Optimal lifting for $BV(S^1, S^1)$

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Abstract

For each $g \in BV(S^1, S^1)$, we solve the following variational problem

$$E(g) = \inf \left\{ \int_{S^1} |\dot{\varphi}| \, : \, \varphi \in BV(S^1, \mathbb{R}), e^{i\varphi} = g \text{ a.e. on } S^1 \right\}$$

and we show that $E(g) \leq 2|g|_{BV}$.

1 Introduction

Let $g \in BV(S^1, S^1)$, i.e. $g \in BV(S^1, \mathbb{R}^2)$ and $|g(y)| = 1$ for a.e. $y \in S^1$. The aim of this paper is to compute the total variation of an optimal lifting BV of g, i.e.

$$E(g) = \inf \left\{ \int_{S^1} |\dot{\varphi}| \, : \, \varphi \in BV(S^1, \mathbb{R}), \, e^{i\varphi} = g \text{ a.e. on } S^1 \right\} \tag{1}$$

(here "$\,\dot{}\,$" stands for the tangential derivative operator). It is easy to see that the above infimum is achieved and it is equal to the relaxed energy

$$E_{\text{rel}}(g) = \inf \left\{ \liminf_{n \to \infty} \int_{S^1} |\dot{g}_n| \, \mathrm{d}\mathcal{H}^1 \, : \, g_n \in C^\infty(S^1, S^1), \, \deg g_n = 0, \, g_n \to g \, \text{ a.e. on } S^1 \right\}$$

(see Remark 1).

In what follows, we will always identify g with its precise representative, which is a Borel function such that

$$g(x) = \frac{g(x+) + g(x-)}{2}, \; \forall x \in S^1.$$

The orientation on the circle S^1 is taken to be counterclockwise.

In order to state the main results, we need to introduce some notations. We decompose the finite Radon measure $\dot{g}$ as

$$\dot{g} = (\dot{g})_{ac} + (\dot{g})_C + (\dot{g})_J,$$
$$\text{with } \; (\dot{g})_J = \sum_{a \in A(g)} (g(a+) - g(a-))\delta_a.$$

Here, $(\dot{g})_{ac}, (\dot{g})_C$ and $(\dot{g})_J$ are the absolutely continuous part, the Cantor part and the jump part of $\dot{g}$. Note that the set of jump points of g

$$A(g) = \{a \in S^1 : \dot{g}(\{a\}) \neq 0\}$$

is at most countable. For any $a \in A(g)$, let $d_a(g) \in (-\pi, \pi] \setminus \{0\}$ be the argument of the unit complex number $\dfrac{g(a+)}{g(a-)}$, i.e.

$$e^{i\,d_a(g)} = \frac{g(a+)}{g(a-)}.$$

Thus, $|d_a(g)| = d_{S^1}(g(a+), g(a-))$ where d_{S^1} is the geodesic distance on S^1. We denote

$$\begin{aligned} S(g) &= \sum_{a \in A(g)} d_a(g) \\ L(g) &= \int_{S^1} g \wedge \Big((\dot{g})_{ac} + (\dot{g})_C \Big) \\ m(g) &= \frac{S(g) + L(g)}{2\pi} \end{aligned}$$

where $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \wedge \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = u_1 \mu_2 - u_2 \mu_1$.

A preliminary result is the following:

Lemma 1 $m(g) \in \mathbb{Z}$, $\forall g \in BV(S^1, S^1)$.

The reason to introduce $m(g)$ is the following: if $a \in A(g)$ and φ is a lifting BV of g, then $|\varphi(a+) - \varphi(a-)| \geq |d_a(g)|$. It turns out that $m(g)$ is related to the number of times where the above inequality is strict.

Set

$$\tilde{E}(g) = \int_{S^1} \Big(|(\dot{g})_{ac}| + |(\dot{g})_C| \Big).$$

This quantity represents the total variation of the diffuse part of the derivative of g. It also plays the role of the total variation of the diffuse part of the derivative of a lifting of g.

If $A(g) = \varnothing$, set $E_J(g) = |L(g)|$; otherwise (i.e. $A(g) \neq \varnothing$), set

$$E_J(g) = \min_{\substack{\alpha_a \in \mathbb{Z},\, a \in A(g) \\ \#\{a \in A(g)\,:\,\alpha_a \neq 0\} < \infty \\ \sum_{a \in A(g)} \alpha_a = m(g)}} \sum_{a \in A(g)} |d_a(g) - 2\pi \alpha_a|. \tag{2}$$

As we will see, $E_J(g)$ is the total variation of the jump part of an optimal lifting of g. The analytic formula for $E_J(g)$ (when g has jumps) is given by:

Lemma 2 *If $A(g) \neq \varnothing$, then*

$$E_J(g) = \begin{cases} \operatorname{sgn}(m(g)) L(g) + 2 \displaystyle\min_{\substack{B \subset A(g) \\ \#B = \min(|m(g)|, \#A(g))}} \sum_{\substack{a \in A(g) \setminus B \\ \operatorname{sgn}(d_a(g)) = \operatorname{sgn}(m(g))}} |d_a(g)| & \text{if } m(g) \neq 0 \\ \displaystyle\sum_{a \in A(g)} |d_a(g)| & \text{if } m(g) = 0 \end{cases}.$$

Our first main result is

Theorem 1 *For every $g \in BV(S^1, S^1)$, we have*

$$E(g) = \tilde{E}(g) + E_J(g).$$

If g is a continuous function of bounded variation, then Theorem 1 gives

$$E(g) = \tilde{E}(g) + 2\pi|\deg(g)|. \tag{3}$$

The formula (3) was already proved by Bourgain-Brezis-Mironescu [BBM]. In the general case, our result can presumably be proved using the theory of Cartesian Currents of Giaquinta-Modica-Soucek[GMS].

The next result yields an estimate of $E(g)$ in terms of the BV-seminorm of g. It is a straightforward variant of the result of Dávila-Ignat[DI] for $BV(\Omega, S^1)$ functions (where $\Omega \subset \mathbb{R}^N$ is a bounded open set):

Theorem 2 *For every $g \in BV(S^1, S^1)$, we have*

$$E(g) \leq 2 \int_{S^1} |\dot{g}|. \tag{4}$$

The constant 2 in the above inequality is optimal (see the examples in Section 5). We present two different proofs for Theorem 2. The first one relies on the explicit formula obtained in Theorem 1, combined with the following trigonometric inequality:

Lemma 3 *Let γ be the unique solution on $(0, \frac{\pi}{2})$ of the equation*

$$3 \sin \frac{\gamma + \pi}{3} = 2\frac{\gamma + \pi}{3} \qquad (\gamma = 1.345752051076...).$$

For p integer, let $x_k \in [0, \frac{\pi}{2}]$, $\forall k \geq 1$ such that $\sum_{k \geq 1} x_k \leq p\,\pi + \gamma$. Then

$$\sum_{k\geq 1} \sin x_k \geq \sum_{k \geq 1} x_k - \max_{\substack{B \subset \mathbb{N} \\ \#B = p}} \sum_{k \in B} x_k.$$

The second proof of Theorem 2 is a straightforward adaptation of the proof given in [DI]; the idea is to use a special class of liftings of g. We discuss in Section 4 some properties of this class. The striking fact is that, although the lifting obtained using the technique in [DI] is not optimal in general (i.e. this lifting is not a minimizer in (1)), inequality (4) is easier to prove using this lifting rather than using an optimal one.

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2 Optimal lifting of $g \in BV(S^1, S^1)$

In this section we prove Lemma 1, Lemma 2 and Theorem 1; we also construct an optimal lifting of g. Finally, we present an estimate of $E(g)$ in terms of a more natural BV-seminorm $|g|_{BVS^1}$, defined below.

First, following [BMP], let us make some remarks about $E(g)$ and $E_{\text{rel}}(g)$:

Remark 1 i) $E(g) < \infty$ and $E_{\text{rel}}(g) < \infty$ (the existence of a lifting BV for g is shown in the proof of Lemma 1);

ii) The infimum (1) is achieved; indeed, let $\varphi_n \in BV(S^1,\mathbb{R})$, $e^{i\varphi_n} = g$ a.e. on S^1 be such that

$$\lim_{n\to\infty} \int_{S^1} |\dot\varphi_n| = E(g) < \infty.$$

By Poincaré's inequality, there exists an universal constant $C > 0$ such that

$$\int_{S^1} |\varphi_n - \fint_{S^1} \varphi_n| \, \mathrm{d}\mathcal{H}^1 \le C \int_{S^1} |\dot\varphi_n|, \, \forall n \in \mathbb{N}$$

(where $\fint_{S^1}$ stands for the average). Therefore, by subtracting a suitable 2π integer multiple, we may assume that $(\varphi_n)_{n\in\mathbb{N}}$ is bounded in $BV(S^1,\mathbb{R})$. Up to a further subsequence, we may assume that $\varphi_n \to \varphi$ a.e. and L^1 for some $\varphi \in BV(S^1,\mathbb{R})$. It follows that φ is a lifting of g on S^1 and

$$E(g) = \lim_{n\to\infty} \int_{S^1} |\dot\varphi_n| \ge \int_{S^1} |\dot\varphi| \ge E(g);$$

iii) $E(g) = E_{\rm rel}(g)$. For "$\le$", take $g_n \in C^\infty(S^1,S^1), \forall n \in \mathbb{N}$ such that $\deg g_n = 0$, $g_n \to g$ a.e. on S^1 and $\sup_{n\in\mathbb{N}} \int_{S^1} |\dot g_n| \, \mathrm{d}\mathcal{H}^1 < \infty$. Then there exists $\varphi_n \in C^\infty(S^1,\mathbb{R})$ such that $e^{i\varphi_n} = g_n$. Since $\int_{S^1} |\dot\varphi_n| \, \mathrm{d}\mathcal{H}^1 = \int_{S^1} |\dot g_n| \, \mathrm{d}\mathcal{H}^1$, using the same argument as above, we may assume that $\varphi_n \to \varphi$ a.e. and L^1 for some $\varphi \in BV(S^1,\mathbb{R})$. Therefore, $e^{i\varphi} = g$ a.e. on S^1 and

$$E(g) \le \int_{S^1} |\dot\varphi| \le \liminf_{n\to\infty} \int_{S^1} |\dot\varphi_n| \, \mathrm{d}\mathcal{H}^1 = \liminf_{n\to\infty} \int_{S^1} |\dot g_n| \, \mathrm{d}\mathcal{H}^1.$$

For "$\ge$", consider a BV lifting φ of g and take an approximating sequence $\varphi_n \in C^\infty(S^1,\mathbb{R})$ such that $\varphi_n \to \varphi$ a.e. and $|\dot\varphi|(S^1) = \lim_{n\to\infty} \int_{S^1} |\dot\varphi_n| \, \mathrm{d}\mathcal{H}^1$. With $g_n = e^{i\varphi_n} \in C^\infty(S^1,S^1)$, we have $\deg g_n = 0$, $g_n \to g$ a.e. on S^1 and

$$E_{\rm rel}(g) \le \lim_{n\to\infty} \int_{S^1} |\dot g_n| \, \mathrm{d}\mathcal{H}^1 = \lim_{n\to\infty} \int_{S^1} |\dot\varphi_n| \, \mathrm{d}\mathcal{H}^1 = \int_{S^1} |\dot\varphi|.$$

We next prove Lemmas 1 and 2:

Proof of Lemma 1. If $(\dot g)_J = 0$, i.e. $A(g) = \varnothing$, then g is continuous on S^1. We claim that $m(g) = \deg g \in \mathbb{Z}$. This is clear when g is smooth; the general case is obtained by approximating g with a sequence $(g_n)_n \subset C^\infty(S^1,S^1)$ such that $g_n \to g$ uniformly and $\dot g_n \rightharpoonup \dot g$ weakly* as $n \to \infty$. Otherwise, let a_1 be a jump point of g on S^1. Consider $S^1 \setminus \{a_1\}$ as an interval and on that interval take $\varphi^{ac}, \varphi^C, \varphi^J$ the BV functions (unique up to constants) having as derivatives in $S^1 \setminus \{a_1\}$ the finite Radon measures $g \wedge (\dot g)_{ac}$, $g \wedge (\dot g)_C$ and $\sum_{a\in A(g)\setminus\{a_1\}} d_a(g)\delta_a$. Let $\varphi = \varphi^{ac} + \varphi^C + \varphi^J$. By the chain rule (see [AD] or [AFP]), we have

$$\dot{(ge^{-i\varphi})} = 0 \text{ on } S^1 \setminus \{a_1\}$$

so that, up to a constant, φ is a lifting of g, i.e. $g = e^{i\varphi}$ a.e. on S^1. Clearly, $\varphi \in BV(S^1,\mathbb{R})$, $\varphi(a_1+) - \varphi(a_1-) = d_{a_1}(g) + 2\pi\alpha$, $\alpha \in \mathbb{Z}$ and

$$\dot\varphi = \dot\varphi\Big|_{S^1\setminus\{a_1\}} + (\varphi(a_1+) - \varphi(a_1-))\delta_{a_1} = g \wedge (\dot g)_{ac} + g \wedge (\dot g)_C + \sum_{a\in A(g)} d_a(g)\delta_a + 2\pi\alpha\delta_{a_1}.$$

Since $\int_{S^1} \dot{\varphi} = 0$ we conclude that $S(g) + L(g) = -2\pi\alpha \in 2\pi\mathbb{Z}$. ∎

Proof of Lemma 2. : After passing to the conjugate function $\bar{g}$ of g if necessary, we may assume that $m(g) \geq 0$. We start by noting that

$$\inf_{\substack{\alpha_a \in \mathbb{Z},\, a \in A(g) \\ \#\{a \in A(g)\,:\,\alpha_a \neq 0\} < \infty \\ \sum_{a \in A(g)} \alpha_a = m(g)}} \sum_{a \in A(g)} |d_a(g) - 2\pi\alpha_a| = \inf_{\substack{\alpha_a \in \mathbb{Z},\, a \in A(g) \\ \#\{a \in A(g)\,:\,\alpha_a \neq 0\} < \infty \\ \sum_{a \in A(g)} \alpha_a = m(g) \\ |\alpha_a - \alpha_b| \leq 1,\, \forall a, b \in A(g)}} \sum_{a \in A(g)} |d_a(g) - 2\pi\alpha_a|. \tag{5}$$

Indeed, it suffices to observe that, if $d_1, d_2 \in (-\pi, \pi], \alpha_1, \alpha_2 \in \mathbb{Z}$ such that $\alpha_1 - \alpha_2 \geq 2$, then

$$|d_1 - 2\pi\alpha_1| + |d_2 - 2\pi\alpha_2| \geq |d_1 - 2\pi(\alpha_1 - 1)| + |d_2 - 2\pi(\alpha_2 + 1)|. \tag{6}$$

We distinguish in our analysis the following cases:
Case 1: $m(g) \geq \#A(g) > 0$. Then, by (5), we have that $\alpha_a \geq 1$, $\forall a \in A(g)$. It follows that $|d_a(g) - 2\pi\alpha_a| = 2\pi\alpha_a - d_a(g)$, $\forall a \in A(g)$. Therefore,

$$E_J(g) = 2\pi m(g) - S(g) = L(g) \geq 0.$$

The minimum is achieved in (2); consider, for example, the choice

$$(\alpha_a)_{a \in A(g)} = \Big(1, \ldots, 1, m(g) - \#A(g) + 1\Big).$$

Case 2: $0 < m(g) < \#A(g)$. By (5), we must have $\alpha_a \in \{0, 1\}$, $\forall a \in A(g)$. Therefore, the RHS of (5) is equal to

$$\inf_{\substack{\alpha_a \in \{0,1\} \\ \#\{a \in A(g)\,:\,\alpha_a \neq 0\} < \infty \\ \sum_{a \in A(g)} \alpha_a = m(g)}} \sum_{a \in A(g)} |d_a(g) - 2\pi\alpha_a| = L(g) + 2 \inf_{\substack{B \subset A(g) \\ \#B = m(g)}} \sum_{\substack{a \in A(g) \setminus B \\ d_a(g) > 0}} d_a(g);$$

this follows by noting that the a's for which $\alpha_a = 1$ have to be the ones with the largest positive jumps $d_a(g)$. The infimum is achieved in (5). Indeed, set

$$\tilde{A}(g) = \{a \in A(g) : d_a(g) > 0\}.$$

If $\#\tilde{A}(g) \geq m(g)$, then choose

$$B = \{a_1, \ldots, a_{m(g)}\} \subset \tilde{A}(g)$$

such that $d_{a_1}(g), \ldots, d_{a_{m(g)}}(g)$ are the biggest $m(g)$ elements of the set $\{d_a(g) \,:\, a \in \tilde{A}(g)\}$. If $\#\tilde{A}(g) < m(g)$, then choose $B \subset A(g)$ such that $\#B = m(g)$ and $\tilde{A}(g) \subset B$. Then an optimal choice is

$$\alpha_a = \begin{cases} 1 & \text{if } a \in B \\ 0 & \text{if } a \in A(g) \setminus B \end{cases}.$$

Case 3: $m(g) = 0$. Here the RHS of (5) is equal to $\sum_{a \in A(g)} |d_a(g)|$ and the infimum (5) is achieved for $\alpha_a = 0, \forall a \in A(g)$. ∎

Proof of Theorem 1.

"$\geq$": Let $\varphi \in BV(S^1, \mathbb{R})$ be a lifting of g on S^1, i.e. $g = e^{i\varphi}$ a.e. on S^1. Then, by the chain rule,

$$(\dot{\varphi})_{ac} + (\dot{\varphi})_C = g \wedge ((\dot{g})_{ac} + (\dot{g})_C)$$

$$\text{and} \quad (\dot\varphi)_J = \sum_{a\in A(g)} (\varphi(a+) - \varphi(a-))\delta_a + \sum_{b\in B}(\varphi(b+) - \varphi(b-))\delta_b.$$

Here,

1. $B \subset S^1$ is some finite set such that $A(g) \cap B = \varnothing$,
2. $\varphi(a+) - \varphi(a-) = d_a(g) - 2\pi\alpha_a$ with $\alpha_a \in \mathbb{Z}, \forall a \in A(g)$,
3. $\varphi(b+) - \varphi(b-) = -2\pi\alpha_b$ where $\alpha_b \in \mathbb{Z}, \forall b \in B$.

Clearly

$$\#\{a \in A(g) : \alpha_a \neq 0\} < \infty.$$

Since $\displaystyle\int_{S^1} \dot\varphi = 0$, we get $\displaystyle\sum_{a\in A(g)\cup B} \alpha_a = \frac{L(g) + S(g)}{2\pi} = m(g)$. We have

$$|\dot\varphi|(S^1) = \int_{S^1} \Big(|(\dot\varphi)_{ac}| + |(\dot\varphi)_C|\Big) + \sum_{a\in A(g)} |d_a(g) - 2\pi\alpha_a| + 2\pi \sum_{b\in B} |\alpha_b|.$$

If $A(g) = \varnothing$, then

$$|\dot\varphi|(S^1) \geq \tilde E(g) + 2\pi\Big|\sum_{b\in B} \alpha_b\Big| = \tilde E(g) + |L(g)|,$$

which is the desired inequality. Otherwise, take $a_1 \in A(g)$ and observe that

$$|\dot\varphi|(S^1) \geq \tilde E(g) + \sum_{a\in A(g)\setminus\{a_1\}} |d_a(g) - 2\pi\alpha_a| + |d_{a_1}(g) - 2\pi\tilde\alpha_{a_1}|$$

where $\tilde\alpha_{a_1} = \alpha_{a_1} + \displaystyle\sum_{b\in B} \alpha_b$. Therefore, we conclude that

$$E(g) \geq \tilde E(g) + E_J(g).$$

"$\leq$" (*The construction of an optimal lifting*): If $A(g) = \varnothing$, then g is continuous on the simply connected set $S^1\setminus\{1\}$ and so there is a unique (up to $2\pi\mathbb{Z}$ constants) lifting $\varphi \in BV(S^1\setminus\{1\}, \mathbb{R}) \cap C^0$ of g on $S^1 \setminus \{1\}$. Moreover, $\varphi(1-) - \varphi(1+) = L(g)$ and we conclude that

$$|\dot\varphi|(S^1) = \tilde E(g) + |L(g)|.$$

Otherwise, take $a_1 \in A(g)$. By Lemma 2, we may take integers $\alpha_a \in \mathbb{Z}, \forall a \in A(g)$ (all zero except a finite number) such that $\displaystyle\sum_{a\in A(g)} \alpha_a = m(g)$ and (2) holds, i.e.

$$\sum_{a\in A(g)} |d_a(g) - 2\pi\alpha_a| = E_J(g).$$

As in the proof of Lemma 1, construct a lifting $\varphi \in BV(S^1, S^1)$ of g satisfying on $S^1 \setminus \{a_1\}$

$$(\dot\varphi)_{ac} = g \wedge (\dot g)_{ac}$$

$$(\dot\varphi)_C = g \wedge (\dot g)_C$$

$$\text{and} \quad (\dot\varphi)_J\Big|_{S^1\setminus\{a_1\}} = \sum_{a\in A(g)\setminus\{a_1\}} (d_a(g) - 2\pi\alpha_a)\delta_a \;\text{ on }\; S^1 \setminus \{a_1\}.$$

Since $\displaystyle\int_{S^1} \dot\varphi = 0$, we find that $\varphi(a_1+) - \varphi(a_1-) = d_{a_1}(g) - 2\pi\alpha_{a_1}$ which implies that $|\dot\varphi|(S^1) = \tilde E(g) + E_J(g)$. ■

Note that the optimal lifting is not unique modulo 2π; indeed, if

$$g(e^{it}) = \begin{cases} 1 & \text{if } t \in (0,\pi) \\ -1 & \text{if } t \in (\pi, 2\pi) \end{cases}$$

then

$$\varphi_1(e^{it}) = \begin{cases} 0 & \text{if } t \in (0,\pi) \\ -\pi & \text{if } t \in (\pi, 2\pi) \end{cases} \quad \text{and} \quad \varphi_2(e^{it}) = \begin{cases} 0 & \text{if } t \in (0,\pi) \\ \pi & \text{if } t \in (\pi, 2\pi) \end{cases}$$

are optimal liftings.

Remark 2 As we have proved, $E(g)$ depends on $(d_a(g))_{a\in A(g)}$ where $d_a(g)$ is the unique argument of the complex number $\frac{g(a+)}{g(a-)}$ in $(-\pi,\pi]$. Consider now, for each $a \in A(g)$, an arbitrary argument $d'_a(g)$ of $\frac{g(a+)}{g(a-)}$ such that $\displaystyle\sum_{a\in A(g)} |d'_a(g)| < \infty$. It is easy to see that

$$m'(g) = \frac{L(g) + \sum_{a\in A(g)} d'_a(g)}{2\pi} \in \mathbb{Z}.$$

Observe that if $A(g) \neq \varnothing$, then

$$E_J(g) = \min_{\substack{\alpha_a \in \mathbb{Z},\, a\in A(g) \\ \#\{a\in A(g)\,:\,\alpha_a \neq 0\} < \infty \\ \sum_{a\in A(g)} \alpha_a = m'(g)}} \sum_{a\in A(g)} |d'_a(g) - 2\pi\alpha_a|.$$

The analytic formula for $E_J(g)$ in Lemma 2 still holds for the $(d'_a(g))_{a\in A(g)}$ and $m'(g)$ provided $d'_a \in [-2\pi, 2\pi], \forall a \in A(g)$ and $|d'_a(g) - d'_b(g)| \le 2\pi$, $\forall a, b \in A(g)$. This is a consequence of the fact that (6) holds if $d_1, d_2 \in [-2\pi, 2\pi]$ and $|d_1 - d_2| \le 2\pi$.

As an immediate consequence of Lemma 2 and Theorem 1, we have:

Corollary 1 *For every $g \in BV(S^1, S^1)$,*

$$E(g) \le 2|g|_{BV\,S^1}$$

where $\displaystyle |g|_{BV\,S^1} = \int_{S^1} \Big(|(\dot g)_{ac}| + |(\dot g)_C|\Big) + \sum_{a\in A(g)} d_{S^1}(g(a+), g(a-))$.

Remark that $|\cdot|_{BV\,S^1}$ is a seminorm equivalent to the standard BV-seminorm $|\cdot|_{BV}$; in fact, we have

$$|g|_{BV} \le |g|_{BV\,S^1} \le \frac{\pi}{2}|g|_{BV},\ \forall g \in BV(S^1, S^1).$$

Therefore, Theorem 2 is an improvement of the above corollary.

3 First proof of Theorem 2

We start by stating some trigonometric inequalities:

Lemma 4 *Let $n \geq p \geq 1$ be two integers and let $x_k \in [0, \frac{\pi}{2}]$, $k = 1, \dots, n$, be such that $\sum_{k=1}^{n} x_k \leq p\pi + \gamma$. Then*

$$\sum_{k=1}^{n} \sin x_k \geq \sum_{k=1}^{n} x_k - \max_{\substack{B \subset \{1,\dots,n\} \\ \#B = p}} \sum_{k \in B} x_k. \tag{7}$$

Proof. If $n = p$ then the conclusion is straightforward. Suppose now that $n > p$. By symmetry, we can assume that $B = \{x_{n-p+1}, \dots, x_n\}$ contains the biggest p terms among $\{x_1, \dots, x_n\}$. Set $z = \min_{n-p+1 \leq k \leq n} x_k$. It is sufficient to prove that

$$\sum_{k=1}^{n-p} (\sin x_k - x_k) + p \sin z \geq 0.$$

Define the smooth symmetric function

$$f : \mathbb{R}^{n-p} \to \mathbb{R},\ f(x_1, \dots, x_{n-p}) = \sum_{k=1}^{n-p} (\sin x_k - x_k) + p \sin z.$$

Then f is a concave function. We want to find the minimum of f over the convex compact polyhedron

$$D = \{(x_1, \dots, x_{n-p}) \in [0, z]^{n-p} \ : \ \sum_{k=1}^{n-p} x_k \leq p(\pi - z) + \gamma\}.$$

Since f is concave, its minimum on D is achieved in one of the extremal points (i.e. corners) of D. By a permutation of the coordinates, a corner $(x_1, \dots, x_{n-p})$ of D has the following form: either

$$x_i \in \{0, z\},\ \forall k = 1, \dots, n-p$$

or

$$x_k \in \{0, z\},\ \forall k = 1, \dots, n-p-1 \ \text{ and } x_{n-p} = \gamma + p(\pi - z) - \sum_{k=1}^{n-p-1} x_k.$$

In order to prove that $f \geq 0$ on these points of D, we check that: if $k, p \geq 1$ are two integer numbers, $I_1 = [0, \frac{p\pi+\gamma}{k+p}] \cap [0, \frac{\pi}{2}]$ and $I_2 = [\frac{p\pi+\gamma}{k+p+1}, \frac{p\pi+\gamma}{k+p}] \cap [0, \frac{\pi}{2}]$, then

$$(k+p)\sin z - kz \geq 0,\ \forall z \in I_1 \tag{8}$$

and

$$(k+p)\sin z - kz + \sin\big(\gamma + p\pi - (k+p)z\big) - p\pi - \gamma + (k+p)z \geq 0,\ \forall z \in I_2. \tag{9}$$

Indeed, remark that the two LHS of each inequality from above represent concave functions in z and therefore, it is sufficient to show that they are positive on the extremities of the given intervals I_1 and I_2.

For (8), let us denote

$$h(z) = (k+p)\sin z - kz,\ \forall z \in I_1.$$

Case 1: $I_1 = [0, \frac{p\pi+\gamma}{k+p}]$, i.e. $\frac{p\pi+\gamma}{k+p} \le \frac{\pi}{2}$. Then $k \ge p+1$. We have that $h(0) = 0$ and it remains to check that

$$h(\frac{p\pi + \gamma}{k+p}) \ge 0.$$

If $p = 1$ and $k = 2$ then $h(\frac{\pi+\gamma}{3}) = 0$. If $p = 1$ and $k \ge 3$, then the inequality

$$\sin z \ge z - \frac{z^3}{6} \tag{10}$$

yields, for $z = \frac{\pi+\gamma}{k+1}$,

$$h(\frac{\pi+\gamma}{k+1}) \ge \frac{\pi+\gamma}{k+1}\left(1 - \frac{(\pi+\gamma)^2}{6(k+1)}\right) \ge 0.$$

Otherwise, $p \ge 2$ and applying (10) for $z = \frac{p\pi+\gamma}{k+p}$, we obtain

$$h(\frac{p\pi+\gamma}{k+p}) \ge \frac{p\pi+\gamma}{k+p}\left(p - \frac{(p\pi+\gamma)^2}{6(k+p)}\right) \ge \frac{p\pi+\gamma}{k+p}\left(p - \frac{(p\pi+\gamma)^2}{6(2p+1)}\right) \ge 0.$$

Case 2: $I_1 = [0, \frac{\pi}{2}]$, i.e. $k \le p$. Then $h(\frac{\pi}{2}) = k + p - k\frac{\pi}{2} \ge (2 - \frac{\pi}{2})k \ge 0$.

The proof of (9) follows along the same lines. ∎

Remark 3 γ is optimal for the inequality (7) (consider $n = 3$, $p = 1$, $x_1 = x_2 = x_3 = \frac{\pi+\gamma}{3}$).

Proof of Lemma 3. We can assume that $B = \{x_1, \dots, x_p\}$ contains the biggest p terms among $\{x_k : k \ge 1\}$. Let $\varepsilon > 0$. There exists $n > p$ such that $\sum\limits_{k>n} x_k < \varepsilon$. By Lemma 4, we know that

$$\sum_{k=1}^{n} \sin x_k \ge \sum_{k=p+1}^{n} x_k.$$

Therefore,

$$\sum_{k\ge 1} \sin x_k \ge \sum_{k=1}^{n} \sin x_k \ge \sum_{k>p} x_k - \varepsilon.$$

Letting now $\varepsilon \to 0$, the conclusion follows. ∎

We now present:

First proof of Theorem 2. It suffices to prove that

$$E_J(g) \le \int_{S^1} \Big(|(\dot g)_{ac}| + |(\dot g)_C|\Big) + 2\sum_{a\in A(g)} |g(a+) - g(a-)|. \tag{11}$$

If $A(g) = \varnothing$, the conclusion follows using the inequality

$$|L(g)| \le \int_{S^1} \Big(|(\dot g)_{ac}| + |(\dot g)_C|\Big).$$

If $m(g) = 0$, (11) is a consequence of the fact that

$$|d_a(g)| \le \frac{\pi}{2}|g(a+) - g(a-)|,\ \forall a \in A(g).$$

Suppose now that $A(g) \neq \varnothing$ and $m(g) \neq 0$; as before, we may assume $m(g) > 0$. As in the proof of Lemma 2, consider

$$\tilde{A}(g) = \{a \in A(g) : d_a(g) > 0\}.$$

If $\#\tilde{A}(g) \leq m(g)$ then, by Lemma 2, $E_J(g) = |L(g)|$ and so (11) holds.

Otherwise, we have $\#\tilde{A}(g) > m(g) \geq 1$. Rewrite $S(g) + L(g) = 2\pi m(g)$ as

$$\sum_{a\in\tilde{A}(g)} d_a(g) - \sum_{a\in A(g)\setminus\tilde{A}(g)} |d_a(g)| + L(g) = 2\pi m(g). \tag{12}$$

Let $B \subset \tilde{A}(g)$ consist of the largest $m(g)$ elements of the set $\{d_a(g) : a \in \tilde{A}(g)\}$. For each $a \in \tilde{A}(g)$, set

$$x_a = \frac{d_a(g)}{2} \in [0, \frac{\pi}{2}].$$

Then $|g(a+) - g(a-)| = 2\sin x_a$. We distinguish two cases:

Case 1:

$$\sum_{a\in A(g)\setminus\tilde{A}(g)} |d_a(g)| - L(g) \leq 2\gamma$$

where γ is given in Lemma 3. By (12), $\displaystyle\sum_{a\in\tilde{A}(g)} d_a(g) \leq 2\pi m(g) + 2\gamma$. By Lemma 3, we have

$$\sum_{a\in\tilde{A}(g)\setminus B} d_a(g) \leq \sum_{a\in\tilde{A}(g)} |g(a+) - g(a-)|.$$

Using Lemma 2, we find that

$$E_J(g) = L(g) + 2\sum_{a\in\tilde{A}(g)\setminus B} d_a(g) \leq |L(g)| + 2\sum_{a\in A(g)} |g(a+) - g(a-)|.$$

Case 2:

$$\sum_{a\in A(g)\setminus\tilde{A}(g)} |d_a(g)| - L(g) > 2\gamma, \text{ i.e. } \sum_{a\in\tilde{A}(g)} d_a(g) > 2\pi m(g) + 2\gamma. \tag{13}$$

The following two situations can occur:

i) There exists $A_1 \subset \tilde{A}(g)$ such that $B \subset A_1$ and

$$2\pi m(g) \leq \sum_{a\in A_1} d_a(g) \leq 2\pi m(g) + 2\gamma. \tag{14}$$

By (14), using Lemma 3, we infer that

$$\sum_{a\in A_1\setminus B} d_a(g) \leq \sum_{a\in A_1} |g(a+) - g(a-)|. \tag{15}$$

With $A_2 = \tilde{A}(g) \setminus A_1$, it follows from (12) and (14) that

$$\sum_{a\in A_2} d_a(g) - \sum_{a\in A(g)\setminus\tilde{A}(g)} |d_a(g)| + L(g) \leq 0.$$

By adding $\displaystyle\sum_{a\in A_2} d_a(g)$, we obtain

$$2\sum_{a\in A_2} d_a(g) + L(g) \leq \sum_{a\in A_2\cup(A(g)\setminus\tilde{A}(g))} |d_a(g)| \leq \frac{\pi}{2} \sum_{a\in A_2\cup(A(g)\setminus\tilde{A}(g))} |g(a+) - g(a-)|. \tag{16}$$

Combining (15) and (16), we deduce

$$E_J(g) = L(g) + 2 \sum_{a \in \tilde{A}(g) \setminus B} d_a(g) \le 2 \sum_{a \in A(g)} |g(a+) - g(a-)|.$$

ii) There exist $A_1 \subset \tilde{A}(g)$ and $\tilde{a} \in \tilde{A}(g) \setminus A_1$ such that $B \subset A_1$ and

$$2\pi m(g) + 2\gamma - d_{\tilde{a}}(g) < \sum_{a \in A_1} d_a(g) < 2\pi m(g).$$

Set $A_2 = \tilde{A}(g) \setminus (A_1 \cup \{\tilde{a}\})$. By (12), we have

$$\sum_{a \in A_2} d_a(g) - \sum_{a \in A(g) \setminus \tilde{A}(g)} |d_a(g)| + L(g) \le -2\gamma. \tag{17}$$

By adding $\sum_{a \in A_2} d_a(g)$ to (17), we find that

$$\begin{aligned} 2 \sum_{a \in A_2} |d_a(g)| + L(g) &\le -2\gamma + \sum_{a \in A_2 \cup (A(g) \setminus \tilde{A}(g))} |d_a(g)| \\ &= -2\gamma + \frac{4}{\pi} \sum_{a \in A_2 \cup (A(g) \setminus \tilde{A}(g))} |d_a(g)| - (\frac{4}{\pi} - 1) \sum_{a \in A_2 \cup (A(g) \setminus \tilde{A}(g))} |d_a(g)|. \end{aligned}$$

From (13), we get that

$$2\gamma + L(g) \le \sum_{a \in A(g) \setminus \tilde{A}(g)} |d_a(g)| \le \sum_{a \in A_2 \cup (A(g) \setminus \tilde{A}(g))} |d_a(g)|.$$

Therefore,

$$\begin{aligned} 2 \sum_{a \in A_2} d_a(g) + L(g) &\le -2\gamma + \frac{4}{\pi} \sum_{a \in A_2 \cup (A(g) \setminus \tilde{A}(g))} |d_a(g)| - (\frac{4}{\pi} - 1)\Big(2\gamma + L(g)\Big) \\ &\le \frac{4}{\pi} \sum_{a \in A_2 \cup (A(g) \setminus \tilde{A}(g))} |d_a(g)| + |L(g)| - \frac{8\gamma}{\pi} \\ &\le 2 \sum_{a \in A_2 \cup (A(g) \setminus \tilde{A}(g))} |g(a+) - g(a-)| + |L(g)| - \frac{8\gamma}{\pi}; \end{aligned} \tag{18}$$

here, we have used the fact that $|d_a(g)| \le \dfrac{\pi}{2}|g(a+) - g(a-)|$, $\forall a \in A(g)$. On the other side, Lemma 3 yields

$$\sum_{a \in A_1 \setminus B} d_a(g) \le \sum_{a \in A_1} |g(a+) - g(a-)|. \tag{19}$$

Remark also that

$$d_{\tilde{a}}(g) \le |g(\tilde{a}+) - g(\tilde{a}-)| + (\pi - 2); \tag{20}$$

this amounts to the inequality $x \le 2 \sin \frac{x}{2} + \pi - 2$, $\forall x \in [0, \pi]$. By combining (18), (19) and (20), we obtain

$$\begin{aligned} E_J(g) &= L(g) + 2 \sum_{a \in A_2} d_a(g) + 2 \sum_{a \in A_1 \setminus B} d_a(g) + 2 d_{\tilde{a}}(g) \\ &\le |L(g)| + 2 \sum_{a \in A(g)} |g(a+) - g(a-)| + 2(\pi - 2) - \frac{8\gamma}{\pi}. \end{aligned}$$

Since $2(\pi - 2) - \frac{8\gamma}{\pi} < 0$, the conclusion follows. ∎

4 Another proof of Theorem 2

We adapt below the proof in [DI]. In that paper, we proved the estimate (4) for $BV(\Omega, S^1)$ functions, where $\Omega \subset \mathbb{R}^N$ is a bounded open set. The idea is to consider the function $f : S^1 \to \mathbb{R}$ defined by

$$f(e^{i\theta}) = \theta \quad \text{for every } \theta \in [-\pi, \pi)$$

and to show that for an appropriate $\alpha \in \mathbb{R}$, the lifting

$$\varphi = f(e^{i\alpha} g) - \alpha$$

satisfies $|\varphi|_{BV} \le 2|g|_{BV}$. For that, one can repeat the same arguments in [DI] and prove that

$$\int_0^{2\pi} |f(e^{i\alpha} g) - \alpha|_{BV} \, \mathrm{d}\alpha \le 4\pi |g|_{BV}; \tag{21}$$

conclude by noting that $\alpha \mapsto |f(e^{i\alpha} g)|_{BV}$ is lower semi-continuous with values in $[0, \infty]$.

Remark 4 i) Set $\mathscr{C}(g) = \{f(e^{i\alpha} g) - \alpha \, : \, \alpha \in \mathbb{R}\}$. Then $\mathscr{C}(g)$ need not be contained in BV. Here is an example. Consider the step function $g \in BV(S^1, S^1)$ defined by

$$g(e^{2\pi i t}) = e^{i x_k}, \, t \in \left(\frac{1}{2^k}, \frac{1}{2^{k-1}}\right), \, k = 1, 2, \dots$$

where $x_k = (-1)^k 2^{-[\frac{k+1}{2}]}\pi$. It is easy to see that

$$|f(e^{i\pi} g) - \pi|_{BV} = |x_1 + 2\pi| + \sum_{k \ge 1} |x_{k+1} - x_k + (-1)^k 2\pi| = \infty.$$

ii) It follows from (21) that, for a.e. $\alpha \in [0, 2\pi]$, $f(e^{i\alpha} g) - \alpha \in BV(S^1, \mathbb{R})$; clearly, the same holds for a.e. $\alpha \in \mathbb{R}$.

iii) There exist some functions $g \in BV(S^1, S^1)$ such that no lifting in $\mathscr{C}(g)$ is optimal. For example, consider the step function $g : S^1 \to S^1$ be defined as:

$$g(e^{\pi i t}) = \begin{cases} 1 & \text{if } \ t \in (0, \frac{1}{7}) \\ e^{i\frac{\pi}{2}} & \text{if } \ t \in (\frac{1}{7}, \frac{2}{7}) \\ e^{i\frac{3\pi}{4}} & \text{if } \ t \in (\frac{2}{7}, \frac{3}{7}) \\ e^{i\frac{(k+2)\pi}{4}} & \text{if } \ t \in (\frac{k}{7}, \frac{k+1}{7}), \, k = 3, \dots, 13 \end{cases}.$$

So g has 2 jumps of length $\frac{\pi}{2}$ (with respect to d_{S^1}) and 12 jumps of length $\frac{\pi}{4}$. Then $m(g) = 2$ and

$$E(g) = E_J(g) = 12 \cdot \frac{\pi}{4} + 2 \cdot (2\pi - \frac{\pi}{2}).$$

Remark now that for every $\alpha \in \mathbb{R}$, the cut $\{z \in \mathbb{C} \, : \, arg(z) = \pi - \alpha \ (\text{mod } 2\pi)\}$ of the function $z \to f(e^{i\alpha} z) - \alpha$ will affect two jumps of g and at least one of them has the size $\frac{\pi}{4}$ (with respect to the geodesic distance d_{S^1} on S^1). Therefore,

$$|f(e^{i\alpha} g) - \alpha|_{BV} \ge \frac{\pi}{2} + 11 \cdot \frac{\pi}{4} + (2\pi - \frac{\pi}{4}) + (2\pi - \frac{\pi}{2}) > E(g).$$

5 Some examples

We present some examples showing that the constant 2 in (4) is optimal (see also Brezis-Mironescu-Ponce [BMP]).

1. Let $g = Id : S^1 \to S^1$. Then $g \in BV(S^1, S^1) \cap C^0$. Remark that $(\dot{g})_C = (\dot{g})_J = 0$. Thus, $\deg g = 1$, $\tilde{E}(g) = E_J(g) = |g|_{BV} = 2\pi$ and so $E(g) = 2|g|_{BV}$.

2. Let $f : [0,1] \to [0,1]$ be the standard Cantor function. Define $g : S^1 \to S^1$ as

$$g(e^{2\pi i\, t}) = e^{2\pi i\, f(t)}, \forall t \in [0,1].$$

Clearly, $g \in BV(S^1, S^1) \cap C^0$, $(\dot{g})_{ac} = (\dot{g})_J = 0$ and $\deg g = 1$. As above, $\tilde{E}(g) = E_J(g) = |g|_{BV} = 2\pi$ and $E(g) = 2|g|_{BV}$.

3. For each $n \geq 2$, take $g_n(e^{2\pi i\, t}) = e^{2\pi i\, k/n}$ for $\frac{k}{n} \leq t < \frac{k+1}{n}$, $k = 0, 1, \ldots, n-1$. Then $g_n \in BV(S^1, S^1)$ and $(\dot{g}_n)_{ac} = (\dot{g}_n)_C = 0$. We have that $\tilde{E}(g_n) = 0$, $m(g_n) = 1$, $E_J(g_n) = 4\pi(1 - \frac{1}{n})$ and $|g_n|_{BV} = 2n \sin \frac{\pi}{n}$. We deduce that

$$\lim_{n \to \infty} \frac{E(g_n)}{|g_n|_{BV}} = 2.$$

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