Pohozaev type identities for an elliptic equation

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Abstract

We present some Pohozaev identities for the equation $-\Delta u = |u|^{p-1}u - \lambda u$ and as an application, we prove some nonexistence results.

1 Introduction

In this paper we present some Pohozaev type identities for the following nonlinear elliptic equation:

$$-\Delta_g u = |u|^{p-1}u - \lambda u \quad \text{on} \quad M. \tag{1}$$

Here, $p > 1$, $\lambda \in \mathbb{R}$ and M is a ball in $\mathbb{R}^n$ or on the unit sphere S^n, $n \geq 3$, equipped with the standard metric g and Δ_g stands for the Laplace-Beltrami operator on (M, g). The goal is to prove nonexistence results for (1) in different ranges of λ.

Motivated by the study of Brezis and Nirenberg [5], we first consider the Dirichlet problem associated to (1) in the unit ball $B_1 \subset \mathbb{R}^n$, i.e.,

$$\begin{cases} -\Delta u &= |u|^{p-1}u - \lambda u, \quad u \not\equiv 0 & \text{in} \quad B_1, \\ u &= 0 & \text{on} \quad \partial B_1. \end{cases} \tag{2}$$

We prove the following identity:

Lemma 1 *Let $\psi : [0, \infty) \to \mathbb{R}$ be a smooth function (with $\psi'(0) = \psi''(0) = 0$ if $n \geq 4$). If u is a solution of (2), then*

$$\begin{aligned} &\int_{B_1} \left\{ r\psi'''(r) + 3\psi''(r) - \frac{(n-1)(n-3)}{r}\psi'(r) - 4\lambda\big(r\psi'(r) + \psi(r)\big) \right\} \frac{u^2}{2}\, dx \\ &= \psi(1) \int_{\partial B_1} \Big|\frac{\partial u}{\partial r}\Big|^2 d\mathcal{H}^{n-1}(x) + 2\int_{B_1} r\psi'(r) \left(|\nabla u|^2 - \Big|\frac{\partial u}{\partial r}\Big|^2 \right) dx \\ &\quad - \int_{B_1} \left\{ \frac{p+3}{p+1} r\psi'(r) - \big(\frac{p-1}{p+1}n - 2\big)\psi(r) \right\} |u|^{p+1}\, dx, \end{aligned} \tag{3}$$

where $r = |x|$ and $\frac{\partial u}{\partial r}(x) = \frac{x}{|x|} \cdot \nabla u(x)$ stands for the radial derivative of u.

As a consequence, we deduce the nonexistence result of Brezis and Nirenberg [5] for positive solutions of (2) in the supercritical case $p \geq \frac{n+2}{n-2}$:

Theorem 2 *(Brezis and Nirenberg [5]) Let $p \geq \frac{n+2}{n-2}$ and $\lambda_1 := \lambda_1(-\Delta; B_1)$ be the first eigenvalue of the Laplace operator with Dirichlet boundary condition in B_1. If one of the following two conditions is satisfied*

(i)

$$n = 3 \quad \text{and} \quad \lambda \notin \left(-\lambda_1, -\frac{\lambda_1}{4}\right),$$

(ii)

$$n \geq 4 \quad \text{and} \quad \lambda \notin (-\lambda_1, 0),$$

then there is no positive solution of (2).

Remark 1 a) The set of positive and nodal *radial* solutions (regular or singular) of (2) is described by Benguria, Dolbeault and Esteban [2].

b) The question if there is no (nodal) solution of (2) for $n = 3$ and $\lambda \in (-\frac{\lambda_1}{4}, 0)$ is still open.

Next we study the Dirichlet problem associated to (1) on a geodesic ball D_{θ^*} centered at the North pole in S^3 of radius $\theta^* \in (0, \pi)$:

$$\begin{cases} -\Delta_g u = |u|^{p-1}u - \lambda u, \quad u \not\equiv 0 & \text{in } D_{\theta^*}, \\ u = 0 & \text{on } \partial D_{\theta^*}. \end{cases} \tag{4}$$

We want to obtain a similar identity to (3) for any solution u of (4). Using the stereographic projection $\Phi_Q : S^3 \setminus \{Q\} \to \mathbb{R}^3$ with vertex at the South pole Q in S^3, the equation (4) writes as

$$-\frac{1}{\rho^3}\,\mathrm{div}(\rho \nabla U) = |U|^{p-1}U - \lambda U \quad \text{in} \quad B_{R^*} \subset \mathbb{R}^3$$

where $\rho(x) = \frac{2}{1+|x|^2}$, $U(x) = u(\Phi_Q^{-1}(x))$ for every $x \in B_{R^*}$ and $R^* = \tan\frac{\theta^*}{2}$. The transformation

$$v(x) = U(x)\sqrt{\rho(x)}$$

turns (4) into

$$\begin{cases} -\Delta v = \rho(x)^{\frac{5-p}{2}}|v|^{p-1}v + \frac{3-4\lambda}{4}\rho^2 v, \quad v \not\equiv 0 & \text{in } B_{R^*}, \\ v = 0 & \text{on } \partial B_{R^*}. \end{cases} \tag{5}$$

We prove the following identity for a solution v of (5):

Lemma 3 *Let $\psi : [0, \infty) \to \mathbb{R}$ be a smooth function. If v is a solution of* (5)*, then*

$$\begin{aligned} &\int_{B_{R^*}} \left\{ r\psi''' + 3\psi'' + \frac{3-4\lambda}{2}\left[r(1+r^2)\psi' + (1-r^2)\psi\right]\rho^3(x) \right\} \frac{v^2}{2}\,dx \\ &= R^*\psi(R^*)\int_{\partial B_{R^*}} \left|\frac{\partial v}{\partial r}\right|^2 d\mathcal{H}^2(x) + 2\int_{B_{R^*}} r\psi'\left(|\nabla v|^2 - \left|\frac{\partial v}{\partial r}\right|^2\right) dx \\ &- \int_{B_{R^*}} \left\{ (1 + \frac{2}{p+1})r\psi' - \frac{p-5}{p+1}\cdot\frac{1-r^2}{1+r^2}\psi \right\} |v|^{p+1}\rho(x)^{\frac{5-p}{2}}\,dx. \end{aligned} \tag{6}$$

From here, we deduce the result of Bandle and Benguria [1], i.e., to determine the range of values of the parameters θ^* and λ for which there exists no positive solution of (4) in the critical case $p = 5$.

Theorem 4 *(Bandle and Benguria [1]) Let $p = 5$ and $D_{\theta^*} \subset S^3$ be a geodesic ball of radius θ^*. Set $\lambda_1^* = \frac{\pi^2-(\theta^*)^2}{(\theta^*)^2}$ be the first eigenvalue of the Laplace-Beltrami operator with Dirichlet boundary condition in D_{θ^*} and $\mu_1^* = \frac{\pi^2-4(\theta^*)^2}{4(\theta^*)^2}$. If one of the following conditions is satisfied*

(i)

$$\lambda \leq \frac{3}{4} \quad \text{and} \quad \lambda \notin (-\lambda_1^*, -\mu_1^*)\,,$$

(ii)

$$\lambda > \frac{3}{4} \quad \text{and} \quad \theta^* \in (0, \frac{\pi}{2}],$$

then there is no positive solution of (4). Moreover, there exists a curve in the strip $(\theta^*, \lambda) \subset \left(\frac{\pi}{2}, \pi\right) \times \left(\frac{3}{4}, \infty\right)$, denoted by $\nu(\theta^*) = \lambda$ such that $\nu(\theta^*) \to \frac{3}{4}$ as $\theta^* \to \pi$ and for $\lambda \in \left(\frac{3}{4}, \nu(\theta^*)\right)$ there is no radial solution of (4) (see Figure 1).

Remark 2 (a) In [6], Brezis and Peletier proved that for any $\theta^* \in (\pi/2, \pi)$, there exist positive radial solutions of (4) for λ sufficiently large; therefore, $\nu(\theta^*) \to +\infty$ as $\theta^* \to \pi/2$.

(b) The question if there is no solution of (4) in the strip $(\theta^*, \lambda) \subset \left(\frac{\pi}{2}, \pi\right) \times \left(\frac{3}{4}, \infty\right)$ below the curve ν is still open (see discussion in Section 4).

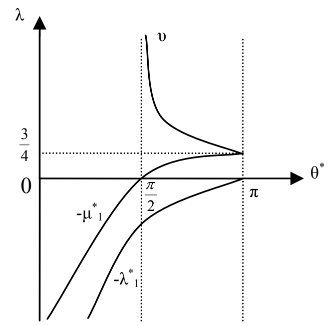

Figure 1: Range of values of λ for nonexistence of positive radial solutions.

Finally, we deal with positive solutions of (1) in the case of the whole unit sphere S^n, $n \geq 3$:

$$\left\{ \begin{array}{rcl} -\Delta_g u & = & u^p - \lambda u \\ u & > & 0 \end{array} \right. \quad \text{on} \quad S^n. \tag{7}$$

For $\lambda \leq 0$, there is no solution of (7) (it directly follows by integration of (7) on S^n). Therefore, we consider the range $\lambda > 0$. The goal is to present a simplified proof of the following Pohozaev type identity due to Gidas and Spruck [9]:

Lemma 5 *(Gidas and Spruck [9]) Let $n \geq 3$ and u be a solution of (7). Set*

$$w = u^{-2/(n-2)} \tag{8}$$

and

$$J(x) = \frac{1}{w^{n-1}} \sum_{i,j=1}^{n} (\nabla_i \tilde{w}_j - \frac{1}{n}\Delta_g w\, \delta_{ij})^2 \geq 0,$$

where $\nabla_i \tilde{w}_j$ *denotes the* j *component of the covariant derivative* ∇_i *of the vector field* $\left(\frac{1}{\rho^2}\partial_j w\right)_{1\le j\le n}$.
For any $\gamma \in \mathbb{R}$ *we have*

$$\begin{aligned}
&\frac{(n-2)^2}{2}\int_{S^n} J(x)u^{-\gamma} \; + \gamma(1-\gamma)\int_{S^n} u^{-\gamma-\frac{2n-2}{n-2}}|du|^4 \\
&+\frac{2(n-1)}{n}\Big(\frac{n+2}{n-2}-p-\gamma\frac{n+2}{2(n-1)}\Big)\int_{S^n} u^{p-\gamma-\frac{n}{n-2}}|du|^2 \\
&+2\left[n-1-\lambda\Big(\frac{4(n-1)}{n(n-2)}-\gamma\frac{n+2}{2n}\Big)\right]\int_{S^n} u^{-\gamma-\frac{2}{n-2}}|du|^2 = 0.
\end{aligned} \tag{9}$$

From here, it follows the following uniqueness result for solutions of (7):

Corollary 6 *Let* $n \ge 3$. *Assume that one of the following conditions holds:*

(i) $1 < p < \frac{n+2}{n-2}$ *and*

$$\begin{cases} 0<\lambda<\min\{\frac{n}{p-1}, a_n\} & \text{if} \quad n<8, \\ 0<\lambda\le\frac{n}{p-1} & \text{if} \quad n\ge 8, \end{cases} \tag{10}$$

where

$$a_n = \frac{2n(n-1)(n-2)}{-n^2+8n-4};$$

(ii) $p = \frac{n+2}{n-2}$ *and* $0 < \lambda < \frac{n(n-2)}{4}$.

Then the only solution of (7) *is the trivial constant solution* $\lambda^{1/(p-1)}$.

This result was extended by M.F. Bidaut-Veron et L. Veron [3] using the Bochner-Lichnerowicz-Weitzenböck formula [10]:

Theorem 7 *(Veron and Veron [3]) Assume that*

$$1 < p \le \frac{n+2}{n-2} \quad \text{and} \quad 0 < \lambda \le \frac{n}{p-1} \tag{11}$$

where at least one of the two inequalities (11) *is strict. Then the only solution of* (7) *is the constant* $\lambda^{1/(p-1)}$.

Recently, Brezis and Li [4] proved Theorem 7 in the case of $\lambda \le n(n-2)/4$ using the theory of moving planes; they also showed that for subcritical exponent p, there exist nonconstant solutions of (7) if $\lambda > \frac{n}{p-1}$ with $|\lambda - \frac{n}{p-1}|$ small. For the critical exponent $p = \frac{n+2}{n-2}$, Corollary 6 and Theorem 7 are also sharp since there is a well-known branch of nonconstant solutions if $\lambda = \frac{n(n-2)}{4}$ (see [7]).

The outline of the paper is the following: we start with some preliminaries on the geometry of the unit sphere S^n that we use in the proof of Lemma 5. In Section 3, we prove Lemma 1 and Theorem 2. In Section 4, we show Lemma 3 and Theorem 4. Finally, we give a simplified proof of the Gidas-Spruck result.

2 Preliminaries

In this section we introduce some notations that we use throughout of the paper. Let $P = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ and $Q = (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ be the North and the South pole of S^n and set $\Omega_P = S^n \setminus \{P\}$ and $\Omega_Q = S^n \setminus \{Q\}$. The stereographic projection $\Phi_P : \Omega_P \to \mathbb{R}^n$ of pole P (respectively, $\Phi_Q : \Omega_Q \to \mathbb{R}^n$ of pole Q) is defined as

$$\Phi_P(y) = \left(\frac{y_1}{1-y_{n+1}}, \ldots, \frac{y_n}{1-y_{n+1}}\right), \quad \forall y = (y_1, \ldots, y_{n+1}) \in \Omega_P$$

$$\text{(respectively,} \quad \Phi_Q(y) = \left(\frac{y_1}{1+y_{n+1}}, \ldots, \frac{y_n}{1+y_{n+1}}\right), \quad \forall y = (y_1, \ldots, y_{n+1}) \in \Omega_Q).$$

We easily check that Φ_P (respectively, Φ_Q) is a homeomorphism of Ω_P (respectively, Ω_Q) into $\mathbb{R}^n$ and the inverse function $\Phi_P^{-1} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ (respectively, $\Phi_Q^{-1} : \mathbb{R}^n \to \mathbb{R}^{n+1}$) is given by

$$\Phi_P^{-1}(x) = \rho(x)\left(x_1, \ldots, x_n, \frac{|x|^2-1}{2}\right), \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

$$\text{(respectively,} \quad \Phi_Q^{-1}(x) = \rho(x)\left(x_1, \ldots, x_n, \frac{1-|x|^2}{2}\right), \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n),$$

where

$$\rho(x) = \frac{2}{1+|x|^2}$$

and $|x|^2 = \sum_{i=1}^{n} x_i^2$ for any $x \in \mathbb{R}^n$. In the following we omit the argument of maps. In the local charts (Ω_P, Φ_P) and (Ω_Q, Φ_Q), the standard metric g on S^n writes as

$$g_{ij} = \rho^2 \delta_{ij}, \; 1 \le i, j \le n$$

where δ_{ij} denotes the Kronecker's symbol. For a function $u : S^n \to \mathbb{R}$, we use the following notations:

$$u_i = \partial_i u, \; \tilde{u}_i = \frac{1}{\rho^2} u_i, \; 1 \le i \le n$$

$$|du|^2 = \sum_{i=1}^{n} u_i \tilde{u}_i,$$

$$\Delta_g u = \frac{1}{\rho^n} \sum_{i=1}^{n} \partial_i(\rho^n \tilde{u}_i).$$

For $1 \le i \le n$, let ∇_i be the covariant derivative. The Cristoffel symbols are given by

$$\Gamma_{ij}^k = \begin{cases} x_k \rho & \text{if} \quad i = j \ne k, \\ -x_i \rho & \text{if} \quad i = j = k, \\ -x_j \rho & \text{if} \quad i \ne j, i = k, \\ -x_i \rho & \text{if} \quad i \ne j, j = k. \end{cases}$$

If V_i are the components of a vector field V, we associate the vector field $\tilde{V}$ of components $\tilde{V}_i = \frac{1}{\rho^2} V_i$. Standard computations yield that

$$(\nabla_i V)_j = \partial_i(V_j) + \sum_{k=1}^{n} \Gamma_{ik}^j V_k, \; 1 \le i, j \le n,$$

$$\sum_{i=1}^{n}(\nabla_i V)_i = \frac{1}{\rho^n}\sum_{i=1}^{n}\partial_i(\rho^n V_i), \tag{12}$$

$$\sum_{j=1}^{n}(\nabla_j\nabla_i V - \nabla_i\nabla_j V)_j = (n-1)\rho^2 V_i,\ 1\le i\le n. \tag{13}$$

One can check that

$$\nabla_i\tilde{u}_j = \nabla_j\tilde{u}_i,\ 1\le i,j\le n, \tag{14}$$

$$\Delta_g u = \sum_{i=1}^{n}\nabla_i\tilde{u}_i, \tag{15}$$

$$\partial_i(|du|^2) = 2\sum_{j=1}^{n}u_j\nabla_j\tilde{u}_i,\ 1\le i\le n, \tag{16}$$

$$\sum_{i,j=1}^{n}(\partial_i\tilde{u}_j\nabla_j\tilde{u}_i + \tilde{u}_j\nabla_j\nabla_i\tilde{u}_i) = \sum_{i,j=1}^{n}(\nabla_i\tilde{u}_j\nabla_j\tilde{u}_i + \tilde{u}_j\partial_j(\nabla_i\tilde{u}_i)), \tag{17}$$

where $\nabla_i\tilde{u}_j$ denotes the j component of the covariant derivative ∇_i of the vector field with the components $\tilde{u}_j$.

3 The case of the unit ball in $\mathbb{R}^n$. Proof of Lemma 1 and Theorem 2

We start by proving identity (3):

Proof of Lemma 1. Let u be a solution of (2). Following the ideas of Brezis and Nirenberg [5], we first multiply (2) by $\psi(r)x\cdot\nabla u(x)\in C^1(B_1)$ (without any condition on ψ) and integrating by parts, we obtain:

$$\begin{aligned}-\int_{B_1}[r\psi' - (n-2)\psi]|\nabla u|^2\,dx + \psi(1)\int_{\partial B_1}\left|\frac{\partial u}{\partial r}\right|^2 d\mathcal{H}^{n-1}(x) + 2\int_{B_1}r\psi'\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)dx \\ = \int_{B_1}(r\psi' + n\psi)\left(\frac{2}{p+1}|u|^{p+1} - \lambda u^2\right)dx.\end{aligned} \tag{18}$$

Next we multiply (2) by $[r\psi' - (n-2)\psi]u\in C^1(B_1)$ (since $\psi'(0)=0$ if $n\ge 4$) and it results by integration by parts:

$$\begin{aligned}\int_{B_1}[r\psi' - (n-2)\psi]|\nabla u|^2\,dx - \int_{B_1}\left[r\psi''' + 3\psi'' - \frac{(n-1)(n-3)}{r}\psi'\right]\frac{u^2}{2}\,dx \\ = \int_{B_1}[r\psi' - (n-2)\psi](|u|^{p+1} - \lambda u^2)\,dx.\end{aligned} \tag{19}$$

(Here, we used that $\psi''(0)=0$ if $n\ge 4$.) Combining (18) and (19), the conclusion follows immediately. □

Using identity (3), we show Theorem 2:

Proof of Theorem 2. Let u be a positive solution of (2). Choosing $\psi=1$, (3) becomes the standard Pohozaev identity [11]:

$$-2\lambda\int_{B_1}u^2\,dx - \left(\frac{p-1}{p+1}n - 2\right)\int_{B_1}u^{p+1}\,dx = \int_{\partial B_1}\left|\frac{\partial u}{\partial r}\right|^2 d\mathcal{H}^{n-1}(x).$$

Therefore, no solution exists for (2) if $\lambda \geq 0$ and $p \geq \frac{n+2}{n-2}$. Set φ be a positive eigenfunction associated to the first eigenvalue λ_1 in B_1. First, we prove the nonexistence result for $\lambda \leq -\lambda_1$. Indeed, multiplying (2) by φ, we deduce:

$$\lambda_1 \int_{B_1} u\varphi = -\int_{B_1} u\Delta\varphi = -\int_{B_1} \Delta u\, \varphi = \int_{B_1} (u^p - \lambda u)\varphi.$$

That is,

$$\int_{B_1} [u^p - (\lambda + \lambda_1)u]\varphi = 0;$$

therefore, we get a contradiction with $u > 0$. Now we treat the remaining case: $n = 3$ and $\lambda \in [-\frac{\lambda_1}{4}, 0)$ where $\lambda_1 = \pi^2$. By the symmetry result of Gidas, Ni and Nirenberg [8] applied for positive solutions of (2), we know that u is radial. Then (3) becomes:

$$\begin{aligned}\int_{B_1} &\{r\psi''' + 3\psi'' - 4\lambda(r\psi' + \psi)\} \frac{u^2}{2}\, dx \\ &= \psi(1)\int_{\partial B_1} |\frac{\partial u}{\partial r}|^2\, d\mathcal{H}^2(x) - \int_{B_1} \left\{\frac{p+3}{p+1} r\psi' - \frac{p-5}{p+1}\psi\right\} u^{p+1}\, dx,\end{aligned} \tag{20}$$

for any smooth function $\psi : [0, \infty) \to \mathbb{R}$. Following the argument in [5], we choose the smooth function

$$\psi(r) = \frac{\sin(2\sqrt{|\lambda|}\, r)}{r} > 0,\ \forall r \in [0, 1).$$

Then

$$\begin{aligned} &r\psi''' + 3\psi'' - 4\lambda(r\psi' + \psi) = 0, \\ \text{and} \quad &r\psi' = \frac{2\sqrt{|\lambda|}\, r\cos(2\sqrt{|\lambda|}\, r) - \sin(2\sqrt{|\lambda|}\, r)}{r} < 0,\ \forall r \in (0, 1].\end{aligned}$$

Since $p \geq 5$, (20) leads to a contradiction. □

Now we discuss the case $n = 3$ and $p = 5$, i.e.,

$$\begin{cases} -\Delta u = u^5 - \lambda u, \quad u \not\equiv 0 & \text{in} \quad B_1 \subset \mathbb{R}^3, \\ \quad\ u = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{21}$$

Our aim is to present a list of properties for a solution u. We start by proving another identity 'à la Pohozaev' related to (3):

Lemma 8 *Let $\psi : [0, \infty) \to \mathbb{R}$ be a smooth function. If u is a solution of (21), then*

$$\begin{aligned} -\int_{B_1} &\left\{r\psi'''(r) + 5\psi''(r) + \frac{4}{r}\psi'(r) + 4\lambda\big(r\psi'(r) + \psi(r)\big)\right\} \frac{u^2}{2}\, dx \\ &= \psi(1)\int_{\partial B_1} |\frac{\partial u}{\partial r}|^2\, d\mathcal{H}^2(x) - 2\int_{B_1} r\psi'(r)\left(|\frac{\partial u}{\partial r}|^2 + \lambda u^2 - \frac{1}{3}u^6\right) dx.\end{aligned} \tag{22}$$

Proof. First, identity (3) writes in the case $n = 3$ and $p = 5$ as

$$\begin{aligned}\int_{B_1} &\{r\psi'''(r) + 3\psi''(r) - 4\lambda\big(r\psi'(r) + \psi(r)\big)\} \frac{u^2}{2}\, dx \\ &= \psi(1)\int_{\partial B_1} |\frac{\partial u}{\partial r}|^2\, d\mathcal{H}^2(x) + \int_0^1 r\psi'(r)\int_{\partial B_r} \left(2|\nabla_{S^2} u|^2 - \frac{4}{3}u^6\right) d\mathcal{H}^2 dr,\end{aligned} \tag{23}$$

where

$$\nabla_{S^2} u = \frac{\partial u}{\partial \theta} \frac{\vec{\theta}}{r} + \frac{\partial u}{\partial \varphi} \frac{\vec{\varphi}}{r \sin \theta}, \quad \nabla u = \frac{\partial u}{\partial r} \vec{r} + \nabla_{S^2} u$$

are written in the spherical coordinates $(r, \theta, \varphi) \in (0,1) \times (0, \pi) \times (0, 2\pi)$. We compute the last term in (23): multiplying (21) by u and integrating by parts in the variables θ and φ on ∂B_r, $r \in (0,1)$, we obtain:

$$\int_0^1 r\psi'(r) \int_{\partial B_r} \left(2|\nabla_{S^2} u|^2 - \frac{4}{3} u^6 \right) d\mathcal{H}^2 dr = \int_{B_1} r\psi'(r) \left[\frac{2}{3} u^6 - 2\lambda u^2 + \frac{2}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) u \right] dx. \tag{24}$$

If we integrate by parts the last term in (24) with respect to r, we get that

$$\int_{B_1} \frac{2\psi'(r)}{r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) u \, dx = \int_{B_1} \left\{ r\psi'''(r) + 4\psi''(r) + \frac{2}{r} \psi'(r) \right\} u^2 \, dx - 2 \int_{B_1} r\psi'(r) \left| \frac{\partial u}{\partial r} \right|^2 dx. \tag{25}$$

Combining (23), (24) and (25), we conclude with (22). □

By Lemmas 1 and 8, we obtain the following properties:

Proposition 9 *If u is a solution of* (21), *then:*

(i)

$$\int_{\partial B_1} \left| \frac{\partial u}{\partial r} \right|^2 d\mathcal{H}^2(x) = 2\lambda \int_{B_1} u^2 \, dx.$$

(ii) If $\psi : [0, \infty) \to \mathbb{R}$ is a smooth function, then

$$\int_{B_1} \left\{ r\psi'''(r) + 4\psi''(r) + \frac{2}{r} \psi'(r) \right\} u^2 \, dx = 2 \int_{B_1} r\psi'(r) \left(\left|\nabla u\right|^2 + \lambda u^2 - u^6 \right) dx.$$

(iii) If $\lambda < 0$ and $\psi(r) = \frac{\sin(2\sqrt{|\lambda|}\, r)}{r}$, $\forall r \in [0,1)$, then

$$\psi(1) \int_{\partial B_1} \left| \frac{\partial u}{\partial r} \right|^2 d\mathcal{H}^2(x) + 2 \int_{B_1} r\psi'(r) \left(|\nabla_{S^2} u|^2 - \frac{2}{3} u^6 \right) dx = 0.$$

(iv) If $\lambda < 0$ and $\psi(r) = \frac{1}{r} \int_0^r \frac{\sinh(2\sqrt{|\lambda|}\, t)}{t} dt$, $\forall r \in [0,1)$, then

$$\psi(1) \int_{\partial B_1} \left| \frac{\partial u}{\partial r} \right|^2 d\mathcal{H}^2(x) = 2 \int_{B_1} r\psi'(r) \left(\left| \frac{\partial u}{\partial r} \right|^2 + \lambda u^2 - \frac{1}{3} u^6 \right) dx.$$

Proof. The first point *(i)* follows from (3) by taking $\psi \equiv 1$. The identity in *(ii)* comes by subtracting (23) from (22). Notice that the following ODE:

$$r\psi''' + 4\psi'' + \frac{2}{r} \psi' = 0$$

has the solution

$$\psi(r) = c_1 + \frac{c_2}{r} + c_3 \ln r.$$

(If we approximate $\psi = 1/r$ by smooth functions, we obtain by *(ii)* the obvious relation $\int_{B_1} (\left|\nabla u\right|^2 + \lambda u^2 - u^6) \, dx = 0$.) Point *(iii)* follows from (23) since ψ is the solution of the ODE:

$$(r\psi)''' - 4\lambda (r\psi)' = 0.$$

Similarly, *(iv)* comes from (22) since $g = (r\psi)'$ satisfies the ODE:

$$-g'' - \frac{2}{r}g' = 4\lambda g.$$

□

4 The case of caps in S^3. Proof of Lemma 3 and Theorem 4

First we present the proof of identity (6):

Proof of Lemma 3. Let v be a solution of (5). Following the same argument as in the proof of Lemma 1, we multiply (5) by $\psi(r)x\cdot\nabla v(x) \in C^1(B_{R^*})$ and integrating by parts, we obtain:

$$\begin{aligned}
&-\int_{B_{R^*}} (r\psi' - \psi)|\nabla v|^2\,dx + R^*\psi(R^*)\int_{\partial B_{R^*}} \left|\frac{\partial v}{\partial r}\right|^2 d\mathcal{H}^2 + 2\int_{B_{R^*}} r\psi'\left(|\nabla v|^2 - \left|\frac{\partial v}{\partial r}\right|^2\right)dx \\
&\quad = \frac{3-4\lambda}{8}\int_{B_{R^*}} \left[r(1+r^2)\psi' + (3-r^2)\psi\right]\rho^3(x)v^2\,dx \\
&\qquad + \frac{2}{p+1}\int_{B_{R^*}} \left\{r\psi' + \left(3 + \frac{(p-5)r^2}{1+r^2}\right)\psi\right\}|v|^{p+1}\rho(x)^{\frac{5-p}{2}}\,dx.
\end{aligned} \tag{26}$$

Then, multiplying (5) by $(r\psi' - \psi)v \in C^1(B_{R^*})$, we obtain:

$$\begin{aligned}
&\int_{B_{R^*}} (r\psi' - \psi)|\nabla v|^2\,dx - \int_{B_{R^*}} (r\psi''' + 3\psi'')\frac{v^2}{2}\,dx \\
&\quad = \int_{B_{R^*}} (r\psi' - \psi)\left(\rho(x)^{\frac{5-p}{2}}|v|^{p+1} + \frac{3-4\lambda}{4}\rho^2v^2\right)dx.
\end{aligned} \tag{27}$$

By summation of (26) and (27), we get the identity (6). □

As an application, we give the proof of Theorem 4:

Proof of Theorem 4. Let u be a positive solution of (4). As in the proof of Theorem 2, multiplying (4) with a positive eigenfunction associated to λ_1^*, we obtain that no positive solution of (4) exists if $\lambda \le -\lambda_1^*$. Set $v(x) = u(\Phi_Q^{-1}(x))\sqrt{\rho(x)}$ be the corresponding solution of (5). We distinguish the following two cases:

(i) $\lambda \in \left[-\mu_1^*, \frac{3}{4}\right]$. Since $\lambda \le \frac{3}{4}$, by the symmetry result in [8] applied for positive solutions of (5), we deduce that v is radial. Therefore, (6) becomes:

$$\begin{aligned}
&\int_{B_{R^*}} \left\{r\psi''' + 3\psi'' + \frac{3-4\lambda}{2}\left[r(1+r^2)\psi' + (1-r^2)\psi\right]\rho^3(x)\right\}\frac{v^2}{2}\,dx \\
&\quad = R^*\psi(R^*)\int_{\partial B_{R^*}} \left|\frac{\partial v}{\partial r}\right|^2 d\mathcal{H}^2(x) - \frac{4}{3}\int_{B_{R^*}} r\psi' v^6 dx.
\end{aligned} \tag{28}$$

Set

$$w = \sqrt{4(1-\lambda)} \tag{29}$$

and we use the change of variable

$$\theta = 2\arctan r \quad \text{for} \quad r \in [0, R^*].$$

Remark that $w \ge 1$ and the assumption $\lambda \ge -\mu_1^*$ turns into $w\theta^* \le \pi$. Like in the proof of Theorem 2, we choose the smooth function

$$\psi(r) = \frac{\sin(w\theta)}{\sin\theta} > 0,\ \forall\theta \in (0, \theta^*).$$

Then

$$r\psi''' + 3\psi'' + \frac{3-4\lambda}{2}\big[r(1+r^2)\psi' + (1-r^2)\psi\big]\rho^3(r) = 0. \tag{30}$$

Moreover,

$$\psi'(r) = 2\frac{w\cos(w\theta)\sin\theta - \cos\theta\sin(w\theta)}{(1+\tan^2\frac{\theta}{2})\sin^2\theta} \le 0,\ \forall\theta\in(0,\theta^*).$$

Indeed, if we denote

$$F(\theta) = w\cos(w\theta)\sin\theta - \cos\theta\sin(w\theta),$$

then $F(0) = 0$ and $F'(\theta) = (4\lambda-3)\sin\theta\sin(w\theta) \le 0$ for every $\theta \in (0,\theta^*)$; therefore, we conclude that $F(\theta) \le 0$ on $(0,\theta^*)$. Using (28), we obtain a contradiction.

(ii) $\lambda > \frac{3}{4}$ and $\theta^* \in (0, \frac{\pi}{2}]$, i.e., $R^* \le 1$. If we take $\psi = 1$, (6) writes as:

$$\frac{3-4\lambda}{4}\int_{B_{R^*}}(1-r^2)\rho^3(x)v^2\,dx = R^*\int_{\partial B_{R^*}}\Big|\frac{\partial v}{\partial r}\Big|^2\,d\mathcal{H}^2(x).$$

Therefore, the nonexistence result also follows in this situation.

Using a similar argument as Bandle and Benguria [1], we prove the nonexistence of radial (nodal) solutions u below a curve $\lambda = \nu(\theta^*)$ in the region $(\lambda,\theta^*) \in (\frac{3}{4},\infty)\times(\frac{\pi}{2},\pi)$. We distinguish three cases:

(I) $\lambda = 1$. We consider $m(\theta) := a\theta - \theta^2$ and

$$\psi(r) = \frac{m(\theta)}{\sin\theta} \tag{31}$$

where $a \in (\pi/2,\pi)$ is to be chosen in such a way that

$$\psi(r) \ge 0 \quad \text{and} \quad \psi'(r) \le 0 \tag{32}$$

for every $0 < \theta < a$. This is equivalent with $G(\theta) > 0$, $\forall\theta \in (0,a)$ where

$$G(\theta) = \cos\theta m(\theta) - \sin\theta m'(\theta). \tag{33}$$

Since $G(0) = 0$, we ask that $G'(\theta) = \sin\theta(\theta^2 - a\theta + 2) \ge 0$ for every $\theta \in (0,a)$; for example, $a = 2\sqrt{2} = 2.828\ldots$. Bandle and Benguria [1] numerically found a better value $a = 3.042$. Since (30) holds, by (28), we get a contradiction. Therefore, for the largest a we set $\nu(a) = 1$.

(II) $\lambda \in (\frac{3}{4}, 1)$. Let $w \in (0,1)$ be given by (29). We consider

$$m(\theta) := \sin(w\theta) - a\cos(w\theta) + a$$

and ψ be defined as in (31) where a is to be chosen in such a way that (32) holds for the largest range of θ. Denote it by $\nu^{-1}(\lambda)$. Then (30) is satisfied and by (28), we deduce that no radial (nodal) solution of (4) exists if $\theta^* < \nu^{-1}(\lambda)$. Let us check that

$$\nu^{-1}(\lambda) > \frac{\pi}{2}.$$

For that, we take a negative a in the interval

$$a \in \left(\frac{\sin(w\pi/2)}{\cos(w\pi/2) - \frac{1}{1-w^2}}, -\frac{\cos(w\pi/2)}{\sin(w\pi/2)}\right). \tag{34}$$

Notice that a is well defined since $w^2 + \cos(w\pi/2) \le 1$ for every $w \in (0,1)$. We want to prove that (32) holds in $(0,\theta_a)$ for some $\theta_a > \pi/2$. We have that

$$m'(\theta) = w\big(\cos(w\theta) + a\sin(w\theta)\big) \quad \text{and} \quad m''(\theta) = w^2\big(a\cos(w\theta) - \sin(w\theta)\big).$$

Therefore, $m''(\theta) \leq 0$ for $\theta \in [0, \pi/2]$, i.e., m is concave. Since $m(0) = 0$ and $m(\pi/2) > 0$ (by (34)), we get that $m \geq 0$ in $[0, \theta_a)$ for $\theta_a > \pi/2$ and close to $\pi/2$. The same argument yields that

$$(m + m'')(\theta) = (1 - w^2)\big(\sin(w\theta) - a\cos(w\theta)\big) + a$$

is concave in $[0, \pi/2]$. The choice (34) leads to $(m + m'')(\pi/2) > 0$. Since $(m + m'')(0) = aw^2 < 0$, we deduce that $m + m''$ changes sign just once in $(0, \pi/2)$. Define G as in (33); then

$$G'(\theta) = -\sin\theta\,(m + m'')(\theta).$$

Hence, G' also changes sign once in $(0, \pi/2)$ and $G' \geq 0$ for θ close to 0. Since $G(0) = 0$ and $G(\pi/2) = -m'(\pi/2) > 0$ (by (34)), we obtain

$$G(\theta) \geq \min\{G(0), G(\pi/2)\} \geq 0, \, \forall\theta \in (0, \theta_a),$$

i.e., (32) is satisfied in $(0, \theta_a)$. Finally, we check that

$$\nu^{-1}(\lambda) \to \pi \quad \text{as} \quad \lambda \downarrow \frac{3}{4}.$$

Indeed, let $\varepsilon > 0$ be very small. We consider λ be close to $\frac{3}{4}$ such that $1 - w^2 = O(\varepsilon^2)$. Choose $a < 0$ with $|a| = O(\varepsilon)$. Then $m''(\theta) \geq 0$ in an interval $(0, \theta_\varepsilon)$ with $\theta_\varepsilon \to \pi$ as $\varepsilon \to 0$. Therefore m is concave in $(0, \theta_\varepsilon)$, and eventually by shrinking that interval, we can assume that $m(\theta_\varepsilon) > 0$, and thus, m is positive in $(0, \theta_\varepsilon)$. Now notice that $m + m'' < 0$ in $(0, \pi)$ and hence, G is positive in $(0, \pi)$. We conclude that (32) holds in the interval $(0, \theta_\varepsilon)$ that tends to $(0, \pi)$ as $\varepsilon \to 0$.

(III) $\lambda \in (1, \infty)$. Set $w = \sqrt{-4(1 - \lambda)}$. Consider

$$m(\theta) := \sinh(w\theta) - a\cosh(w\theta) + a$$

and ψ be as in (31) where a is to be chosen in such a way that (32) holds for the largest range of θ. Denote it by $\nu^{-1}(\lambda)$. The same argument as before gives that $\nu^{-1}(\lambda) > \pi/2$, i.e., ν is well-defined; for that, it suffices to choose a positive a in the interval

$$a \in \left(\frac{\cosh(w\pi/2)}{\sinh(w\pi/2)}, \frac{\sinh(w\pi/2)}{\cosh(w\pi/2) - \frac{1}{1+w^2}}\right).$$

Since (30) is satisfied, we conclude by (28) that no radial (nodal) solution of (4) exists if $\theta^* < \nu^{-1}(\lambda)$. □

Notice that for $\theta^* \in (\pi/2, \pi)$, we don't know in general if a positive solution of (4) is radial; moreover, for λ large enough, non-radial solutions do exist as announced by Bandle and Wei. As mentioned in Remark 2, it would be interesting to see if no solution of (4) exists below the curve ν in the strip $(\theta^*, \lambda) \subset \big(\frac{\pi}{2}, \pi\big) \times \big(\frac{3}{4}, \infty\big)$. We believe that the answer to this question is related to the open question raised in Remark 1.

5 The simplified proof of the Gidas-Spruck result

In the following we present the proof of Lemma 5:

Proof of Lemma 5. The relation (8) between w and u leads to

$$w_i = -\frac{2}{n-2}u^{-n/(n-2)}u_i \quad \text{and} \quad |dw|^2 = \frac{4}{(n-2)^2}u^{-2n/(n-2)}|du|^2. \tag{35}$$

By (7), w satisfies

$$-\frac{1}{n}w\Delta_g w+\frac{1}{2}|dw|^2=\frac{2}{n(n-2)}\left(\lambda w^2-w^{\frac{n-2}{2}\left(\frac{n+2}{n-2}-p\right)}\right) \tag{36}$$

which writes in terms of u as

$$\Delta_g w=\frac{2n}{(n-2)^2}u^{-2(n-1)/(n-2)}|du|^2+\frac{2}{n-2}\left(u^{p-\frac{n}{n-2}}-\lambda u^{-\frac{2}{n-2}}\right). \tag{37}$$

We will use the vector field defined in [9] that has the components

$$V_i=\frac{1}{w^{n-1}}\left(\frac{1}{2}\partial_i(|dw|^2)-\frac{1}{n}w_i\Delta_g w\right). \tag{38}$$

Using the equations (35) and (37), the expression of V_i in function of u writes as

$$\begin{aligned}V_i=\frac{2}{(n-2)^2}\Big[&\partial_i\big(u^{-2/(n-2)}|du|^2\big)+(n-2)\partial_i\big(u^{-2/(n-2)}\big)|du|^2\\&+\frac{2}{n}\big(u^{p-\frac{2}{n-2}}-\lambda u^{\frac{n-4}{n-2}}\big)u_i\Big].\end{aligned} \tag{39}$$

Notice that by (14) and (15), we have that

$$J(x)=\frac{1}{w^{n-1}}\left(\sum_{i,j=1}^n\nabla_i\tilde w_j\nabla_j\tilde w_i-\frac{1}{n}(\Delta_g w)^2\right).$$

Now we compute the co-differential of the vector $\tilde V=\frac{1}{\rho^2}V$:

$$\begin{aligned}\sum_{i=1}^n\nabla_i\tilde V_i&=\sum_{i=1}^n\nabla_i\left[\frac{1}{w^{n-1}}\big(\frac{1}{2\rho^2}\partial_i(|dw|^2)-\frac{1}{n}\tilde w_i\Delta_g w\big)\right]\\
&\overset{(15),(16)}{=}\frac{1}{w^{n-1}}\left[\sum_{i,j=1}^n\big(\tilde w_j\nabla_i\nabla_j\tilde w_i+\partial_i(\tilde w_j)\nabla_j\tilde w_i\big)-\frac{1}{n}(\Delta_g w)^2-\frac{1}{n}\sum_{i=1}^n\tilde w_i\partial_i(\Delta_g w)\right]\\
&\quad+\sum_{i=1}^n\partial_i\big(\frac{1}{w^{n-1}}\big)\left[\frac{1}{2\rho^2}\partial_i(|dw|^2)-\frac{1}{n}\tilde w_i\Delta_g w\right]\\
&\overset{(13),(17)}{=}J(x)+\frac{n-1}{w^{n-1}}\rho^2\sum_{j=1}^n\tilde w_j^2+\frac{n-1}{nw^{n-1}}\sum_{i=1}^n\tilde w_i\partial_i(\Delta_g w)\\
&\quad-\sum_{i=1}^n\frac{(n-1)w_i}{w^n}\left[\frac{1}{2\rho^2}\partial_i(|dw|^2)-\frac{1}{n}\tilde w_i\Delta_g w\right]\\
&\overset{(36)}{=}J(x)+\frac{n-1}{w^{n-1}}|dw|^2\\
&\quad+\frac{2(n-1)}{n(n-2)}\left[\big(\frac{n+2}{2}-p\frac{n-2}{2}\big)w^{-\frac{n}{2}-p\frac{n-2}{2}}-2\lambda w^{1-n}\right]|dw|^2.\end{aligned}$$

In terms of u, the co-differential of $\tilde V$ becomes

$$\begin{aligned}\sum_{i=1}^n\nabla_i\tilde V_i\overset{(35)}{=}&J(x)+\frac{4(n-1)}{(n-2)^2}u^{-2/(n-2)}|du|^2\\&+\frac{8(n-1)}{n(n-2)^3}\left[\big(\frac{n+2}{2}-p\frac{n-2}{2}\big)u^{p-\frac{n}{n-2}}-2\lambda u^{-2/(n-2)}\right]|du|^2.\end{aligned} \tag{40}$$

Now let $\gamma \in \mathbb{R}$. Integration by parts yields

$$0 = \int_{S^n} \sum_{i=1}^n \nabla_i (u^{-\gamma} \tilde{V})_i = \int_{S^n} u^{-\gamma} \sum_{i=1}^n \nabla_i \tilde{V}_i - \gamma \int_{S^n} u^{-\gamma-1} \sum_{i=1}^n u_i \tilde{V}_i. \tag{41}$$

By (39), integrating by parts, we deduce:

$$\begin{aligned} \frac{(n-2)^2}{2} \int_{S^n} u^{-\gamma-1} \sum_{i=1}^n u_i \tilde{V}_i &= - \int_{S^n} \sum_{i=1}^n \nabla_i (u^{-\gamma-1} \tilde{u}_i) u^{-2/(n-2)} |du|^2 - 2 \int_{S^n} u^{-\gamma-2(n-1)/(n-2)} |du|^4 \\ &\quad + \frac{2}{n} \int_{S^n} \big(u^{p-\gamma-\frac{n}{n-2}} - \lambda u^{-\gamma-\frac{2}{n-2}} \big) |du|^2 \\ &\overset{(7)}{=} (\gamma - 1) \int_{S^n} u^{-\gamma-2(n-1)/(n-2)} |du|^4 \\ &\quad + \frac{n+2}{n} \int_{S^n} \big(u^{p-\gamma-\frac{n}{n-2}} - \lambda u^{-\gamma-\frac{2}{n-2}} \big) |du|^2. \end{aligned}$$

The conclusion follows by (40) and (41). □

Corollary 6 is a trivial consequence of Lemma 5:

Proof of Corollary 6. Suppose that hypothesis *(i)* holds. Set

$$\gamma_0 = \frac{2(n-1)}{n+2} \left(\frac{n+2}{n-2} - p \right).$$

If $n \geq 8$, then $0 < \gamma_0 < 1$ and

$$\lambda \Big(\frac{4(n-1)}{n(n-2)} - \gamma_0 \frac{n+2}{2n} \Big) \leq n - 1 \tag{42}$$

provided that $\lambda \leq \frac{n}{p-1}$. Applying (9) for γ_0, we conclude that the second term in (9) must vanish, that means u is constant. If $3 \leq n < 8$, an easy computation shows that

$$a_n \geq \frac{n}{p-1} \quad \Leftrightarrow \quad p \geq \frac{n(n+2)}{2(n-1)(n-2)} \quad \Leftrightarrow \quad \gamma_0 \leq 1.$$

Therefore, if $a_n \geq \frac{n}{p-1}$, we choose $\gamma = \gamma_0$ in (9) and we deduce that the last term in (9) must be zero, i.e., u is constant provided that $\lambda < \frac{n}{p-1}$. Otherwise, (9) for $\gamma = 1$ also yields that the last term vanishes and the conclusion follows. Now suppose that *(ii)* holds. Then for $\gamma = 0$, (9) shows that the last term is zero, that is u must be constant provided that $\lambda < \frac{n}{p-1}$. □

Remark 3 When $\big\{ n \geq 8,\, p \in (1, \frac{n+2}{n-2}) \big\}$ and $\big\{ 3 \leq n < 8,\, p \in (\frac{n(n+2)}{2(n-1)(n-2)}, \frac{n+2}{n-2}) \big\}$, Corollary 6 is sharp.

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