Pattern formation in micromagnetics

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We present several non-local variational models leading to rich pattern formation. These models arise mainly in micromagnetics and we are interested in developing an asymptotic analysis based on an entropy method coming from scalar conservation laws.

1. The Aviles-Giga model. Let $\Omega \subset \mathbf{R}^2$ be an open domain. For vector fields $u \in H^1_{div}(\Omega, \mathbf{R}^2)$ of vanishing divergence $\nabla \cdot u = 0$ in Ω, the following energy functional is defined:

$$AG_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega (1 - |u|^2)^2 \, dx,$$

for a small parameter $\varepsilon > 0$. The question of $\Gamma-$convergence of $\{AG_\varepsilon\}_{\varepsilon \downarrow 0}$ was intensively studied. The compactness of configurations $\{u_\varepsilon\}_{\varepsilon \downarrow 0}$ of uniformly bounded energy $AG_\varepsilon(u_\varepsilon) \leq C$ was proved (in strong L^2-topology) by Ambrosio, De Lellis and Mantegazza [2] and DeSimone, Kohn, Müller and Otto [7]. The limiting configurations u_0 satisfy

$$|u_0| = 1 \quad \text{ and } \nabla \cdot u_0 = 0 \quad \text{ in } \Omega. \tag{1}$$

Moreover, De Lellis and Otto [6] proved the $\mathcal{H}^1-$rectifiability of the jump set J of u_0, even if u_0 is in general not BV (see [2]). It is expected that the limit energy of $\{AG_\varepsilon(u_\varepsilon)\}_{\varepsilon \downarrow 0}$ concentrates on the jump set J and has the following form (first stated by Aviles and Giga [3]):

$$AG_0(u_0) = \frac{1}{3} \int_J |u_0^+(x) - u_0^-(x)|^3 \, d\mathcal{H}^1.$$

In fact, AG_0 is a lower-bound of $\{AG_\varepsilon\}_{\varepsilon \downarrow 0}$ (see Aviles and Giga [4], Jin and Kohn [11]). The difficulty consists in the upper bound construction for limiting configurations u_0: recovery sequences have been constructed *only* for BV configurations u_0 (see Conti and De Lellis [5] and Poliakovsky [12]).

Entropies. One of the main tool of this study consists in the concept of entropies coming from the scalar conservation law hidden in (1). Indeed, writing $u_0 = (v, h(v))$ for the flux $h(v) = \pm\sqrt{1 - v^2}$, the divergence-free condition in u_0 turns into the nonlinear transport equation:

$$\partial_t v + \partial_s [h(v)] = 0, \tag{2}$$

where $(t, s) := (x_1, x_2)$ correspond to (time, space) variables. The notion of entropy solution is introduced via the pair (entropy, entropy-flux), i.e., a couple of scalar functions (η, q) such that $\frac{dq}{dv} = \frac{dh}{dv}\frac{d\eta}{dv}$ which entails that every smooth solution v of (2) has vanishing entropy production, i.e.,

$$\partial_t [\eta(v)] + \partial_s [q(v)] = 0. \tag{3}$$

More general, an entropy solution v has the property that for every pair (η, q), the entropy production is a (signed) measure that concentrates on lines (corresponding to "shocks" of v). It suggests the interest of using "global" quantities $\Phi(u_0) :=$

$(\eta(v), q(v))$ to detect "local" line-singularities of u_0. Indeed, we will say that $\Phi \in C^\infty(\mathbf{R}^2, \mathbf{R}^2)$ is a $DKMO-$entropy (see [7]) if

$$\Phi(0) = 0,\ D\Phi(0) = 0 \text{ and } \ z \cdot D\Phi(z) z^\perp = 0 \ \text{ holds for all } \ z \in \mathbf{R}^2.$$

In particular, if u_0 is a smooth vector field satisfying (1), then $\nabla \cdot [\Phi(u_0)] = 0$ (similarly to (3)). More general, the family of entropy productions $\{\nabla \cdot [\Phi(u_\varepsilon)]\}_{\varepsilon \downarrow 0}$ is asymptotically bounded as measure for every family $\{u_\varepsilon\}_{\varepsilon \downarrow 0} \subset H^1_{div}(\Omega, \mathbf{R}^2)$ of uniformly bounded energy: there exists a constant $C_\Phi > 0$ such that

$$\limsup_{\varepsilon \to 0} \left| \int_\Omega \nabla \cdot [\Phi(u_\varepsilon)] \zeta \, dx \right| \le C_\Phi \|\zeta\|_\infty \limsup_{\varepsilon \to 0} AG_\varepsilon(u_\varepsilon), \quad \text{for every } \ \zeta \in C_c^\infty(\Omega).$$

This is the starting point in proving the L^2-compactness result and the fine structure of the limiting configurations u_0 (see [7, 6]).

2. The Bloch wall model. Let us now discuss a rather more "geometric" and non-convex model coming from micromagnetics: For S^2-valued vector fields $m = (u, m_3) \in H^1_{div}(\Omega, S^2)$ with $\nabla \cdot u = 0$ in $\Omega \subset \mathbf{R}^2$, we define the functional:

$$E_\varepsilon(m) = \varepsilon \int_\Omega |\nabla m|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega m_3^2 \, dx,$$

for a small parameter $\varepsilon > 0$. As before, the aim is to analyze the asymptotic behavior of E_ε as $\varepsilon \to 0$. First, note that E_ε dominates the Aviles-Giga energy AG_ε, i.e., $AG_\varepsilon(u) \le E_\varepsilon(m)$, since $|\nabla u| \le |\nabla m|$ and $(1 - |u|^2)^2 = m_3^4 \le m_3^2$. Therefore, the L^2-strong compactness holds for uniformly bounded energy configurations $E_\varepsilon(m_\varepsilon) \le C$; the limiting configurations m_0 are in-plane, i.e., $m_0 = (u_0, 0)$ with (1) and a $\mathcal{H}^1-$rectifiable jump set J of u_0 can be defined. It is conjectured that the transition layers (at level $\varepsilon > 0$) corresponding to a jump (u_0^-, u_0^+) are one-dimensional and that the Γ-limit of $\{E_\varepsilon\}_{\varepsilon \downarrow 0}$ is given by

$$E_0(m_0) = \int_J |u_0^+(x) - u_0^-(x)|^2 \, d\mathcal{H}^1.$$

In a joint work with Merlet (see [9]), we obtained several partial results. In order to deal with the expected quadratic cost of jumps, we analyze the following class of Lipschitz entropies: $\Phi \in Lip(S^2, \mathbf{R}^2)$ such that for $\varepsilon \downarrow 0$,

$$\nabla \cdot [\Phi(m)] \le \varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} m_3^2 + o(1) \quad \text{in } \Omega, \quad \forall m \in C^\infty_{div}(\Omega, S^2) \tag{4}$$

with the condition that $[\Phi(u_0^+) - \Phi(u_0^-)] \cdot \nu = |u_0^+ - u_0^-|^2$ for jumps (u_0^-, u_0^+) of normal direction $\nu := \mathbf{e}_1$. We find such an entropy for the biggest jump $(0, \pm 1, 0)$ proving that the one-dimensional layer is optimal in this case. Even if we find entropies for each jump (u_0^-, u_0^+) satisfying (4) but in a restricted class of configurations m, we prove that the entropy method doesn't work in general for small angles. However, we show in a second paper [8] that E_0 is lower semicontinuous (in L^2 topology), enforcing the expectation that no microstructure appears for the Bloch wall model.

3. A zigzag wall model. We study now the following energy functional:

$$F_\epsilon(m) = \int_\Omega \left(\varepsilon|\nabla m|^2 + \frac{1}{\varepsilon}m_2^2\right)\,dx + \frac{1}{\varepsilon^s}\|\nabla\cdot m\|^2_{\dot H^{-1}(\Omega)}$$

for $m = (m_1, m_2, m_3) \in H^1(\Omega, S^2)$ where the constraint $\nabla\cdot(m_1, m_2) \neq 0$ is penalized in $\dot H^{-1}$-seminorm by the energy where $s \in (1, 2)$. The penalization of m_2 (instead of m_3 as previously) generates loss of coercivity of F_ε: configurations of uniformly bounded energy are in general no longer compact in strong L^2-topology due to possible oscillations in x_2-direction. The main idea of a joint work with Moser [10] is to study the quantity

$$\psi = \sin\vartheta - \vartheta\cos\vartheta,$$

where $\vartheta := \arctan\frac{m_3}{m_1}$ in the hemisphere where $|\vartheta| \le \frac{\pi}{2}$. We show that as long as ϑ remains sufficiently small, the functional

$$F_0(\psi) = 2\int_\Omega \left|\frac{\partial\psi}{\partial x_1}\right|\,dx$$

is the Γ-limit energy of $\{F_\varepsilon\}_{\varepsilon\downarrow 0}$. In general, the wall energy given by F_0 is *not* achieved by a one-dimensional transition between two limiting states $m^\pm = (\cos\theta, 0, \pm\sin\theta)$ of normal direction $\nu := \mathbf{e}_1$. Instead, in order to obtain the optimal limiting energy given by F_0, a transition with an additional zigzag structure is required. The matching with the upper bound (coming from the zigzag wall construction) is fulfilled via a lower bound based on generalized entropies. More precisely, as in (4), we study the entropies $\Phi \in Lip(S^2, \mathbf{R}^2)$ such that for $\varepsilon \downarrow 0$,

$$\nabla\cdot[\Phi(m)] \le \varepsilon|\nabla m|^2 + \frac{1}{\varepsilon}m_2^2 \quad \text{in } \Omega, \quad \forall m \in C^\infty_{div}(\Omega, S^2)$$

with the condition that $[\Phi(m^+) - \Phi(m^-)]\cdot\nu = 4\psi(\theta)$ for jumps $m^\pm = (\cos\theta, 0, \pm\sin\theta)$ of normal direction $\nu := \mathbf{e}_1$. In contrast with the Bloch wall model, we succeed to find such entropies for small angles θ and we prove that no entropy exists for the biggest jump $(0, 0, \pm 1)$. There is another situation where the Γ-limit is explicitly known for a problem involving similar microstructures: the problem leading to cross-tie walls in thin ferromagnetic films [13, 14, 1]. The cross-tie wall consists in a mixture of vortices and Néel walls (one-dimensional transition layers similar to Bloch walls, but taking values only in S^1). Remarkably, the function $\psi(\theta) = \sin\theta - \theta\cos\theta$ plays an important role in that context as well, although this may be a mere coincidence.

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