Interaction energy between vortices of vector fields on Riemannian surfaces

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Abstract

We study a variational Ginzburg-Landau type model depending on a small parameter $\varepsilon > 0$ for (tangent) vector fields on a 2-dimensional Riemannian surface. As $\varepsilon \to 0$, the vector fields tend to be of unit length and will have singular points of a (non-zero) index, called vortices. Our main result determines the interaction energy between these vortices as a Γ-limit (at the second order) as $\varepsilon \to 0$.

1 Introduction

Let (S, g) be a closed (i.e., compact, connected without boundary) 2-dimensional Riemannian manifold of genus $\mathfrak{g}$. We will focus on (tangent) vector fields

$$u : S \to TS, \quad \text{i.e., } u(x) \in T_x S \text{ for every } x \in S$$

where $TS = \cup_{x \in S} T_x S$ is the tangent bundle of S. It is well known that there are no smooth vector fields $\mathcal{X}(S)$ (or more generally, of Sobolev regularity $\mathcal{X}^{1,2}(S)$) of unit length $|u|_g = 1$ on S (unless $\mathfrak{g} = 1$). In fact, vector fields of unit length have in general singular points with a (non-zero) index. Our aim is to determine the interaction energy between these singular points in a variational model of Ginzburg-Landau type depending on a small parameter $\varepsilon > 0$ where the penalty $|u|_g = 1$ in S is relaxed.

Model. For vector fields $u : S \to TS$, we define the energy functional

$$E_\varepsilon(u) = \int_S e_\varepsilon(u) \, \mathrm{vol}_g, \quad e_\varepsilon(u) := \frac{1}{2}|Du|_g^2 + \frac{1}{4\varepsilon^2} F(|u|_g^2),$$

where $|Du|_g^2 := |D_{\tau_1} u|_g^2 + |D_{\tau_2} u|_g^2$ in S, vol_g is the volume 2-form on (S, g) and D_v denotes covariant differentiation (with respect to the Levi-Civita connection) of u in direction v and $\{\tau_1, \tau_2\}$ is any local orthonormal basis of TS. The potential $F : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with $F(1) = 0$ and there exists some $c > 0$ such that $F(s^2) \geq c(1 - s)^2$ for every $s \geq 0$; in particular, 1 is the unique zero of F. The parameter $\varepsilon > 0$ is small penalizing $|u|_g \neq 1$ in S; the goal is to analyse the asymptotic behaviour of E_ε in the framework of Γ-convergence (at first and second order) in the limit $\varepsilon \to 0$. This is a "toy" problem for some physical models arising for thin shells

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in micromagnetics or nematic liquid crystals (see e.g., [4, 5]).

Connection 1-form. On an open subset $O \subset S$, a moving frame is a pair of smooth, properly oriented, orthonormal vector fields $\tau_k \in \mathcal{X}(O)$, $k = 1, 2$, i.e., $(\tau_k, \tau_l)_g = \delta_{k\ell}$, $k, l = 1, 2$, and $\mathrm{vol}_g(\tau_1, \tau_2) = 1$ in O, where $(\cdot, \cdot)_g$ is the scalar product on TS. (We will use the same notation $(\cdot, \cdot)_g$ for the inner product associated to k-forms, $k = 0, 1, 2$.) Defining $i : TS \to TS$ such that i is an isometry of T_xS to itself for every $x \in S$ satisfying

$$i^2 w = -w, \qquad (iw, v)_g = -(w, iv)_g \; = \; \mathrm{vol}_g(w, v),$$

then every smooth vector field $\tau \in \mathcal{X}(O)$ of unit length provides a moving frame $\{\tau_1, \tau_2\} := \{\tau, i\tau\}$ on O. Moreover, if $\{\tau_1, \tau_2\}$ is any moving frame in O, then $\tau_2 = i\tau_1$. [1] Given a moving frame $\{\tau_1, \tau_2\}$ on an open subset $O \subset S$, the *connection 1-form* A associated to $\{\tau_1, \tau_2\}$ is defined for every smooth vector field $v \in \mathcal{X}(O)$:

$$A(v) := (D_v \tau_2, \tau_1)_g = -(D_v \tau_1, \tau_2)_g \qquad \text{in } O.$$

In particular, $D_v \tau_1 = -A(v)\tau_2$ and $D_v \tau_2 = A(v)\tau_1$ in O. In complex notation, it yields for any smooth complex-valued function ϕ on O:

$$D_v(\phi \tau_1) = (d\phi(v) - iA(v)\phi)\tau_1 \qquad \text{in } O.$$

The definition of A depends on the choice of the moving frame. However, the exterior derivative dA of the connection 1-form is independent of the moving frame, in particular, the following identity holds

$$dA = \kappa \; \mathrm{vol}_g,$$

where κ is the Gaussian curvature of S (see [7] Proposition 2, Chapter 5.3). We recall the Gauss-Bonnet theorem that states

$$\int_S \kappa \, \mathrm{vol}_g = 2\pi \chi(S),$$

where $\chi(S)$ is the Euler characteristic, related to the genus $\mathfrak{g}$ of S by $\chi(S) = 2 - 2\mathfrak{g}$.

Vortices. We will identify vortices of a vector field u with small geodesic balls centered at some points around which u has a (non-zero) index. To be more precise, we introduce the Sobolev space $\mathcal{X}^{1,p}(S)$ of vector fields $u : S \to TS$ such that $|u|_g$ and $|Du|_g$ belong to $L^p(S)$ (with respect to the volume 2-form), $p \geq 1$. Given $u \in \mathcal{X}^{1,p}(S) \cap L^q(S)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$, we define the 1-form $j(u)$ by [2]

$$j(u) = (Du, iu)_g.$$

In particular, $j(u)$ is a well-defined 1-form in $L^1(S)$ if $u \in \mathcal{X}^{1,1}(S)$ with $|u|_g = 1$ almost everywhere in S; the same is true if $u \in \mathcal{X}^{1,p}(S)$ for $p \geq \frac{4}{3}$. To introduce the notion of index, we assume that O is a simply connected open subset of S and $u \in \mathcal{X}^{1,2}(N)$ is a vector field in a neighborhood N of ∂O such that $|u|_g \geq \frac{1}{2}$ a.e. in N; then the *index* (or winding number) of u along ∂O is defined by

$$\deg(u; \partial O) := \frac{1}{2\pi} \left(\int_{\partial O} \frac{j(u)}{|u|_g^2} + \int_O \kappa \; \mathrm{vol}_g \right)$$

[1] In general a moving frame exists only locally on S.

[2] Note that if $\{\tau_1, \tau_2\}$ is a moving frame on an open set $O \subset S$, then the connection 1-form A associated to the moving frame is given by $A = -j(\tau_1)$ on O. In particular, $dj(u) = -k \, \mathrm{vol}_g$ in O for every smooth $u \in \mathcal{X}(O)$ of unit length.

(see [7] Chapter 6.1). In particular, if u is defined in O and has unit length on ∂O, then one has $\int_O \omega(u) = 2\pi \deg(u;\partial O)$ where $\omega(u)$ is the *vorticity* associated to the vector field u:

$$\omega(u) := dj(u) + \kappa \,\mathrm{vol}_g. \tag{1}$$

Sometimes we can identify the index of u at a point $P \in S$ with the index of u along a curve around P. Note that every smooth vector field $u \in \mathcal{X}(O)$ (or more generally, $u \in \mathcal{X}^{1,2}(O)$) of unit length in O has $\deg(u;\partial O) = 0$; moreover, a vortex with non-zero index will carry infinite energy E_ε as $\varepsilon \to 0$.

We will prove a Γ-convergence result (at the second order) of E_ε as $\varepsilon \to 0$. In particular, at the level of minimizers u_ε of E_ε, we show that u_ε converges in $\mathcal{X}^{1,1}(S)$ (for a subsequence) to a canonical harmonic vector field u^* of unit length that is smooth [3] away from $n = |\chi(S)|$ distinct singular points $a_1, \dots, a_n$, each singular point a_k carrying the same index $d_k = \operatorname{sign} \chi(S)$ so that [4]

$$\sum_{k=1}^n d_k = \chi(S). \tag{2}$$

The vorticity $\omega(u^*)$ detects the singular points $\{a_k\}_{k=1}^n$ of u^*:

$$\omega(u^*) = 2\pi \sum_{k=1}^n d_k \delta_{a_k} \qquad \text{in } S, \tag{3}$$

where δ_{a_k} is the Dirac measure (as a 2-form) at a_k. The expansion of the minimal energy E_ε at the second order is given by

$$E_\varepsilon(u_\varepsilon) = n\pi \log\frac{1}{\varepsilon} + \lim_{r\to 0}\left(\int_{S\setminus\cup_{k=1}^n B_r(a_k)} \frac{1}{2}|Du^*|_g^2 \,\mathrm{vol}_g + n\pi\log r\right) + n\gamma_F + o(1), \text{ as } \varepsilon \to 0,$$

where $\gamma_F > 0$ is a constant depending only on the potential F and $B_r(a_k)$ is the geodesic ball centered at a_k of radius r. The second term in the above RHS is called the *renormalized energy* between the vortices $a_1, \dots, a_n$ and governs the optimal location of these singular points as in the Euclidian case (see the seminal book [3]). In particular, if S is the unit sphere in $\mathbb{R}^3$ endowed with the standard metric g, then $n = 2$ and a_1 and a_2 are two diametrically opposed points on S.

Outline of the note. The note is divided as follows. Section 2 is devoted to characterize canonical harmonic vector fields of unit length. In Section 3, we determine the renormalized energy between singular points of canonical harmonic vector fields. The main Γ-convergence result is stated in the last section. The proofs of these results are part of our forthcoming article [9].

2 Canonical harmonic vector fields of unit length

We will say that a canonical harmonic vector field of unit length having the singular points $a_1, \dots, a_n \in S$ of index $d_1, \dots, d_n \in \mathbb{Z}$ for some $n \geq 1$, is a vector field $u^* \in \mathcal{X}^{1,1}(S)$ such that $|u^*|_g = 1$ in S, (3) holds and

$$d^* j(u^*) = 0 \qquad \text{in } S. \tag{4}$$

[3] In the case of a surface (S, g) with genus 1 (i.e., homeomorphic with the flat torus), then $n = 0$ and u^* is smooth in S.

[4] In fact, $\deg(u^*;\gamma) = d_k$ for every closed simple curve γ around a_k and lying near a_k.

Here, d^* is the adjoint of the exterior derivative d, i.e., $d^* j(u^*)$ is the unique 0-form on S such that

$$\int_S \big(d^* j(u^*), \zeta\big)_g \operatorname{vol}_g = \int_S \big(j(u^*), d\zeta\big)_g \operatorname{vol}_g \quad \text{for every smooth 0-form } \zeta.$$

If u^* satisfies (3), then the Gauss-Bonnet theorem combined with (1) imply that necessarily (2) holds.

We will see that condition (2) is also sufficient. Indeed, if (2) holds, we will construct solutions of (3) and (4), as follows: let $\psi = \psi(a, d)$ be the unique 2-form on S solving:

$$-\Delta\psi = -\kappa \operatorname{vol}_g + 2\pi \sum_{k=1}^{n} d_k \delta_{a_k} \qquad \text{in } S, \qquad\qquad \int_S \psi = 0, \tag{5}$$

with the sign convention that $-\Delta = dd^* + d^*d$. The idea is to find u^* such that $j(u^*) - d^*\psi$ is an harmonic 1-form, i.e.,

$$Harm^1(S) = \{\text{integrable 1-forms } \eta \text{ on } S \ : \ d\eta = d^*\eta = 0 \text{ as distributions}\}.$$

The dimension of the space $Harm^1(S)$ is twice the genus (i.e., $2\mathfrak{g}$) of (S, g) and we fix an orthonormal basis $\eta_1, \dots, \eta_{2\mathfrak{g}}$ of $Harm^1(S)$ such that

$$\int_S (\eta_k, \eta_l)_g \ \operatorname{vol}_g = \delta_{kl} \quad \text{ for } k, l = 1, \dots, 2\mathfrak{g}.$$

Therefore, it is expected that

$$j(u^*) = d^*\psi + \sum_{k=1}^{2\mathfrak{g}} \Phi_k \eta_k \qquad \text{in } S \tag{6}$$

for some constant vector $\Phi = (\Phi_1, \dots, \Phi_{2\mathfrak{g}}) \in \mathbb{R}^{2\mathfrak{g}}$. These constants are called *flux integrals* as they can be recovered by

$$\Phi_k = \int_S (j(u^*), \eta_k)_g \operatorname{vol}_g, \qquad \text{for } k = 1, \dots, 2\mathfrak{g}.$$

Note that (6) combined with (5) automatically yield (3) and (4). One important point is to characterize for which values of Φ the RHS of (6) arises as $j(u^*)$ for some vector field u^* of unit length in S. For that condition, we need to recall the following theorem of Federer-Fleming [8]: there exist $2\mathfrak{g}$ simple closed geodesics γ_ℓ on S, $\ell = 1, \dots, 2\mathfrak{g}$, such that for any closed Lipschitz curve γ on S, one can find integers $c_1 \dots, c_{2\mathfrak{g}}$ such that

$$\gamma \text{ is homologous to } \sum_{\ell=1}^{2\mathfrak{g}} c_\ell \gamma_\ell$$

i.e., there exists an integrable function $f : S \to \mathbb{Z}$ such that

$$\int_\gamma \zeta - \sum_{\ell=1}^{2\mathfrak{g}} c_\ell \int_{\gamma_\ell} \zeta \ = \ \int_S f \, d\zeta \quad \text{ for all smooth 1-forms } \zeta.$$

Having chosen the geodesic curves $\{\gamma_\ell\}_{\ell=1}^{2\mathfrak{g}}$ and the harmonic 1-forms $\{\eta_k\}_{k=1}^{2\mathfrak{g}}$, we fix the notation

$$\alpha_{\ell k} := \int_{\gamma_\ell} \eta_k, \quad k, \ell = 1, \dots, 2\mathfrak{g}. \tag{7}$$

Theorem 1 *Let $n \geq 1$ and $d = (d_1, \dots, d_n) \in \mathbb{Z}^n$ satisfy* (2)*. Then for every $a = (a_1, \dots, a_n) \in S^n$, there exists*

$$\zeta_\ell = \zeta_\ell(a; d) \in \mathbb{R}/2\pi\mathbb{Z}, \qquad \ell = 1, \dots, 2\mathfrak{g}$$

such that if a vector field $u^* \in \mathcal{X}^{1,1}(S)$ of unit length solves (3) *and* (4)*, then $j(u^*)$ has the form* (6) *for constants $\Phi_1, \dots, \Phi_{2\mathfrak{g}}$ such that*

$$\sum_{k=1}^{2\mathfrak{g}} \alpha_{\ell k} \Phi_k + \zeta_\ell(a, d) \in 2\pi\mathbb{Z}, \qquad \ell = 1, \dots, 2\mathfrak{g}, \tag{8}$$

where $(\alpha_{\ell k})$ were defined in (7)*. Conversely, given any $\Phi_1, \dots, \Phi_{2\mathfrak{g}}$ satisfying* (8)*, there exists a vector field $u^* \in \mathcal{X}^{1,1}(S)$ of unit length solving* (3) *and* (4) *and such that $j(u^*)$ satisfies* (6)*. In addition, the following hold:*

1) $\zeta_\ell(\cdot; d)$ depends continuously on $a \in S^n$ for every $\ell = 1, \dots, 2\mathfrak{g}$. More generally, if [5]

$$\mu^t := 2\pi \sum_{l=1}^{n_t} d_l^t \delta_{a_l^t} \to \mu^0 := 2\pi \sum_{l=1}^{n_0} d_l^0 \delta_{a_l^0} \qquad \text{in } W^{-1,1} \qquad \text{as } t \downarrow 0,$$

$\{d_l^t\}_l$ are integers with (2) *and $\sum_{l=1}^{n_t} |d_l^t|$ is uniformly bounded in t, then $\zeta_\ell(a^t, d^t) \to \zeta_\ell(a^0, d^0)$ as $t \downarrow 0$.*

2) any u^* solving (3) *and* (4) *belongs to $\mathcal{X}^{1,p}(S)$ for all $1 \leq p < 2$, and is smooth away from $\{a_k\}_{k=1}^n$.*

3) If $u^*, \tilde{u}^*$ both satisfy (6) *for the same (a, d) and the same $\{\Phi_k\}_{k=1}^{2\mathfrak{g}}$, then $\tilde{u}^* = e^{i\beta} u^*$ for some $\beta \in \mathbb{R}$.*

The constants $\{\zeta_\ell(a; d)\}_{\ell=1}^{2\mathfrak{g}}$ are determined as follows. For every $\ell = 1, \dots, 2\mathfrak{g}$, we let λ_ℓ be some smooth simple closed curve such that λ_ℓ is homologous to γ_ℓ (the geodesics fixed in (7)) so that $\{a_k\}_{k=1}^n$ is disjoint from λ_ℓ; for example, λ_ℓ is either γ_ℓ or, if γ_ℓ intersects some a_k, a small perturbation thereof. We now define $\zeta_\ell(a, d)$ to be the element of $\mathbb{R}/2\pi\mathbb{Z}$ such that

$$\zeta_\ell(a, d) := \int_{\lambda_\ell} (d^*\psi + A) \mod 2\pi, \quad \ell = 1, \dots, 2\mathfrak{g}, \tag{9}$$

where $\psi = \psi(a, d)$ is the 2-form given by (5) and A is the connection 1-form associated to any moving frame defined in a neighborhood of λ_ℓ. The integral in (9) is independent, modulo $2\pi\mathbb{Z}$, of the choice of moving frame and of the curve λ_ℓ homologous to γ_ℓ. In examples in which it can be explicitly computed, in general $\zeta_\ell(a, d) \neq 0 \mod 2\pi$ for $\ell = 1, \dots, 2\mathfrak{g}$.

3 Renormalized energy

For any $n \geq 1$, we consider n **distinct** points $a = (a_1, \dots, a_n) \in S^n$. Let $d = (d_1, \dots, d_n) \in \mathbb{Z}^n$ satisfying (2), $\{\zeta_\ell(a; d)\}_{\ell=1}^{2\mathfrak{g}}$ be given in Theorem 1 and $\Phi \in \mathbb{R}^{2\mathfrak{g}}$ be a constant vector inside the set:

$$\mathcal{L}(a, d) := \{\Phi = (\Phi_1, \dots, \Phi_{2\mathfrak{g}}) \in \mathbb{R}^{2\mathfrak{g}} \;:\; \sum_{k=1}^{2\mathfrak{g}} \alpha_{\ell k} \Phi_k + \zeta_\ell(a, d) \in 2\pi\mathbb{Z},\, \ell = 1, \dots, 2\mathfrak{g}\}.$$

[5] If μ is a 2-form (possibly measure-valued) then we write for $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$\|\mu\|_{W^{-1,p}} := \sup \left\{ \int_S f\mu \;:\; f \in W^{1,q}(S; \mathbb{R}),\, \|f\|_{W^{1,q}} := \|f\|_{L^q} + \|df\|_{L^q} \leq 1 \right\}.$$

We define the *renormalized energy* between the vortices a of indices d by

$$W(a,d,\Phi) := \lim_{r\to 0}\left(\int_{S\setminus\cup_{k=1}^n B_r(a_k)} \frac{1}{2}|Du^*|_g^2 \ \mathrm{vol}_g + \pi \log r \sum_{k=1}^n d_k^2\right),$$

where $u^* = u^*(a,d,\Phi)$ is the unique (up to a multiplicative complex number) canonical harmonic vector field given in Theorem 1 and $B_r(a_k)$ is the geodesic ball centered at a_k of radius r. Our arguments show that the above limit indeed exists. As in the euclidian case (see [3]), we can compute the renormalized energy by using the Green's function. For that, let $G(x,y)$ be the unique function on $S\times S$ such that

$$-\Delta_x(G(\cdot,y)\ \mathrm{vol}_g) = \delta_y - \frac{\mathrm{vol}_g}{\mathrm{Vol}_g(S)} \ \text{ distributionally in } S, \qquad \int_S G(x,y)\,\mathrm{vol}_g(x) = 0 \quad \text{for every } y\in S.$$

with $\mathrm{Vol}_g(S) := \int_S \mathrm{vol}_g$. Then G may be represented in the form (see [2] Chapter 4.2):

$$G(x,y) = G_0(x,y) + H(x,y), \qquad \text{with } H\in C^1(S\times S),$$

where G_0 is smooth away from the diagonal, with

$$G_0(x,y) = -\frac{1}{2\pi}\log(dist(x,y)) \ \text{ if the geodesic distance } dist(x,y) < \frac{1}{2}(\text{injectivity radius of } S).$$

The 2-form $\psi = \psi(a,d)$ defined at (5) can be written as:

$$\psi = 2\pi\sum_{k=1}^n d_k G(\cdot,a_k)\,\mathrm{vol}_g \ + \psi_0\,\mathrm{vol}_g \qquad \text{in } S,$$

where $\psi_0 \in C^\infty(S)$ has zero average on S and solves

$$-\Delta\psi_0 = -\kappa + \bar{\kappa}, \qquad \text{for } \bar{\kappa} = \frac{1}{\mathrm{Vol}(S)}\int_S \kappa\,\mathrm{vol}_g = \frac{2\pi\chi(S)}{\mathrm{Vol}(S)}. \tag{10}$$

In other words, the 2-form $x\mapsto \psi(x) + d_k \log \mathrm{dist}(x,a_k)\,\mathrm{vol}_g$ is C^1 in a neighborhood of a_k for every $1\le k\le n$. We have the following expression of the renormalized energy:

Proposition 2 *Given $n\ge 1$ distinct points $a_1,\dots,a_n\in S$, integers $d_1,\dots,d_n$ with* (2) *and $\Phi\in\mathcal{L}(a,d)$, then*

$$W(a,d,\Phi) = 4\pi^2\sum_{l\neq k} d_l d_k G(a_l,a_k) + 2\pi\sum_{k=1}^n\left[\pi d_k^2 H(a_k,a_k) + d_k\psi_0(a_k)\right] + \frac{1}{2}|\Phi|^2 + \int_S \frac{|d\psi_0|^2}{2}\, vol_g\ , \tag{11}$$

where ψ_0 is defined in (10).

In the case of the unit sphere S in $\mathbb{R}^3$ endowed with the standard metric (in particular, ψ_0 vanishes in S), if $n=2$ and $d_1=d_2=1$, then the second term in the RHS of (11) is independent of a_k (as $x\mapsto H(x,x)$ is constant, see [14]); moreover, $\Phi=0$ and so, minimizing W is equivalent by minimizing the Green's function $G(a_1,a_2)$ over the set of pairs (a_1,a_2) in $S\times S$, namely, the minimizing pairs are diametrically opposed.

4 Γ-convergence

Given the potential F in Section 1, we compute the energy E_ε of the radial profile of a vortex of index 1 inside a geodesic ball of radius $R > 0$:

$$I_F(R,\varepsilon) := \inf\left\{\pi\int_0^R \left[f'(r)^2 + \frac{f(r)^2}{r^2} + \frac{1}{2\varepsilon^2}F(f(r)^2)\right] rdr \ : f(0)=0, f(R)=1\right\}.$$

Then $I_F(R,\varepsilon) = I_F(\lambda R, \lambda\varepsilon) = I_F(1, \frac{\varepsilon}{R}) =: I_F(\frac{\varepsilon}{R})$ for every $\lambda > 0$, and the following limit exists (see [3]):

$$\gamma_F := \lim_{t\to 0}(I_F(t) + \pi \log t).$$

We state our main result:

Theorem 3 *The following Γ-convergence result holds.*

1) *(Compactness) Let $(u_\varepsilon)_{\varepsilon\downarrow 0}$ be a family of vector fields on S satisfying $E_\varepsilon(u_\varepsilon) \leq N\pi|\log\varepsilon| + C$ for some integer $N \geq 0$ and a constant $C > 0$. Denoting by*

$$\Phi(u_\varepsilon) := \left(\int_S (j(u_\varepsilon), \eta_1)_g\, vol_g, \ldots, \int_S (j(u_\varepsilon), \eta_{2\mathfrak{g}})_g\, vol_g\right) \in \mathbb{R}^{2\mathfrak{g}},$$

then there exists a sequence $\varepsilon \downarrow 0$ such that

$$\omega(u_\varepsilon) \longrightarrow 2\pi\sum_{k=1}^n d_k\delta_{a_k} \quad \text{in } W^{-1,1}, \ \text{ as } \varepsilon \to 0, \tag{12}$$

where $\{a_k\}_{k=1}^n$ are distinct points in S and $\{d_k\}_{k=1}^n$ are nonzero integers satisfying (2) *and $\sum_{k=1}^n |d_k| \leq N$. Moreover, if $\sum_{k=1}^n |d_k| = N$, then $n = N$, $|d_k| = 1$ for every $k = 1, \ldots, n$ and for a subsequence, there exists $\Phi \in \mathcal{L}(a,d)$ such that $\Phi(u_\varepsilon) \to \Phi$ as $\varepsilon \to 0$ (in particular, $n = \chi(S)$ modulo 2).*

2) *(Γ-liminf inequality) Assume that the vector fields $u_\varepsilon \in \mathcal{X}^{1,2}(S)$ satisfy* (12) *for n distinct points $\{a_k\}_{k=1}^n \in S^n$ and $|d_k| = 1$, $k = 1, \ldots n$ that satisfy* (2) *and $\Phi \in \mathcal{L}(a,d)$. Then*

$$\liminf_{\varepsilon\to 0}\,[E_\varepsilon(u_\varepsilon) - n\pi|\log\varepsilon|] \ \geq \ W(a,d,\Phi) + n\gamma_F.$$

3) *(Γ-limsup inequality) For every n distinct points $a_1, \ldots, a_n \in S$ and $d_1, \ldots, d_n \in \{\pm 1\}$ satisfying* (2) *and every $\Phi \in \mathcal{L}(a,d)$ there exists a sequence of vector fields u_ε on S such that* (12) *holds and*

$$E_\varepsilon(u_\varepsilon) - n\pi|\log\varepsilon| \longrightarrow W(a,d,\Phi) + n\gamma_F \quad \text{as } \varepsilon \to 0.$$

This theorem is the generalization of the Γ-convergence result for E_ε in the euclidian case (see [6, 11, 13, 1]) and it is based on topological methods for energy concentration (vortex ball construction, vorticity estimates etc.) as introduced in [10, 12].

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