Uniqueness of degree-one Ginzburg-Landau vortex in the unit ball in dimensions $N \geq 7$

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Abstract

For $\varepsilon > 0$, we consider the Ginzburg-Landau functional for $\mathbb{R}^N$-valued maps defined in the unit ball $B^N \subset \mathbb{R}^N$ with the vortex boundary data x on ∂B^N. In dimensions $N \geq 7$, we prove that for every $\varepsilon > 0$, there exists a unique global minimizer u_ε of this problem; moreover, u_ε is symmetric and of the form $u_\varepsilon(x) = f_\varepsilon(|x|)\frac{x}{|x|}$ for $x \in B^N$.

Keywords: uniqueness, symmetry, minimizers, Ginzburg-Landau.
MSC: 35A02, 35B06, 35J50.

1 Introduction and main results

In this note, we consider the following Ginzburg-Landau type energy functional

$$E_\varepsilon(u) = \int_{B^N} \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2}W(1-|u|^2)\right] dx,$$

where $\varepsilon > 0$, B^N is the unit ball in $\mathbb{R}^N$, $N \geq 2$, and the potential $W \in C^1((-\infty, 1];\mathbb{R})$ satisfies

$$W(0) = 0,\ W(t) > 0 \text{ for all } t \in (-\infty, 1] \setminus \{0\}, \text{ and } W \text{ is convex.} \tag{1}$$

We investigate the global minimizers of the energy E_ε in the set

$$\mathscr{A} := \{u \in H^1(B^N; \mathbb{R}^N) : u(x) = x \text{ on } \partial B^N = \mathbb{S}^{N-1}\}.$$

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The requirement that $u(x) = x$ on $\mathbb{S}^{N-1}$ is sometimes referred in the literature as the vortex boundary condition.

We note that in our analysis the convexity of W needs not be strict; compare [6] where strict convexity is assumed.

The direct method in the calculus of variations yields the existence of a global minimizer u_ε of E_ε over $\mathscr{A}$ for all range of $\varepsilon > 0$. Moreover, any minimizer u_ε belongs to $C^1(\overline{B^N}; \mathbb{R}^N)$ and satisfies $|u_\varepsilon| \leq 1$ and the system of PDEs (in the sense of distributions)

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon \, W'(1 - |u_\varepsilon|^2) \quad \text{in } B^N. \tag{2}$$

The goal of this note is to give a short proof of the uniqueness and symmetry of the global minimizer of E_ε in $\mathscr{A}$ for *all* $\varepsilon > 0$ in dimensions $N \geq 7$. We prove that, in these dimensions, the global minimizer is unique and given by the unique radially symmetric critical point of E_ε defined by

$$u_\varepsilon(x) = f_\varepsilon(|x|) \frac{x}{|x|} \quad \text{for all } x \in B^N, \tag{3}$$

where the radial profile $f_\varepsilon : [0, 1] \to \mathbb{R}_+$ is the unique solution of

$$\begin{cases} -f_\varepsilon'' - \frac{N-1}{r} f_\varepsilon' + \frac{N-1}{r^2} f_\varepsilon = \frac{1}{\varepsilon^2} f_\varepsilon \, W'(1 - f_\varepsilon^2) & \text{for } r \in (0, 1), \\ f_\varepsilon(0) = 0, f_\varepsilon(1) = 1. \end{cases} \tag{4}$$

Moreover, $f_\varepsilon > 0$ and $f_\varepsilon' > 0$ in $(0, 1)$ (see e.g. [4]).

Theorem 1. *Assume that W satisfies* (1)*. If $N \geq 7$, then for every $\varepsilon > 0$, u_ε given in* (3) *is the unique global minimizer of E_ε in $\mathscr{A}$.*

To our knowledge, the question about the uniqueness of minimizers/critical points of E_ε in $\mathscr{A}$ for any $\varepsilon > 0$ was raised in dimension $N = 2$ in the book of Bethuel, Brezis and Hélein [1, Problem 10, page 139], and in general dimensions $N \geq 2$ and also for the blow-up limiting problem around the vortex (when the domain is the whole space $\mathbb{R}^N$ and by rescaling, ε can be assumed equal to 1) in an article of Brezis [2, Section 2].

It is well known that uniqueness is present for large enough $\varepsilon > 0$ for any $N \geq 2$. Indeed, for any $\varepsilon > (W'(1)/\lambda_1)^{1/2}$ where λ_1 is the first eigenvalue of $-\Delta$ in B^N with zero Dirichlet boundary condition, E_ε is strictly convex in $\mathscr{A}$ and thus has a unique critical point in $\mathscr{A}$ (that is the global minimizer of our problem).

For *sufficiently small* $\varepsilon > 0$ all results regarding uniqueness question available in the literature are in the affirmative. In particular, we have:

(i) Pacard and Rivière [11, Theorem 10.2] showed in dimension $N = 2$ that, for small $\varepsilon > 0$, E_ε has in fact a unique critical point in $\mathscr{A}$.

(ii) Mironescu [10] showed in dimension $N = 2$ that, when B^2 is replaced by $\mathbb{R}^2$ and $\varepsilon = 1$, a local minimizer of E_ε subjected to a degree-one boundary condition at infinity is unique (up to translation and suitable rotation). This was generalized to dimension $N = 3$ by Millot and Pisante [9] and dimensions $N \geq 4$ by Pisante [12], also in the case of the blow-up limiting problem on $\mathbb{R}^N$ and $\varepsilon = 1$.

These results should be compared to those for the limit problem on the unit ball obtained by sending $\varepsilon \to 0$. In this limit, the Ginzburg-Landau problem 'converges' to the harmonic map problem from B^N to $\mathbb{S}^{N-1}$. It is well known that, the vortex boundary condition gives rise to a unique minimizing harmonic map $x \mapsto \frac{x}{|x|}$ if $N \geq 3$; see Brezis, Coron and Lieb [3] in dimension $N = 3$, Jäger and Kaul [7] in dimensions $N \geq 7$, and Lin [8] in dimensions $N \geq 3$.

We highlight that, in contrast to the above, our result holds for *all* $\varepsilon > 0$, provided that $N \geq 7$. The method of our proof deviates somewhat from that in the aforementioned works. In fact it is reminiscent of our recent work [6] on the (non-)uniqueness and symmetry of minimizers of the Ginzburg-Landau functionals for $\mathbb{R}^M$-valued maps defined on N-dimensional domains, where M is not necessarily the same as N. However we note that the results in [6] do not directly apply to the present context, as in [6] it is required that W be *strictly convex*. Furthermore, a priori, it is not clear why non-strict convexity of the potential W is sufficient to ensure uniqueness of global minimizers.

We exploit the convexity of W to lower estimate the 'excess' energy by a suitable quadratic energy which can be handled by the factorization trick à la Hardy. Indeed, the positivity of the excess energy is then related to the validity of a Hardy-type inequality, which explains our restriction of $N \geq 7$. This echoes our observation made in [6] that a result of Jäger and Kaul [7] on the minimality of the equator map in these dimensions is related to a certain inequality involving the sharp constant in the Hardy inequality.

We expect that our result remains valid in dimensions $2 \leq N \leq 6$, but this goes beyond the scope of this note and remains for further investigation.

2 Proof of Theorem 1

Theorem 1 will be obtained as a consequence of a stronger result on the uniqueness of global minimizers of for the $\mathbb{R}^M$-valued Ginzburg-Landau functional with $M \geq N$. By a slight abuse of notation, we consider the energy functional

$$E_\varepsilon(u) = \int_{B^N} \Big[\frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2)\Big]\,dx,$$

where u belongs to

$$\mathscr{A} := \{u \in H^1(B^N; \mathbb{R}^M) : u(x) = x \text{ on } \partial B^N = \mathbb{S}^{N-1} \subset \mathbb{R}^M\}.$$

Theorem 2. *Assume that W satisfies* (1). *If $M \geq N \geq 7$, then for every $\varepsilon > 0$, u_ε given in* (3) *is the unique global minimizer of E_ε in $\mathscr{A}$.*

When W is strictly convex, the above theorem is proved in [6]; see Theorem 1.7. The argument therein uses the strict convexity in a crucial way.

Proof. The proof will be done in several steps. First, we consider the difference between the energies of the critical point u_ε, defined in (3), and an arbitrary competitor $u_\varepsilon + v$ and show that this difference is controlled from below by some quadratic energy functional $F_\varepsilon(v)$. Second, we employ the positivity of the radial profile f_ε in (4) and apply the Hardy decomposition method in order to show that $F_\varepsilon(v) \geq 0$, which proves in particular that u_ε is a global minimizer of E_ε. Finally, we characterise the situation when this difference is zero and conclude to the uniqueness of the global minimizer u_ε.

Step 1: Lower bound for energy difference. For any $v \in H_0^1(B^N; \mathbb{R}^M)$, we have

$$\begin{aligned} E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) = &\int_{B^N} \left[\nabla u_\varepsilon \cdot \nabla v + \frac{1}{2}|\nabla v|^2\right] dx \\ &+ \frac{1}{2\varepsilon^2} \int_{B^N} \left[W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2)\right] dx. \end{aligned}$$

Using the convexity of W, we have

$$W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \geq -W'(1 - |u_\varepsilon|^2)(|u_\varepsilon + v|^2 - |u_\varepsilon|^2).$$

The last two relations imply that

$$\begin{aligned} E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq &\int_{B^N} \left[\nabla u_\varepsilon \cdot \nabla v - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2) u_\varepsilon \cdot v\right] dx \\ &+ \int_{B^N} \left[\frac{1}{2}|\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2)|v|^2\right] dx. \end{aligned}$$

Moreover, by (2), we obtain

$$E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_{B^N} \left[\frac{1}{2}|\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2)|v|^2\right] dx =: \frac{1}{2} F_\varepsilon(v) \tag{5}$$

for all $v \in H_0^1(B^N; \mathbb{R}^M)$.

Step 2: A rewriting of $F_\varepsilon(v)$ using the decomposition $v = f_\varepsilon w$ for every scalar test function $v \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$. We consider the operator

$$L_\varepsilon := \frac{1}{2} \nabla_{L^2} F_\varepsilon = -\Delta - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2).$$

Using the decomposition

$$v = f_\varepsilon w$$

for the scalar function $v \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$, we have (see e.g. [5, Lemma A.1]):

$$\begin{aligned} F_\varepsilon(v) = \int_{B^N} L_\varepsilon v \cdot v \, dx &= \int_{B^N} w^2 L_\varepsilon f_\varepsilon \cdot f_\varepsilon \, dx + \int_{B^N} f_\varepsilon^2 |\nabla w|^2 \, dx \\ &= \int_{B^N} f_\varepsilon^2 \left(|\nabla w|^2 - \frac{N-1}{r^2} w^2 \right) dx, \end{aligned}$$

because (4) yields $L_\varepsilon f_\varepsilon \cdot f_\varepsilon = -\frac{N-1}{r^2} f_\varepsilon^2$ in B^N.

Step 3: We prove that $F_\varepsilon(v) \geq 0$ for every scalar test function $v \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$. Within the notation $v = f_\varepsilon w$ of Step 2 with $v, w \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$, we use the decomposition

$$w = \varphi g$$

with $\varphi = |x|^{-\frac{N-2}{2}}$ being the first eigenfunction of the Hardy's operator $-\Delta - \frac{(N-2)^2}{4|x|^2}$ in $\mathbb{R}^N \setminus \{0\}$ and $g \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R})$. We compute

$$|\nabla w|^2 = |\nabla \varphi|^2 g^2 + |\nabla g|^2 \varphi^2 + \frac{1}{2} \nabla(\varphi^2) \cdot \nabla(g^2).$$

As $|\nabla \varphi|^2 = \frac{(N-2)^2}{4|x|^2} \varphi^2$ and φ^2 is harmonic in $B^N \setminus \{0\}$, integration by parts yields

$$\begin{aligned} F_\varepsilon(v) &= \int_{B^N} f_\varepsilon^2 \left(|\nabla g|^2 \varphi^2 + \frac{(N-2)^2}{4r^2} \varphi^2 g^2 - \frac{N-1}{r^2} \varphi^2 g^2 \right) dx - \frac{1}{2} \int_{B^N} \nabla(\varphi^2) \cdot \nabla(f_\varepsilon^2) g^2 \, dx \\ &\geq \int_{B^N} f_\varepsilon^2 |\nabla g|^2 \varphi^2 \, dx + \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{f_\varepsilon^2}{r^2} \varphi^2 g^2 \, dx \\ &\geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} \, dx \geq 0, \end{aligned} \tag{6}$$

where we have used $N \geq 7$ and $\frac{1}{2} \nabla(\varphi^2) \cdot \nabla(f_\varepsilon^2) = 2\varphi \varphi' f_\varepsilon f_\varepsilon' \leq 0$ in $B^N \setminus \{0\}$.

Step 4: We prove that $F_\varepsilon(v) \geq 0$ for every $v \in H_0^1(B^N; \mathbb{R}^M)$ meaning that u_ε is a global minimizer of E_ε over $\mathscr{A}$; moreover, $F_\varepsilon(v) = 0$ if and only if $v = 0$. Let $v \in H_0^1(B^N; \mathbb{R}^M)$. As a point has zero H^1 capacity in $\mathbb{R}^N$, a standard density argument implies the existence of a sequence $v_k \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}^M)$ such that $v_k \to v$ in $H^1(B^N, \mathbb{R}^M)$ and a.e. in B^N. On the one hand, by definition (5) of F_ε, since $W'(1 - f_\varepsilon^2) \in L^\infty$, we deduce that $F_\varepsilon(v_k) \to F_\varepsilon(v)$ as $k \to \infty$. On the other hand, by (6) and Fatou's lemma, we deduce

$$\begin{aligned} \liminf_{k \to \infty} F_\varepsilon(v_k) &\geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \liminf_{k \to \infty} \int_{B^N} \frac{v_k^2}{r^2} \, dx \\ &\geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} \, dx. \end{aligned}$$

Therefore, we conclude that

$$F_\varepsilon(v) \geq \left(\frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} \, dx \geq 0, \quad \forall v \in H_0^1(B^N; \mathbb{R}^M),$$

implying by (5) that u_ε is a minimizer of E_ε over $\mathscr{A}$. Moreover, $F_\varepsilon(v) = 0$ if and only if $v = 0$.

Step 5: Conclusion. We have shown that u_ε is a global minimizer. Assume that $\tilde{u}_\varepsilon$ is another global minimizer of E_ε over $\mathscr{A}$. If $v := \tilde{u}_\varepsilon - u_\varepsilon$, then $v \in H_0^1(B^N; \mathbb{R}^M)$ and by Steps 1 and 4, we have that $0 = E_\varepsilon(\tilde{u}_\varepsilon) - E_\varepsilon(u_\varepsilon) \geq F_\varepsilon(v) \geq 0$, which yields $F_\varepsilon(v) = 0$. Step 4 implies that $v = 0$, i.e., $\tilde{u}_\varepsilon = u_\varepsilon$. □

Remark 3. *Recall that in the case $M \geq N \geq 7$, Jäger and Kaul [7] proved the uniqueness of global minimizer for harmonic map problem*

$$\min_{u \in \mathscr{A}_*} \int_{B^N} |\nabla u|^2 \, dx,$$

where $\mathscr{A}_* = \{u \in H^1(B^N; \mathbb{S}^{M-1}) : u(x) = x$ on $\partial B^N = \mathbb{S}^{N-1} \subset \mathbb{S}^{M-1}\}$. This can also be seen by the method above as observed in our earlier paper [6]. We give the argument here for readers' convenience: Take a perturbation $v \in H_0^1(B^N, \mathbb{R}^M)$ of the harmonic map $u_*(x) = \frac{x}{|x|}$ such that $|u_*(x) + v(x)| = 1$ a.e. in B^N. Then, by [6, Proof of Theorem 5.1],

$$\int_{B^N} \Big[|\nabla(u_*+v)|^2 - |\nabla u_*|^2\Big] \, dx = \int_{B^N} \Big[|\nabla v|^2 - |\nabla u_*|^2 |v|^2\Big] \, dx = \int_{B^N} \Big[|\nabla v|^2 - (N-1)\frac{|v|^2}{|x|^2}\Big] \, dx.$$

Using Hardy's inequality in dimension N we arrive at

$$\int_{B^N} \Big[|\nabla(u_* + v)|^2 - |\nabla u_*|^2\Big] \, dx \geq \left(\frac{(N-2)^2}{4} - (N-1)\right) \int_{B^N} \frac{|v|^2}{|x|^2} \, dx.$$

The result follows since $N \geq 7$.

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