}}^0 = \{\tilde{B}^0_j\}$ of closed pairwise disjoint balls that cover the set where $|u_\varepsilon|_g \le \frac{1}{2}$, and such that*

$$\sum_j \tilde{r}^0_j \le C\varepsilon \int_S e^{in}_\varepsilon(|u_\varepsilon|_g)\ vol_g, \qquad \text{where } \tilde{r}^0_j \text{ denotes the radius of } \tilde{B}^0_j.$$

Proof. Let $\rho := |u_\varepsilon|_g$. Then $\frac{1}{2}|d\rho|_g^2 + \frac{1}{4\varepsilon^2}F(\rho^2) \ge \frac{1}{\varepsilon\sqrt{2}}|d\rho|_g\sqrt{F(\rho^2)} \ge \frac{c}{\varepsilon}|1-\rho|\,|d\rho|_g$, by (1). Thus the coarea formula, which remains valid on a smooth manifold, implies that

$$\varepsilon\int_S e^{in}_\varepsilon(\rho)\operatorname{vol}_g \ge c\int_0^\infty |1-s|\mathcal{H}^1(\rho^{-1}(s))\,ds \ge c\int_{1/2}^{3/4}\mathcal{H}^1(\rho^{-1}(s))\,ds.$$

In particular we may find some $\alpha \in [\frac{1}{2}, \frac{3}{4}]$, a regular value of ρ, such that $\mathcal{H}^1(\rho^{-1}(\alpha)) \le C\varepsilon\int_S e^{in}_\varepsilon(\rho)\operatorname{vol}_g$. Following standard arguments, we may start with an efficient finite cover of $\rho^{-1}(\alpha)$ and then merge balls to find a collection of closed *pairwise disjoint* balls that cover $\rho^{-1}(\alpha)$ and whose radii sum to at most $2\mathcal{H}^1(\rho^{-1}(\alpha))$. This is $\tilde{\mathcal{B}}^0$. The complement of the union of these balls is connected as long as ε is small enough, so on the complement, either $\rho > \alpha$ or $\rho < \alpha$ everywhere. The latter case is impossible by (51) and (123), if ε is small enough, and this proves the lemma. □

A few more definitions are needed before we prove Proposition 8.2. W.l.o.g., we may assume that $\frac{1}{2}$ is a regular value of $\rho = |u|_g$. First, we set

$$\begin{aligned} Z &:= \{x \in S : |u(x)|_g \le \frac{1}{2}\}, \\ Z_E &:= \cup\{\text{connected components } Z_l \text{ of } Z \ : \ \deg(u; \partial Z_l) \ne 0\}. \end{aligned}$$

Next, for any set $V \subset S$ such that $\partial V \cap Z_E = \emptyset$ we define the generalized degree

$$\operatorname{dg}(u; \partial V) := \sum\{\deg(u; \partial Z_l) \ : \ \text{components } Z_l \text{ of } Z_E \text{ such that } Z_l \subset\subset V\}.$$

Note that $\operatorname{dg}(u; \partial V) = \deg(u, \partial V)$ if ∂V is C^1, say, and $|u|_g > \frac{1}{2}$ on ∂V. Finally we define

$$\Lambda_\varepsilon(\sigma) := \int_0^\sigma \lambda_\varepsilon(r)\,dr.$$

It is straightforward to check that

$$\Lambda_\varepsilon(\sigma) \ge \pi\log(1 + \frac{\sigma}{c_4\varepsilon}) - C\sigma^2 \ge \pi(\log\frac{\sigma}{\varepsilon} - C) \qquad \text{for } 0 \le \sigma \le r_0(S). \tag{127}$$

We record several other properties. First, since $\Lambda_\varepsilon(\cdot)$ is the integral over $[0,\sigma]$ of a positive nonincreasing function, it is easy to see that

$$\Lambda_\varepsilon(\sigma_1+\sigma_2) \le \Lambda_\varepsilon(\sigma_1)+\Lambda_\varepsilon(\sigma_2), \qquad \sigma \mapsto \frac{1}{\sigma}\Lambda_\varepsilon(\sigma) \text{ is nonincreasing.}$$

Finally, consider two radii $r_1 < r_2$ such that $\varepsilon \le r_j \le r_0$ for $j = 1,2$, and assume that $x \in S$ is a point such that Z_E does not intersect the annulus $B_{r_2} \setminus B_{r_1}(x)$. Then $\mathrm{dg}(u;\partial B_r(x)) = \mathrm{dg}(u;\partial B_{r_1}(x))$ for all $r \in (r_1,r_2)$, so one may use the coarea formula and integrate (124) from r_1 to r_2 to find that

$$\int_{B_{r_2}\setminus B_{r_1}(x)} e_\varepsilon^{in}(u)\,\mathrm{vol}_g \ge |d|\left[\Lambda_\varepsilon\left(\frac{r_2}{|d|}\right)-\Lambda_\varepsilon\left(\frac{r_1}{|d|}\right)\right], \qquad d := \mathrm{dg}(u;\partial B_{r_1}(x)). \tag{128}$$

Proof of Proposition 8.2. We divide the proof in several steps:

Step 1. An initial covering of Z_E. We claim first that there exists a collection $\mathcal{B}^0 = \{B_{l,0}\}_{k=1}^K$ of closed, pairwise disjoint balls with centers $a_{l,0}$ and radii $r_{l,0} \ge \varepsilon$ for all l, such that $Z_E \subset \cup B_k^0$, and (after possibly decreasing the constant c_2 in the definition (125) of λ_ε, in a way that depends only on the geometry of S)

$$\int_{B_{l,0}} e_\varepsilon^{in}(u)\,\mathrm{vol}_g \ \ge \frac{c_2}{4\varepsilon} r_{l,0} \ge \ \Lambda_\varepsilon(r_{l,0}) \qquad \text{for every } l. \tag{129}$$

We first cover Z_E with balls that satisfy (129). Indeed, for every $x \in Z_E$, this estimate holds for $B_r(x)$, for the smallest $r \ge \varepsilon$ such that $\min_{\partial B_r(x)} \rho \ge 1/2$. This is a result of Lemma A.2, if its $r \le 2\varepsilon$, and otherwise it follows from (123) and the coarea formula. One can then choose a finite subcover. The balls obtained in this fashion may overlap. If so, they may be combined into pairwise disjoint balls that still satisfy (129), by exactly the arguments in [24], proof of Proposition 3.3, where the same result is proved in the Euclidean setting. This argument involves a slightly more careful choice of balls (so that no center is contained in any other ball) and use of the Besicovitch covering lemma. For our present purposes, we may appeal to Federer [16], sections 2.8.9 - 2.8.14, for a difficult but doubtless correct version of the covering lemma that is valid on a smooth compact Riemannian manifold, and indeed in much greater generality. Adjustments to the constant c_2 depend on constants appearing in this covering lemma, which are explicitly described in the above reference.

Step 2. Growing and merging balls. Now let $d_{l,0} := \mathrm{dg}(u;\partial B_{l,0})$. We will assume for this discussion that $d_{l,0} \ne 0$ for some l, as the other case is both easier and less relevant for our main results. Using Lemma A.1 and associated properties of Λ_ε, such as those in (129), we may now follow the algorithm from [24], proof of Proposition 4.1, to which one may refer for the details omitted here. We describe it briefly. First, define

$$\sigma_0 := \min_{\mathcal{B}^0} r_{l,0}/|d_{l,0}| \overset{(51),(129)}{\le} C\ \varepsilon|\log\varepsilon|\ .$$

We claim that for $\sigma \in (\sigma_0, r_0(S))$, there exists a finite collection of pairwise disjoint closed balls $\mathcal{B}^\sigma = \{B_{l,\sigma}\}_{l=1}^{K_\sigma}$ with centers $a_{l,\sigma}$ and radii $r_{l,\sigma}$, such that

$$Z_E \subset \cup B_{l,\sigma}\,, \qquad \int_{B_{l,\sigma}} e_\varepsilon^{in}(u)\,\mathrm{vol}_g \ge \frac{r_{l,\sigma}}{\sigma}\Lambda_\varepsilon(\sigma)\,, \ \text{ and } \ r_{l,\sigma} \ge \sigma|d_{l,\sigma}| \quad \text{for all } l, \tag{130}$$

where $d_{l,\sigma} = \mathrm{dg}(u;B_{l,\sigma})$. We take $\mathcal{B}^{\sigma_0}$ to be the collection found in Step 1 above. Given any $\sigma_1 \ge \sigma_0$ for which such a collection exists, we say that the *minimizing balls* are those for which $r_{l,\sigma_1} = \sigma_1|d_{l,\sigma_1}|$. Since the balls are closed and pairwise disjoint, there is some $\delta > 0$ such that for $\sigma_1 \le \sigma \le \sigma + \delta$, we can expand the minimizing balls, while leaving the centers fixed, by enclosing them in pairwise disjoint annuli chosen so that, for every σ, the equality $r_{l,\sigma} = \sigma|d_{l,\sigma}|$ holds for all minimizing balls. We add balls to the collection of minimizing balls as σ increases, when necessary. This preserves (130) due to properties of Λ_ε summarized above, such as (128). At certain values of σ, for example $\sigma = \sigma_1 + \delta$, the expansion process will lead to two or more balls colliding. When this occurs, one can regroup them into larger, pairwise disjoint balls in a way that preserves the properties (130). (Details of all these assertions can be found in [24].) This process can be continued as long as every minimizing ball has radius at most $r_0(S)$, which happens as long as $\sigma < r_0(S)$.

Step 3. Stopping the process, and covering all of Z. Recalling that $n > T-1$ by hypothesis, we fix $q \in (0, 1 - \frac{T}{n+1})$, which implies that $\frac{T}{1-q} < n+1$. It then follows from (130), (127), and (51) that if $\varepsilon^q \le \sigma < r_0(S)$, then

$$\sigma \sum |d_{l,\sigma}| \le \sum r_{l,\sigma} \le \sigma \frac{T\pi|\log\varepsilon| + C}{\pi|\log(\sigma/\varepsilon)| - C} \le \sigma\Big(\frac{T}{1-q} + \frac{C}{|\log\varepsilon|}\Big). \tag{131}$$

Thus there exists $\varepsilon_0 > 0$ (depending on S, q, and the constant in (51)) such that if $0 < \varepsilon < \varepsilon_0$, then $\sum |d_{l,\sigma}| < n+1$, and thus $\sum |d_{l,\sigma}| \le n$.

These balls have all the desired properties (the bound (57) on the sum of the radii follows from (131)) except that they cover Z_E rather than all of Z. To rectify this, recall from Lemma A.3 that $Z \setminus Z_E$ can be covered by a finite collection of balls whose radii sum to at most $C\varepsilon|\log\varepsilon|$. We can add these balls to $\mathcal{B}^\sigma$, merging as necessary to obtain a pairwise disjoint collection (still denoted $\mathcal{B}^\sigma$) that covers all of Z, and with the sum of the radii increased by at most $C\varepsilon|\log\varepsilon|$. Since $\frac{T}{1-q} < n+1$, it follows from (131) that these still satisfy $\sum r_{l,\sigma} < \sigma(n+1)$ for $0 < \varepsilon < \varepsilon_0$. The bound on the total degrees (56) and the energy lower bound (58) for this modified collection of balls are directly inherited from the previous collection.

□

Acknowledgment. R.I. acknowledges partial support by the ANR project ANR-14-CE25-0009-01. The work of R.J. was partially supported by the Natural Sciences and Engineering Research Council of Canada under operating Grant 261955.

References

[1] R. Alicandro, M. Ponsiglione, *Ginzburg-Landau functionals and renormalized energy: a revised* Γ-*convergence approach*, J. Funct. Anal. **266** (2014), 4890-4907.
[2] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer-Verlag, Berlin, 1998.
[3] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg-Landau vortices*, Birkhäuser, Boston, 1994.
[4] F. Bethuel, X.M. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Funct. Anal. **80** (1988), 60–75.
[5] A. Boutet de Monvel-Berthier, V. Georgescu, R. Purice, *A boundary value problem related to the Ginzburg-Landau model*, Comm. Math. Phys. **142** (1991), 1–23.
[6] H. Brezis, J.M. Coron, E.H. Lieb, *Harmonic maps with defects*, Comm. Math. Phys., **107** (1986) 649–705.
[7] H. Brezis, L. Nirenberg, *Degree theory and BMO. I. Compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), 197–263.
[8] G. Canevari, A. Segatti, *Defects in Nematic Shells: a* Γ-*convergence discrete-to-continuum approach*, Arch. Rat. Mech. Anal. **229** (2018) no. 1, 125-186.
[9] G. Canevari, A. Segatti, M. Veneroni, *Morse's index formula in VMO for compact manifolds with boundary*. J. Funct. Anal. **269** (2015), no. 10, 3043-3082.
[10] G. Carbou, *Thin layers in micromagnetism*, Math. Models Methods Appl. Sci. **11** (2001), 1529-1546.
[11] D. Chouchkov, N. M. Ercolani, S. Rayan, I. M. Sigal, *Ginzburg-Landau equations on Riemann surfaces of higher genus*, arXiv 1704.03422.

[12] J.E. Colliander, R.L. Jerrard, *Ginzburg-Landau vortices: weak stability and Schrödinger equation dynamics*, J. Anal. Math. **77** (1999), 129-205.

[13] A. Contreras, *On the first critical field in Ginzburg-Landau theory for thin shells and manifolds*, Arch. Ration. Mech. Anal. **200** (2011), 563-611.

[14] A. Contreras, P. Sternberg, *Γ-convergence and the emergence of vortices for Ginzburg-Landau on thin shells and manifolds*, Calc. Var. Partial Differential Equations **38** (2010), 243-274.

[15] M.P. do Carmo, Differential forms and applications, Springer-Verlag, Berlin, 1994.

[16] H. Federer, "Geometric Measure Theory" *Die Grundlehren der mathematischen Wissenschaften, Band 153*, Springer-Verlag, New York 1969.

[17] H. Federer, W.H. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458-520.

[18] G. Gioia, R. D. James, *Micromagnetics of very thin films*, Proc. Roy. Soc. London A **453** (1997), 213-223.

[19] R. Ignat, *A survey of some new results in ferromagnetic thin films*, Séminaire: Équations aux Dérivées Partielles. 2007–2008, Exp. No. VI, 21 pp., Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2009.

[20] R. Ignat, R.L. Jerrard, *Interaction energy between vortices of vector fields on Riemannian surfaces*, C. R. Math. Acad. Sci. Paris **355** (2017), 515-521.

[21] R. Ignat, L. Nguyen, V. Slastikov, A. Zarnescu, *Uniqueness of degree-one Ginzburg-Landau vortex in the unit ball in dimensions $N \geq 7$*, C. R. Math. Acad. Sci. Paris **356** (2018), 922-926.

[22] R. Ignat, L. Nguyen, V. Slastikov, A. Zarnescu, *On the uniqueness of minimisers of Ginzburg-Landau functionals*, Ann. Sci. Éc. Norm. Supér. **53** (2020), 589-613.

[23] R. Ignat, F. Otto, *A compactness result for Landau state in thin-film micromagnetics*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **28** (2011), 247-282.

[24] R.L. Jerrard, *Lower bounds for generalized Ginzburg-Landau functionals*, SIAM J. Math. Anal. **30** (1999), 721-746.

[25] R.L. Jerrard, H.M. Soner, *The Jacobian and the Ginzburg-Landau energy,* Calc. Var. PDE **14** (2002), 151-191.

[26] R.L. Jerrard, D. Spirn, *Refined Jacobian estimates for Ginzburg-Landau functionals*, Indiana Univ. Math. Jour. **56** (2007), 135-186.

[27] R.L. Jerrard, D. Spirn, *Refined Jacobian estimates and Gross-Pitaevsky vortex dynamics*, Arch. Ration. Mech. Anal. **190** (2008), 425-475.

[28] S. Jimbo, Y. Morita and J. Zhai, *Ginzburg-Landau equation and stable steady state solutions in a non-trivial domain.* Comm. Partial Differential Equations **20** (1995), no. 11-12, 2093-2112.

[29] H. Karcher, *Riemannian center of mass and mollifier smoothing*, Comm. Pure Appl. Math. **30** (1977), 509-541.

[30] P. Mironescu, *Local minimizers for the Ginzburg-Landau equation are radially symmetric*, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), 593-598.

[31] G. Napoli, L. Vergori. *Equilibrium of nematic vesicles.* J. Phys. A: Math. Theor., **43** (2010) 445207.

[32] G. Napoli, L. Vergori. *Extrinsic curvature effects on nematic shells.* Phys. Rev. Lett., **108** (2012) 207803.

[33] G. Orlandi, *Asymptotic behavior of the Ginzburg-Landau functional on complex line bundles over compact Riemann surfaces.* Rev. Math. Phys. **8** (1996), no. 3, 457-486.

[34] F. Pacard, T. Rivière, *Linear and nonlinear aspects of vortices. The Ginzburg-Landau model*, Progress in Nonlinear Differential Equations and their Applications, 39. Birkhäuser Boston, Inc., Boston, MA, 2000.

[35] J. Qing, *Renormalized energy for Ginzburg-Landau vortices on closed surfaces.* Math. Z. **225** (1997), no. 1, 1-34.

[36] J. Rubinstein, P. Sternberg, *Homotopy classification of minimizers of the Ginzburg-Landau energy and the existence of permanent currents.* Comm. Math. Phys. **179** (1996), no. 1, 257-263.

[37] E. Sandier, *Lower bounds for the energy of unit vector fields and applications* J. Funct. Anal. **152** (1998), 379-403.

[38] E. Sandier, S. Serfaty, *Vortices in the magnetic Ginzburg-Landau model*, Birkhäuser, 2007.

[39] A. Segatti, M. Snarski, M. Veneroni, *Analysis of a variational model for nematic shells.* Math. Models Methods Appl. Sci. **26** (2016), no. 10, 1865-1918.

[40] J. Steiner, *A geometrical mass and its extremal properties for metrics on $\mathbb{S}^2$*, Duke Math. J. **129** (2005), 63-86.

[41] M. Struwe, *On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2 dimensions*, Differential Integral Equations **7** (1994), 1613–1624.

[42] C.M. Wood, *On the energy of a unit vector field*, Geom. Dedicata **64** (1997), no. 3, 319-330.

Radu Ignat
Institut de Mathématiques de Toulouse & Institut Universitaire de France
UMR 5219, Université de Toulouse, CNRS, UPS, IMT
F-31062 Toulouse Cedex 9, France
radu.ignat@math.univ-toulouse.fr

Robert L. Jerrard
Department of Mathematics
University of Toronto
Toronto, Ontario, M5S 2E4
rjerrard@math.toronto.edu