Lifting of BV functions with values in S^1

Relèvement des fonctions BV à valeurs sur le cercle S^1

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Abstract

We show that for every $u \in \mathrm{BV}(\Omega; S^1)$, there exists a bounded variation function $\varphi \in \mathrm{BV}(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$ a.e. on Ω and $|\varphi|_{\mathrm{BV}} \leq 2|u|_{\mathrm{BV}}$. The constant 2 is optimal in dimension $n > 1$. *To cite this article: J Dávila, R. Ignat, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

On montre que pour tout $u \in \mathrm{BV}(\Omega; S^1)$, il existe une fonction à variation bornée $\varphi \in \mathrm{BV}(\Omega; \mathbb{R})$ telle que $u = e^{i\varphi}$ p.p. dans Ω et $|\varphi|_{\mathrm{BV}} \leq 2|u|_{\mathrm{BV}}$. La constante 2 est optimale en dimension $n > 1$. *Pour citer cet article : J Dávila, R. Ignat, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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Soit $\Omega \subset \mathbb{R}^n$ un ouvert et $u : \Omega \to S^1$ une fonction mesurable. Un relèvement de u est une fonction mesurable $\varphi : \Omega \to \mathbb{R}$ telle que

$$u(x) = e^{i\varphi(x)}$$

pour presque tout $x \in \Omega$. Une question naturelle est de savoir s'il existe un relèvement φ qui préserve la régularité de la fonction u. Par exemple, si Ω est simplement connexe et u est continue, alors on sait qu'on peut trouver un relèvement φ continu. Motivée par l'étude de l'équation de Ginzburg-Landau et

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la théorie du degré, on constate une recherche assidue sur ce problème de l'existence du relèvement dans les espaces de Sobolev, BMO et VMO (voir [BZ,BBM1,CM,BN]).

Dans ce papier on étudie le cas des fonctions à variation bornée :

Théorème 0.1 *Soit* $u \in \mathrm{BV}(\Omega; S^1)$*. Alors il existe un relèvement* $\varphi \in \mathrm{BV}(\Omega;\mathbb{R})$ *de* u *tel que* $|\varphi|_{\mathrm{BV}} \leq 2|u|_{\mathrm{BV}}$.

Si $n \geq 2$ la constante 2 est optimale ; on donne un exemple dans la Section 4. En dimension $n = 1$ on peut trouver un relèvement φ tel que $|\varphi|_{\mathrm{BV}} \leq \frac{\pi}{2}|u|_{\mathrm{BV}}$. L'existence du relèvement BV est montré aussi dans [GMS], mais sans contrôle sur $|\varphi|_{\mathrm{BV}}$.

L'idée de la démonstration du Théorème 0.1 est de considérer la fonction $L : S^1 \to \mathbb{R}$, $L(e^{i\theta}) = \theta$ $\forall -\pi \leq \theta < \pi$. Alors $\varphi = L(u)$ est un relèvement (mesurable) de u, ainsi que toutes les fonctions $L(e^{i\alpha}u) - \alpha$, $\forall \alpha \in \mathbb{R}$. En suite on montre que la fonction $\alpha \mapsto |L(e^{i\alpha}u)|_{\mathrm{BV}}$ est mesurable et qu'on a $\int_0^{2\pi} |L(e^{i\alpha}u)|_{\mathrm{BV}}\, d\alpha \leq 4\pi|u|_{\mathrm{BV}}$.

Corollaire 0.2 *Soit* $u \in \mathrm{BV}(\Omega; S^1)$*. Alors il existe une suite* $u_k \in C^\infty(\Omega; S^1) \cap \mathrm{BV}(\Omega)$ *telle que* $u_k \to u$ *p.p. et* $\limsup_{k\to\infty} |u_k|_{\mathrm{BV}} \leq 2|u|_{\mathrm{BV}}$.

Remarque 1 Si on note $SBV(\Omega, \mathbb{R}^m) = \{u \in \mathrm{BV}(\Omega; \mathbb{R}^m) \,:\, D^c u \equiv 0\}$ *(où* $D^c u$ *est la partie Cantor de la différentielle* Du*), alors pour tout* $u \in SBV(\Omega; S^1)$ *il existe un relèvement* $\varphi \in SBV(\Omega;\mathbb{R})$ *tel que* $|\varphi|_{\mathrm{BV}} \leq 2|u|_{\mathrm{BV}}$.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to S^1$ a measurable function. A lifting of u is a measurable function $\varphi : \Omega \to \mathbb{R}$ such that

$$u(x) = e^{i\varphi(x)}$$

for a.e. $x \in \Omega$. If u has some regularity one may ask whether or not φ can be chosen with some regularity as well. For example, if Ω is simply connected and u is continuous then it is well known that φ can be chosen to be continuous.

We are concerned in this note with the case when u has bounded variation, and by this we mean that $u \in L^1_{loc}(\Omega; \mathbb{R}^2)$, $|u(x)| = 1$ for a.e. $x \in \Omega$ and its BV seminorm is finite, i.e.

$$|u|_{\mathrm{BV}} = \sup\left\{ \int_\Omega \sum_{i=1}^{2} u_i \operatorname{div} g_i \, dx \;:\; g_i \in C_0^\infty(\Omega;\mathbb{R}^n),\ \sum_{i=1}^{2} |g_i|^2 \leq 1 \text{ in } \Omega \right\} < \infty,$$

where the norm in $\mathbb{R}^n$ is the Euclidean norm.

Remark 1 *Throughout the paper we will say that* $v \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ *if* $v \in L^1_{loc}(\Omega)$ *and its standard BV seminorm* $|v|_{\mathrm{BV}}$ *is finite. We adopt this convention, because in the case that the Lebesgue measure of* Ω *is infinite, with the standard definition of BV where it is required that* $v \in L^1(\Omega)$*, there would not be any* S^1*-valued BV function.*

Our main result is the following.

Theorem 2 *Let* $u \in \mathrm{BV}(\Omega; S^1)$*. Then there exists a lifting* $\varphi \in \mathrm{BV}(\Omega;\mathbb{R})$ *of* u *such that*

$$|\varphi|_{\mathrm{BV}} \leq 2|u|_{\mathrm{BV}}. \tag{1}$$

Remark 3 *1) If* $n \geq 2$ *the constant 2 appearing in (1) is optimal. We present an example in Section 4.*

2) The case of dimension $n = 1$ *is simple, and in fact one can find a lifting* φ *with*

$$|\varphi|_{\mathrm{BV}} \leq \frac{\pi}{2}|u|_{\mathrm{BV}}.$$

3) In [GMS], the authors show the existence of a lifting BV but they don't control its BV seminorm.

4) If u belongs to the Sobolev space $W^{1,1}(\Omega; S^1)$ and $\Omega \subset \mathbb{R}^2$ is smooth, bounded and simply connected it was already known that u has a lifting $\varphi \in \mathrm{BV}(\Omega;\mathbb{R})$ which satisfies (1) (private communication of H. Brezis and P. Mironescu).

Regarding other function spaces there has been recently much research, specially motivated by the study of the Ginzburg-Landau equation. Firstly, Bethuel and Zheng [BZ] proved that if Ω is bounded and simply connected and $u \in W^{1,p}(\Omega; S^1)$ with $p \geq 2$ then u has a lifting $\varphi \in W^{1,p}(\Omega;\mathbb{R})$; this result is false in general if $n \geq 2$ and $1 \leq p < 2$. A complete description for the existence of the lifting in general Sobolev spaces $W^{s,p}(\Omega; S^1)$, $0 < s < \infty$ and $1 < p < \infty$ was given later by Bourgain, Brezis and Mironescu [BBM1]. There are also results in the space BMO and VMO, see Coifman and Meyer [CM] and Brezis and Nirenberg [BN].

The idea for the proof of Theorem 2 is to consider the function $L : S^1 \to \mathbb{R}$ defined by

$$L(e^{i\theta}) = \theta \quad \forall - \pi \leq \theta < \pi. \tag{2}$$

Then $\varphi = L(u)$ is a lifting of u, in the sense that $e^{i\varphi(x)} = u(x)$ for all $x \in \Omega$. We would like to have $|\varphi|_{\mathrm{BV}} \leq 2|u|_{\mathrm{BV}}$, but this is far from true. It may even happen that $L(u)$ does not belong to BV (classical results for composition of functions assert only that if $f : S^1 \to \mathbb{R}$ is Lipschitz then $f(u)$ is BV). There is a way to remedy this situation. Indeed, observe that for fixed $\alpha \in \mathbb{R}$ the function $L(e^{i\alpha}u) - \alpha$ is also a lifting of u. We shall prove

Theorem 4 *The function $\alpha \mapsto |L(e^{i\alpha}u)|_{\mathrm{BV}}$ is measurable and*

$$\int_0^{2\pi} |L(e^{i\alpha}u)|_{\mathrm{BV}}\, d\alpha \leq 4\pi |u|_{\mathrm{BV}}. \tag{3}$$

Remark 5 *Inequality (3) can be viewed as a sort of co-area inequality. In particular it implies that for a.e. $\alpha \in \mathbb{R}$, $L(e^{i\alpha}u) \in \mathrm{BV}$. The constant 4π in (3) is sharp; see the example in Section 4.*

To prove Theorem 2, the mean value theorem yields from (3) that there is α_0 such that $|L(e^{i\alpha_0}u)|_{\mathrm{BV}} \leq 2|u|_{\mathrm{BV}}$. Therefore, $\varphi = L(e^{i\alpha_0}u) - \alpha_0$ is a lifting of u that satisfies (1).

Corollary 6 *Let $u \in \mathrm{BV}(\Omega; S^1)$. Then there exists a sequence $u_k \in C^\infty(\Omega; S^1) \cap \mathrm{BV}(\Omega)$ such that $u_k \to u$ a.e. and in L^1_{loc} and*

$$\limsup_{k\to\infty} |u_k|_{\mathrm{BV}} \leq 2|u|_{\mathrm{BV}}.$$

2. Preliminaries about the space BV

The material that we present next is standard and can be found in the book [AFP]. Let $v \in \mathrm{BV}(\Omega;\mathbb{R}^m)$. Its jump set $S(v)$ is defined by the requirement that $x \in \Omega \setminus S(v)$ if and only if there exists $\tilde{v}(x) \in \mathbb{R}^m$ such that $\tilde{v}(x) = \operatorname*{ap-lim}_{y\to x} v(y)$, that is:

$$\forall \varepsilon > 0 \quad \lim_{r\to 0} \frac{\mathrm{meas}\big(B_r(x) \cap \{y \in \Omega \,:\, |v(y) - \tilde{v}(x)| > \varepsilon\}\big)}{\mathrm{meas}(B_r(x))} = 0.$$

It can be proved (see [AFP]) that for $\mathcal{H}^{n-1}$-a.e. $x \in S(v)$ there exist $v^+(x), v^-(x) \in \mathbb{R}^m$ and a unit vector $\nu_v(x)$ such that

$$\operatorname*{ap-lim}_{y\to x,\ \langle y-x,\nu_v(x)\rangle>0} v(y) = v^+(x), \qquad \operatorname*{ap-lim}_{y\to x,\ \langle y-x,\nu_v(x)\rangle<0} v(y) = v^-(x). \tag{4}$$

Dv is a matrix valued Radon measure which can be decomposed as $Dv = D^a v + D^j v + D^c v$, where $D^a v$ is defined as the absolutely continuous part of Dv with respect to the Lebesgue measure, while $D^j v$ and $D^c v$ are defined as

$$D^j v = Dv \llcorner S(v), \quad D^c v = (Dv - D^a v) \llcorner (\Omega \setminus S(v)).$$

$D^j v$ is called the jump part and $D^c v$ the Cantor part of Dv. It can be proved that

$$D^j v = (v^+ - v^-) \otimes \nu_v \, \mathcal{H}^{n-1} \llcorner S(v).$$

Since we use just the local behavior of BV functions, throughout the paper we consider the precise representative $v^* : \Omega \mapsto \mathbb{R}^m$ of each $v \in \mathrm{BV}$ i.e.

$$v^*(x) = \lim_{r \to 0} \frac{1}{\mathrm{meas}(B_r(x))} \int_{B_r(x)} v \, dy$$

if this limit exists, and $v^*(x) = 0$ otherwise. Remark that v^* specifies the values of v except on a $\mathcal{H}^{n-1}$-negligible set.

It is well known that if $v \in \mathrm{BV}(\Omega; \mathbb{R}^m)$ and $f : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz then $f \circ v$ belongs to BV, and Ambrosio and Dal Maso [AD] proved a chain rule in this context. The following lemma is a slight modification of this chain rule for u in BV with values in S^1 (see also Theorem 3.99 in [AFP] for the case of scalar BV functions):

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in \mathrm{BV}(\Omega; S^1)$. Let $f : S^1 \to \mathbb{R}$ be a Lipschitz function. Then $v = f \circ u$ belongs to $\mathrm{BV}(\Omega; \mathbb{R})$, f is differentiable at $u(x)$ for $(|D^a u| + |D^c u|)$-a.e. x and*

$$Dv = \boldsymbol{f_\tau}(u)(D^a u + D^c u) + (f(u^+) - f(u^-))\nu_u \, \mathcal{H}^{n-1} \llcorner S(u), \tag{5}$$

where $\boldsymbol{f_\tau}$ denotes the tangential derivative of f.

3. Proof of Theorem 4

Let $u \in \mathrm{BV}(\Omega; S^1)$. For the proof of this theorem we consider a sequence of Lipschitz functions that approximate L (defined in (2)), and carry out the computations with this approximation. For small $\varepsilon > 0$ we let $L_\varepsilon : S^1 \to \mathbb{R}$ denote the following function

$$L_\varepsilon(e^{i\theta}) = \begin{cases} \theta & \text{if } 0 \le \theta \le \pi - \varepsilon, \\ \dfrac{\pi - \varepsilon}{\varepsilon}(\pi - \theta) & \text{if } \pi - \varepsilon \le \theta \le \pi + \varepsilon, \\ \theta - 2\pi & \text{if } \pi + \varepsilon \le \theta \le 2\pi. \end{cases}$$

Let $\alpha \in \mathbb{R}$ and define $\phi_{\alpha,\varepsilon} : S^1 \to \mathbb{R}$ by

$$\phi_{\alpha,\varepsilon}(e^{i\theta}) = L_\varepsilon(e^{i(\alpha+\theta)}).$$

Then $\phi_{\alpha,\varepsilon}$ is Lipschitz and therefore $\phi_{\alpha,\varepsilon}(u) \in \mathrm{BV}$. We use now the chain rule from Lemma 2.1 to compute the derivative of $\phi_{\alpha,\varepsilon}(u)$:

$$D\phi_{\alpha,\varepsilon}(u) = (\boldsymbol{L}_\varepsilon)_\tau(e^{i\alpha}u)(D^a u + D^c u) + (\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-))\nu_u \, \mathcal{H}^{n-1} \llcorner S(u).$$

Since the measures in the expression above are mutually singular, for the total variation of the corresponding measures we have

$$|D\phi_{\alpha,\varepsilon}(u)| \le |(\boldsymbol{L}_\varepsilon)_\tau(e^{i\alpha}u)| (|D^a u| + |D^c u|) + |\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)| \, \mathcal{H}^{n-1} \llcorner S(u).$$

Integrating this total variation over Ω we get

$$|\phi_{\alpha,\varepsilon}(u)|_{\mathrm{BV}} \le \int_\Omega |(\boldsymbol{L}_\varepsilon)_\tau(e^{i\alpha}u)| \, d(|D^a u| + |D^c u|) + \int_{S(u)} |\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)| \, d\mathcal{H}^{n-1}. \tag{6}$$

Observe that the map $\alpha \mapsto |\phi_{\alpha,\varepsilon}(u)|_{\mathrm{BV}}$ is lower semi-continuous because it is the supremum over a family of continuous functions in α. In particular $\alpha \mapsto |\phi_{\alpha,\varepsilon}(u)|_{\mathrm{BV}}$ is measurable. Integrating (6) with respect to α over $[0, 2\pi]$ we get

$$\int_0^{2\pi} |\phi_{\alpha,\varepsilon}(u)|_{\mathrm{BV}}\, d\alpha \le \int_0^{2\pi} \int_\Omega |(\boldsymbol{L}_\varepsilon)_\tau(e^{i\alpha}u)|\, d(|D^a u| + |D^c u|)\, d\alpha + \int_0^{2\pi} \int_{S(u)} |\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)|\, d\mathcal{H}^{n-1}\, d\alpha.$$

Let us consider the first term on the right hand side above; by Fubini's theorem

$$\int_0^{2\pi} \int_\Omega |(\boldsymbol{L}_\varepsilon)_\tau(e^{i\alpha}u)|\, d(|D^a u| + |D^c u|)\, d\alpha = \int_\Omega \int_0^{2\pi} |(\boldsymbol{L}_\varepsilon)_\tau(e^{i\alpha}u)|\, d\alpha\, d(|D^a u| + |D^c u|).$$

But an easy computation shows that for any fixed x, $\int_0^{2\pi} |(\boldsymbol{L}_\varepsilon)_\tau(e^{i\alpha}u(x))|\, d\alpha = 4(\pi - \varepsilon)$. So

$$\int_0^{2\pi} \int_\Omega |(\boldsymbol{L}_\varepsilon)_\tau(e^{i\alpha}u)|\, d(|D^a u| + |D^c u|)\, d\alpha = 4(\pi - \varepsilon)(|D^a u|(\Omega) + |D^c u|(\Omega)). \tag{7}$$

On the other hand, using the explicit formula for L_ε it is not hard to verify that if $|\theta_1 - \theta_2| \le \pi$ then

$$\int_0^{2\pi} |L_\varepsilon(e^{i(\alpha+\theta_1)}) - L_\varepsilon(e^{i(\alpha+\theta_2)})|\, d\alpha = 2\frac{\pi - \varepsilon}{\pi}|\theta_1 - \theta_2|(2\pi - |\theta_1 - \theta_2|) \le 8(\pi - \varepsilon)\sin(|\theta_1 - \theta_2|/2).$$

Observe that if $e^{i\theta_1} = u^+(x)$ and $e^{i\theta_2} = u^-(x)$ with $|\theta_1 - \theta_2| \le \pi$, then $|u^+(x) - u^-(x)| = 2\sin(|\theta_1 - \theta_2|/2)$. Hence, for any fixed $x \in S(u)$ we obtain

$$\int_0^{2\pi} |\phi_{\alpha,\varepsilon}(u^+(x)) - \phi_{\alpha,\varepsilon}(u^-(x))|\, d\alpha \le 4(\pi - \varepsilon)|u^+(x) - u^-(x)|.$$

Integrating over $S(u)$ and combining the result with (7) we establish that

$$\int_0^{2\pi} |\phi_{\alpha,\varepsilon}(u)|_{\mathrm{BV}}\, d\alpha \le 4(\pi - \varepsilon)|u|_{\mathrm{BV}}. \tag{8}$$

To finish the proof note that $\alpha \mapsto |L(e^{i\alpha}u)|_{\mathrm{BV}}$ is measurable with values in $[0, \infty]$, because

$$|L(e^{i\alpha}u)|_{\mathrm{BV}} = \sup_{g \in C_0^\infty\ |g| \le 1} \int_\Omega L(e^{i\alpha}u)\, \mathrm{div}\, g\, dx$$

and for fixed g the map $\alpha \mapsto \int_\Omega L(e^{i\alpha}u)\, \mathrm{div}\, g\, dx$ is measurable. Also observe that for all except a countable set of $\alpha \in \mathbb{R}$ we have $\mathrm{meas}(\{y \in \Omega : u(y) = -e^{-i\alpha}\}) = 0$, and for these values of α, $L_\varepsilon(e^{i\alpha}u) \to L(e^{i\alpha}u)$ a.e. in Ω as $\varepsilon \to 0$. This implies that for a.e. α, $|L(e^{i\alpha}u)|_{\mathrm{BV}} \le \liminf_{\varepsilon \to 0} |L_\varepsilon(e^{i\alpha}u)|_{\mathrm{BV}}$. Hence, using (8) and Fatou's lemma

$$\int_0^{2\pi} |L(e^{i\alpha}u)|_{\mathrm{BV}}\, d\alpha \le \liminf_{\varepsilon \to 0} \int_0^{2\pi} |L_\varepsilon(e^{i\alpha}u)|_{\mathrm{BV}} d\alpha \le 4\pi |u|_{\mathrm{BV}}. \qquad \square$$

Remark 7 *Recall the space of special functions with bounded variation*

$$SBV(\Omega; \mathbb{R}^m) = \{u \in \mathrm{BV}(\Omega; \mathbb{R}^m) \mid D^c u \equiv 0 \text{ in } \Omega\}.$$

We say that $u \in SBV(\Omega; S^1)$ if $u \in SBV(\Omega; \mathbb{R}^2)$ and $|u(x)| = 1$ for a.e. $x \in \Omega$. Then each $u \in SBV(\Omega; S^1)$ has a lifting $\phi \in SBV(\Omega; \mathbb{R})$ satisfying (1). *Indeed, by Theorem 2, there exists a lifting*

$\phi \in \mathrm{BV}(\Omega;\mathbb{R})$ *such that* (1) *holds. By the chain rule for BV functions applied to the relation* $u = e^{i\phi}$ *we obtain*

$$Du = iu(D^a\phi + D^c\phi) + (e^{i\phi^+} - e^{i\phi^-})\nu_\phi\, \mathcal{H}^{n-1} \llcorner S(\phi). \tag{9}$$

Since $D^c u = 0$ *we see that* $D^c\phi = 0$ *and so* $\phi \in SBV$.

4. The constant 2 is optimal

The following result is a consequence of the paper [BBM2]:

Lemma 4.1 *Let* Ω *be the unit disc in* $\mathbb{R}^2$. *Define* $u : \Omega \setminus \{0\} \mapsto S^1$, $u(x) = \frac{x}{|x|}$ *for every* $x \in \Omega \setminus \{0\}$. *Let* $\phi \in \mathrm{BV}(\Omega;\mathbb{R})$ *be a lifting of* u. *Then* $|D\phi|(\Omega) \geq 4\pi = 2|u|_{\mathrm{BV}}$.

Proof. Firstly remark that $u \in \mathrm{W}^{1,p}(\Omega)$ for all $p \in [1, 2)$ and $\int_\Omega |\nabla u|\, dx = 2\pi$. Take $\phi_n \in W^{1,1} \cap C^\infty(\Omega;\mathbb{R})$ such that $\phi_n \to \phi$ a.e. on Ω and $\int_\Omega |\nabla\phi_n|\, dx \to |D\phi|(\Omega)$ as $n \to \infty$. Set $u_n = e^{i\phi_n} \in W^{1,1} \cap C^\infty(\Omega; S^1)$. For every $r \in (0, 1)$ denote $S_r = \{x \in \mathbb{R}^2 : |x| = r\}$. Up to a subsequence, for a.e. $r \in (0, 1)$ we have $u_n \to u$ $\mathcal{H}^1$-a.e. in S_r and $\sup_n \int_{S_r} |\nabla u_n|\, d\mathcal{H}^1 < \infty$; for those r, by Lemma 18 of [BBM2]

$$\liminf_{n\to\infty} \int_{S_r} |\nabla u_n \cdot \tau|\, d\mathcal{H}^1 \geq \int_{S_r} |\nabla u \cdot \tau|\, d\mathcal{H}^1 + 2\pi = \int_{S_r} |\nabla u|\, d\mathcal{H}^1 + 2\pi$$

(here τ is the tangent vector in each point of S_r). Therefore, by Fatou's lemma,

$$|D\phi|(\Omega) = \liminf_{n\to\infty} \int_\Omega |\nabla u_n|\, dx \geq \int_0^1 \liminf_{n\to\infty} \int_{S_r} |\nabla u_n|\, d\mathcal{H}^1\, dr \geq \int_\Omega |\nabla u| + 2\pi. \qquad \square$$

Remark 8 *For dimension* $n \geq 3$, *we consider the cylinder* $\Omega = B^2 \times (0, 1)^{n-2} \subset \mathbb{R}^n$ *where* B^2 *is the unit disc in* $\mathbb{R}^2$ *and we repeat the above argument for the function* $v(z, x_3, \ldots, x_N) = u(z)$.

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