A necessary condition in a De Giorgi type conjecture for elliptic systems in infinite strips

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May 28, 2019

Dedicated to Haïm Brezis on his seventy-fifth anniversary with esteem

Abstract

Given a bounded Lipschitz domain $\omega \subset \mathbb{R}^{d-1}$ and a lower semicontinuous function $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ that vanishes on a finite set and that is bounded from below by a positive constant at infinity, we show that every map $u : \mathbb{R} \times \omega \to \mathbb{R}^N$ with

$$\int_{\mathbb{R}\times\omega} \left(|\nabla u|^2 + W(u)\right) \mathrm{d}x_1 \, \mathrm{d}x' < +\infty$$

has a limit $u^{\pm} \in \{W = 0\}$ as $x_1 \to \pm\infty$. The convergence holds in $L^2(\omega)$ and almost everywhere in ω. We also prove a similar result for more general potentials W in the case where the considered maps u are divergence-free in Ω with ω being the $(d-1)$-torus and $N = d$.

Keywords. Nonlinear elliptic PDEs; De Giorgi conjecture; Energy estimates; Geodesic distance.

1 Introduction

Let $N \geq 1$, $d \geq 2$ and $\Omega = \mathbb{R} \times \omega$ be an infinite cylinder in $\mathbb{R}^d$, where $\omega \subset \mathbb{R}^{d-1}$ is an open connected bounded set with Lipschitz boundary. For a lower semicontinuous potential $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$, we consider the functional

$$E(u) = \int_\Omega \Big(|\nabla u|^2 + W(u)\Big) \, \mathrm{d}x, \quad u \in \dot{H}^1(\Omega, \mathbb{R}^N), \tag{1.1}$$

where $|\cdot|$ is the Euclidean norm and

$$\dot{H}^1(\Omega, \mathbb{R}^N) = \left\{u \in H^1_{loc}(\Omega, \mathbb{R}^N) \ : \ \nabla u = (\partial_j u_i)_{1\leq i \leq N, 1 \leq j \leq d} \in L^2(\Omega, \mathbb{R}^{N\times d})\right\}.$$

A natural problem consists in studying optimal transition layers for the functional E between two wells $u^{\pm}$ of W (i.e., $W(u^{\pm}) = 0$). In particular, motivated by the De Giorgi conjecture, one aim

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is to analyse under which conditions on the potential W and on the dimensions d and N, every minimizer u of E connecting $u^\pm$ as $x_1 \to \pm\infty$ is one-dimensional, i.e., depending only on x_1. Obviously, such one-dimensional transition layers u coincide with their x'-average $\overline{u} : \mathbb{R} \to \mathbb{R}^N$ defined as

$$\overline{u}(x_1) := \fint_\omega u(x_1, x')\,\mathrm{d}x', \quad x_1 \in \mathbb{R}, \tag{1.2}$$

where $x' = (x_2, \ldots, x_d)$ denotes the $d-1$ variables in ω and the x'-average symbol is denoted by $\fint_\omega = \frac{1}{|\omega|}\int_\omega$.

1.1 Main results

The purpose of this note is to prove a necessary condition for finite energy configurations u provided that W satisfies the following two conditions:

(H1) W has a finite number of wells, i.e., $\mathrm{card}(\{z \in \mathbb{R}^N : W(z) = 0\}) < \infty$;

(H2) $\liminf_{|z|\to\infty} W(z) > 0$.

More precisely, we prove that under these assumptions, there exist two wells $u^\pm$ of W such that $u(x_1, \cdot)$ converges to $u^\pm$ in L^2 and a.e. in ω as $x_1 \to \pm\infty$; in particular, the x'-average $\overline{u}$ (as a continuous map in $\mathbb{R}$) admits the limits $\overline{u}(\pm\infty) = u^\pm$ as $x_1 \to \pm\infty$. Here, $u(x_1, \cdot)$ stands for the trace of the Sobolev map $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$ on the section $\{x_1\} \times \omega$ for every $x_1 \in \mathbb{R}$.

Theorem 1. *Let $\Omega = \mathbb{R} \times \omega$, where $\omega \subset \mathbb{R}^{d-1}$ is an open connected bounded set with Lipschitz boundary. If $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semicontinuous potential satisfying* **(H1)** *and* **(H2)***, then every $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$ with $E(u) < \infty$ connects two wells*[1] *$u^\pm \in \mathbb{R}^N$ of W at $x_1 = \pm\infty$ (i.e., $W(u^\pm) = 0$) in the sense that*

$$\lim_{x_1\to\pm\infty} \|u(x_1, \cdot) - u^\pm\|_{L^2(\omega,\mathbb{R}^N)} = 0 \quad \textit{and} \quad \lim_{x_1\to\pm\infty} u(x_1, \cdot) = u^\pm \quad \textit{a.e. in } \omega. \tag{1.3}$$

In particular,

$$\lim_{x_1\to\pm\infty} \fint_\omega u(x_1, x')\ \mathrm{d}x' = u^\pm.$$

Remark 2. i) As a consequence of the Poincaré-Wirtinger inequality[2], for $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$ with $\bar{u}(\pm\infty) = u^\pm$, there exist two sequences $(R_n^+)_{n\in\mathbb{N}}$ and $(R_n^-)_{n\in\mathbb{N}}$ such that $(R_n^\pm)_{n\in\mathbb{N}} \to \pm\infty$ and

$$\|u(R_n^\pm, \cdot) - u^\pm\|_{H^1(\omega,\mathbb{R}^N)} \underset{n\to\infty}{\longrightarrow} 0 \tag{1.4}$$

(see [24, Lemma 3.2]).

ii) Theorem 1 also holds true if ω is a closed (i.e., compact, connected without boundary) Riemannian manifold.

iii) Theorem 1 also applies for maps u taking values into a closed set $\mathcal{N} \subset \mathbb{R}^N$ (e.g., $\mathcal{N}$ could be a compact manifold embedded in $\mathbb{R}^N$). More precisely, if the potential $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies **(H1)**, **(H2)** and $\mathcal{N} := \{z \in \mathbb{R}^N : W(z) < +\infty\}$ is a closed set such that $W_{|\mathcal{N}} : \mathcal{N} \to \mathbb{R}_+$ is lower semicontinuous, then Theorem 1 handles the case where the nonlinear constraint $u \in \mathcal{N}$ is present.

[1] u^- and u^+ could be equal.

[2] The assumption that ω is connected with Lipschitz boundary is needed for the Poincaré-Wirtinger inequality.

The result in Theorem 1 extends to slightly more general potentials W in the following context of divergence-free maps. For that, let $d = N$ and $\Omega = \mathbb{R} \times \omega$ with $\omega = \mathbb{T}^{d-1}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ being the flat torus. We consider maps $u \in H^1_{loc}(\Omega, \mathbb{R}^d)$ periodic in $x' \in \omega$ and divergence-free, i.e.,

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega.$$

Then the x'-average $\bar{u} : \mathbb{R} \to \mathbb{R}^d$ is continuous and its first component is constant, i.e., there is $a \in \mathbb{R}$ such that

$$\bar{u}_1(x_1) = a \quad \text{for every} \quad x_1 \in \mathbb{R}$$

(see [24, Lemma 3.1]). For such maps u, we consider potentials W satisfying the following two conditions:

$\mathbf{(H1)}_a$ $W(a, \cdot)$ has a finite number of wells, i.e., $\mathrm{card}(\{z' \in \mathbb{R}^{d-1} : W(a, z') = 0\}) < \infty$;

$\mathbf{(H2)}_a$ $\liminf_{z_1 \to a,\, |z'| \to \infty} W(z_1, z') > 0.$

In this context, we have proved in our previous paper [24] that the x'-average map $\bar{u}$ admits limits $u^\pm$ as $x_1 \to \pm\infty$, where $u_1^\pm = a$ and they are two wells of $W(a, \cdot)$, see [24, Lemma 3.7]. As in Theorem 1, we will prove that $u(x_1, \cdot)$ converges to $u^\pm$ in L^2 and a.e. in ω as $x_1 \to \pm\infty$.

Theorem 3. *Let $\Omega = \mathbb{R} \times \omega$ with $\omega = \mathbb{T}^{d-1}$ the $(d-1)$-dimensional torus and $u \in H^1_{loc}(\Omega, \mathbb{R}^d)$ such that $E(u) < \infty$ and $\bar{u}_1 = a$ in $\mathbb{R}$ for some $a \in \mathbb{R}$. If $W : \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semicontinuous potential satisfying* $\mathbf{(H1)}_a$ *and* $\mathbf{(H2)}_a$*, then there exist two wells $u^\pm \in \mathbb{R}^d$ of W such that* (1.3) *holds true and $u_1^\pm = a$. In particular, $\bar{u}(\pm\infty) = u^\pm$.*

Note that we don't assume that u is divergence-free in Theorem 3, only the assumption that $\bar{u}_1$ is constant.

1.2 Motivation

Our main result is motivated by the well-known De Giorgi conjecture that consists in investigating the one-dimensional symmetry of critical points of the functional E, i.e., solutions $u : \Omega \to \mathbb{R}^N$ to the nonlinear elliptic system

$$\begin{cases} \Delta u = \frac{1}{2} \nabla W(u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega = \mathbb{R} \times \partial\omega, \end{cases} \tag{1.5}$$

where W is assumed to be locally Lipschitz in (1.5) and ν is the unit outer normal vector field at $\partial\omega$. Theorem 1 states in particular that solutions u of finite energy satisfy the boundary condition (1.3) for two wells $u^\pm$ of W. A natural question related to the De Giorgi conjecture arises in this context:

Question: Under which assumptions on the potential W and the dimensions d and N, is it true that every global minimizer u of E connecting two wells[3] of W is one-dimensional symmetric, i.e., $u = u(x_1)$?

Link with the Gibbons and De Giorgi conjectures. i) In the scalar case $N = 1$ (d is arbitrary) and $W(u) = \frac{1}{2}(1 - u^2)^2$, the answer to the above question is positive provided that the limits (1.3) are replaced by uniform convergence (see [12, 17]); within these uniform boundary conditions, the problem is called the Gibbons conjecture. We mention that many articles have been written on

[3] We say that u connects two wells $u^\pm$ of W if (1.3) is satisfied.

Gibbons' conjecture in the case of the entire space $\Omega = \mathbb{R}^d$: more precisely, if a solution[4] $u : \mathbb{R}^d \to \mathbb{R}$ of the PDE

$$\Delta u = \frac{1}{2}\frac{dW}{du}(u) \quad \text{in} \quad \mathbb{R}^d \tag{1.6}$$

satisfies the convergence $\lim_{x_1\to\pm\infty} u(x_1, x') = \pm 1$ uniformly in $x' \in \mathbb{R}^{d-1}$ and $|u| \le 1$ in $\mathbb{R}^d$, then u is one-dimensional (see [5, 6, 11, 18]).

Let us now speak about the long standing De Giorgi conjecture in the scalar case $N = 1$. It predicts that any bounded solution u of (1.6) that is monotone in the x_1 variable is one-dimensional in dimension $d \le 8$, i.e., the level sets $\{u = \lambda\}$ of u are hyperplanes. The conjecture has been solved in dimension $d = 2$ by Ghoussoub-Gui [21], using a Liouville-type theorem and monotonicity formulas. Using similar techniques, Ambrosio-Cabré [4] extended these results to dimension $d = 3$, while Ghoussoub-Gui [22] showed that the conjecture is true for $d = 4$ and $d = 5$ under some antisymmetry condition on u. The conjecture was finally proved by Savin [31] in dimension $d \le 8$ under the additional condition $\lim_{x_1\to\pm\infty} u(x_1, x') = \pm 1$ pointwise in $x' \in \mathbb{R}^{d-1}$, the proof being based on fine regularity results on the level sets of u. Lately, Del Pino-Kowalczyk-Wei [13] gave a counterexample to the De Giorgi conjecture in dimension $d \ge 9$, which satisfies the pointwise limit conditions $\lim_{x_1\to\pm\infty} u(x_1, x') = \pm 1$ for a.e. $x' \in \mathbb{R}^{d-1}$. It would be interesting to investigate whether these results transfer (or not) to the context of the strip $\Omega = \mathbb{R} \times \omega$ as stated in Question. Theorem 1 proves that the pointwise convergence as $x_1 \to \pm\infty$ is a necessary condition in the context of a strip $\mathbb{R} \times \omega$ and for finite energy configurations.

ii) Less results are available for the vector-valued case $N \ge 2$. In the case $\Omega = \mathbb{R}^d$, $N = 2$ and $W(u_1, u_2) = \frac{1}{2}(u_1^2 - 1)^2 + \frac{1}{2}(u_2^2 - 1)^2 + \Lambda u_1^2 u_2^2 - \frac{1}{2}$ with $\Lambda \ge 1$ (so $W \ge 0$ and W has exactly four wells $\{(0, \pm 1), (\pm 1, 0)\}$, thus, **(H1)** and **(H2)** are satisfied), the Gibbons and De Giorgi conjectures corresponding to the system (1.5) are discussed in [19]. Several other phase separation models (e.g., arising in a binary mixture of Bose-Einstein condensates) are studied in the vectorial case where W has a non-discrete set of zeros (see e.g., [7, 8, 20]).

We recall that in the study of the De Giorgi conjecture for (1.6), i.e., $N = 1$, there is a link between monotonicity of solutions (e.g., the condition $\partial_1 u > 0$), stability (i.e., the second variation of the corresponding energy at u is nonnegative), and local minimality of u (in the sense that the energy does not decrease under compactly supported perturbations of u). We refer to [2, Section 4] for a fine study of these properties. In particular, it is shown that the monotonicity condition in the De Giorgi conjecture implies that u is a local minimizer of the energy (see [2, Theorem 4.4]). Therefore, it is natural to study Question under the monotonicity condition in x_1 (instead of the global minimality condition on u).

Link with micromagnetic models. We have studied Question in the context of divergence-free maps $u : \mathbb{R} \times \omega \to \mathbb{R}^N$ where $d = N$ and $\omega = \mathbb{T}^{d-1}$ is the $(d-1)$-dimensional torus, see [24]. By developing a theory of calibrations, we have succeeded to give sufficient conditions on the potential W in order that the answer to Question is positive, in particular in the case where $\mathbf{(H1)}_a$ and $\mathbf{(H2)}_a$ are satisfied, see [24, Theorem 2.11]. In that context, Question is related to some reduced model in micromagnetics in the regime where the so-called stray-field energy is strongly penalized favoring the divergence constraint $\nabla \cdot u = 0$ of the magnetization u (the unit-length constraint on u being relaxed in the system). In the theory of micromagnetics, a challenging question concerns the symmetry of domain walls. Indeed, much effort has been devoted lately to identifying on the one hand, the domain walls that have one-dimensional symmetry, such as the so-called symmetric Néel and symmetric Bloch walls (see e.g. [14, 26, 23]), and on the other hand, the domain walls involving microstructures, such as the so-called cross-tie walls (see e.g., [3, 30]), the zigzag walls

[4] Here, u needs not be a global minimizer of E within the boundary condition (1.3), nor monotone in x_1, i.e., $\partial_1 u > 0$. Obviously, this result applies also to global minimizers, as $|u| \le 1$ in $\mathbb{R}^d$ by the maximum principle.

(see e.g., [25, 29]) or the asymmetric Néel / Bloch walls (see e.g. [16, 15]). Thus, answering to Question would give a general approach in identifying the anisotropy potentials W for which the domain walls are one-dimensional in the elliptic system (1.5).

Link with heteroclinic connections. One dimensional [5] solutions $u = u(x_1)$ of the system (1.5) are called heteroclinic connections. Given two wells $u^\pm$ of a potential W satisfying **(H1)** and **(H2)**, it is known that there exists a heteroclinic connection $\gamma : \mathbb{R} \to \mathbb{R}^N$ obtained by minimizing $\int_{\mathbb{R}} |\frac{d}{dx_1}\gamma|^2 + W(\gamma)\, dx_1$ under the condition $\gamma(\pm\infty) = u^\pm$ (see [27, 33, 34]). In the vectorial case $N \geq 2$, this connection may not be unique in the sense that there could exist two (minimizing) heteroclinic connections γ_1, γ_2 such that $\gamma_i(\pm\infty) = u^\pm$ for $i = 1, 2$ but $\gamma_1(\cdot)$ and $\gamma_2(\cdot - \tau)$ are distinct for every $\tau \in \mathbb{R}$. If this is the case, at least in dimension $d = 2$ and $\Omega = \mathbb{R}^2$, there also exists a solution u to $\Delta u = \frac{1}{2}\nabla W(u)$ which realizes an interpolation between γ_1 and γ_2 in the following sense (see [32, 1, 28]):

$$\begin{cases} u(x_1, x_2) \to u^\pm & \text{as } x_1 \to \pm\infty \text{ uniformly in } x_2, \\ u(x_1, x_2) \to \gamma_1(x_1) & \text{as } x_2 \to -\infty \text{ uniformly in } x_1, \\ u(x_1, x_2) \to \gamma_2(x_1) & \text{as } x_2 \to +\infty \text{ uniformly in } x_1. \end{cases}$$

Moreover, this solution is energy local minimizing, i.e., the energy cannot decrease by compactly supported perturbations of u. Solutions to the system $\Delta u = \frac{1}{2}\nabla W(u)$ naturally arise when looking at the local behavior of a transition layer near a point at the interface between two wells $u^\pm$; solutions satisfying the preceding boundary conditions correspond to the case of an interface point where the 1D connection passes from γ_1 to γ_2. The existence of such stable entire solutions to the Allen-Cahn system makes a significative difference with the scalar case, i.e. $N = 1$, where only 1D solutions are present by the De Giorgi conjecture.

2 Pointwise convergence and convergence of the x'-average

In this section we prove that under the assumptions in Theorem 1, the x'-average $\overline{u}$ (as a continuous map in $\mathbb{R}$) has limits $\overline{u}(\pm\infty) = u^\pm$ as $x_1 \to \pm\infty$ corresponding to two wells of W. For that, we will follow the strategy that we developed in our previous paper (see [24, Section 3.1]). The idea consists in introducing an "averaged" potential V in $\mathbb{R}^N$ with $W \geq V \geq 0$ and $\{V = 0\} = \{W = 0\}$ (see Lemma 4), and a new functional E_V associated to the x'-average $\overline{u}$ of a map u such that $\frac{1}{|\omega|}E(u) \geq E_V(\bar{u})$. This can be seen as a dimension reduction technique since the new map $\bar{u}$ has only one variable. We will prove that every transition layer $\bar{u}$ connecting two wells $u^\pm$ has the energy $E_V(\bar{u})$ bounded from below by the geodesic pseudo-distance geod_V between the wells $u^\pm$ (see Lemma 6). As the Euclidean distance in $\mathbb{R}^N$ is absolutely continuous with respect to geod_V (see Lemma 5), we will conclude that $\bar{u}$ admits limits at $\pm\infty$ given by two wells of W (see Lemma 7). Note that in Section 3, we will give a second proof of the claim $\bar{u}(\pm\infty) = u^\pm$ without using the geodesic pseudo-distance geod_V.

We first introduce the energy functional E (defined in (1.1)) restricted to appropriate subsets $A \subset \Omega$ (e.g., A can be a subset of the form $I \times \omega$ for an interval $I \subset \mathbb{R}$, or a section $\{x_1\} \times \omega$): for every map $u \in \dot{H}^1(A, \mathbb{R}^N)$, we set

$$E(u, A) := \int_A |\nabla u|^2 + W(u)\, \mathrm{d}x,$$

so that for $A = \Omega$, we have $E(u) = E(u, A)$. For any interval $I \subset \mathbb{R}$, the Jensen inequality yields

$$E(u, I \times \omega) = \int_I \int_\omega \big(|\partial_1 u|^2 + |\nabla' u|^2 + W(u)\big)\, \mathrm{d}x'\, \mathrm{d}x_1 \geq |\omega| \int_I \Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|^2 + e(u(x_1, \cdot))\, \mathrm{d}x_1,$$

[5] If $u = u(x_1)$, the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ is automatically satisfied.

where $\nabla' = (\partial_2, \ldots, \partial_d)$, $\bar{u}$ is the x'-average of u given in (1.2) and the x'-average energy e is defined by

$$e(v) := \fint_\omega \big(|\nabla' v|^2 + W(v)\big)\,\mathrm{d}x' \quad \text{for all } v \in H^1(\omega, \mathbb{R}^N).$$

Introducing the averaged potential $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ defined for all $z \in \mathbb{R}^N$ by

$$V(z) := \inf\left\{ e(v) \ : \ v \in H^1(\omega, \mathbb{R}^N),\ \fint_\omega v\,\mathrm{d}x' = z \right\} \geq 0, \tag{2.1}$$

we have

$$E(u, I\times\omega) \geq |\omega| \int_I \left(\left|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\right|^2 + V(\overline{u}(x_1)) \right) \mathrm{d}x_1. \tag{2.2}$$

This observation is the starting point in the proof of the following lemma:

Lemma 4. *Let $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function satisfying* **(H2)**. *Then the averaged potential $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ defined in* (2.1) *satisfies the following:*

1. *V is lower semicontinuous in $\mathbb{R}^N$,*
2. *for all $z \in \mathbb{R}^N$, $V(z) \leq W(z)$, the infimum in* (2.1) *is achieved and*[6] $\Big[V(z) = 0 \Leftrightarrow W(z) = 0\Big]$,
3. $V_\infty := \liminf_{|z|\to\infty} V(z) > 0$,
4. *for every interval $I \subset \mathbb{R}$ and for every $u \in \dot{H}^1(I\times\omega, \mathbb{R}^N)$, one has*

$$\frac{1}{|\omega|} E(u, I\times\omega) \geq E_V(\overline{u}, I), \quad E_V(\overline{u}, I) := \int_I \left|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\right|^2 + V(\overline{u}(x_1))\,\mathrm{d}x_1.$$

The new energy $E_V(\bar{u}) := E_V(\overline{u}, \mathbb{R})$ associated to the x'-average $\bar{u}$ will play an important role for proving the existence of the two limits $\bar{u}(\pm\infty)$.

Proof of Lemma 4. The claim *4* follows from (2.2). We divide the rest of the proof in three steps.

Step 1: proof of claim *2*. Clearly, for all $z \in \mathbb{R}^N$, one has $V(z) \leq e(z) = W(z)$. By the compact embedding $H^1(\omega) \hookrightarrow L^1(\omega)$, the lower semicontinuity of W, Fatou's lemma and the lower semicontinuity of the L^2 norm in the weak L^2-topology (see [9]), we deduce that e is lower semicontinuous in the weak $H^1(\omega, \mathbb{R}^N)$-topology. Then the direct method in the calculus of variations implies that the infimum is achieved in (2.1) (infimum that could be equal to $+\infty$ as W can take the value $+\infty$).

If $W(z) = 0$, then $V(z) = 0$ (as $0 \leq V \leq W$ in $\mathbb{R}^N$). Conversely, if $V(z) = 0$ with $z \in \mathbb{R}^N$, then a minimizer $v \in H^1(\omega, \mathbb{R}^N)$ in (2.1) satisfies $V(z) = e(v) = 0$ so that $v \equiv z$ and $W(z) = 0$.

Step 2: V is lower semicontinuous in $\mathbb{R}^N$. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence converging to z in $\mathbb{R}^N$. We need to show that

$$V(z) \leq \liminf_{n\to\infty} V(z_n).$$

Without loss of generality, one can assume that $(V(z_n))_{n\in\mathbb{N}}$ is a bounded sequence that converges to $\liminf_{n\to\infty} V(z_n)$. By Step 1, for each $n \in \mathbb{N}$, there exists $v_n \in H^1(\omega, \mathbb{R}^N)$ such that

$$\fint_\omega v_n\,\mathrm{d}x' = z_n \quad \text{and} \quad e(v_n) = V(z_n).$$

[6] In particular, if W satisfies **(H1)**, then V satisfies **(H1)**, too.

Since $(z_n)_{n\in\mathbb{N}}$ and $(e(v_n))_{n\in\mathbb{N}}$ are bounded, we deduce that $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\omega,\mathbb{R}^N)$ by the Poincaré-Wirtinger inequality. Thus, up to extraction, one can assume that $(v_n)_{n\in\mathbb{N}}$ converges weakly in H^1, strongly in L^1 and a.e. in ω to a limit $v\in H^1(\omega,\mathbb{R}^N)$. In particular, $⨍_\omega v\,\mathrm{d}x' = z$. Since e is lower semicontinuous in weak $H^1(\omega,\mathbb{R}^N)$-topology (by Step 1), we conclude

$$V(z)\le e(v)\le \liminf_{n\to\infty} e(v_n) = \liminf_{n\to\infty} V(z_n).$$

STEP 3: PROOF OF CLAIM *3*. Assume by contradiction that there exists a sequence $(z_n)_{n\in\mathbb{N}}\subset\mathbb{R}^N$ such that $|z_n|\to\infty$ and $V(z_n)\to 0$ as $n\to\infty$. Then, there exists a sequence of maps $(w_n)_{n\in\mathbb{N}}$ in $H^1(\omega,\mathbb{R}^N)$ satisfying

$$\int_\omega w_n(x')\,\mathrm{d}x' = 0 \quad \text{for each } n\in\mathbb{N} \quad \text{and} \quad e(z_n+w_n)\underset{n\to\infty}{\longrightarrow} 0.$$

By the Poincaré-Wirtinger inequality, we have that $(w_n)_{n\in\mathbb{N}}$ is bounded in H^1. Thus, up to extraction, one can assume that it converges weakly in H^1, strongly in L^1 and a.e. to a map $w\in H^1(\omega,\mathbb{R}^N)$. We claim that w is constant since

$$⨍_\omega |\nabla' w|^2\,\mathrm{d}x' \le \liminf_{n\to\infty} ⨍_\omega |\nabla' w_n|^2\,\mathrm{d}x' \le \liminf_{n\to\infty} e(z_n+w_n) = 0.$$

We deduce $w\equiv 0$ since $\int_\omega w = \lim_{n\to\infty}\int_\omega w_n = 0$. Thus $w_n\to 0$ a.e and **(H2)** implies that for a.e. $x'\in\omega$,

$$\liminf_{n\to\infty} W(z_n+w_n(x')) \ge \liminf_{|z|\to\infty} W(z) > 0,$$

which contradicts the fact that $e(z_n+w_n)\to 0$. □

For every lower semicontinuous function $W:\mathbb{R}^N\to\mathbb{R}_+\cup\{+\infty\}$ satisfying **(H1)** and **(H2)**, we introduce the geodesic pseudo-distance geod_W in $\mathbb{R}^N$ endowed with the singular pseudo-metric $4Wg_0$, g_0 being the standard Euclidean metric in $\mathbb{R}^N$; this geodesic pseudo-distance (that can take the value $+\infty$) is defined for every $x,y\in\mathbb{R}^N$ by

$$\mathrm{geod}_W(x,y) := \inf\left\{\int_{-1}^1 2\sqrt{W(\sigma(t))}|\dot\sigma|(t)\,\mathrm{d}t \;:\; \sigma\in \mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N),\ \sigma(-1)=x,\ \sigma(1)=y\right\}, \tag{2.3}$$

where $\mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N)$ is the set of continuous and **piecewise locally Lipschitz** curves [7] on $[-1,1]$:

$$\mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N) := \Big\{\sigma\in\mathcal{C}^0([-1,1],\mathbb{R}^N) \;:\text{there is a partition } -1=t_1<\dots<t_{k+1}=1, \\ \text{with } \sigma\in \mathrm{Lip}_{loc}((t_i,t_{i+1})) \text{ for every } 1\le i\le k\Big\}.$$

By *pseudo-distance*, we mean that geod_W satisfies all the axioms of a distance; the only difference with respect to the standard definition is that a pseudo-distance can take the value $+\infty$. We will prove that geod_W yields a lower bound for the energy E (see Lemma 6); this plays an important role in the proof of our claim $\overline{u}(\pm\infty)=u^\pm$.

We start by proving some elementary facts about the pseudo-metric structure induced by geod_W on $\mathbb{R}^N$:

[7] In general, we cannot hope that a minimizing sequence in (2.3) is better than piecewise locally Lipschitz because W is not assumed locally bounded ($\dot\sigma$ is the derivative of σ). However, in the case of a locally bounded W, we could use a regularization procedure in order to restrict to Lipschitz curves σ.

Lemma 5. *Let* $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ *be a lower semicontinuous function satisfying* **(H1)** *and* **(H2)***. Then the function* $\mathrm{geod}_W : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ *defines a pseudo-distance over* $\mathbb{R}^N$ *and the Euclidean distance is absolutely continuous with respect to* geod_W*, i.e., for every* $\delta > 0$*, there exists* $\varepsilon > 0$ *such that for every* $x, y \in \mathbb{R}^N$ *with* $\mathrm{geod}_W(x, y) < \varepsilon$*, we have* $|x - y| < \delta$*.*

Proof of Lemma 5. In proving that $\mathrm{geod}_W : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ defines a pseudo-distance over $\mathbb{R}^N$, the only non-trivial axiom to check is the non-degeneracy, i.e., $\mathrm{geod}_W(x, y) > 0$ whenever $x \neq y$. In fact, we prove the stronger property that for every $\delta > 0$, there exists $\varepsilon > 0$ such that for every $x, y \in \mathbb{R}^N$, $|x - y| \geq \delta$ implies $\mathrm{geod}_W(x, y) \geq \varepsilon$ which also yields the absolute continuity of the Euclidean distance with respect to geod_W. For that, we recall that the set $\{W = 0\}$ is finite (by **(H1)**); therefore, w.l.o.g. we can assume that $\delta > 0$ is small enough so that the open balls $B(p, \delta/2)$, for $p \in \{W = 0\}$, are disjoint. We consider the following disjoint union of balls

$$\Sigma_\delta := \bigsqcup_{p \in \{W=0\}} B(p, \frac{\delta}{4}),$$

the distance between each ball being larger than $\delta/2$. We now take two points $x, y \in \mathbb{R}^N$ with $|x - y| \geq \delta$. In order to obtain a lower bound on $\mathrm{geod}_W(x, y)$, we take an arbitrary continuous and piecewise locally Lipschitz curve $\sigma : [-1, 1] \to \mathbb{R}^N$ such that $\sigma(-1) = x$ and $\sigma(1) = y$. As $|x - y| \geq \delta$ (so no ball in Σ_δ can contain both x and y), by connectedness, the image $\sigma([-1, 1])$ cannot be contained in Σ_δ. Thus, there exists $t_0 \in [-1, 1]$ with $\sigma(t_0) \notin \Sigma_\delta$. It implies that $B(\sigma(t_0), \delta/8) \cap \Sigma_{\delta/2} = \emptyset$. Moreover, since $|x - y| \geq \delta$, we have either $|\sigma(t_0) - x| \geq \delta/2$ or $|\sigma(t_0) - y| \geq \delta/2$; w.l.o.g., we may assume that $|\sigma(t_0) - y| \geq \delta/2$. Then the (continuous) curve $\sigma\big|_{[t_0,1]}$ has to get out of the ball $B(\sigma(t_0), \delta/8)$; in particular, it has length larger than $\delta/8$ and

$$\int_{-1}^{1} 2\sqrt{W(\sigma(t))}|\dot{\sigma}|(t)\,\mathrm{d}t \geq \frac{\delta}{4} \inf_{z \in B(\sigma(t_0), \delta/8)} \sqrt{W(z)} \geq \frac{\delta}{4} \inf_{z \in \mathbb{R}^N \setminus \Sigma_{\delta/2}} \sqrt{W(z)}.$$

Since W is lower semicontinuous and bounded from below at infinity (by **(H2)**), we deduce that W is bounded from below by a constant $c_\delta > 0$ on $\mathbb{R}^N \setminus \Sigma_{\delta/2}$. Taking the infimum over curves $\sigma \in \mathrm{Lip}_{ploc}([-1, 1], \mathbb{R}^N)$ connecting x to y, we deduce from the preceding lower bound that

$$\mathrm{geod}_W(x, y) \geq \frac{\delta\sqrt{c_\delta}}{4} > 0.$$

This finishes the proof of the result. □

We now use a regularization argument to derive the following lower bound on the energy:

Lemma 6. *Let* $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ *be a lower semicontinuous function. Then, for every interval* $I \subset \mathbb{R}$ *and every map* $\sigma \in \dot{H}^1(I, \mathbb{R}^N)$ *having limits* $\sigma(\inf I)$ *and* $\sigma(\sup I)$ *at the endpoints of* I*, we have*

$$E_W(\sigma, I) := \int_I \Big(|\dot{\sigma}(t)|^2 + W(\sigma(t))\Big)\,\mathrm{d}t \geq \mathrm{geod}_W\big(\sigma(\inf I), \sigma(\sup I)\big). \tag{2.4}$$

Proof of Lemma 6. W.l.o.g. we assume that I is an open interval. Since $\dot{H}^1(I, \mathbb{R}^N) \subset W^{1,1}_{loc}(I, \mathbb{R}^N)$, we can define the arc-length $s : I \to J := s(I) \subset \mathbb{R}$ by

$$s(t) := \int_{t_0}^{t} |\dot{\sigma}|(x_1)\,\mathrm{d}x_1, \quad t \in I,$$

where $t_0 \in I$ is fixed. Thus s is a nondecreasing continuous function with $\dot{s} = |\dot{\sigma}|$ a.e. in I. Then the arc-length reparametrization of σ, i.e.

$$\tilde{\sigma}(s(t)) := \sigma(t), \quad t \in I,$$

is well-defined and provides a Lipschitz curve $\tilde{\sigma} : J \to \mathbb{R}^N$ with constant speed on the interval J, i.e. $|\dot{\tilde{\sigma}}| = 1$ a.e., and such that $\tilde{\sigma}(\inf J) = \sigma(\inf I)$ and $\tilde{\sigma}(\sup J) = \sigma(\sup I)$. W.l.o.g. we may assume that σ is not constant, so J has a nonempty interior. Then we consider an arbitrary function $\varphi \in \mathrm{Lip}_{loc}((-1,1), \mathrm{int}J)$ which is nondecreasing and surjective onto the interior of the interval J and we set

$$\gamma(t) := \tilde{\sigma}(\varphi(t)), \quad t \in (-1,1).$$

So γ is a locally Lipschitz map that is continuous on $[-1,1]$ as $\tilde{\sigma}$ admits limits at $\inf J$ and $\sup J$; thus, $\gamma \in \mathrm{Lip}_{ploc}([-1,1], \mathbb{R}^N)$. The changes of variable $s := \varphi(t)$, resp. $s := s(t)$, yield

$$\int_{-1}^{1} 2\sqrt{W(\gamma(t))}|\dot{\gamma}|(t)\,\mathrm{d}t = \int_J 2\sqrt{W(\tilde{\sigma}(s))}|\dot{\tilde{\sigma}}|(s)\,\mathrm{d}s = \int_I 2\sqrt{W(\sigma(t))}\,|\dot{\sigma}|(t)\,\mathrm{d}t.$$

Combined with $\gamma(-1) = \sigma(\inf I)$ and $\gamma(1) = \sigma(\sup I)$, the definition of geod_W and the Young inequality imply

$$E_W(\sigma, I) \geq \int_I 2\sqrt{W(\sigma(t))}\,|\dot{\sigma}|(t)\,\mathrm{d}t = \int_{-1}^{1} 2\sqrt{W(\gamma(t))}|\dot{\gamma}|(t)\,\mathrm{d}t \geq \mathrm{geod}_W\big(\sigma(\inf I), \sigma(\sup I)\big).$$

This completes the proof. □

The convergence of the x'-average in Theorem 1 stating that $\overline{u}(\pm\infty) = u^{\pm}$ is a consequence of the following lemma:

Lemma 7. *Let $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function satisfying* **(H1)** *and* **(H2)**. *Then for every map $\sigma \in \dot{H}^1(\mathbb{R}, \mathbb{R}^N)$ such that $E_W(\sigma, \mathbb{R}) < +\infty$ with E_W defined at* (2.4), *there exist two wells u^-, $u^+ \in \{W = 0\}$ such that $\lim_{t \to \pm\infty} \sigma(t) = u^{\pm}$.*

Proof of Lemma 7. We use the fact that the energy bound $E_W(\sigma, \mathbb{R}) < +\infty$ yields a bound on the total variation of $\sigma : \mathbb{R} \to \mathbb{R}^N$ where $\mathbb{R}^N$ is endowed with the pseudo-metric geod_W. More precisely, for every sequence $t_1 < \cdots < t_k$ in $\mathbb{R}$, we have by Lemma 6:

$$\sum_{i=1}^{k} \mathrm{geod}_W(\sigma(t_{i+1}), \sigma(t_i)) \leq \sum_{i=1}^{k} E_W(\sigma, [t_i, t_{i+1}]) \leq E_W(\sigma, \mathbb{R}) < +\infty.$$

In particular, for every $\varepsilon > 0$, there exists $R > 0$ such that for all $t, s \in \mathbb{R}$ with $t, s \geq R$ or $t, s \leq -R$, one has $\mathrm{geod}_W(\sigma(t), \sigma(s)) < \varepsilon$. Since by Lemma 5, smallness of $\mathrm{geod}_W(x,y)$ implies smallness of $|x-y|$, we deduce that σ has a limit $u^{\pm} \in \mathbb{R}^N$ at $\pm\infty$. Since $W(\sigma(\cdot))$ is integrable in $\mathbb{R}$, we have furthermore that $W(u^{\pm}) = 0$. □

Now we can prove the convergence of the x'-average $\bar{u}$ at $\pm\infty$ as stated in Theorem 1:

Proof of the convergence in x'-average in Theorem 1. By Lemma 4, we have $E_V(\overline{u}, \mathbb{R}) < +\infty$ for the lower semicontinuous function $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying **(H1)** and **(H2)**. By Lemma 7 applied to E_V, we deduce that there exists $u^{\pm} \in \{V = 0\} = \{W = 0\}$ such that $\lim_{t \to \pm\infty} \overline{u}(t) = u^{\pm}$. □

The pointwise convergence of $u(x_1,\cdot)$ as $x_1 \to \pm\infty$ stated in Theorem 1 is proved in the following:

Proof of the pointwise convergence in Theorem 1. We prove that $u(x_1,\cdot)$ converges a.e. in ω to $u^\pm \in \{W=0\}$ as $x_1 \to \pm\infty$, where $u^\pm$ are the limits $\bar{u}(\pm\infty)$ of the x'-average $\bar{u}$ proved above. For that, we have by Fubini's theorem:

$$E(u) \geq \int_\Omega |\partial_1 u|^2 + W(u)\, \mathrm{d}x \geq \int_\omega E_W(u(\cdot,x'),\mathbb{R})\, \mathrm{d}x'$$

with the usual notation

$$E_W(\sigma,\mathbb{R}) = \int_\mathbb{R} |\dot{\sigma}|^2 + W(\sigma)\, \mathrm{d}x_1, \quad \sigma \in \dot{H}^1(\mathbb{R},\mathbb{R}^N).$$

As $E(u) < \infty$, we deduce that $E_W(u(\cdot,x'),\mathbb{R}) < \infty$ for a.e. $x' \in \omega$. By Lemma 7, we deduce that for a.e. $x' \in \omega$, there exist two wells $u^\pm(x')$ of W such that

$$\lim_{x_1 \to \pm\infty} u(x_1,x') = u^\pm(x'). \tag{2.5}$$

By (1.4), as $\bar{u}(\pm\infty) = u^\pm$, we know that $\|u(R_n^\pm,\cdot) - u^\pm\|_{L^2(\omega,\mathbb{R}^N)} \to 0$ as $n \to \infty$ for two sequences $R_n^\pm \to \pm\infty$. Up to a subsequence, we deduce that $u(R_n^\pm,\cdot) \to u^\pm$ a.e. in ω as $n \to \infty$. By (2.5), we conclude that $u^\pm(x') = u^\pm$ for a.e. $x' \in \omega$. □

3 The L^2 convergence

In this section, we prove that $u(x_1,\cdot)$ converges in $L^2(\omega,\mathbb{R}^N)$ to $u^\pm$ as $x_1 \to \pm\infty$. The idea is to go beyond the averaging procedure in Section 2 and keep the full information given by the x'-average energy e introduced at Section 2 over the set $H^1(\omega,\mathbb{R}^N)$. More precisely, we extend e to the space $L^2(\omega,\mathbb{R}^N)$ as follows

$$e(v) = \begin{cases} \fint_\omega \Big(|\nabla' v|^2 + W(v)\Big)\, \mathrm{d}x' & \text{if } v \in H^1(\omega,\mathbb{R}^N), \\ +\infty & \text{if } v \in L^2(\omega,\mathbb{R}^N) \setminus H^1(\omega,\mathbb{R}^N). \end{cases} \tag{3.1}$$

In particular, we have for every $u \in \dot{H}^1(\Omega,\mathbb{R}^N)$,

$$E(u) = \int_\mathbb{R} \Big(\|\partial_1 u(x_1,\cdot)\|^2_{L^2(\omega,\mathbb{R}^N)} + |\omega| e(u(x_1,\cdot))\Big)\, \mathrm{d}x_1. \tag{3.2}$$

In the sequel, we will also need the following properties of the energy e:

Lemma 8. *If $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semicontinuous function satisfying* **(H2)**, *then*

1. *e is lower semicontinuous in $L^2(\omega,\mathbb{R}^N)$,*
2. *the sets of zeros of e and W coincide; moreover $\Sigma := \{e=0\} = \{W=0\} \subset \mathbb{R}^N$ is compact,*
3. *for every $\varepsilon > 0$, we have*

$$k_\varepsilon := \inf\big\{e(v) \,:\, v \in L^2(\omega,\mathbb{R}^N) \text{ with } d_{L^2}(v,\Sigma) \geq \varepsilon\big\} > 0.$$

Proof. We divide the proof in several steps:

STEP 1. LOWER SEMICONTINUITY OF e IN $L^2(\omega, \mathbb{R}^N)$. Indeed, let $v_n \to v$ in $L^2(\omega, \mathbb{R}^N)$. W.l.o.g., we may assume that $(e(v_n))_n$ is bounded, in particular, $(v_n)_n$ is bounded in $H^1(\omega, \mathbb{R}^N)$; thus, $(v_n)_n$ converges to v weakly in $H^1(\omega, \mathbb{R}^N)$. By Step 1 in the proof of Lemma 4, we know that $e\big|_{H^1(\omega,\mathbb{R}^N)}$ is lower semicontinuous w.r.t. the weak H^1 topology and the conclusion follows.

STEP 2. ZEROS OF e. The equality of the zero sets of e and W is straightforward thanks to the connectedness of ω. Thanks to the assumption **(H2)**, the set of zeros Σ of W is bounded and by the lower semicontinuity and non-negativity of W, the set of zeros Σ of W is closed; thus, Σ is compact in $\mathbb{R}^N$.

STEP 3. WE PROVE THAT $k_\varepsilon > 0$. Assume by contradiction that $k_\varepsilon = 0$ for some $\varepsilon > 0$. Then there exists a minimizing sequence $v_n \in L^2(\omega, \mathbb{R}^N)$ such that $d_{L^2}(v_n, \Sigma) \geq \varepsilon$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} e(v_n) = 0$. W.l.o.g., we may assume that $v_n \in H^1(\omega, \mathbb{R}^N)$ for every n as $\|v_n\|_{\dot{H}^1} \to 0$. Denoting $\overline{v_n}$ the (x'-)average of v_n, the Poincaré-Wirtinger inequality implies that the sequence $(w_n := v_n - \overline{v_n})_n$ converges in $H^1(\omega, \mathbb{R}^N)$ to 0. Up to extracting a subsequence, we may assume that $w_n \to 0$ for a.e. $x' \in \omega$.

Claim: The sequence $(\overline{v_n})_n$ is bounded in $\mathbb{R}^N$.

Indeed, assume by contradiction that there exists a subsequence of $(\overline{v_n})_n$ (still denoted by $(\overline{v_n})_n$) such that $|\overline{v_n}| \to \infty$ as $n \to \infty$. As W is l.s.c. and $w_n \to 0$ for a.e. $x' \in \omega$, the assumption **(H2)** implies

$$\liminf_{n\to\infty} W(v_n(x')) = \liminf_{n\to\infty} W(w_n(x') + \overline{v_n}) \geq \liminf_{|z|\to\infty} W(z) > 0 \quad \text{for a.e. } x' \in \omega$$

which by integration over $x' \in \omega$ contradicts the assumption $e(v_n) \to 0$. This finishes the proof of the claim.

As a consequence of the claim, we deduce that $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\omega, \mathbb{R}^N)$. In particular, $(v_n)_{n\in\mathbb{N}}$ has a subsequence that converges in $L^2(\omega, \mathbb{R}^N)$ to a map $v \in H^1(\omega, \mathbb{R}^N)$ and we deduce $d_{L^2}(v, \Sigma) \geq \varepsilon$, in particular, v is not a zero of e, i.e., $e(v) > 0$. As e is l.s.c. in $L^2(\omega, \mathbb{R}^N)$, we have $0 = \lim_{n\to\infty} e(v_n) \geq e(v)$, which contradicts that $e(v) > 0$. □

Now we prove the L^2-convergence of $u(x_1, \cdot)$ to $u^\pm$ as $x_1 \to \pm\infty$:

Proof of the L^2-convergence in Theorem 1. Take $u \in H^1_{loc}(\Omega, \mathbb{R}^N)$ such that $E(u) < +\infty$ and set $\sigma(t) := u(t, \cdot) \in H^1(\omega, \mathbb{R}^N)$ for a.e. $t \in \mathbb{R}$. We prove that $\sigma(t)$ converges in $L^2(\omega, \mathbb{R}^N)$ to a limit that is a zero in Σ as $t \to +\infty$ (the proof of the convergence as $t \to -\infty$ is similar). Moreover, we will see that these limits are in fact the zeros $u^\pm$ of W given by the x'-average $\bar{u}$ and the a.e. convergence of $u(x_1, \cdot)$ as $x_1 \to \pm\infty$.

STEP 1: CONTINUITY. We prove that $t \in \mathbb{R} \mapsto \sigma(t) \in L^2(\omega, \mathbb{R}^N)$ is continuous in $\mathbb{R}$, and moreover, it is a $\frac{1}{2}$-Hölder map. Indeed, for a.e. $t, s \in \mathbb{R}$, we have

$$d_{L^2}(\sigma(t), \sigma(s))^2 = \int_\omega \Big| \int_t^s \partial_{x_1} u(x_1, x')\, \mathrm{d}x_1 \Big|^2 \mathrm{d}x' \leq |t - s| \|\partial_{x_1} u\|^2_{L^2(\Omega,\mathbb{R}^N)}.$$

STEP 2: CONVERGENCE OF A SUBSEQUENCE $(\sigma(t_n))_n$ TO SOME $u^+ \in \Sigma$. Since $e(\sigma(\cdot)) \in L^1(\mathbb{R})$ by (3.2), there is a sequence $(t_n)_{n\in\mathbb{N}} \to +\infty$ such that $\lim_{n\to\infty} e(\sigma(t_n)) = 0$. Exactly like in Step 3 in the proof of Lemma 8, we deduce that $(\sigma(t_n))_{n\in\mathbb{N}}$ has a subsequence that converges strongly in $L^2(\omega, \mathbb{R}^N)$ to some map $\sigma_\infty \in L^2(\omega, \mathbb{R}^N)$ (the assumption **(H2)** is essential here). Since e is l.s.c. in L^2 and $e \geq 0$ in L^2, we deduce that $e(\sigma_\infty) = 0$ and so, there exists $u^+ \in \Sigma$ such that $\sigma_\infty \equiv u^+$.

Step 3: Convergence to u^+ in L^2 as $t \to +\infty$. Assume by contradiction that $\sigma(t)$ does not converge in $L^2(\omega, \mathbb{R}^N)$ to u^+ as $t \to \infty$. Then there is a sequence $(s_n)_{n\in\mathbb{N}} \to +\infty$ such that $\varepsilon := \inf_{n\in\mathbb{N}} d_{L^2}(\sigma(s_n), u^+) > 0$. Now, by Step 1, the curve $t \in [s_n, +\infty) \mapsto \sigma(t) \in L^2(\omega, \mathbb{R}^N)$ is continuous. Moreover, $\sigma(s_n)$ doesn't belong to the L^2-ball centered at u^+ with radius $\frac{3\varepsilon}{4}$. By Step 2, it has to enter (at some time $t > s_n$) in the L^2-ball centered at u^+ with radius $\frac{\varepsilon}{4}$. Therefore, the curve $\sigma_{|(s_n,+\infty)}$ has to cross the ring $\mathcal{R} := B_{L^2}(u^+, \frac{3\varepsilon}{4}) \setminus B_{L^2}(u^+, \frac{\varepsilon}{4})$, so it has L^2-length larger than $\frac{\varepsilon}{2}$, i.e.,

$$\int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}\}} \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)} \,\mathrm{d}t = \int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}\}} \|\dot\sigma\|_{L^2(\omega,\mathbb{R}^N)} \,\mathrm{d}t \geq \frac{\varepsilon}{2}.$$

Moreover, by the third claim in Lemma 8, we know that $e(\sigma(t)) \geq k_{\varepsilon/4}$ if $\sigma(t) \in \mathcal{R}$ (up to lowering ε, we may assume that the other zeros of Σ are placed at distance larger than 2ε from u^+, the assumption **(H1)** is essential here). We obtain

$$\begin{aligned}\int_{s_n}^{+\infty} \sqrt{e(u(t,\cdot))}\, \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)} \,\mathrm{d}t &\geq \int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}\}} \sqrt{e(u(t,\cdot))}\, \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)} \,\mathrm{d}t \\ &\geq \frac{\varepsilon}{2}\sqrt{k_{\varepsilon/4}}.\end{aligned} \tag{3.3}$$

This is a contradiction with the assumption $E(u) < +\infty$ implying by (3.2):

$$\begin{aligned}2|\omega|^{\frac{1}{2}} \int_{s_n}^{+\infty} \sqrt{e(u(t,\cdot))}\, \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)} \,\mathrm{d}t &\leq \int_{s_n}^{+\infty} \Big(|\omega| e(u(t,\cdot)) + \|\partial_{x_1} u(t,\cdot)\|^2_{L^2(\omega,\mathbb{R}^N)}\Big) \,\mathrm{d}t \\ &\underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Step 4: The L^2 limits $u^\pm$ coincide with the average limits $\bar{u}(\pm\infty)$. This is clear as L^2 convergence implies convergence in average. □

Remark 9. i) The above proof does not use (so, it is independent of) the almost everywhere convergence of $u(x_1, \cdot)$ as $x_1 \to \pm\infty$ or the convergence of the x'-average $\bar{u}$. Therefore, thanks to this proof, one can obtain as a direct consequence the convergence of the x'-average $\bar{u}$ as well as the almost everywhere convergence of $u(x_1, \cdot)$ as $x_1 \to \pm\infty$.[8]

ii) Also, the above proof applies to Lemma 7 leading to a second method that does not use the geodesic distance geod_W.

iii) Behind the above proof, the notion of geodesic distance over $L^2(\omega, \mathbb{R}^N)$ with the degenerate weight $\sqrt{e}$ is hidden (see (3.3)). Therefore, one could repeat the arguments in the first proof of Theorem 1 based on this geodesic distance.

The above argument can also be used directly to obtain a second proof for the existence of limits of $\bar{u}$ at $\pm\infty$ without using the geodesic pseudo-distance geod_W (as presented in the proof in Section 2). For completeness, we redo the proof in the sequel:

Second proof of the convergence in x'-average in Theorem 1. Let $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$ such that $E(u) < \infty$. We want to prove that the x'-average $\bar{u}$ admits a limit u^+ as $x_1 \to \infty$ and $W(u^+) = 0$ (the proof of the convergence as $x_1 \to -\infty$ is similar). Let V and E_V given by Lemma 4. Recall that $\Sigma := \{V = 0\} = \{W = 0\}$ and $E_V(\bar{u}) \leq \frac{1}{|\omega|} E(u) < \infty$.

[8] As the L^2-convergence implies almost everywhere convergence of $u(x_1, \cdot)$ only up to a subsequence, one should repeat the argument in the proof of the a.e. convergence in Theorem 1 at page 10.

Step 1. We prove that for every $\varepsilon > 0$,

$$\kappa_\varepsilon := \inf\left\{V(z)\ :\ z \in \mathbb{R}^N,\ d_{\mathbb{R}^N}(z,\Sigma) \geq \varepsilon\right\} > 0.$$

Assume by contradiction that there exists a sequence $(z_n)_n$ such that $V(z_n) \to 0$ and $d_{\mathbb{R}^N}(z_n,\Sigma) \geq \varepsilon$. By the third claim in Lemma 4, we deduce that $(z_n)_n$ is bounded, so that, up to a subsequence, $z_n \to z$ for some $z \in \mathbb{R}^N$ yielding $d_{\mathbb{R}^N}(z,\Sigma) \geq \varepsilon$ and $V(z) = 0$, i.e., $z \in \Sigma$ (since V is l.s.c. and $V \geq 0$) which is a contradiction.

Step 2. There exists a sequence $(\bar{u}(t_n))_n$ converging to a well $u^+ \in \Sigma$. Indeed, as $V(\bar{u}) \in L^1(\mathbb{R})$, there exists a sequence $t_n \to \infty$ with $V(\bar{u}(t_n)) \to 0$. By **(H2)**, $(\bar{u}(t_n))_n$ is bounded, so that up to a subsequence, $\bar{u}(t_n) \to u^+$ as $n \to \infty$ for some point $u^+ \in \mathbb{R}^N$. As V is l.s.c. and $V \geq 0$, we deduce that $V(u^+) = 0$, i.e., $u^+ \in \Sigma$.

Step 3: Convergence of $\bar{u}$ to u^+ as $x_1 \to +\infty$. Assume by contradiction that $\bar{u}(x_1)$ does not converge to u^+ as $x_1 \to \infty$. Then there is a sequence $(s_n)_{n\in\mathbb{N}} \to +\infty$ such that $\varepsilon := \inf_{n\in\mathbb{N}} d_{\mathbb{R}^N}(\bar{u}(s_n), u^+) > 0$. As $\bar{u} : [s_n, +\infty) \to \mathbb{R}^N$ is continuous, by Step 2, it has to get out of the ball $B(\bar{u}(s_n), \varepsilon/4)$ and it has to enter in the ball $B(u^+, \varepsilon/4)$. Therefore, $\bar{u}$ has to cross the ring $\mathcal{R} := B(u^+, \frac{3\varepsilon}{4}) \setminus B(u^+, \frac{\varepsilon}{4}) \subset \mathbb{R}^N$. Moreover, by Step 1, we know that $V(\bar{u}(x_1)) \geq \kappa_{\varepsilon/4}$ if $\bar{u}(x_1) \in \mathcal{R}$ (where we assumed w.l.o.g. that $\varepsilon > 0$ is small enough so that the other zeros of Σ are placed at distance larger than 2ε from u^+). We obtain

$$\int_{s_n}^{+\infty} \sqrt{V(\bar{u}(x_1))}\,\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\bar{u}(x_1)\Big|\ \mathrm{d}x_1 \geq \int_{\{x_1\in(s_n,+\infty)\,:\,\bar{u}(x_1)\in\mathcal{R}\}} \sqrt{V(\bar{u}(x_1))}\,\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\bar{u}(x_1)\Big|\ \mathrm{d}x_1 \geq \frac{\varepsilon}{2}\sqrt{\kappa_{\varepsilon/4}}.$$

This is a contradiction with the assumption $E_V(\bar{u}) < +\infty$ implying

$$2\int_{s_n}^{+\infty} \sqrt{V(\bar{u}(x_1))}\,\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\bar{u}(x_1)\Big|\,\mathrm{d}x_1 \leq \int_{s_n}^{+\infty} \Big(\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\bar{u}(x_1)\Big|^2 + V(\bar{u}(x_1))\Big)\,\mathrm{d}x_1 \underset{n\to\infty}{\longrightarrow} 0.$$

□

4 Proof of Theorem 3

In this section, we consider $d = N$, $\Omega = \mathbb{R} \times \omega$ with $\omega = \mathbb{T}^{d-1}$ and $u \in H^1_{loc}(\Omega, \mathbb{R}^d)$ periodic in $x' \in \omega$ with $\bar{u}_1 = a$ in $\mathbb{R}$ for some constant $a \in \mathbb{R}$ (recall that $\bar{u}$ is the x'-average of u). Note that $|\omega| = 1$. We set

$$L^2_a(\omega, \mathbb{R}^d) := \left\{v = (v_1, \ldots, v_d) \in L^2(\omega, \mathbb{R}^d)\ :\ \int_\omega v_1\, dx' = a\right\}$$

and $H^1_a(\omega, \mathbb{R}^d) := H^1 \cap L^2_a(\omega, \mathbb{R}^d)$. Note that for a.e. $x_1 \in \mathbb{R}$, $u(x_1, \cdot) \in H^1_a(\omega, \mathbb{R}^d)$. We define the following energy e_a on the convex closed subset $L^2_a(\omega, \mathbb{R}^d)$ of $L^2(\omega, \mathbb{R}^d)$:

$$e_a(v) = \begin{cases} \displaystyle\int_\omega \Big(|\nabla' v|^2 + W(v)\Big)\,\mathrm{d}x' & \text{if } v \in H^1_a(\omega, \mathbb{R}^d), \\ +\infty & \text{if } v \in L^2_a(\omega, \mathbb{R}^d) \setminus H^1(\omega, \mathbb{R}^d). \end{cases} \tag{4.1}$$

In particular, we have for every $u \in \dot{H}^1(\Omega, \mathbb{R}^d)$ with $\bar{u}_1 = a$:

$$E(u) = \int_{\mathbb{R}} \Big(\|\partial_1 u(x_1, \cdot)\|^2_{L^2(\omega,\mathbb{R}^d)} + e_a(u(x_1, \cdot))\Big)\,\mathrm{d}x_1. \tag{4.2}$$

The aim is to adapt the proof of Theorem 1 given in Section 3 to Theorem 3. We start by transfering the properties of the energy e in Lemma 8 to the energy e_a defined in $L^2_a(\omega,\mathbb{R}^d)$. More precisely, if $W:\mathbb{R}^d\to\mathbb{R}_+\cup\{+\infty\}$ is a lower semicontinuous function, then e_a is lower semicontinuous in $L^2_a(\omega,\mathbb{R}^d)$ endowed with the strong L^2-norm and the sets of zeros of e_a and $W(a,\cdot)$ coincide, i.e.,

$$\Sigma^a := \{v\in L^2_a(\omega,\mathbb{R}^d) \,:\, e_a(v)=0\} = \{z=(a,z')\in\mathbb{R}^d \,:\, W(a,z')=0\}.$$

If in addition W satisfies $\mathbf{(H2)}_a$, then Σ^a is compact in $\mathbb{R}^d$ and for every $\varepsilon>0$, we have

$$k^a_\varepsilon := \inf\big\{e_a(v) \,:\, v\in L^2_a(\omega,\mathbb{R}^d) \text{ with } d_{L^2}(v,\Sigma^a)\geq\varepsilon\big\}>0$$

(the proof of these properties follows by the same arguments presented in the proof of Lemma 8).

Proof of Theorem 3. Let $u\in H^1_{loc}(\Omega,\mathbb{R}^d)$ such that $E(u)<+\infty$ and $\bar{u}_1=a$ in $\mathbb{R}$. We set $\sigma(t):=u(t,\cdot)\in H^1_a(\omega,\mathbb{R}^d)$ for a.e. $t\in\mathbb{R}$. We prove that $\sigma(t)$ converges in $L^2(\omega,\mathbb{R}^d)$ to a limit that is a zero in Σ^a as $t\to+\infty$ (the proof of the convergence as $t\to-\infty$ is similar). As in Steps 1 and 2 in the proof of the L^2-convergence in Theorem 1, we have that $t\in\mathbb{R}\mapsto\sigma(t)\in L^2_a(\omega,\mathbb{R}^d)$ is a $\frac{1}{2}$-Hölder continuous map in $\mathbb{R}$ and there is a sequence $(t_n)_{n\in\mathbb{N}}\to+\infty$ such that $\sigma(t_n)\to u^+$ in $L^2(\omega,\mathbb{R}^d)$ for a well $u^+\in\Sigma^a$ (the assumption $\mathbf{(H2)}_a$ is essential here). In order to prove the convergence of $\sigma(t)$ to u^+ in L^2 as $t\to+\infty$, we argue by contradiction. If $\sigma(t)$ does not converge in $L^2(\omega,\mathbb{R}^d)$ to u^+ as $t\to\infty$, then there is a sequence $(s_n)_{n\in\mathbb{N}}\to+\infty$ such that $\varepsilon:=\inf_{n\in\mathbb{N}} d_{L^2}(\sigma(s_n),u^+)>0$. We repeat the argument in Step 3 in the proof of the L^2-convergence in Theorem 1 by restricting ourselves to $L^2_a(\omega,\mathbb{R}^d)$ endowed by the strong L^2 topology. More precisely, the continuous curve $t\in[s_n,+\infty)\mapsto\sigma(t)\in L^2_a(\omega,\mathbb{R}^d)$ has to cross the ring $\mathcal{R}_a:=\big(B_{L^2}(u^+,\frac{3\varepsilon}{4})\setminus B_{L^2}(u^+,\frac{\varepsilon}{4})\big)\cap L^2_a(\omega,\mathbb{R}^d)$, so it has L^2-length larger than $\frac{\varepsilon}{2}$, i.e.,

$$\int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}_a\}}\|\partial_{x_1}u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^d)}\,\mathrm{d}t=\int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}_a\}}\|\dot{\sigma}\|_{L^2(\omega,\mathbb{R}^d)}\,\mathrm{d}t\geq\frac{\varepsilon}{2}.$$

As $e(\sigma(t))\geq k^a_{\varepsilon/4}$ if $\sigma(t)\in\mathcal{R}_a$ (up to lowering ε, we may assume that the other zeros of Σ^a are placed at distance larger than 2ε from u^+, the assumption $\mathbf{(H1)}_a$ is essential here), we obtain

$$\int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}_a\}}\sqrt{e_a(u(t,\cdot))}\,\|\partial_{x_1}u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^d)}\,\mathrm{d}t\geq\frac{\varepsilon}{2}\sqrt{k^a_{\varepsilon/4}}.$$

This is a contradiction with (4.2):

$$2\int_{s_n}^{+\infty}\sqrt{e_a(u(t,\cdot))}\,\|\partial_{x_1}u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^d)}\,\mathrm{d}t\leq\int_{s_n}^{+\infty}\Big(e_a(u(t,\cdot))+\|\partial_{x_1}u(t,\cdot)\|^2_{L^2}\Big)\,\mathrm{d}t\underset{n\to\infty}{\longrightarrow}0.$$

Clearly, the L^2 convergence implies also the convergence in average of $\sigma(t)$ over ω as $t\to\infty$ as well as the a.e. convergence $\sigma(t)\to u^+$ in ω but only up to a subsequence. For the full almost everywhere convergence of $u(x_1,\cdot)\to u^+$, we proceed as follows. First, by the Poincaré-Wirtinger inequality on $\omega=\mathbb{T}^{d-1}$, we have for a.e. $x_1\in\mathbb{R}$,

$$\int_\omega|\nabla' u_1(x_1,x')|^2\,\mathrm{d}x'\geq 4\pi^2\int_\omega|u_1(x_1,x')-\bar{u}_1(x_1)|^2\,\mathrm{d}x'=4\pi^2\int_\omega|u_1(x_1,x')-a|^2\,\mathrm{d}x'.$$

By Fubini's theorem, we deduce that

$$E(u)\geq\int_\Omega\big(|\partial_1 u|^2+|\nabla' u_1|^2+W(u)\big)\,\mathrm{d}x\geq\int_{\mathbb{T}^{d-1}}E_{W_a}(u(\cdot,x'),\mathbb{R})\,\mathrm{d}x',$$

where $W_a(z) := W(z) + 4\pi^2|z_1 - a|^2$ and, as usual,

$$E_{W_a}(\sigma, \mathbb{R}) = \int_{\mathbb{R}} \left(|\dot{\sigma}|^2 + W_a(\sigma)\right) \mathrm{d}x_1, \quad \sigma \in \dot{H}^1(\mathbb{R}, \mathbb{R}^N).$$

Hence, $E_{W_a}(u(\cdot, x'), \mathbb{R}) < \infty$ for a.e. $x' \in \omega$. Note that W_a is lower semicontinuous and satisfies assumptions **(H1)** (the set of zeros of W_a coincides with Σ^a, which is finite by $\mathbf{(H1)}_a$) and the coercivity condition **(H2)** (thanks to $\mathbf{(H2)}_a$). Thus, Lemma 7 implies that for a.e. $x' \in \omega$, there exist two wells $u^\pm(x')$ of W_a such that

$$\lim_{x_1 \to \pm\infty} u(x_1, x') = u^\pm(x'). \tag{4.3}$$

By (1.4), as $\bar{u}(\pm\infty) = u^\pm$, we know that $\|u(R_n^\pm, \cdot) - u^\pm\|_{L^2(\omega, \mathbb{R}^N)} \to 0$ as $n \to \infty$ for two sequences $(R_n^\pm)_{n\in\mathbb{N}} \to \pm\infty$. Up to a subsequence, we deduce that $u(R_n^\pm, \cdot) \to u^\pm$ a.e. in ω as $n \to \infty$. By (4.3), we conclude that $u^\pm(x') = u^\pm$ for a.e. $x' \in \omega$. □

Acknowledgment. R.I. acknowledges partial support by the ANR project ANR-14-CE25-0009-01.

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