Lower bound for the energy of Bloch walls in micromagnetics

Radu Ignat [*]   Benoît Merlet [†]

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Abstract

We study a $2D$ nonconvex and nonlocal variational model in micromagnetics. It consists in a free-energy functional defined over vector fields with values into the unit sphere S^2. This energy depends on two small parameters β and ε penalizing the divergence of the vector field and its vertical component, respectively. We are interested in the analysis of the asymptotic regime $\beta \ll \varepsilon \ll 1$ through the method of $\Gamma-$convergence. Finite energy configurations tend to become in-plane in the magnetic sample except in some small regions of length scale ε (called Bloch walls) where the magnetization varies rapidly between two directions on S^2. The limiting magnetizations are in-plane unit vector fields of vanishing divergence having an $\mathcal{H}^1-$rectifiable jump set. We prove that the $\Gamma-$limit energy concentrates on the jump set of the limiting configurations and the energetic cost of a jump is quadratic in the size of the jump. The exact charge of the jump is computed by a $\Gamma-$convergence analysis for $1D$ transition layers. Using the concept of entropies, we find lower bounds for the $2D$ model that coincide with the $\Gamma-$limit in $1D$ in some particular cases. Finally, we show that entropies are not appropriate in general for the $2D$ model in order to obtain the full $\Gamma-$limit.

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1 Introduction

In this paper, we consider a simple model for Bloch walls in micromagnetics. Micromagnetics is a variational principle for ferromagnetic samples of small size. The state of a ferromagnetic sample occupying a region $\Omega \subset \mathbf{R}^3$ is characterized by its magnetization

$$m \ : \ \Omega \to \mathbf{R}^3.$$

The magnitude of the magnetization is considered to be constant (for a fixed temperature); therefore, in the nondimensionalized form, m satisfies the nonconvex constraint

$$|m| = 1 \text{ in } \Omega.$$

The micromagnetic principle states that the magnetization m corresponds to a (local) minimizer of the following free-energy functional (written here in the absence of external magnetic field):

$$\mathcal{E}^{3D}(m) \quad := \quad d^2 \int_\Omega |\nabla m|^2 + \int_\Omega \varphi(m) + \int_{\mathbf{R}^3} |H(m)|^2. \tag{1}$$

[*]Laboratoire de Mathématiques, Université Paris-Sud 11, Bât. 425, 91405 Orsay, France (e-mail: Radu.Ignat@math.u-psud.fr)
[†]Université Paris Nord - Institut Galilée LAGA (Laboratoire d'Analyse, Géométrie et Applications), Avenue J.B. Clément, 93430 Villetaneuse, France (e-mail: merlet@math.univ-paris13.fr)

The first term is called the exchange energy and penalizes variations of m. The characteristic constant d is called the exchange length and is an intrinsic parameter of the material (of order of nanometers).

The second term represents the anisotropy energy. It favors some easy axes for the magnetization corresponding to global minima of $\varphi : S^2 \to \mathbf{R}_+$.

The last term in (1) is the magnetostatic or stray-field energy. The stray-field $H(m) : \mathbf{R}^3 \to \mathbf{R}^3$ is a $3D$ vector field induced by the magnetization via the static Maxwell equation:

$$\begin{cases} \nabla \times H(m) = 0 & \text{in } \mathbf{R}^3, \\ \nabla \cdot H(m) = -\nabla \cdot \Big(m \mathbf{1}_\Omega \Big) & \text{in } \mathbf{R}^3, \end{cases}$$

that is, $H(m) = \nabla(-\Delta)^{-1}\nabla \cdot \Big(m\mathbf{1}_\Omega \Big)$. Therefore, the stray field is generated both by volume charges (given by the divergence $\nabla \cdot m$ of m inside the sample Ω) and surface charges (carried by the normal component $m \cdot \boldsymbol{n}$ of the magnetization on the boundary $\partial\Omega$). It implies that a stable state favors flux-closure configurations in order that the stray field energy is avoided (that is the principle of pole avoidance). For more details, see the books of Brown [7] and Hubert and Schäfer [13].

The difficulty of the variational principle comes from the nonconvex constraint on the magnetization and on the nonlocal character of the stray field interaction. Together with the multi-scale nature of the system, it leads to a rich pattern formation for the magnetization. Generically, a pattern of a stable state consists in large uniformly magnetized regions (called magnetic domains) that are separated by narrow transition layers (domain walls) where the direction of the magnetization varies quickly.

Physical experiments put in evidence these different behaviors of the ferromagnets. The variety of the transition layers is explained by the competition between the three energy terms of (1) (and, in some cases, an additional term due to an applied external field). From the mathematical point of view, it is natural to study various asymptotic regimes accounting for the differences between the leading order of the energy terms (see e.g. DeSimone, Kohn, Müller and Otto [10], Rivière and Serfaty [21], Alouges, Rivière and Serfaty [2] and the overview of DeSimone, Kohn, Müller and Otto [12]). Our goal is to study one of the transition layers of the magnetization, called the Bloch wall, in a special asymptotic regime through a $\Gamma-$convergence analysis.

1.1 Our model

We consider a ferromagnetic sample corresponding to an infinite cylinder $\Omega = \omega \times \mathbf{R}$ where $\omega \subset \mathbf{R}^2$ is a two-dimensional bounded domain with Lipschitz boundary. Let $\ell = \mathrm{diam}\,(\omega)$ be the length scale of the domain ω and let $\boldsymbol{n}$ be the unit normal vector at $\partial\omega$. Here, we discuss the case of a surface anisotropy of the form

$$\varphi(m) \quad = \quad Qm_3^2,$$

where the easy plane is the horizontal one. The quality factor $Q > 0$ is an intrinsic and nondimensionalized parameter of the magnetic material that spans six orders of magnitude (e.g., from $2,5 \times 10^{-4}$ in Permalloy to 38 in $SmCo_5$). We also assume that m does not depend on the x_3-direction, i.e,

$$m = m(x_1, x_2) \quad \text{and} \quad m \in H^1(\omega, S^2).$$

We are led to study the following two-dimensional functional corresponding to the energy (1) per unit length in the x_3-direction:

$$\mathcal{E}^{2D}(m) \quad := \quad d^2 \int_\omega |\nabla m|^2 + Q \int_\omega m_3^2 + \int_{\mathbf{R}^2} |h(m')|^2.$$

Throughout the paper, we always use the notation $m = (m', m_3)$ with $m' = (m_1, m_2)$ and the differential operator

$$\nabla = (\partial_1, \partial_2).$$

The two-dimensional stray field $h(m') : \mathbf{R}^2 \to \mathbf{R}^2$ is defined by

$$\begin{cases} \nabla \times h(m') = 0 & \text{in } \mathbf{R}^2, \\ \nabla \cdot h(m') = -\nabla \cdot \Big(m' \mathbf{1}_\omega \Big) & \text{in } \mathbf{R}^2, \end{cases}$$

and corresponds to $h(m') = \nabla(-\Delta)^{-1}\nabla \cdot (m'\mathbf{1}_\omega)$; therefore,

$$\int_{\mathbf{R}^2} |h(m')|^2 = \|\nabla \cdot (m'\mathbf{1}_\omega)\|^2_{\dot{H}^{-1}(\mathbf{R}^2)} := \int_{\mathbf{R}^2} \frac{1}{|\xi|^2} |\mathcal{F}(\nabla \cdot (m'\mathbf{1}_\omega))|^2 \, d\xi,$$

where the Fourier transform of a function $v : \mathbf{R}^2 \to \mathbf{R}$ is denoted by $\mathcal{F}(v)(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}^2} e^{-i\xi \cdot x} v(x)\, dx$, $\forall \xi \in \mathbf{R}^2$.

We nondimensionalize all the quantities in order to identify the different scales in the energy terms. Setting $\tilde{x} := \frac{x}{\ell}$, $\tilde{\omega} := \frac{\omega}{\ell}$, $\tilde{m}(\tilde{x}) := m(x)$, $\tilde{h}(\tilde{m}') = h(m')$, $\varepsilon := \frac{d}{\sqrt{Q}\ell}$ and $\beta := 2\sqrt{Q}\frac{d}{\ell}$, we will focus on the renormalized energy $\tilde{E}_{\varepsilon,\beta}(\tilde{m}) := \frac{1}{2\sqrt{Q}\ell d} \mathcal{E}^{2D}(m)$, i.e.,

$$\tilde{E}_{\varepsilon,\beta}(\tilde{m}) \quad := \quad \frac{\varepsilon}{2} \int_{\tilde{\omega}} |\tilde{\nabla} \tilde{m}|^2 \, d\tilde{x} + \frac{1}{2\varepsilon} \int_{\tilde{\omega}} \tilde{m}_3^2 \, d\tilde{x} + \frac{1}{\beta} \int_{\mathbf{R}^2} |\tilde{h}(\tilde{m}')|^2 \, d\tilde{x}. \tag{2}$$

In the following, we omit the tilde $\tilde{}$ for our variables.

We are interested in the following asymptotic regime:

$$\varepsilon \ll 1 \qquad \text{and} \qquad \beta \ll \varepsilon. \tag{3}$$

We expect the limiting states of the magnetization to satisfy the flux-closure constraint as $\varepsilon \downarrow 0$ (and by (3), $\beta \downarrow 0$), i.e,

$$\nabla \cdot (m'\mathbf{1}_\omega) \quad \equiv \quad 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^2) \tag{4}$$

and to be in-plane vector fields ($m_3 = 0$), i.e.,

$$m' \in S^1 \quad \text{ a.e. in } \ \omega. \tag{5}$$

(In the sequel, we will always identify the plane $\mathbf{R}^2$ with $\mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$; in particular, we identify the unit circle $S^1 \subset \mathbf{R}^2$ and $S^1 \times \{0\} \subset S^2$.)

Due to (3), the leading order term in (2) is the magnetostatic energy so that for a minimizer of $E_{\varepsilon,\beta}$, the stray-field energy (penalizing the constraint (4)) is asymptotically stronger than the planar anisotropy (leading to (5)). This regime is different than the one considered in [2, 21] where $\varepsilon \ll \beta$, i.e., the anisotropy was more expensive than the stray field energy.

Our aim is to study the asymptotic of the energy (2) in the regime (3) in order to deduce the limit energy in the spirit of $\Gamma-$convergence. More precisely, we consider families of maps $\{m_\varepsilon\}_{\varepsilon\downarrow 0} \subset H^1(\omega, S^2)$ such that the following condition holds true for $\beta = \beta(\varepsilon) \ll \varepsilon$:

$$\limsup_{\varepsilon \downarrow 0} \, E_{\varepsilon,\beta}(m_\varepsilon) \ < \ \infty. \tag{6}$$

We first analyze the limiting configurations m_0 of such families of magnetizations $\{m_\varepsilon\}$ as $\varepsilon \downarrow 0$. Then we compute a lower bound energy E_0 that satisfies the inequality

$$E_0(m_0) \le \liminf_{\varepsilon \downarrow 0} \, E_{\varepsilon,\beta}(m_\varepsilon).$$

Clearly, every strong L^1-limit m_0 of a family $\{m_\varepsilon\}_{\varepsilon\downarrow 0}$ of uniformly bounded energy $E_{\varepsilon,\beta}(m_\varepsilon) \le C$ in the regime (3) must satisfy (4) & (5). The problem associated to these two conditions is rather rigid for smooth solutions. Indeed, the condition (4) implies that

$$\nabla \cdot m_0 = 0 \text{ in } \omega \text{ and } m_0 \cdot \boldsymbol{n} = 0 \text{ on } \partial\omega$$

and thus, there exists a stream function ψ such that $m_0 = \nabla^\perp \psi := (-\partial_2\psi, \partial_1\psi)$. The constraint (5) means that ψ satisfies the eikonal equation $|\nabla\psi| = 1$ with a constant Dirichlet boundary condition, i.e., $\partial_{\boldsymbol{\tau}}\psi = 0$ on $\partial\omega$ (because of (4)). The method of characteristics implies that $\nabla\psi$ generates line-singularities. Therefore, we expect that m_0 should be smooth away from an $\mathcal{H}^1-$rectifiable set J oriented by a unit normal vector $\boldsymbol{\nu}$. It is important to observe that the normal component of m_0 does not jump across the singular set J because of (4), i.e.,

$$m_0^+ \cdot \boldsymbol{\nu} = m_0^- \cdot \boldsymbol{\nu} \text{ on } J,$$

where m_0^+ and m_0^- are the one-sided traces of m_0 on J (see Theorem 1 for a precise definition). Therefore, each jump singularity is determined by the angle $\theta = 2\arccos(m_0^+ \cdot \boldsymbol{\nu})$. The line-singularities of m_0 have a physical meaning: they represent an idealization of domain walls of the magnetization at the mesoscopic level. At the microscopic level, these one-dimensional singularities are replaced by narrow two-dimensional regions (called *Bloch walls*) where the magnetization behaves like a smooth transition layer that quickly varies in S^2 between two given states $m_0^\pm$ of angle θ (called *wall angle*).

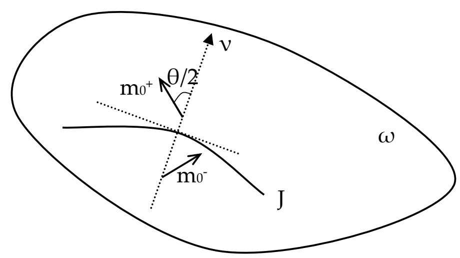

Figure 1: Jump set of a limit magnetization m_0.

For such limiting configurations m_0, it is expected that the asymptotic energy $E_0(m_0)$ of the family $\{E_{\varepsilon,\beta}(m_\varepsilon)\}_{\varepsilon\downarrow 0}$ concentrates on the singular set J. Assuming that the transition layers have a $1D$ structure across a wall, an appropriate candidate for E_0 can be deduced by analyzing the one-dimensional problem associated to our model. Indeed, we prove that the Γ-limit of $\{E_{\varepsilon,\beta}\}_{\varepsilon\downarrow 0}$ in the one-dimensional case is the following functional (see Section 3):

$$E_0(m_0) \quad := \quad \frac{1}{2}\int_J |m_0^+(x) - m_0^-(x)|^2 \, d\mathcal{H}^1(x). \tag{7}$$

The main issue of this paper is to study whether this asymptotic lower bound stands true in the two-dimensional case.

A special context for our model is given by smooth divergence free magnetizations corresponding to the limit case $\beta \downarrow 0$. Then we are led to consider the family of energies

$$E_\varepsilon(m) \quad := \quad \frac{\varepsilon}{2}\int_\omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_\omega m_3^2, \tag{8}$$

defined for magnetizations $m \in H^1(\omega, S^2)$ satisfying the constraint (4). As before, we study the asymptotic behavior of families of magnetizations $\{m_\varepsilon\} \subset H^1(\omega, S^2)$ such that

$$\nabla \cdot (m_\varepsilon' \mathbf{1}_\omega) \equiv 0 \quad \text{and} \quad \limsup_{\varepsilon\downarrow 0} E_\varepsilon(m_\varepsilon) \ < \ \infty. \tag{9}$$

We emphasize that (9) is a particular case of (6). On the other hand, in the regime (3), the situation (6) is a small perturbation of situation (9). Thus, we expect that the limiting behavior of the family of magnetizations $\{m_\varepsilon\}$ and of the energies $\{E_\varepsilon(m_\varepsilon)\}$ in (9) when $\varepsilon \downarrow 0$ is the same as in the situation (6).

We conjecture that the transition layers are essentially one-dimensional. This conjecture is supported by the partial results of Section 2 and also by numerical simulations which are briefly detailed at the end of the same section.

1.2 A related model

The study of the energy E_ε over divergence-free configurations is rather similar to the Aviles-Giga model that arises in several physical applications such as smectic liquid crystals, film blisters or convective pattern formation (see e.g. Aviles and Giga [5], Jin and Kohn [18]). It consists in associating to a function $\psi \in H_0^1(\omega) \cap H^2(\omega)$ the following energy functional:

$$AG_\varepsilon(\nabla\psi) \quad := \quad \frac{\varepsilon}{2}\int_\omega |\nabla\nabla\psi|^2 + \frac{1}{2\varepsilon}\int_\omega (1-|\nabla\psi|^2)^2.$$

Writing $m' := \nabla^\perp\psi : \omega \to \mathbf{R}^2$, the constraint $\nabla \cdot m' = 0$ is satisfied and we have

$$AG_\varepsilon(m') \quad := \quad \frac{\varepsilon}{2}\int_\omega |\nabla m'|^2 + \frac{1}{2\varepsilon}\int_\omega (1-|m'|^2)^2. \tag{10}$$

Notice that our functional E_ε dominates the Aviles-Giga energy AG_ε; indeed, if $m \in H^1(\omega, S^2)$ satisfies (4) then the inequalities $|\nabla m'| \le |\nabla m|$ and $(1-|m'|^2)^2 = m_3^4 \le m_3^2$ yield

$$AG_\varepsilon(m') \quad \le \quad E_\varepsilon(m). \tag{11}$$

The question of $\Gamma-$convergence of $\{AG_\varepsilon\}_{\varepsilon\downarrow 0}$ was intensively studied. The compactness of configurations $\{m'_\varepsilon\}_{\varepsilon\downarrow 0}$ of uniformly bounded energy $AG_\varepsilon(m'_\varepsilon) \le C$ was proved by Ambrosio, De Lellis and Mantegazza [3] and DeSimone, Kohn, Müller and Otto [11]. The limiting configurations m_0 satisfy (4) & (5). Moreover, De Lellis and Otto [9] proved the $\mathcal{H}^1-$rectifiability of the jump set J of m_0 (see Theorem 1), even if m_0 is in general not BV (see [3]). It is expected that the $\Gamma-$limit energy $AG_0(m_0)$ of the family $\{AG_\varepsilon(m'_\varepsilon)\}_{\varepsilon\downarrow 0}$ concentrates on the jump set J and has the following form (first stated by Aviles and Giga [5]):

$$AG_0(m_0) \quad := \quad \frac{1}{6}\int_J |m_0^+(x) - m_0^-(x)|^3 \, d\mathcal{H}^1.$$

In fact, AG_0 is a lower-bound of $\{AG_\varepsilon\}_{\varepsilon\downarrow 0}$ (see Aviles and Giga [6], Jin and Kohn [18]). The difficulty consists in the upper bound construction for admissible configurations m_0: recovery sequences have been constructed *only* for BV configurations m_0 (see Conti and De Lellis [8] and Poliakovsky [20]).

We emphasize that the difference between the line-energy density associated to jumps of m_0 in E_0 and AG_0 comes from the two different anisotropy terms: $\frac{1}{2\varepsilon}m_3^2$ for $E_{\varepsilon,\beta}$ and $\frac{1}{2\varepsilon}m_3^4 = \frac{1}{2\varepsilon}(1-|m'|^2)^2$ for AG_ε, respectively. In particular, the energetic cost of a jump in the Aviles-Giga model is cubic so that small jumps are less penalized than in our setting where this cost is expected to be a quadratic function of the size of the jump.

1.3 Entropies

The use of the concept of entropies from scalar conservation laws is suggested by the structure of the limiting configurations m_0 satisfying (4) & (5). Indeed, (5) implies that one can write $m_0 = (\cos\theta_0, \sin\theta_0)$ in terms of the phase θ_0 so that (4) reads as a conservation law:

$$\partial_1 \cos\theta_0 + \partial_2 \sin\theta_0 = 0.$$

Then, following De Simone, Kohn, Müller and Otto [11] and De Lellis and Otto [9], entropies are introduced as:

Definition 1. *(De Lellis and Otto [9]) A smooth compactly supported map* $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ *is called entropy if for every smooth map* $m' : \omega \to \mathbf{R}^2$ *we have*

$$\big(\nabla \cdot m' = 0 \quad \text{and} \quad |m'| = 1\big) \qquad \Longrightarrow \qquad \nabla \cdot \big\{\Phi(m')\big\} = 0.$$

In other words, Φ *is an entropy if the following relation is satisfied on the unit circle:*

$$z \cdot D\Phi(z) \cdot z^{\perp} \;=\; 0, \; \forall z \in S^1, \tag{12}$$

where $D\Phi$ *denotes the matrix* $(\partial_j \Phi_i)_{1\le i,j\le 2}$ *and* $z^{\perp} := (-z_2, z_1)$ *for* $z \in \mathbf{R}^2$.

The relation (12) suggests a suitable continuation of the entropy in the whole space $\mathbf{R}^2$. That gives the following definition of a particular class of entropies introduced by DeSimone, Kohn, Müller and Otto [11]:

Definition 2. *(DeSimone, Kohn, Müller and Otto [11]) A smooth compactly supported map* $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ *is called* $DKMO-$*entropy if*

$$\Phi(0) = 0, \qquad D\Phi(0) = 0 \qquad \text{and} \qquad z \cdot D\Phi(z) \cdot z^{\perp} = 0, \forall\, z \in \mathbf{R}^2. \tag{13}$$

The $DKMO-$entropies were used in [11] for proving the relative compactness of a family $\{m'_\varepsilon\}$ with uniformly bounded energy $AG_\varepsilon(m'_\varepsilon) \le C$ as $\varepsilon \downarrow 0$. The method of $DKMO-$entropies may lead to similar compactness results for more general energies with the nonlocal term $\int_{\mathbf{R}^2} |h(m')|^2$ (e.g., see Jabin, Otto and Perthame [17]). More precisely, the following energy functional is considered

$$F_\varepsilon(m'_\varepsilon) \quad := \quad \int_\omega \varepsilon |\nabla m'_\varepsilon|^2 + \frac{1}{\varepsilon}\int_\omega (1 - |m'_\varepsilon|^2)^2 + \frac{1}{\varepsilon}\int_{\mathbf{R}^2} |h(m'_\varepsilon)|^2, \tag{14}$$

for vector fields $m'_\varepsilon \in H^1(\omega, \mathbf{R}^2)$. As stated in [9], one can adapt the technique of [11] for proving compactness of a families $\{m'_\varepsilon\}$ satisfying $F_\varepsilon(m'_\varepsilon) \le C$ as $\varepsilon \downarrow 0$. The main ingredient is the inequality:

$$\left| \int_\omega \nabla \cdot \big\{\Phi(m'_\varepsilon)\big\}\, \zeta \right| \quad \le \quad \tilde{C}_\Phi \left(F_\varepsilon(m'_\varepsilon) \|\zeta\|_\infty + \varepsilon^{1/2} F_\varepsilon(m'_\varepsilon)^{1/2} \|\nabla \zeta\|_{L^2(\omega)} \right), \tag{15}$$

where $\tilde{C}_\Phi > 0$ is a positive constant depending on the $C^{1,1}$-norm of a $DKMO-$entropy Φ and ζ is an arbitrary test function.

If m_0 is a limiting configuration of the family $\{m'_\varepsilon\}$ of uniformly bounded energy $F_\varepsilon(m'_\varepsilon) \le C$, then inequality (15) implies that the entropy production $\nabla \cdot \{\Phi(m_0)\}$ is a measure for every $DKMO-$entropy Φ. De Lellis and Otto [9] characterized this class of vector fields where the entropy production is a measure for every entropy. Essentially, every limiting configuration m_0 shares some structure properties of maps of bounded variation $BV(\omega)$, in particular it is possible to give a rigorous definition of the jump set J. (A similar result was independently obtained by Ambrosio, Kirchheim, Lecumberry and Rivière [4] using the characterization of m_0 in terms of its phase θ_0.)

Theorem 1. *(De Lellis and Otto [9])*

(I) For every strong L^1-*limit* m_0 *of a family* $\{m'_\varepsilon\}_{\varepsilon\downarrow 0}$ *satisfying* $\limsup_{\varepsilon\downarrow 0} F_\varepsilon(m'_\varepsilon) < \infty$, *the distribution*

$$\mu_\Phi \quad := \quad \nabla \cdot \{\Phi(m_0)\} \tag{16}$$

is a measure of finite total mass for every entropy Φ.

(II) Let $A(\omega)$ *be the set of maps* $m_0 : \omega \to \mathbf{R}^2$ *such that* (4) *&* (5) *hold and* μ_Φ *be defined by* (16) *is a measure of locally finite total variation for every entropy* Φ. *If* $m_0 \in A(\omega)$, *there exists a set* $J \subset \omega$ *(called jump set) such that*

(a) J *is* $\mathcal{H}^1$ σ*–finite and rectifiable;*

(b) for $\mathcal{H}^1-$ a.e. $x \notin J$, m_0 is of vanishing mean oscillation at x, i.e.,

$$\lim_{r\downarrow 0} \frac{1}{r^2} \int_{B_r(x)} \left| m_0(y) - \fint_{B_r(x)} m_0 \right| dy \quad = \quad 0;$$

(c) for $\mathcal{H}^1-$ a.e. $x \in J$, there exist the traces $m_0^+(x), m_0^-(x) \in S^1$ with

$$\lim_{r\downarrow 0} \frac{1}{r^2} \left\{ \int_{B_r^+(x)} |m_0(y) - m_0^+(x)|\, dy + \int_{B_r^-(x)} |m_0(y) - m_0^-(x)|\, dy \right\} \quad = \quad 0,$$

where $B_r^{\pm}(x) := \{y \in B_r(x) \,|\, \pm y \cdot \boldsymbol{\nu}(x) > 0\}$ and $\boldsymbol{\nu}(x)$ is a unit normal vector on J at x;

(d) for every entropy Φ,

$$\begin{aligned} \mu_\Phi \llcorner J &= \left[\boldsymbol{\nu} \cdot (\Phi(m^+) - \Phi(m^-))\right] \llcorner J, \\ \mu_\Phi \llcorner K &= 0 \quad \text{for any } K \subset \omega \setminus J \quad \text{with } \mathcal{H}^1(K) < \infty. \end{aligned}$$

Observe that the limiting configurations in our model satisfy the same properties since the energy $E_{\varepsilon,\beta}$ dominates F_ε. Indeed, in the regime (3), we have for ε small enough,

$$F_\varepsilon(m'_\varepsilon) \quad \leq \quad 2E_{\varepsilon,\beta}(m_\varepsilon). \tag{17}$$

Therefore, the jump set J of m_0 and the limit energy E_0 are well defined in (7).

Another particular class of entropies was used by Jin and Kohn [18] in order to obtain lower bounds for the Aviles-Giga model. The idea also comes from scalar conservation laws where the entropy production through shocks is asymptotically cubic in the limit of small jumps. Therefore, smooth entropies seem to be adapted for the energy AG_ε. Indeed, let $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ be the following smooth entropy:

$$\Phi(m') = (m_2(1-m_1^2) - \frac{1}{3}m_2^3\ ,\ m_1(1-m_2^2) - \frac{1}{3}m_1^3), \forall m' \in \mathbf{R}^2. \tag{18}$$

(Notice that Φ is not a $DKMO-$entropy.) Then the entropy production is estimate by the Aviles-Giga energy density (up to a small perturbation), i.e., for smooth maps $m' : \mathbf{R}^2 \to \mathbf{R}^2$ with $\nabla \cdot m' = 0$ in $\mathbf{R}^2$, one has

$$\begin{aligned} \nabla \cdot \{\Phi(m')\} &= (1-|m'|^2)(\partial_1 m_2 + \partial_2 m_1) \\ &\leq \frac{(1-|m'|^2)^2}{2\varepsilon} + \frac{\varepsilon}{2}|\nabla m'|^2 + \varepsilon \nabla \cdot \begin{pmatrix} m_2 \partial_2 m_1 \\ -m_2 \partial_1 m_1 \end{pmatrix}. \end{aligned}$$

Moreover, the entropy production is the limit energy density associated to AG_0:

$$\{\Phi(m^+) - \Phi(m^-)\} \cdot \boldsymbol{e}_1 = \frac{|m^+ - m^-|^3}{6},$$

for every jump configuration $m_0 : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$m_0(x) = \begin{cases} m^- & \text{if } x_1 < 0 \\ m^+ & \text{if } x_1 > 0 \end{cases} \quad \text{with} \quad m^{\pm} := (\overline{m}_1, \pm\overline{m}_2, 0),\ \overline{m}_1{}^2 + \overline{m}_2{}^2 = 1. \tag{19}$$

In our model, the energetic cost of a jump configuration is expected to be quadratic in the size of the jump. Therefore, smooth entropies are no longer adapted here. The idea is to use entropies with discontinuous gradients. More precisely, we show that a special class of Lipschitz continuous entropies can detect the quadratic charges over the singular set of limiting configurations. It comes via an improvement of inequality (15) where the constant $\tilde{C}_\Phi > 0$ will depend only on the Lipschitz norm of a $DKMO-$entropy Φ (see (51)). The main ingredient consists in the control of total variation $\|m_3^2\|_{BV}$ by the energy $E_{\varepsilon,\beta}$ through the Young inequality:

$$|\nabla(m_3^2)| \leq \varepsilon |\nabla m_3|^2 + \frac{m_3^2}{\varepsilon}.$$

2 Main results

We start our analysis with the one-dimensional case associated to our model. It corresponds to the blow-up problem around a jump point for $1D$ transition layers. We discuss the optimal profile of a Bloch wall and we prove $\Gamma-$convergence of the $1D-$energy $E_{\varepsilon,\beta}$ to the limit energy E_0.

Then we study the two-dimensional case. First we prove relative compactness of families of magnetizations of uniformly bounded energy (6). Then we find a lower bound corresponding to the limit energy E_0 (up to a multiplicative constant) for the family of energies $\{E_{\varepsilon,\beta}\}$. Even if the constant is not the optimal one, this lower bound proves that the energetic cost of jumps in $2D$ is quadratic as indicated in the $1D$ case. The proof is based on the construction of a $DKMO-$entropy that has a jump in the gradient.

We also have optimal results for the lower bound E_0. More precisely, we localize the problem by considering periodic configurations in the x_2- direction in the domain $\omega := \mathbf{R} \times \mathbf{R}/\mathbf{Z}$ with a transition imposed by boundary conditions at $x_1 = \pm\infty$. We search for appropriate maps that are generalizations of the special entropy (18) used by Jin and Kohn [18]. We find such a map Φ that is adapted to Bloch walls of $180°$; in other words, the optimal $2D$ transition layer for $180°$ Bloch walls has asymptotically the same energy per unit length as the optimal one-dimensional structure. We also define suitable maps Φ for general wall angles; then the optimal lower bound is proved for energies $E_{\varepsilon,\beta}(m_\varepsilon)$ if the configurations m_ε take values on a certain spherical cap defined by the wall angle. However, we prove that in general there is no map Φ suitable for a wall angle when the configurations m_ε are allowed to take values into the whole sphere S^2.

2.1 One dimensional analysis

Let us present the $\Gamma-$convergence result in the one-dimensional case. For that, let $\overline{m}_1 \in (-1, 1)$ and $\overline{m}_2 \in [0, 1]$ be such that $\overline{m}_1{}^2 + \overline{m}_2{}^2 = 1$. As in (19), we denote by

$$m^\pm := (\overline{m}_1, \pm\overline{m}_2, 0)$$

two possible mesoscopic states of the magnetization across a wall of normal direction $\boldsymbol{e_1}$. ($\overline{m}_1$ and $\overline{m}_2$ represent the normal and the tangential component of the mesoscopic transition, respectively.)

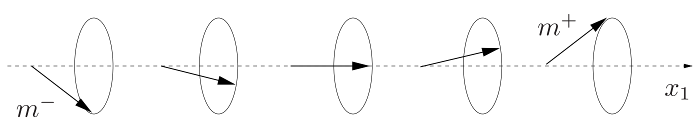

Figure 2: Element of M^{1D}

We consider the set of one-dimensional transition layers:

$$M^{1D} \quad := \quad \left\{ m \in H^1_{loc}(\mathbf{R}, S^2) \;:\; \lim_{x_1 \to \pm\infty} m_1(x_1) = \overline{m}_1 \right\}$$

and the following one-dimensional energy corresponding to $E_{\varepsilon,\beta}$ per unit length in the tangential direction of the wall:

$$E^{1D}_{\varepsilon,\beta}(m) \quad := \quad \frac{\varepsilon}{2} \int_{\mathbf{R}} \left| \frac{dm}{dx_1} \right|^2 + \frac{1}{2\varepsilon} \int_{\mathbf{R}} m_3^2 + \frac{1}{\beta} \int_{\mathbf{R}} (m_1 - \overline{m}_1)^2.$$

We have the following compactness result that also gives the structure of the limiting one-dimensional configurations. They are piecewise constant maps with a finite number of jumps of the same wall angle.

Theorem 2. *Consider a family of maps* $\{m_\varepsilon\}_\varepsilon \subset M^{1D}$ *such that*

$$\limsup_{\varepsilon\downarrow 0} E^{1D}_{\varepsilon,\beta}(m_\varepsilon) < \infty, \tag{20}$$

where $\beta = \beta(\varepsilon)$ *satisfies* (3). *Then the family* $\{m_\varepsilon\}_{\varepsilon\downarrow 0}$ *is relatively compact in* $L^1(\omega)$. *Moreover, any accumulation point* $m_0 : \mathbf{R} \to S^1$ *is of bounded total variation, takes exactly two values* $\{m^\pm\}$ *and can be written as:*

$$m_0 = \sum_{n=1}^{N+1} \begin{pmatrix} \overline{m}_1 \\ (-1)^{n+p}\overline{m}_2 \\ 0 \end{pmatrix} \mathbf{1}_{(t_{n-1},t_n)}, \tag{21}$$

where $N \geq 0$ *is an integer,* $p \in \{0,1\}$ *and* $-\infty = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$.

Notice that the limit configurations remain in M^{1D}. However, in general, boundary constraints of type $m^\pm$ at $\pm\infty$ are not conserved in the limit; for example, one could imagine a transition layer whose center moves to ∞ so that the limit map is a constant.

Let us denote by $\mathcal{A}^{1D}$ the set of all limiting configurations given by (21). For such a configuration $m_0 \in \mathcal{A}^{1D}$, we define the following one-dimensional energy corresponding to E_0:

$$E^{1D}_0(m_0) \ := \ \frac{1}{2}|m^+ - m^-|^2 \cdot \Big(\text{ number of jumps of } m_0 \Big), \tag{22}$$

where the number N of jumps of m_0 in (21) corresponds to the number of limiting walls. We show that E^{1D}_0 represents the $\Gamma-$limit of energies $E^{1D}_{\varepsilon,\beta}$:

Theorem 3. *Let* $\beta = \beta(\varepsilon)$ *satisfies* (3). *Then*

$$E^{1D}_{\varepsilon,\beta} \xrightarrow{\Gamma} E^{1D}_0 \text{ under the } L^1_{loc}(\mathbf{R}, S^2)-\text{topology as } \varepsilon \downarrow 0, \text{ i.e.,}$$

(i) If $\{m_\varepsilon\}_\varepsilon \subset M^{1D}$ *satisfies* (20) *and* $m_\varepsilon \overset{\varepsilon\downarrow 0}{\to} m_0$ *in* $L^1_{loc}(\mathbf{R}, S^2)$, *then* $m_0 \in \mathcal{A}^{1D}$ *and*

$$\liminf_{\varepsilon\downarrow 0} E^{1D}_{\varepsilon,\beta}(m_\varepsilon) \geq E^{1D}_0(m_0); \tag{23}$$

(ii) For every $m_0 \in \mathcal{A}^{1D}$, *there exist smooth maps* $\{m_\varepsilon\}_\varepsilon \subset M^{1D}$ *such that* $m_\varepsilon - m_0$ *has compact support in* $\mathbf{R}$ *for all* ε, $m_\varepsilon \overset{\varepsilon\downarrow 0}{\to} m_0$ *in* $L^1_{loc}(\mathbf{R}, S^2)$ *and*

$$\lim_{\varepsilon\downarrow 0} E^{1D}_{\varepsilon,\beta}(m_\varepsilon) = E^{1D}_0(m_0).$$

Obviously, the same $\Gamma-$convergence result stands true for the corresponding $1D$ energy E_ε (defined in (8)) over configurations $\{m_\varepsilon\} \subset M^{1D}$ of vanishing divergence (when the normal component of m_ε is a constant function equal to $\overline{m}_1$). In the case of in-plane transition layers (called Néel walls), a similar result was obtained by Ignat [14] where the energetic cost of a transition is quartic in the size of the jump.

2.2 Compactness

We now turn our attention to the two-dimensional case. First we prove a compactness result for a family of magnetizations of uniformly bounded energy $E_{\varepsilon,\beta}$. It is a generalization of the compactness result for the Aviles-Giga model.

Theorem 4. *Let* $\omega \subset \mathbf{R}^2$ *be a bounded domain. We consider a family of maps* $\{m_\varepsilon\}_\varepsilon \subset H^1(\omega, S^2)$ *such that*

$$\limsup_{\varepsilon\downarrow 0} E_{\varepsilon,\beta}(m_\varepsilon) \quad < \quad \infty,$$

where $\beta = \beta(\varepsilon)$ *satisfies* (3). *Then the family* $\{m_\varepsilon\}_{\varepsilon\downarrow 0}$ *is relatively compact in* $L^1(\omega)$. *Every* L^1-*strong limit* m_0 *of* $\{m_\varepsilon\}_{\varepsilon\downarrow 0}$ *satisfies* (4) *&* (5) *and belongs to* $A(\omega)$.

The proof of this theorem adapts the technique of [11] where the planar configurations $\{m'_\varepsilon\}$ were of vanishing divergence. The method is based on the theory of Young measures and the application of the div-curl lemma of Murat and Tartar (see e.g. [22, 19]) to families $\{\Phi(m'_\varepsilon)\wedge \tilde\Phi(m'_\varepsilon)\}_{\varepsilon\downarrow 0}$ where $\Phi, \tilde\Phi \in C^\infty(\mathbf{R}^2, \mathbf{R}^2)$ are two arbitrary $DKMO-$entropies. Incidentally we establish an improved version of inequality (15) for Lipschitz $DKMO-$entropies as well as for general Lipschitz entropies (see Remark 4.2).

2.3 A lower bound for $\{E_{\varepsilon,\beta}\}_{\varepsilon\downarrow 0}$

We show the following lower bound for (6):

Theorem 5. *Let $\omega \subset \mathbf{R}^2$ be a bounded open set. Assume that the family of maps $\{m_\varepsilon\}_{\varepsilon\downarrow 0} \subset H^1(\omega, S^2)$ converges to m_0 in $L^1(\omega)$. If $\beta = \beta(\varepsilon)$ satisfies (3), then*

$$E_0(m_0) \quad \le \quad C \liminf_{\varepsilon\downarrow 0} E_{\varepsilon,\beta}(m_\varepsilon),$$

for some universal constant $C > 1$.

Actually, we prove the result for the non-optimal constant $C = 2\sqrt{4+\pi^2}$. The proof is based on the construction of a Lipschitz $DKMO-$entropy Φ_0 that is adapted to the quadratic cost of a jump, i.e., the entropy production through a jump configuration $m_0 : \mathbf{R}^2 \to \mathbf{R}^2$ defined by (19) is given by the expected limit density of energy E_0:

$$\nabla \cdot \{\Phi_0(m_0)\} = \frac{1}{2}|m_0^+ - m_0^-|^2\, \mathcal{H}^1 \llcorner \{x_1 = 0\} \qquad \text{in } \mathcal{D}'(\mathbf{R}^2).$$

Even if we do not obtain the optimal constant $C = 1$ in Theorem 5, the role of this result is to show that the energetic cost of the jumps of limiting configurations has a quadratic behavior in our model (as indicated by the one-dimensional analysis).

2.4 Partial results for the optimal lower bound

We prove the optimal limit behavior of the family of energies $\{E_{\varepsilon,\beta}(m_\varepsilon)\}$ in some particular cases. More precisely, we focus on the periodic situation

$$\omega = \mathbf{R} \times \mathbf{R}/\mathbf{Z}$$

and we consider periodic magnetizations which are periodic in the tangential direction to the wall with transitions imposed by the limit condition at infinity:

$$M \quad := \quad \Big\{ m \in H^1_{loc}(\omega, S^2) \ : \ m(\lambda \cdot, \cdot) \overset{\lambda\uparrow\infty}{\longrightarrow} m^\infty \quad \text{in } L^1_{loc}(\omega) \Big\},$$

where m^∞ is the map defined by $m^\infty(x_1, x_2) := m^\pm$ for $\pm x_1 > 0$ with $m^\pm$ given by (19).[1] The associated two-dimensional stray field $h(m')$ is considered to be x_2-periodic and the stray field energy per-unit length in x_2-direction is given by:

$$\int_\omega |h(m')|^2 = \|\nabla \cdot m'\|^2_{\dot H^{-1}(\omega)}. \tag{24}$$

Here, we will always use the periodic stray field energy (24) as the last term in the energy $E_{\varepsilon,\beta}$:

$$E_{\varepsilon,\beta}(m) = \frac{\varepsilon}{2}\int_\omega |\nabla m|^2\,dx + \frac{1}{2\varepsilon}\int_\omega m_3^2\,dx + \frac{1}{\beta}\int_\omega |h(m')|^2\,dx.$$

In order to show that the optimal constant for the lower bound in Theorem 5 is $C = 1$, one should prove in the periodic case that for any family $\{m_\varepsilon\} \subset M$, we have

$$\frac{1}{2}|m^+ - m^-|^2 = 2\overline{m}_2{}^2 \quad \le \quad \liminf_{\varepsilon\downarrow 0} E_{\varepsilon,\beta}(m_\varepsilon) \tag{25}$$

[1] This limit condition is more general than asking $\lim_{x_1\to\pm\infty} m(x_1, \cdot) = m^\pm$ in $L^2(\mathbf{R}/\mathbf{Z})$.

in the regime (3).

We introduce a class of maps Φ that are a generalization of the entropies (18) used by Jin and Kohn [18] for the Aviles-Giga model. More precisely, we define the Lipschitz continuous maps $\Phi = (\varphi, \psi) \in Lip(S^2, \mathbf{R}^2)$ and $\alpha \in Lip(S^2)$ such that

$$\varphi(m^+) - \varphi(m^-) \;=\; [\Phi(m^+) - \Phi(m^-)] \cdot \mathbf{e}_1 \;=\; \frac{1}{2}|m^+ - m^-|^2 \tag{26}$$

and for every smooth $m \in C^\infty(\omega, S^2)$, the following inequality holds:

$$\nabla \cdot \{\Phi(m)\} + \alpha(m)\nabla \cdot m' \;\le\; \frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}m_3^2 + \nabla \cdot \{a_\varepsilon(m)\nabla m\} \quad \text{a.e. in } \omega, \tag{27}$$

where $\varepsilon > 0$ is a small parameter and for every $x \in S^2$, $a_\varepsilon(x) \in \mathcal{L}\big((T_xS^2)^2, \mathbf{R}^2\big)$ is a linear operator of two variables in the tangent plane T_xS^2. In the language of differential geometry, $x \mapsto a_\varepsilon(x)$ is a section of the vector bundle

$$\mathcal{B} := \{(x, a) \;:\; x \in S^2,\, a \in \mathcal{L}((T_xS^2)^2, \mathbf{R}^2)\}$$

based on S^2 with fiber $\mathcal{L}(\mathbf{R}^4, \mathbf{R}^2)$. Using the natural differential structure, $\mathcal{B}$ is locally diffeomorphic to $\mathbf{R}^2 \times \mathcal{L}(\mathbf{R}^4, \mathbf{R}^2)$. With the induced topology, we will always assume that the section $x \mapsto a_\varepsilon(x)$ is Lipschitz (in order that (27) makes sense). Moreover, the inequality (27) holds true for every point $x \in \omega$ such that $m(x)$ is a Lebesgue point of $D\Phi$ and Da_ε.

This class of generalized maps Φ are in fact Lipschitz entropies. Indeed, the following Proposition describes the link between (12) and inequality (27).

Proposition 1. *Let $\Phi \in Lip(S^2, \mathbf{R}^2)$, $\alpha \in Lip(S^2)$ and a_ε be a Lipschitz section of $\mathcal{B}$ such that (27) holds for every $m \in C^\infty(\omega, S^2)$. Then (12) holds in the sense that*

$$z \cdot D\Phi(z) \cdot z^\perp \;=\; 0, \quad \textit{for almost every } z \in S^1. \tag{28}$$

(Notice that since Φ is Lipschitz the tangential derivative $D\Phi(z) \cdot z^\perp$ exists for a.e. $z \in S^1 = S^1 \times \{0\}$.)

Conversely, let $\Phi \in C^\infty(S^2, \mathbf{R}^2)$ satisfying (12) and $\partial_{m_3}\Phi \equiv 0$ on S^1 (m_3-symmetric entropies $\Phi(m', m_3) = \Phi(m', -m_3)$ do satisfy this condition). Then there exist $c > 0$ and $\alpha \in C^\infty(S^2)$ such that $c\Phi$ satisfies (27) with $a_\varepsilon \equiv 0$ for every $m \in C^\infty(\omega, S^2)$ and every $\varepsilon > 0$.

Therefore we are still looking for maps Φ in the class of entropies as in the previous section. The main difference is that here we want an estimate of $\int \nabla \cdot \{\Phi(m)\}$ by the energy (with the optimal multiplicative constant $C = 1$) and allowing a perturbation $\nabla \cdot \{a_\varepsilon(m)\nabla m\}$ in the RHS of (27).

The existence of a triplet $(\Phi = (\varphi, \psi), \alpha)$ satisfying (27) would solve (25). Indeed, let $m \in M$. First, notice that

$$\left|\int_\omega \alpha(m)\nabla \cdot m'\right| \le \|\nabla \cdot m'\|_{\dot{H}^{-1}(\omega)} \|\nabla[\alpha(m)]\|_{L^2(\omega)} \le \|\nabla\alpha\|_{L^\infty} \left(\frac{2\beta}{\varepsilon}\right)^{1/2} E_{\varepsilon,\beta}(m).$$

Then integrating (27) on ω and taking into account the boundary conditions (26), we would deduce (25) in the regime (3) (see details in the proof of Proposition 2). This justifies the following definition:

Definition 3. *For $0 < \overline{m}_2 \le 1$ and $\overline{m}_1 = \sqrt{1 - \overline{m}_2{}^2}$, let $m^\pm$ be given by (19). We will say that a triplet $(\Phi = (\varphi, \psi), \alpha) \in Lip(S^2, \mathbf{R}^2) \times Lip(S^2)$ is adapted to the jump (m^-, m^+) if (26) holds and there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ one can construct a Lipschitz section a_ε of $\mathcal{B}$ for which (27) holds for every map $m \in C^\infty(\omega, S^2)$.*

For the 180° Bloch wall (i.e., the biggest possible jump), we have a positive answer.

Proposition 2. *There exists a smooth triplet $(\Phi = (\varphi, \psi), \alpha)$ adapted to the jump $(-\mathbf{e}_2, \mathbf{e}_2)$. Consequently, (25) holds for $\overline{m}_2 = 1$.*

For smaller jumps, we only have a partial result. For $0 < \overline{m}_2 < 1$, we define the spherical cap

$$S_{\overline{m}_2} \quad := \quad \left\{ m \in S^2 \;:\; m_1 \geq \overline{m}_1 = \sqrt{1 - \overline{m}_2{}^2} \right\}$$

and the set of magnetizations taking values in this cap:

$$M_{\overline{m}_2} \quad := \quad \{ m \in M \;:\; m(x) \in S_{\overline{m}_2} \text{ for a.e. } x \in \omega \} .$$

We show that one can find a triplet $(\Phi = (\varphi, \psi), \alpha)$ that is adapted to a jump (m^-, m^+) if we restrict to configurations of $M_{\overline{m}_2}$.

Proposition 3. *For every $0 < \overline{m}_2 < 1$ and every $\varepsilon > 0$, there exists $\Phi_{\overline{m}_2} = (\varphi_{\overline{m}_2}, \psi_{\overline{m}_2}) \in C^\infty(S_{\overline{m}_2}, \mathbf{R}^2)$, $\alpha_{\overline{m}_2} \in C^\infty(S_{\overline{m}_2})$ and a smooth section a_ε of $\mathcal{B}$ such that (26) and (27) hold for every $m \in C^\infty(\omega, S_{\overline{m}_2})$. Consequently if $\{m_\varepsilon\} \subset M_{\overline{m}_2}$, then (25) stands true.*

In the proofs of Propositions 1, 2 and 3 below, we exhibit adapted triplets (φ, ψ, α). The construction of an adapted triplet is derived by some necessary conditions. Indeed, in the following lemma we state that condition (27) yields some necessary pointwise bounds for an admissible triplet.

Lemma 1. *Let $\varepsilon > 0$, $(\Phi = (\varphi, \psi), \alpha) \in Lip(S^2, \mathbf{R}^2) \times Lip(S^2)$ and a_ε be a Lipschitz section of $\mathcal{B}$ satisfying (27) for every map $m \in C^\infty(\omega, S^2)$. Then for almost every $m \in S^2$, we have*

$$|\nabla\varphi(m) + \alpha(m)\Pi_m \mathbf{e}_1| \quad \leq \quad |m_3|, \tag{29}$$

$$|\nabla\psi(m) + \alpha(m)\Pi_m \mathbf{e}_2| \quad \leq \quad |m_3|, \tag{30}$$

where Π_m denotes the orthogonal projection on $T_m S^2$, for $m \in S^2$.

Despite Propositions 2 & 3, we will prove that for small jumps, inequalities (29) & (30) are not compatible with condition (26). Consequently, there is no triplet $(\Phi = (\varphi, \psi), \alpha)$ adapted to a fixed jump for general configurations (when the magnetizations cover the entire sphere S^2):

Theorem 6. *There exists $\eta > 0$ such that for $0 < \overline{m}_2 < \eta$, there is no triplet $(\Phi = (\varphi, \psi), \alpha)$ adapted to the jump (m^-, m^+).*

However, we strongly believe that the optimal constant in Theorem 5 is indeed $C = 1$, in particular (25) holds for every wall angle. We have performed numerical simulations in the periodic two-dimensional context indicating that the microscopic transition layers are one-dimensional.

Let us briefly describe the numerical method we have used. Let $\theta \in (0, 2\pi)$ be a wall angle and let $m^\pm = (\overline{m}_1, \pm\overline{m}_2, 0)$ with $\overline{m}_1 = \cos\theta/2$, $\overline{m}_2 = \sin\theta/2$. We want to observe the transition between the left and right mesoscopic states m^- and m^+ (the transition must be in the direction $\nu = e_1$ since the divergence free condition on the limit magnetization implies $(m^+ - m^-) \cdot \nu = 0$). For this, we set $\omega := \mathbf{R} \times \mathbf{R}/\mathbf{Z}$ and we minimize the energy (8) for $m \in H^1_{loc}(\omega, S^2)$ satisfying the constraint $\nabla \cdot m' = 0$ in $\mathcal{D}'(\omega)$ and the boundary conditions $m(x_1, \cdot) = m^\pm$ for $\pm x_1 > 1$. After rescaling we are led to minimize the energy

$$\frac{1}{2}\int_{\omega_\varepsilon} |\nabla m|^2 + \frac{1}{2}\int_{\omega_\varepsilon} m_3^2,$$

where $\omega_\varepsilon := \mathbf{R} \times \mathbf{R}/\varepsilon^{-1}\mathbf{Z}$. The rescaled magnetizations must satisfy $m \in H^1_{loc}(\omega_\varepsilon, S^2)$, $\nabla \cdot m' = 0$ in $\mathcal{D}'(\omega_\varepsilon)$ and $m(x_1, \cdot) = m^\pm$ for $\pm x_1 > \varepsilon^{-1}$.
Next for numerical purpose, we relax the constraint on $\nabla \cdot m'$ and replace it by a penalizing term; leading to the functional

$$J(m) \;:=\; \frac{1}{2}\int_{\omega_\varepsilon} |\nabla m|^2 + \frac{1}{2}\int_{\omega_\varepsilon} m_3^2 + \frac{\lambda}{2}\int_{\omega_\varepsilon} |\nabla \cdot m'|^2,$$

for some large parameter $\lambda \gg 1$. Then this energy is discretized by standard Finite Difference approximation. Finally, the discretized energy is optimized by applying the method of [1] to our functional.
We have performed several numerical simulations for various values of θ and ε. We always observe purely one-dimensional transition layers $m^h = m^h(x_1)$ which are close (for small ε and large λ) to the exact transition layer computed in Section 3 (namely $m^{1D}(x_1) = \big(\overline{m}_1, \overline{m}_2 \tanh x_1, \overline{m}_2 (\sinh x_1)^{-1}\big)$). An example of these computations is given Figure 3.

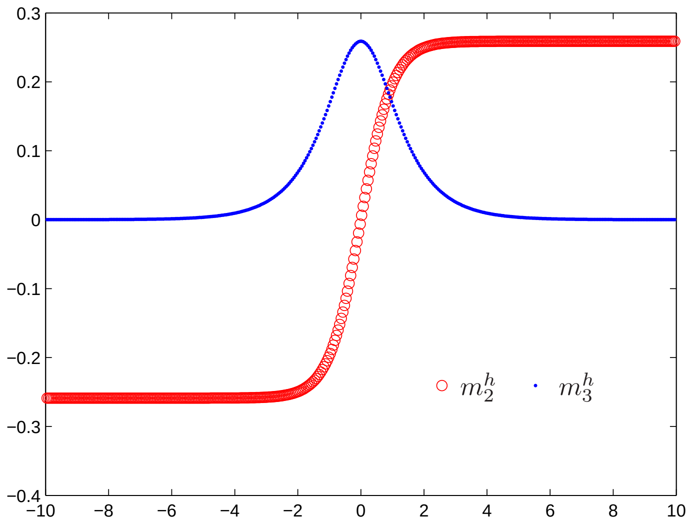

Figure 3: 1D profile of the numerical minimizer for $\theta = \pi/6$, $\varepsilon = 10^{-2}$, $\lambda = 10^6$. The step of the two dimensional grid is $h_{x_1} = h_{x_2} = 5 \cdot 10^{-2}$. We have $\|m^h - m^{1D}\|_\infty \approx 9.4 \cdot 10^{-5}$ and $\|m^h(x_1, x_2) - m^h(x_1, \tilde{x}_2)\|_{L^\infty_{x_1,x_2,\tilde{x}_2}} \approx 1.6 \cdot 10^{-10}$.

The paper is organized as follows. In Section 3, we solve the $\Gamma-$convergence problem in the one-dimensional case. In Section 4 we prove the compactness result stated in Theorem 4. In Section 5, we prove a lower bound of the energy $E_{\varepsilon,\beta}$ in the two-dimensional case that is given in Theorem 5. Proposition 1, Propositions 2 and 3, Lemma 1 and Theorem 6 are proved in Sections 6.1— 6.4, respectively.

3 $\Gamma-$convergence in the one-dimensional case

We start with some remarks about the one-dimensional case. Let $m = m(x_1) \in M^{1D}$ where x_1 is the normal direction to the wall. Then the stray field h only depends on the x_1-axis and satisfies the equations:

$$\frac{dh_1}{dx_1} = -\frac{dm_1}{dx_1} \quad \text{and} \quad \frac{dh_2}{dx_1} = 0 \qquad \text{in } \mathbf{R}.$$

The unique solution of this system vanishing as $|x_1| \to \infty$ is given by

$$h = (\overline{m}_1 - m_1, 0).$$

That explains the form of the stray field energy in $E^{1D}_{\varepsilon,\beta}(m)$.

Suppose that $m \in M^{1D}$ is a configuration of finite energy, i.e., $E^{1D}_{\varepsilon,\beta}(m) < \infty$. Then $m_1 - \overline{m}_1, m_3 \in H^1(\mathbf{R})$. Therefore, m_1 and m_3 are continuous functions with the following behavior at infinity:

$$\lim_{|x_1|\to\infty} m_1 = \overline{m}_1 \quad \text{and} \quad \lim_{|x_1|\to\infty} m_3 = 0.$$

Moreover, m_2 is a continuous function with $\dfrac{dm_2}{dx_1} \in L^2(\mathbf{R})$ and since m takes values in S^2, we have

$$\lim_{|x_1|\to\infty} |m_2| = \ \overline{m}_2 \ := \ \sqrt{1 - \overline{m}_1{}^2}.$$

Now we prove that the limiting $1D$ configurations correspond to a finite number of Bloch walls of the same angle that are transversal to the x_1-axis:

Proof of Theorem 2. Let $m \in M^{1D}$ with $E^{1D}_{\varepsilon,\beta}(m) < \infty$. We start with some estimates on m needed for the compactness result. Let us denote

$$u = \sqrt{1 - m_1^2}.$$

Then u is a continuous nonnegative function and the set $\{u > 0\}$ is a countable union of disjoint open intervals. If $I \subset \{u > 0\}$ is an interval, then

$$\frac{1}{u}(m_2, m_3) \in C^0(I, S^1).$$

Hence, there exists a continuous phase $\theta \in C^0(I, \mathbf{R})$ such that

$$m_2 = u\cos\theta \quad \text{and} \quad m_3 = u\sin\theta \quad \text{in} \quad I \tag{31}$$

and one computes that

$$\left|\frac{dm_2}{dx_1}\right|^2 + \left|\frac{dm_3}{dx_1}\right|^2 = \left|\frac{du}{dx_1}\right|^2 + u^2\left|\frac{d\theta}{dx_1}\right|^2 \quad \text{a.e. in} \quad I. \tag{32}$$

On the set where u vanishes, one can set $\theta \equiv 0$ in $\{u = 0\}$. Then (31) and (32) stand true a.e. in $\mathbf{R}$. Indeed, since $m, u \in H^1_{loc}(\mathbf{R})$ and $\{u = 0\} = \{m_2 = 0\} \cap \{m_3 = 0\}$, it follows that $\frac{dm_2}{dx_1} = \frac{dm_3}{dx_1} = \frac{du}{dx_1} = 0$ a.e. in $\{u = 0\}$. Therefore, we have

$$\left|\frac{dm}{dx_1}\right|^2 = \frac{1}{m_1^2}\left|\frac{du}{dx_1}\right|^2 + u^2\left|\frac{d\theta}{dx_1}\right|^2 \quad \text{a.e. in} \quad \mathbf{R}.$$

By Young inequality, we have the following estimates on m:

$$\int_{\mathbf{R}} (m_1 - \overline{m}_1)^2 \ \le \ \beta E^{1D}_{\varepsilon,\beta}(m), \tag{33}$$

$$\int_{\mathbf{R}} \left|\frac{d(m_1 - \overline{m}_1)^2}{dx_1}\right| \ \le \ \sqrt{\beta\varepsilon}\int_{\mathbf{R}}\left|\frac{dm_1}{dx_1}\right|^2 + \frac{1}{\sqrt{\beta\varepsilon}}\int_{\mathbf{R}}(m_1 - \overline{m}_1)^2 \ \le \ 2\sqrt{\frac{\beta}{\varepsilon}}E^{1D}_{\varepsilon,\beta}(m),$$

$$\int_{\mathbf{R}} m_3^2 \ \le \ 2\varepsilon E^{1D}_{\varepsilon,\beta}(m), \tag{34}$$

$$\int_{\mathbf{R}} \left|\frac{d(m_3^2)}{dx_1}\right| \ \le \ \varepsilon\int_{\mathbf{R}}\left|\frac{dm_3}{dx_1}\right|^2 + \frac{1}{\varepsilon}\int_{\mathbf{R}} m_3^2 \ \le \ 2E^{1D}_{\varepsilon,\beta}(m).$$

Using the inequality $\frac{\overline{m}_2}{2}\left|u - \overline{m}_2\right| \le |m_1 - \overline{m}_1|$, we also obtain via Young's inequality that

$$\frac{\overline{m}_2}{2}\int_{\mathbf{R}}\left|\frac{d(u - \overline{m}_2)^2}{dx_1}\right| \ \le \ \frac{\sqrt{\beta\varepsilon}}{2}\int_{\mathbf{R}}\left|\frac{du}{dx_1}\right|^2 + \frac{1}{2\sqrt{\beta\varepsilon}}\int_{\mathbf{R}}(m_1 - \overline{m}_1)^2 \ \le \ \sqrt{\frac{\beta}{\varepsilon}}\,E^{1D}_{\varepsilon,\beta}(m), \tag{35}$$

$$\int_{\mathbf{R}} u^2\left|\frac{d\cos\theta}{dx_1}\right| \ \le \ \frac{\varepsilon}{2}\int_{\mathbf{R}} u^2\left|\frac{d\theta}{dx_1}\right|^2 + \frac{1}{2\varepsilon}\int_{\mathbf{R}} m_3^2 \ \le \ E^{1D}_{\varepsilon,\beta}(m). \tag{36}$$

Let $\{m_\varepsilon = (m_{\varepsilon,1}, m_{\varepsilon,2}, m_{\varepsilon,3})\}_\varepsilon \subset M^{1D}$ be such that (20) holds. By (3), (33) and (34), it follows that

$$m_{\varepsilon,1} - \overline{m}_1 \to 0 \quad \text{and} \quad m_{\varepsilon,3} \to 0 \quad \text{in} \quad L^2(\mathbf{R}). \tag{37}$$

Since $|m_\varepsilon| = 1$, we have

$$|m_{\varepsilon,2}| \to \overline{m}_2 \quad \text{in} \quad L^1_{loc}(\mathbf{R}).$$

If $\overline{m}_2 = 0$, i.e. $\overline{m}_1 = \pm 1$, then we conclude that $m_{\varepsilon,2} \to 0$ in $L^1_{loc}(\mathbf{R})$, that means

$$m_\varepsilon \to (\overline{m}_1, 0, 0) \quad \text{in} \quad L^1_{loc}(\mathbf{R}).$$

Otherwise, $\overline{m}_2 > 0$ and it remains to prove that $\{m_{\varepsilon,2}\}_{\varepsilon \downarrow 0}$ is relatively compact in $L^1_{loc}(\mathbf{R})$. Using notations (31), it results that

$$u_\varepsilon = \sqrt{1 - m_{\varepsilon,1}^2} \to \overline{m}_2 \quad \text{and} \quad |\cos\theta_\varepsilon| \to 1 \quad \text{in} \quad L^1_{loc}(\mathbf{R}). \tag{38}$$

Since $\lim_{|x_1|\to\infty} u_\varepsilon = \overline{m}_2$, combining (3), (20) and (35), we obtain

$$\begin{aligned} u_\varepsilon(x_1) &\geq \overline{m}_2 - \left|u_\varepsilon(x_1) - \overline{m}_2\right| = \overline{m}_2 - \left(\int_{-\infty}^{x_1} \frac{d(u_\varepsilon - \overline{m}_2)^2}{dx_1}\right)^{1/2} \\ &\geq \overline{m}_2 - \left(\frac{2}{\overline{m}_2}\left(\frac{4\beta}{\varepsilon}\right)^{1/2} E^{1D}_{\varepsilon,\beta}(m_\varepsilon)\right)^{1/2} = \overline{m}_2 + o(1),\ \forall x_1 \in \mathbf{R}. \end{aligned}$$

Then (36) leads to

$$E^{1D}_{\varepsilon,\beta}(m_\varepsilon) \geq \int_{\mathbf{R}} \left(\overline{m}_2{}^2 + o(1)\right) \left|\frac{d\cos\theta_\varepsilon}{dx_1}\right|. \tag{39}$$

Since $\overline{m}_2 > 0$, (20) implies that $\{\cos\theta_\varepsilon\}_{\varepsilon<\varepsilon_0}$ has uniformly bounded variation in $\mathbf{R}$. Combining with (38), we deduce that any limit function of $\{\cos\theta_\varepsilon\}_{\varepsilon\downarrow 0}$ in L^1_{loc} is of bounded variation and takes the values ± 1. Therefore, $\{m_{\varepsilon,2} = u_\varepsilon \cos\theta_\varepsilon\}$ is relatively compact in L^1_{loc} and any accumulation point in L^1_{loc} has the form

$$\sum_{n=1}^{N+1} (-1)^{n+p} \overline{m}_2 \mathbf{1}_{(t_{n-1}, t_n)},$$

where $N \geq 0$ is an integer, $p \in \{0, 1\}$ and $-\infty = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$. The constraint that $m_{\varepsilon,2}$ has the limits $\pm\overline{m}_2$ at $\pm\infty$ for every ε is not conserved in general in the limit $\varepsilon \downarrow 0$. Therefore, N can vanish as well as p can take both values 0 or 1. ■

We prove the first assertion in Theorem 3 for the lower bound of the energy $E^{1D}_{\varepsilon,\beta}$:

Proof of (i) in Theorem 3. By Theorem 2, we know that $m_0 \in \mathcal{A}^{1D}$, i.e.,

$$m_0 = \begin{pmatrix} m_1 \\ m_2 \\ 0 \end{pmatrix} = \sum_{n=1}^{N+1} \begin{pmatrix} \overline{m}_1 \\ (-1)^{n+p}\overline{m}_2 \\ 0 \end{pmatrix} \mathbf{1}_{(t_{n-1}, t_n)},$$

where $N \geq 0$ is an integer, $p \in \{0, 1\}$ and $-\infty = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$. Notice that if $\overline{m}_2 = 0$ or $N = 0$, then $E^{1D}_0(m_0) = 0$ and inequality (23) is trivial. Therefore, we assume that $N \geq 1$ and $\overline{m}_2 > 0$. Since $m_\varepsilon \to m_0$ in L^1_{loc}, using notations (31), we deduce

$$m_{\varepsilon,2} \to m_2, \qquad u_\varepsilon = \sqrt{1 - m_{\varepsilon,1}^2} \to \overline{m}_2 \qquad \text{and} \qquad \cos\theta_\varepsilon \to \frac{m_2}{\overline{m}_2} \qquad \text{in} \qquad L^1_{loc}(\mathbf{R}).$$

Therefore,

$$\liminf_{\varepsilon\downarrow 0} \int_{\mathbf{R}} \left|\frac{d\cos\theta_\varepsilon}{dx_1}\right| \geq \frac{1}{\overline{m}_2} \int_{\mathbf{R}} \left|\frac{dm_2}{dx_1}\right| = 2N.$$

Together with (39), the conclusion follows:

$$\liminf_{\varepsilon\downarrow 0} E^{1D}_{\varepsilon,\beta}(m_\varepsilon) \ \geq\ \liminf_{\varepsilon\downarrow 0}\int_{\mathbf{R}} (\overline{m}_2{}^2 + o(1))\left|\frac{d\cos\theta_\varepsilon}{dx_1}\right| \ \geq\ 2N\overline{m}_2{}^2 = N\frac{|m^+ - m^-|^2}{2}.$$

■

Before showing the second issue *(ii)* in Theorem 3, let us now discuss about the optimal profile of a transition layer, the so called *Bloch wall*. It corresponds to the minimizer m_ε of $E^{1D}_{\varepsilon,\beta}$ over the configurations of M^{1D} that are of vanishing divergence, i.e.,

$$E^{1D}_{\varepsilon,\beta}(m_\varepsilon) = \min_{\substack{m\in M^{1D}\\ m_1\equiv\overline{m}_1}} E^{1D}_{\varepsilon,\beta}(m). \tag{40}$$

In this case, if $m \in M^{1D}$ and $m_1 \equiv \overline{m}_1$, there exists $\theta \in H^1_{loc}(\mathbf{R})$ (*the transition angle*) such that

$$m(t) = (\overline{m}_1, \overline{m}_2\cos\theta(t), \overline{m}_2\sin\theta(t)), \tag{41}$$

with $\lim_{t\to\pm\infty}\cos\theta(t) = \pm 1$. Then (40) turns into the following Cahn-Hilliard type problem:

$$E^{1D}_{\varepsilon,\beta}(m_\varepsilon) = \overline{m}_2{}^2 \min_{\substack{\theta\in H^1_{loc}(\mathbf{R})\\ \cos\theta(t)\to\pm1,\, t\to\pm\infty}} \left\{\frac{\varepsilon}{2}\int_{\mathbf{R}}\left|\frac{d\theta}{dt}\right|^2 + \frac{1}{2\varepsilon}\int_{\mathbf{R}}\sin^2\theta\right\}. \tag{42}$$

One can solve the Euler-Lagrange equation corresponding to m_ε in terms of its transition angle θ_ε which is the Cauchy problem associated to the first order ODE:

$$\frac{d\theta_\varepsilon}{dt} = \frac{1}{\varepsilon}\sin\theta_\varepsilon, \quad \text{with} \ \ \cos\theta_\varepsilon(t) \to \pm1,\ t\to\pm\infty.$$

It follows that the unique one-dimensional transition layer between $m^\pm$ centered in the origin is given by (41) with the transition angle:

$$\theta_\varepsilon(t) = 2\arctan e^{-t/\varepsilon}. \tag{43}$$

We denote by v the following smooth increasing odd function:

$$v(t) = \cos\theta_1(t) = \tanh(t), \qquad \forall t\in\mathbf{R}. \tag{44}$$

Then one can check that

$$E^{1D}_{\varepsilon,\beta}(m_\varepsilon) = \frac{\overline{m}_2{}^2}{2}\left\{\int_{\mathbf{R}}\frac{1}{1-v^2}\left|\frac{dv}{dt}\right|^2 + \int_{\mathbf{R}}(1-v^2)\right\} = 2\overline{m}_2{}^2. \tag{45}$$

Now we construct recovery families for every limiting configuration:

Proof of (ii) in Theorem 3. Let $m_0 \in \mathcal{A}^{1D}$, i.e.,

$$m_0 = \sum_{n=1}^{N+1}\begin{pmatrix}\overline{m}_1\\ (-1)^{n+p}\overline{m}_2\\ 0\end{pmatrix}\mathbf{1}_{(t_{n-1},t_n)},$$

where $N \geq 0$ is an integer, $p \in \{0,1\}$ and $-\infty = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$. We want to construct smooth transition layers m_ε such that $m_\varepsilon - m_0$ has compact support in $\mathbf{R}$, $m_\varepsilon \to m_0$ in $L^1_{loc}(\mathbf{R}, S^2)$ and

$$\limsup_{\varepsilon\downarrow 0} E^{1D}_{\varepsilon,\beta}(m_\varepsilon) \leq E^{1D}_0(m_0). \tag{46}$$

In the case where $\overline{m}_2 = 0$ or $N = 0$, i.e., m_0 is a constant map, then $E_0^{1D}(m_0) = E_{\varepsilon,\beta}^{1D}(m_0) = 0$ and hence, we may consider the recovery family $m_\varepsilon := m_0$ for every $\varepsilon > 0$.

Otherwise, $N \geq 1$ and $\overline{m}_2 > 0$. Let

$$\gamma := \frac{1}{5}\begin{cases} \min\limits_{2\leq n\leq N}\{|t_n - t_{n-1}|, 1\} & \text{if } N \geq 2, \\ 1 & \text{if } N = 1. \end{cases}$$

We approximate the Bloch wall profile $(\overline{m}_1, \overline{m}_2\cos\theta_\varepsilon, \overline{m}_2\sin\theta_\varepsilon)$ with θ_ε given in (43) by a localized transition layer around the origin on the interval $[-\gamma,\gamma]$. More precisely, we consider the following transition layer

$$\tilde{m}_\varepsilon = (\overline{m}_1, \overline{m}_2 v_\varepsilon, \overline{m}_2 w_\varepsilon) : \mathbf{R} \to S^2$$

where

$$v_\varepsilon(t) \;:=\; \begin{cases} v\left(\frac{2\gamma}{\pi\varepsilon}\tan\left(\frac{\pi t}{2\gamma}\right)\right) & \text{if } t \in [-\gamma,\gamma], \\ \pm 1 & \text{if } \pm t \geq \gamma \end{cases} \qquad \text{and} \qquad w_\varepsilon \;:=\; \sqrt{1 - v_\varepsilon^2},$$

and v is defined by (44). Then v_ε is an increasing continuous odd function in $\mathbf{R}$. Using the change of variable $s = \frac{2\gamma}{\pi\varepsilon}\tan\left(\frac{\pi t}{2\gamma}\right)$ and the fact that $1 - v^2(s) = \frac{1}{1-v^2}\left|\frac{dv}{ds}(s)\right|^2 = \frac{1}{\cosh^2(s)}$, we compute:

$$\begin{aligned} E_{\varepsilon,\beta}^{1D}(\tilde{m}_\varepsilon) &= \frac{\overline{m}_2^{\,2}}{2}\left\{\int_{-\gamma}^{\gamma} \frac{\varepsilon}{1-v_\varepsilon^2}\left|\frac{dv_\varepsilon}{dt}\right|^2 dt \;+\; \int_{-\gamma}^{\gamma}\frac{(1-v_\varepsilon^2)}{\varepsilon}\,dt\right\} \\ &= \frac{\overline{m}_2^{\,2}}{2}\left\{\int_{-\infty}^{+\infty}\frac{1}{1-v^2}\left|\frac{dv}{ds}\right|^2\left(1 + (\tfrac{\pi\varepsilon}{2\gamma})^2 s^2\right) ds \;+\; \int_{-\infty}^{+\infty}(1-v^2)\frac{ds}{1+(\frac{\pi\varepsilon}{2\gamma})^2 s^2}\right\} \\ &= \frac{\overline{m}_2^{\,2}}{2}\left\{\int_{\mathbf{R}}\frac{1}{1-v^2}\left|\frac{dv}{ds}\right|^2 \;+\; \int_{\mathbf{R}}(1-v^2)\right\} + o(1) \overset{(45)}{=} 2\overline{m}_2^{\,2} + o(1). \end{aligned} \tag{47}$$

We adapt the transition layer $\tilde{m}_\varepsilon$ for the walls of the limit magnetization m_0. For every $\varepsilon > 0$, we consider the following $C^1(\mathbf{R}, S^2)-$maps

$$m_\varepsilon(t) = (\overline{m}_1, (-1)^{n+p-1}\overline{m}_2 v_\varepsilon(t-t_n), \overline{m}_2 w_\varepsilon(t-t_n)) \text{ if } t \in \left(\frac{t_{n-1}+t_n}{2}, \frac{t_n+t_{n+1}}{2}\right), n = 1,\ldots,N.$$

Then $m_\varepsilon - m_0$ has compact support in $(t_1 - 1, t_N + 1)$ and

$$m_\varepsilon - m_0 \to 0 \quad \text{in } L^1(\mathbf{R}) \text{ as } \varepsilon \downarrow 0.$$

Moreover,

$$E_{\varepsilon,\beta}^{1D}(m_\varepsilon) = N E_{\varepsilon,\beta}^{1D}(\tilde{m}_\varepsilon) \overset{(47)}{=} 2N\overline{m}_2^{\,2} + o(1).$$

■

4 Compactness

In this section we prove Theorem 4. Our proof is based on the compensated compactness method described in [11] where entropies are used jointly with the theory of Young measures and the div-curl lemma of Murat and Tartar. In order to use this program, it is sufficient to prove that for every $DKMO-$entropy Φ,

$$\{\nabla\cdot\{\Phi(m_\varepsilon')\}\}_{\varepsilon\downarrow 0} \qquad \text{is relatively compact in} \qquad H^{-1}(\omega). \tag{48}$$

Let us first recall the following property of $DKMO-$entropies:

Lemma 2. *(DeSimone, Kohn, Müller and Otto [11]) For every DKMO−entropy* Φ*, there exist* $\Psi \in C_0^\infty(\mathbf{R}^2, \mathbf{R}^2)$ *and* $\Xi \in C_0^\infty(\mathbf{R}^2, \mathbf{R})$ *such that*

$$D\Phi(z) \quad = \quad -2\Psi(z) \otimes z + \Xi(z) Id \quad \text{for every } z \in \mathbf{R}^2. \tag{49}$$

Consequently, for every $m' \in H^1(\omega, \mathbf{R}^2)$*, we have*

$$\nabla \cdot \{\Phi(m')\} \quad = \quad \Psi(m') \cdot \nabla(1 - |m'|^2) + \Xi(m') \nabla \cdot m' \qquad \text{a.e. in } \omega. \tag{50}$$

An important ingredient for (48) is the following estimate:

Lemma 3. *Let* $m \in H^1(\omega, S^2)$*,* $\zeta \in H_0^1 \cap L^\infty(\omega)$ *and* Φ *be a DKMO−entropy. With the notations in Lemma 2, we have*

$$\left| \int_\omega \nabla \cdot \{\Phi(m')\} \zeta \right| \leq \|\Psi\|_\infty \int_\omega |\nabla(1 - |m'|^2)||\zeta| \tag{51}$$
$$+ C_\Phi \left(\frac{\beta}{\varepsilon} E_{\varepsilon,\beta}(m) \right)^{1/2} \left(E_{\varepsilon,\beta}(m)^{1/2} \|\zeta\|_\infty + \varepsilon^{1/2} \|\nabla \zeta\|_{L^2(\omega)} \right),$$

where $C_\Phi = \sqrt{2} \max\{\|\Xi\|_\infty, \|\nabla \Xi\|_\infty\}$.

Remark 4.1. *The inequality* (51) *is an improvement of the estimate* (15)*. In the regime* (3)*, one can get* (15) *from* (51) *by observing that:*

$$\int_\omega |\nabla(1 - |m'|^2)| \, |\zeta| \leq \ \|\zeta\|_\infty \int_\omega |\nabla(m_3^2)| \leq \ \|\zeta\|_\infty \int_\omega \left(\varepsilon |\nabla m_3|^2 + \frac{1}{\varepsilon} m_3^2 \right) \leq 2\|\zeta\|_\infty E_{\varepsilon,\beta}(m). \tag{52}$$

The advantage of (51) *consists in having the leading order term only dependent on the* L^∞*-norm of* Ψ *(controlled by the Lipschitz norm of the entropy* Φ*) whereas in* (15) *the constant* $\tilde{C}_\Phi$ *depends on the* $C^{1,1}$*-norm of* Φ*. For this reason, if* Φ_0 *is a Lipschitz continuous map satisfying* (13) *and if* m_0 *is a strong limit of* $\{m_\varepsilon\}$ *satisfying* $\limsup_{\varepsilon \downarrow 0} E_{\varepsilon,\beta}(m_\varepsilon) < \infty$ *in the regime* (3)*, then* μ_{Φ_0} *defined by* (16) *is a measure of finite total mass. The choice of a suitable Lipschitz entropy* Φ_0 *and the inequality* (51) *are essential in the proof of Theorem 5.*

Remark 4.2. *Notice that* (51) *(as well as* (15)*) are not restricted to DKMO−entropies. Indeed if* Φ *is an entropy and* $\rho \in C_0^\infty((0, \infty))$ *is a cut-off function such that* $\rho(1) = 1$ *then* $\tilde{\Phi}$ *defined by*

$$\tilde{\Phi}(m') := \rho(|m'|)\Phi(\frac{m'}{|m'|}) \quad \text{for every} \quad m' \in \mathbf{R}^2 \setminus \{0\}$$

is a DKMO−entropy and thus satisfies (51)*. Now the difference* $\bar{\Phi} := \Phi - \tilde{\Phi}$ *satisfies* $|\bar{\Phi}(m')| \leq C|1 - |m'||$*; then integrating by parts, the Cauchy-Schwarz inequality leads to*

$$\left| \int_\omega \nabla \cdot \{\bar{\Phi}(m')\} \zeta \right| \ \leq \ C\varepsilon^{1/2} E_{\varepsilon,\beta}(m)^{1/2} \|\nabla \zeta\|_{L^2(\omega)}, \qquad \forall m \in H^1(\omega, S^2), \quad \forall \zeta \in H_0^1(\omega).$$

Thus Φ *satisfies* (51)*.*

Proof of Lemma 3. Using the duality $< \cdot, \cdot >_{H^{-1}(\omega), H_0^1(\omega)}$, (50) yields

$$\begin{aligned}
\left| \int_\omega \nabla \cdot \{\Phi(m')\} \zeta \right| &\leq \int_\omega |\nabla(1 - |m'|^2)||\Psi(m')||\zeta| + \left| \int_\omega \nabla \cdot m' \Xi(m') \zeta \right| \\
&\leq \|\Psi\|_\infty \int_\omega |\nabla(1 - |m'|^2)||\zeta| + \|\nabla \cdot m'\|_{H^{-1}(\omega)} \|\nabla[\Xi(m')\zeta]\|_{L^2(\omega)} \\
&\leq \|\Psi\|_\infty \int_\omega |\nabla(1 - |m'|^2)||\zeta| \\
&\quad + \|\nabla \cdot (m' \mathbf{1}_\omega)\|_{\dot{H}^{-1}(\mathbf{R}^2)} \left(\|\Xi\|_\infty \|\nabla \zeta\|_{L^2(\omega)} + \|\nabla \Xi\|_\infty \|\zeta\|_{L^\infty(\omega)} \|\nabla m'\|_{L^2(\omega)} \right)
\end{aligned}$$

and (51) follows. ∎

We then prove that $\{\nabla\cdot[\Phi(m_\varepsilon')]\}$ is relatively compact in $H^{-1}(\omega)$ whenever Φ is a $DKMO-$entropy Φ:

Proof of (48). It is sufficient to show that for every family of test functions $\{\zeta_\varepsilon\}\subset H_0^1(\omega)$ such that $\zeta_\varepsilon \rightharpoonup 0$ in $H_0^1(\omega)$, we have

$$\int_\omega \nabla\cdot\left\{\Phi(m_\varepsilon')\right\}\zeta_\varepsilon \quad \stackrel{\varepsilon\downarrow 0}{\longrightarrow} \quad 0. \tag{53}$$

Let $\{\zeta_\varepsilon\}$ be such a family of test functions. For $\delta>0$, we define the truncated functions (as in [11]):

$$\zeta_\varepsilon^1(x) := \begin{cases} -\delta, & \text{if} \quad \zeta_\varepsilon(x)\le -\delta, \\ \zeta_\varepsilon(x) & \text{if} \quad |\zeta_\varepsilon(x)|<\delta, \\ \delta & \text{if} \quad \zeta_\varepsilon(x)\ge\delta, \end{cases} \qquad \text{and} \quad \zeta_\varepsilon^2 := \zeta_\varepsilon-\zeta_\varepsilon^1.$$

Using this decomposition of ζ_ε and integrating by parts, we compute

$$\left|\int_\omega \nabla\cdot\left\{\Phi(m_\varepsilon')\right\}\,\zeta_\varepsilon\right| \quad\le\quad \left|\int_\omega \Phi(m_\varepsilon')\cdot\nabla\zeta_\varepsilon^2\right| + \left|\int_\omega \nabla\cdot\left\{\Phi(m_\varepsilon')\right\}\,\zeta_\varepsilon^1\right|. \tag{54}$$

For the first term of the RHS of (54), the Cauchy-Schwarz inequality yields that

$$\limsup_{\varepsilon\downarrow 0}\left|\int_\omega \Phi(m_\varepsilon')\cdot\nabla\zeta_\varepsilon^2\right| \quad\le\quad \limsup_{\varepsilon\downarrow 0}\left(\int_\omega |\nabla\zeta_\varepsilon|^2\right)^{1/2}\left(\int_{\{|\zeta_\epsilon|>\delta\}} |\left\{\Phi(m_\varepsilon')\right\}|^2\right)^{1/2} = 0, \tag{55}$$

since $\{\nabla\zeta_\varepsilon\}$ is bounded in $L^2(\omega)$, $\zeta_\epsilon\to 0$ in $L^2(\omega)$ and $\{\|\Phi(m_\varepsilon')\|_\infty\}_{\varepsilon\downarrow 0}$ is uniformly bounded. For the second term of the RHS of (54), we apply (51) and (52) in the regime (3):

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0}\left|\int_\omega \nabla\cdot\left\{\Phi(m_\varepsilon')\right\}\zeta_\varepsilon^1\right| &\stackrel{(50),(52)}{\le} \limsup_{\varepsilon\downarrow 0}\; 2\delta\|\Psi\|_\infty\; E_{\varepsilon,\beta}(m_\varepsilon) \\ &+ \limsup_{\varepsilon\downarrow 0}\; C_\Phi\left(\frac{\beta}{\varepsilon}E_{\varepsilon,\beta}(m)\right)^{1/2}\left(\delta E_{\varepsilon,\beta}(m)^{1/2}+\varepsilon^{1/2}\|\nabla\zeta_\varepsilon\|_{L^2(\omega)}\right) \;\le\; C\delta. \end{aligned}\tag{56}$$

Finally, since $\delta>0$ is arbitrary, (54), (55) and (56) yield (53) which implies (48). ∎

Now we complete the proof of Theorem 4:

Proof of Theorem 4. First of all, for configurations $\{m_\varepsilon=(m_{\varepsilon,1},m_{\varepsilon,2},m_{\varepsilon,3}):\omega\to S^2\}$ of uniformly bounded energy $E_{\varepsilon,\beta}(m_\varepsilon)\le C$, their vertical components vanish asymptotically, i.e., $m_{\varepsilon,3}\to 0$ in $L^2(\omega)$. Then (48) and the compensated compactness program presented in [11] enables us to prove that $\{m_\varepsilon'\}$ is relatively compact in $L^1(\omega)$. Obviously, every strong limit m_0 satisfies (4) & (5). It remains to prove that the limit m_0 belongs to $A(\omega)$. For every fixed $DKMO-$entropy Φ, using (51) and (52) for Φ and m_ε and passing to the limit as $\varepsilon\to 0$ we obtain that μ_Φ is a measure. For a general smooth entropy Φ, we associate a $DKMO-$entropy $\tilde\Phi$ to Φ as in Remark 4.2; since $\Phi(m_0)=\tilde\Phi(m_0)$ for $|m_0|=1$, we conclude that $\mu_\Phi=\mu_{\tilde\Phi}$ is a measure, i.e., $m_0\in A(\omega)$. ∎

Remark 4.3. *The use of entropies seems to be appropriate for proving compactness of magnetizations in asymptotic regimes of thick thin-films micromagnetics. However, for ultrathin-films, other techniques based on the topology of the flow of magnetization are to be used (see Ignat & Otto [15, 16]).*

5 A lower bound

The aim of this section is to prove a lower bound for the energy $E_{\varepsilon,\beta}$. The idea is to define a Lipschitz continuous entropy that is appropriate for the expected quadratic cost of a jump.

5.1 The "$DKMO-$entropy" Φ_0

We introduce a map $\Phi_0 : \mathbf{R}^2 \to \mathbf{R}^2$ that plays the role of a $DKMO-$entropy and is well suited to catch the square of the size of a jump of a limiting configuration. More precisely, we ask for Φ_0 to be (only) a Lipschitz continuous map satisfying (13) a.e. in $\mathbf{R}^2$ and to satisfy

$$\nabla \cdot \{\Phi_0(m_0)\} = 2\sin^2\theta\, \mathcal{H}^1 \llcorner \{x_1 = 0\} = \frac{1}{2}|m_0^+ - m_0^-|^2\, \mathcal{H}^1 \llcorner \{x_1 = 0\} \quad \text{in } \mathcal{D}'(\mathbf{R}^2), \tag{57}$$

for every jump configuration $m_0 \,:\, \mathbf{R}^2 \to S^1$ of the form

$$m_0(x_1, x_2) = m_0^{\pm} \;:= (\cos\theta, \pm\sin\theta) \quad \text{if } \pm x_1 > 0, \qquad \theta \in (0,\pi).$$

The first ansatz is to search Φ_0 of the following form in polar coordinates:

$$\Phi_0(r,\theta) = r^2 g(\theta),$$

where $g = (g_1, g_2) : \mathbf{R} \to \mathbf{R}^2$ is Lipschitz continuous and 2π-periodic. With these assumptions, (13) turns into

$$\cos\theta\, \partial_\theta g_1 + \sin\theta\, \partial_\theta g_2 = 0 \quad \text{for a.e. } \theta \in [-\pi,\pi], \tag{58}$$

while (57) gives

$$g_1(\theta) - g_1(-\theta) = 2\sin^2\theta, \quad \forall \theta \in (0,\pi).$$

The second ansatz is to consider g_1 as an odd function (i.e., $g_1(\theta) = -g_1(-\theta)$ for $\theta \in (0,\pi)$). We find

$$g_1(\theta) = \operatorname{sign}(\theta)\sin^2\theta.$$

The condition (58) suggests that g_2 is even (i.e., $g_2(\theta) = g_2(-\theta)$ for $\theta \in (0,\pi)$) and $\partial_\theta g_2 = -2\cos^2\theta$ for $\theta \in (0,\pi)$. Since g_2 needs to be continuous and periodic, we choose

$$g_2(\theta) = \frac{\pi}{2} - \operatorname{sign}(\theta)\left(\theta + \frac{\sin(2\theta)}{2}\right).$$

(The constant $\pi/2$ is chosen in order to minimize $\|\Psi_0\|_{L^\infty}$ where Ψ_0 is associated to Φ_0 via (49), see below.) That justifies the following choice of our "$DKMO-$entropy": for $r > 0$ and $-\pi \le \theta < \pi$, we set

$$\Phi_0(re^{i\theta}) \quad := \quad r^2 \begin{pmatrix} \operatorname{sign}(\theta)\sin^2\theta \\ \frac{\pi}{2} - \operatorname{sign}(\theta)\left(\theta + \frac{\sin(2\theta)}{2}\right) \end{pmatrix}. \tag{59}$$

In fact Φ_0 is not a proper $DKMO-$entropy since it is not compactly supported and only Lipschitz continuous. But the identity (13) holds for a.e. $z \in \mathbf{R}^2$ and in $\mathcal{D}'(\mathbf{R}^2)$. We compute

$$D\Phi_0(z) = \frac{2}{r^2}\Phi_0(z) \otimes z - \frac{2}{r}\operatorname{sign}(\theta)\cos\theta\, z^\perp \otimes z^\perp.$$

That yields the decomposition

$$D\Phi_0(z) \quad = \quad -2\Psi_0(z) \otimes z + \Xi_0(z) Id,$$

where Ψ_0 and Ξ_0 are given in the following: for $z = re^{i\theta}$, $r > 0$, $-\pi \le \theta < \pi$,

$$\Psi_0(z) \;=\; \left(-\operatorname{sign}(\theta), -\frac{\pi}{2} + |\theta|\right), \qquad \Xi_0(z) \;=\; -2\operatorname{sign}(\theta) r\cos\theta.$$

Moreover, the following equality holds in $L^1(\omega)$ for $m' \in H^1(\omega, \mathbf{R}^2)$:

$$\nabla \cdot \left\{\Phi_0(m')\right\} \quad = \quad \Psi_0(m') \cdot \nabla(1 - |m'|^2) + \Xi_0(m')\nabla \cdot m'. \tag{60}$$

We also have $\|\Psi_0\|_{L^\infty} = \sqrt{1 + \pi^2/4}$.

5.2 Smooth approximation of Φ_0

We can not apply (51) to Φ_0 because of its lack of regularity (recall that (51) is valid only for $C^{1,1}-$entropies while $\Phi_0 \in C^{0,1}$). To overcome this difficulty we introduce smooth and compactly supported approximations of Φ_0. First, let $\{\phi_f\}_{f\in S^1}$ be the family of elementary $DKMO-$entropies (see [11]):

$$\phi_f(z) \quad = \quad \begin{cases} |z|^2 f & \text{for } z\cdot f > 0, \\ 0 & \text{for } z \cdot f \le 0. \end{cases}$$

The maps ϕ_f are not entropies since there are not continuous, but the formula

$$\Phi(z) \quad := \quad \chi(|z|)\int_{S^1} w(f)\phi_f(z)\,df \tag{61}$$

defines a $DKMO-$entropy for any smooth weight $w \;:\; S^1 \to \mathbf{R}$ and any smooth cut-off function χ.

Notice that Φ_0 (defined in (59)) may be obtained by (61) with $\chi \equiv 1$ and the BV-weight:

$$w_0(e^{i\theta}) := \begin{cases} \sin(\theta) & \text{if } -\frac{\pi}{2} \le \theta < \frac{\pi}{2}, \\ -\sin(\theta) & \text{if } \frac{\pi}{2} \le \theta < \frac{3\pi}{2}. \end{cases}$$

This formula comes as follows: taking $z = r\,e^{i\theta}$ and differentiating (61) with respect to θ (for $\chi \equiv 1$), one gets

$$\frac{1}{r^2}\partial_\theta \Phi_0 = \Big(w_0(ie^{i\theta}) + w_0(-ie^{i\theta})\Big)ie^{i\theta}.$$

Then choosing w_0 to be $\pi-$periodic (i.e., $w_0(e^{i\theta}) = w_0(-e^{i\theta})$ for $\theta \in (-\pi,\pi)$), we deduce the above formula for w_0 via definition (59).

Here the behavior of the $DKMO-$entropy $\Phi(m)$ does not count for $|m| > 2$ since our families of maps $\{m_\varepsilon\}$ satisfy $|m'_\varepsilon| \le 1$. Therefore, in the sequel, we fix a cut-off function $\chi \in C_0^\infty(\mathbf{R},\mathbf{R}_+)$ such that $\chi(r) = 1$ for $|r| \le 2$. By mollifying the weight w_0, we can obtain smooth approximations of Φ_0 in the disk $B_2(0) \subset \mathbf{R}^2$. More precisely, let $\rho \in C_0^\infty(\mathbf{R},\mathbf{R}_+)$ be a mollifier with support in $(-\pi,\pi)$ satisfying $\int \rho = 1$. For $1 \ge \eta > 0$, identifying $\mathbf{R}^2$ with the complex plane $\mathbf{C}$, we set

$$\rho_\eta(z) := \eta^{-1}\rho(\theta/\eta), \qquad \text{for } \; z = e^{i\theta} \in S^1, \quad -\pi \le \theta < \pi$$

and

$$w_\eta(z) \quad := \quad \int_{S^1} w_0(z/y)\rho_\eta(y)\,dy.$$

Applying (61) with the weight w_η, we define a smooth $DKMO-$entropy Φ_η. Since w_0 is a BV-function, there exists a positive constant $C > 0$ only depending on ρ, such that

$$\|\Phi_\eta - \Phi_0\|_{L^\infty(B_2(0))} \quad \le \quad C\eta. \tag{62}$$

The decomposition of $D\Phi_\eta$ is given by

$$D\Phi_\eta(z) \quad = \quad -2\Psi_\eta(z)\otimes z + \Xi_\eta(z)Id \qquad \text{for } |z| < 2 \tag{63}$$

with

$$\Psi_\eta(z) \;=\; \int_{S^1} \Psi_0(z/y)\rho_\eta(y)\,dy \quad \text{ and } \quad \Xi_\eta(z) \;=\; \int_{S^1} \Xi_0(z/y)\rho_\eta(y)\,dy.$$

5.3 Local results

We prove local lower bounds for the limiting energy density. Let $\{m_\varepsilon\}_{\varepsilon\downarrow 0}$ be a family of uniformly bounded energy and assume that $m_\varepsilon \to m_0$ in $L^1(\omega)$. With the notations of Theorem 1, we set $\zeta_{x_0,r}$ the following cut-off function around a jump point $x_0 \in J$ of m_0:

$$\zeta_{x_0,r}(x) = \begin{cases} \dfrac{1}{r}\left(1 - \dfrac{|x-x_0|}{r}\right) & \text{if} \quad |x-x_0| < r, \\ 0 & \text{if} \quad |x-x_0| \geq r, \end{cases}$$

for any $r > 0$ such that $d(x_0, \partial\omega) > r$. Let R_{x_0} be the rotation in the plane such that $R_{x_0}\boldsymbol{e}_1 = \boldsymbol{\nu}(x_0)$. We consider the following quantity:

$$q_r(x_0) \quad := \quad -\int_\omega \left\{R_{x_0}\Phi_0(R_{x_0}^{-1}m_0(x))\right\} \cdot \nabla\zeta_{x_0,r}(x)\,dx.$$

The quantity $q_r(x_0)$ is relevant for the concentration of the flow $\nabla \cdot \Phi_0(m_0)$ around the jump point x_0 of m_0 and provides information about the limiting energy density in the disk $B_r(x_0)$. More precisely, we have:

Lemma 4. *For every $x_0 \in J$ and for every $r < d(x_0, \partial\omega)$, we have*

$$|q_r(x_0)| \quad \leq \quad \frac{C}{r}\liminf_{\varepsilon\downarrow 0}\int_{B_r(x_0)} |\nabla(1-|m_\varepsilon'|^2)|,$$

where $C > 0$ is some universal positive constant ($C = \sqrt{1+\pi^2/4}$).

Proof. Let $x_0 \in J$. Up to a rotation, we may assume that $\boldsymbol{e}_1 = \boldsymbol{\nu}(x_0)$ (and then $R_{x_0} = Id$). By our assumption, $m_\varepsilon' \to m_0$ in $L^1(\omega)$. Let $\eta > 0$. By the dominated convergence theorem, we have

$$\int_\omega \Phi_\eta(m_0(x)) \cdot \nabla\zeta_{x_0,r}(x) \quad = \quad \lim_{\varepsilon\downarrow 0}\int_\omega \Phi_\eta(m_\varepsilon'(x)) \cdot \nabla\zeta_{x_0,r}(x). \tag{64}$$

Now, we use (51) to get

$$\left|\int_\omega \Phi_\eta(m_\varepsilon'(x)) \cdot \nabla\zeta_{x_0,r}(x)\,dx\right| \;\leq\; \|\Psi_\eta\|_\infty \int_\omega |\nabla(1-|m_\varepsilon'|^2)||\zeta_{x_0,r}|$$
$$+ C_{\Phi_\eta}\left(\frac{\beta}{\varepsilon}E_{\varepsilon,\beta}(m_\varepsilon)\right)^{1/2}\left(E_{\varepsilon,\beta}(m_\varepsilon)^{1/2}\|\zeta_{x_0,r}\|_\infty + \varepsilon^{1/2}\|\nabla\zeta_{x_0,r}\|_{L^2(\omega)}\right),$$

where $C_{\Phi_\eta} = \sqrt{2}\max\{\|\Xi_\eta\|_\infty, \|\nabla\Xi_\eta\|_\infty\}$. Letting $\varepsilon \downarrow 0$, the second term in the RHS asymptotically vanishes. By (64), inequality $\|\Psi_\eta\|_\infty \leq \|\Psi_0\|_\infty \leq C := \sqrt{1+\pi^2/4}$, identity $\|\zeta_{x_0,r}\|_\infty = r^{-1}$ and $\operatorname{supp}\zeta_{x_0,r} \subset B_r(x_0)$, we are led to

$$\left|\int_\omega \Phi_\eta(m_0(x)) \cdot \nabla\zeta_{x_0,r}(x)\,dx\right| \;\leq\; \frac{C}{r}\liminf_{\varepsilon\downarrow 0}\int_{B_r(x_0)} |\nabla(1-|m_\varepsilon'|^2)|.$$

Finally, letting $\eta \downarrow 0$, the conclusion follows by the dominated convergence theorem. ∎

We then check that the normal component of m_0 does not jump through J for $\mathcal{H}^1$–a.e. $x_0 \in J$.

Lemma 5. *With the notations of Theorem 1, we have*

$$m_0^+(x_0) \cdot \boldsymbol{\nu}(x_0) \quad = \quad m_0^-(x_0) \cdot \boldsymbol{\nu}(x_0) \text{ for } \mathcal{H}^1\text{–a.e. } x_0 \in J.$$

Proof. Let $x_0 \in J$ be such that point *(c)* of Theorem 1 holds. Up to a rotation, we may assume that $\boldsymbol{e}_1 = \boldsymbol{\nu}(x_0)$. Since $\nabla \cdot m_0 = 0$, we have for every $d(x_0, \partial\omega) > r > 0$:

$$0 \;=\; \int_\omega m_0 \cdot \nabla \zeta_{x_0,r} \;=\; \int_{B_r^+(x_0)} m_0 \cdot \nabla \zeta_{x_0,r} + \int_{B_r^-(x_0)} m_0 \cdot \nabla \zeta_{x_0,r}. \tag{65}$$

Writing $m_0(x) = m_0^+(x_0) + (m_0(x) - m_0^+(x_0))$, we compute

$$\int_{B_r^+(x_0)} m_0 \cdot \nabla \zeta_{x_0,r} \;=\; m_0^+(x_0) \cdot \int_{B_r^+(x_0)} \nabla \zeta_{x_0,r} + \int_{B_r^+(x_0)} (m_0 - m_0^+(x_0)) \cdot \nabla \zeta_{x_0,r}.$$

A direct computation shows that

$$\int_{B_r^+(x_0)} \nabla \zeta_{x_0,r} = -\boldsymbol{\nu}(x_0). \tag{66}$$

Since $|\nabla \zeta_{x_0,r}| \le r^{-2}$, we get by point *(c)* of Theorem 1:

$$\int_{B_r^+(x_0)} (m_0 - m_0^+(x_0)) \cdot \nabla \zeta_{x_0,r} \;=\; O\left(r^{-2} \int_{B_r^+(x_0)} |m_0 - m_0^+(x_0)| \right) \xrightarrow{r \downarrow 0} 0.$$

Thus

$$\lim_{r \downarrow 0} \int_{B_r^+(x_0)} m_0 \cdot \nabla \zeta_{x_0,r} \;=\; -m_0^+(x_0) \cdot \nu(x_0).$$

Similarly,

$$\lim_{r \downarrow 0} \int_{B_r^-(x_0)} m_0 \cdot \nabla \zeta_{x_0,r} \;=\; m_0^-(x_0) \cdot \nu(x_0)$$

and the conclusion follows from (65). ∎

Finally, we study the limit of $q_r(x_0)$ as $r \downarrow 0$.

Lemma 6. *For $\mathcal{H}^1$–a.e. $x_0 \in J$, we have*

(a) $\quad |q_r(x_0)| \;\le\; \pi \|\Phi_0\|_{L^\infty(S^1)}, \qquad$ *for* $\quad 0 < r < d(x_0, \partial\omega)$*;*

(b) $\quad \lim_{r \downarrow 0} |q_r(x_0)| \;=\; \frac{1}{2} |m_0^+(x_0) - m_0^-(x_0)|^2.$

Proof. The point (a) is a consequence of the definition of $q_r(x_0)$ since $|m_0| = 1$, $|\nabla \zeta_{x_0,r}| \le r^{-2}$ and $\operatorname{supp} \zeta_{x_0,r} \subset B_r(x_0)$.

To prove (b), we proceed as in Lemma 5. Up to a rotation, we may assume that $\boldsymbol{\nu}(x_0) = \boldsymbol{e}_1$ and that point *(c)* of Theorem 1 holds for $x_0 \in J$. We write

$$-q_r(x_0) = \int_{B_r^+(x_0)} \Phi_0(m_0(x)) \cdot \nabla \zeta_{x_0,r}(x)\, dx + \int_{B_r^-(x_0)} \Phi_0(m_0(x)) \cdot \nabla \zeta_{x_0,r}(x)\, dx, \tag{67}$$

for $0 < r < d(x_0, \partial\omega)$. Since Φ_0 is Lipschitz and $\|\nabla \zeta_{x_0,r}\|_\infty \le r^{-2}$, we have

$$\int_{B_r^+(x_0)} \Phi_0(m_0(x)) \cdot \nabla \zeta_{x_0,r}(x)\, dx \;=\; \Phi_0(m_0^+(x_0)) \cdot \int_{B_r^+(x_0)} \nabla \zeta_{x_0,r}(x)\, dx + O\left(r^{-2} \int_{B_r^+(x_0)} |m_0 - m_0^+(x_0)| \right).$$

Letting $r \downarrow 0$, Theorem 1 (c) and (66) lead to:

$$\lim_{r\downarrow 0} \int_{B_r^+(x_0)} \Phi_0(m_0(x)) \cdot \nabla \zeta_{x_0,r}(x)\, dx = -\Phi_0(m_0^+(x_0)) \cdot \boldsymbol{e}_1.$$

From Theorem 1 and Lemma 5, we may assume that $m_0^+(x_0) = (\cos\theta, \sin\theta)$ and $m_0^-(x_0) = (\cos\theta, -\sin\theta)$ for some $\theta \in [-\pi, \pi)$. We then have

$$\lim_{r\downarrow 0} \int_{B_r^+(x_0)} \Phi_0(m_0(x)) \cdot \nabla \zeta_{x_0,r}(x)\, dx \quad \stackrel{(59)}{=} \quad -\operatorname{sign}(\theta)|\sin\theta|^2.$$

Similarly,

$$\lim_{r\downarrow 0} \int_{B_r^-(x_0)} \Phi_0(m_0(x)) \cdot \nabla \zeta_{x_0,r}(x)\, dx = \Phi_0(m_0^-(x_0)) \cdot \boldsymbol{e}_1 = -\operatorname{sign}(\theta)|\sin\theta|^2.$$

Letting $r \to 0$ in (67), we get

$$\lim_{r\downarrow 0} |q_r(x_0)| \quad = \quad 2\,|\sin\theta|^2 \;\; = \;\; \frac{1}{2}|m_0^+(x_0) - m_0^-(x_0)|^2.$$

■

5.4 End of the proof of Theorem 5

Since J is $\mathcal{H}^1$ σ–finite and rectifiable, there exists an increasing sequence of graphs $\{\Sigma_k\}_{k\in\mathbf{N}}$ such that Σ_k is a finite union of disjoint embedded C^1 curves (of finite length) and $J \subset \cup_k \Sigma_k \cup P$ for some $\mathcal{H}^1$-negligible set P. Theorem 5 is then the consequence of the monotone convergence theorem, inequality (52) and the following result:

Proposition 4. *Let $\Sigma \subset\subset \omega$ be a finite union of closed disjoint embedded C^1-curves (of finite length). Then we have*

$$\frac{1}{2}\int |m_0^+ - m_0^-|^2 d\mathcal{H}^1 \llcorner (J \cap \Sigma) \quad \leq \quad C \liminf_{\varepsilon\downarrow 0} \int_\omega |\nabla(1 - |m'_\varepsilon|^2)|,$$

for some universal constant $C > 0$ ($C = \sqrt{4+\pi^2}$).

Proof. Using Lemma 6 and the dominated convergence theorem, we have

$$\frac{1}{2}\int |m_0^+(x_0) - m_0^-(x_0)|^2 d\mathcal{H}^1 \llcorner (J \cap \Sigma)(x_0) \quad = \quad \lim_{r\downarrow 0} \int_{\Sigma\cap J} |q_r(x_0)|\, dx_0.$$

Then Lemma 4 yields

$$\frac{1}{2}\int |m_0^+(x_0) - m_0^-(x_0)|^2 d\mathcal{H}^1 \llcorner (J \cap \Sigma)(x_0) \\ \leq \sqrt{1+\pi^2/4} \lim_{r\downarrow 0} \liminf_{\varepsilon\downarrow 0} r^{-1} \int_\Sigma \int_{B_r(x_0)} |\nabla(1 - |m'_\varepsilon(x)|^2)|\, dx\, d\mathcal{H}^1(x_0).$$

Since Σ is a finite union of disjoint embedded C^1 curves, for every $\delta > 0$ there exists $r_0 = r_0(\delta) > 0$ such that for $0 < r < r_0$, we have for every $x \in \Sigma$,

$$\mathcal{H}^1(B_r(x) \cap \Sigma) \quad \leq \quad 2(1+\delta) r.$$

Thus, from Fubini's Theorem, we have for every $r < \min\{r_0(\delta), d(\Sigma, \partial\omega)\}$,

$$\frac{1}{2}\int |m_0^+(x_0) - m_0^-(x_0)|^2 d\mathcal{H}^1 \llcorner (J \cap \Sigma)(x_0) \\ \leq (1+\delta)\sqrt{4+\pi^2} \liminf_{\varepsilon\downarrow 0} \int_{\Sigma + B_r(0)} |\nabla(1 - |m'_\varepsilon(x)|^2)|\, dx.$$

The conclusion follows by letting $\delta \downarrow 0$. ■

6 Is the entropy method efficient for the optimal constant problem ?

In this section, we focus on the issue of finding the optimal constant. For simplicity we work in the periodic domain $\omega = \mathbf{R} \times \mathbf{R}/\mathbf{Z}$ and we fix the limit magnetization at $x_1 = \pm\infty$. We believe that the optimal constant is the same as in the one-dimensional case although we are only able to prove partial results in this direction. These results are obtained through the construction of maps Φ such that inequality (27) hold. As stated in Propostion 1, such maps are in fact entropies. This proposition is proved in subsection 6.1 and the partial results in subsection 6.2. In the last subsections we establish that the entropy method can not lead to the general result. The question of the optimal constant is still open.

6.1 Proof of Proposition 1

Assume that $\Phi \in Lip(S^2, \mathbf{R}^2)$ satisfies (27) for every $m \in C^\infty(\omega, S^2)$. We will prove that Φ satisfies (28). Let $z \in S^1$. There exists an open ball $B \subset \omega$ centered at 0 and a map $m' \in C^\infty(\overline{B}, S^1)$ such that

$$m'(0) = z, \qquad Dm'(0) = (\partial_j m'_i)(0) \;=\; z^\perp \otimes z \qquad \text{and} \qquad \nabla \cdot m' \equiv 0 \text{ on } B.$$

For example, m' may be the vortex map centered at $z^\perp$ defined on $B = B(0, |z|/2)$ by

$$m'(x) \;:=\; \left(\frac{x - z^\perp}{|x - z^\perp|}\right)^\perp.$$

Next for every $\lambda \in \mathbf{R}$, $|\lambda| > 1$, there exists a map $m_\lambda \in C^\infty(\omega, S^2)$ such that $m_\lambda(x) = m'(x/\lambda)$ for x in some small neighborhood ω'_λ of 0. Applying (27) to m_λ at $x = 0$ yields

$$\nabla \cdot \{\Phi(m_\lambda)\} \quad\leq\quad \frac{\varepsilon}{2}|\nabla m_\lambda|^2 + \frac{1}{2\varepsilon} m_{\lambda,3}^2 + \nabla \cdot \{a_\varepsilon(m_\lambda)\nabla m_\lambda\} \qquad \text{a.e. in } \omega'_\lambda. \tag{68}$$

Now assume that z is a Lebesgue point of the tangential derivatives $z \mapsto D\Phi(z) \cdot z^\perp$ and $z \mapsto Da_\varepsilon(z) \cdot z^\perp$ on S^1, respectively. Then inequality (68) holds at $x = 0$ and by the definition of m_λ, it reads

$$\lambda^{-1} z \cdot D\Phi(z) \cdot z^\perp \;\leq\; \lambda^{-2}\left(\frac{\varepsilon}{2} + \nabla \cdot (a_\varepsilon(m').\nabla m')(0)\right).$$

Letting λ tend to $\pm\infty$, we obtain (28).

Conversely, assume that Φ satisfies (12) and $\partial_{m_3}\Phi$ vanishes on S^1. As in Remark 4.2, set $\tilde{\Phi}(m') := |m'|^2 \Phi(m'/|m'|)$ for $m' \in \mathbf{R}^2$. The map $\tilde{\Phi}$ is a $DKMO-$entropy and by (50) we have the decomposition

$$\nabla \cdot \{\tilde{\Phi}(m')\} - \Xi(m')\nabla \cdot m' \;=\; \Psi(m') \cdot \nabla(1 - |m'|^2) \quad \text{in } \omega,$$

for every $m \in C^\infty(\omega, S^2)$ where Ψ and Ξ are smooth in $\mathbf{R}^2$. Since $1 - |m'|^2 = m_3^2$, we write

$$\Psi(m') \cdot \nabla(1 - |m'|^2) = \Psi(m') \cdot \nabla(m_3^2) = O(|\nabla m_3||m_3|) = O\left(\frac{\varepsilon}{2}|\nabla m_3|^2 + \frac{1}{2\varepsilon} m_3^2\right).$$

Now the assumption $\partial_{m_3}\Phi \equiv 0$ on S^1 implies that the difference $m \mapsto \Phi(m) - \tilde{\Phi}(m')$ may be written on the form $m_3^2\Theta(m)$, with $\Theta \in C^\infty(S^2, \mathbf{R}^2)$. So we have

$$\nabla \cdot \{\Phi(m) - \tilde{\Phi}(m')\} \;=\; O(|\nabla m||m_3|) \;=\; O\left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon} m_3^2\right).$$

Finally for $c > 0$ small enough, $c\Phi$ satisfies (27) for every $\varepsilon > 0$ with $\alpha = -c\Xi$ and $a_\varepsilon \equiv 0$.

6.2 Optimal lower bound for the local model. Proof of Propositions 2 and 3

To prove Propositions 2 and 3, we adapt the method of Jin and Kohn [18] developed for the family of energies $\{AG_\varepsilon\}$ to our setting. An important difference with respect to [18] is that here the generalized entropies $\Phi_{\overline{m}_2}$ must depend on the size of the jump. Indeed, for the Aviles-Giga model in [18], once the direction of the jump is fixed (here $\mathbf{e}_1$), there exists an entropy leading to the optimal lower bound for every possible jumps (see (18)).

The construction of our map $\Phi_{\overline{m}_2} = (\varphi, \psi)$ for a fixed angle (defined via $\overline{m}_2$) is based on Lemma 1 (that we prove in the next section). For that, let us define $f(\theta) := \varphi(\overline{m}_1, \overline{m}_2 \cos\theta, \overline{m}_2 \sin\theta)$. Inequality (29) yields

$$|f'(\theta)| \quad \leq \quad \overline{m}_2{}^2 |\sin\theta|.$$

On the other hand, (26) yields

$$0 \;=\; 2\overline{m}_2{}^2 + f(\pi) - f(0) \quad = \quad \int_0^\pi \left\{ f'(\theta) + \overline{m}_2{}^2 \sin\theta \right\} d\theta.$$

So the integrand vanishes and we have $f'(\theta) = -\overline{m}_2{}^2 \sin\theta$. Consequently, the function φ is known up to an additive constant on the circle $\{m_1 = \overline{m}_1\} \cap S^2$:

$$\varphi(\overline{m}_1, m_2, m_3) \;=\; \overline{m}_2 m_2 + c, \qquad \text{if } m_2^2 + m_3^2 = \overline{m}_2{}^2. \tag{69}$$

Thus we will look for adapted triplets among triplets satisfying (69).

We now prove Propositions 2 and 3.

Proof of Propositions 2 and 3. We first assume that $\overline{m}_2 = 1$. Condition (69) implies that $\varphi(m)$ is the projection of m on $\mathbf{e_2}$ (up to a constant) when m turns on the circle $\{m_1 = 0\} \cap S^2$. The way to extend Φ to S^2 is the following: for any $m \in S^2$, we set

$$\Phi(m) \quad = \quad (\varphi(m), \psi(m)) \;:=\; \left(m_2(1 - m_1^2)\;,\; m_1(1 - m_2^2) \right) \quad \text{and} \quad \alpha(m) \;:=\; 3 m_1 m_2.$$

Condition (26) is checked since (69) is satisfied. Recall that for $m \in S^2$, the projection $\Pi_m f$ of $f \in \mathbf{R}^3$ on the tangent plane $T_m S^2$ is given by the formula

$$\Pi_m f = f - (f \cdot m) m,$$

we compute for every $m \in S^2$,

$$\begin{aligned} &\Pi_m \{\nabla\varphi(m) + \alpha(m)\mathbf{e}_1\} \;=\; m_3(0, m_3, -m_2) \\ \text{and} \qquad &\Pi_m \{\nabla\psi(m) + \alpha(m)\mathbf{e}_2\} \;=\; m_3(m_3, 0, -m_1). \end{aligned}$$

Combining these identities with the fact that $\partial_i m \in T_m S^2 \;\Rightarrow\; (\mathbf{e}; \partial_i m) = (\Pi_m(\mathbf{e}); \partial_i m)$ for $\mathbf{e} \in \mathbf{R}^3$ and $i = 1, 2$, we have for $m \in M$,

$$\begin{aligned} \nabla \cdot \{\Phi(m)\} + \alpha(m) \nabla \cdot m' &= \Pi_m \{\nabla\varphi(m) + \alpha(m)\mathbf{e}_1\} \cdot \partial_1 m + \Pi_m \{\nabla\psi(m) + \alpha(m)\mathbf{e}_2\} \cdot \partial_2 m \\ &= m_3 \begin{pmatrix} -m_1 \\ -m_2 \\ m_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_2 m_3 \\ \partial_1 m_3 \\ \partial_2 m_1 + \partial_1 m_2 \end{pmatrix}. \end{aligned} \tag{70}$$

By Young's inequality, we obtain

$$\begin{aligned} \nabla \cdot \{\Phi(m)\} + \alpha(m) \nabla \cdot m' &\leq \frac{1}{2\varepsilon} m_3^2 + \frac{\varepsilon}{2} \left((\partial_1 m_3)^2 + (\partial_2 m_3)^2 + (\partial_2 m_1 + \partial_1 m_2)^2 \right) \\ &= \frac{1}{2\varepsilon} m_3^2 + \frac{\varepsilon}{2} \left\{ (\partial_1 m_3)^2 + (\partial_2 m_3)^2 + (\partial_2 m_1)^2 + (\partial_1 m_2)^2 \right\} + \varepsilon \partial_2 m_1 \partial_1 m_2 \\ &\leq \frac{1}{2\varepsilon} m_3^2 + \frac{\varepsilon}{2} |\nabla m|^2 + \varepsilon (\partial_2 m_1 \partial_1 m_2 - \partial_1 m_1 \partial_2 m_2) \\ &= \frac{1}{2\varepsilon} m_3^2 + \frac{\varepsilon}{2} |\nabla m|^2 + \varepsilon \nabla \cdot \begin{pmatrix} m_2 \partial_2 m_1 \\ -m_2 \partial_1 m_1 \end{pmatrix}. \end{aligned}$$

So (27) holds for every $\varepsilon > 0$ where the smooth section a_ε of $\mathcal{B}$ is given by

$$a_\varepsilon(m)(v,\tilde v) = \varepsilon m_2(\tilde v\cdot \mathbf{e_1}, -v\cdot \mathbf{e_2}) \quad \text{for every } m\in S^2, \quad v,\tilde v\in TS^2.$$

It remains to prove that (25) holds for $\overline{m}_2 = 1$, i.e., for any family $\{m_\varepsilon\}\subset M$,

$$2 \le \liminf_{\varepsilon\downarrow 0} E_{\varepsilon,\beta}(m_\varepsilon)$$

in the regime (3). For that, let $\chi : \mathbf{R}\to\mathbf{R}_+$ be a smooth positive cut-off function such that $\chi(x_1) = 1$ for $|x_1| < 1$ and $\chi(x_1) = 0$ for $|x_1| > 2$. Set $\chi_k(x_1) = \chi(\frac{x_1}{k})$ for every $x_1 \in \mathbf{R}$ and $k\in\mathbf{N}$. Then (27) implies

$$\begin{aligned}\int_\omega \Big(\nabla\cdot\{\Phi(m_\varepsilon)\}\,\chi_k(x_1) + \alpha(m_\varepsilon)\nabla\cdot m_\varepsilon'\chi_k(x_1)\Big)\,dx \\ \le \int_\omega \left\{\frac{\varepsilon}{2}|\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon}m_{\varepsilon,3}^2\right\}\chi_k(x_1) - \int_\omega \frac{d\chi_k}{dx_1}(x_1)a_\varepsilon(m_\varepsilon)(\nabla m_\varepsilon)\cdot \boldsymbol{e}_1\,dx. \qquad (71)\end{aligned}$$

First, we pass to the limit as $k\to\infty$. For the first term of the RHS in (71), the dominated convergence theorem yields

$$\lim_{k\to\infty}\int_\omega \left\{\frac{\varepsilon}{2}|\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon}m_{\varepsilon,3}^2\right\}\chi_k(x_1)\,dx = E_{\varepsilon,\beta}(m_\varepsilon)$$

For the second term of the RHS in (71), Cauchy-Schwarz's inequality leads to:

$$\begin{aligned}\left|\int_\omega \frac{d\chi_k}{dx_1}(x_1)a_\varepsilon(m_\varepsilon)(\nabla m_\varepsilon)\cdot\boldsymbol{e}_1\,dx\right| &\le \frac{\sup_{z\in S^2}\|a_\varepsilon(z)\|_{\mathcal{L}\left((T_zS^2)^2,\mathbf{R}^2\right)}}{\sqrt{k}}\left(\int_{\mathbf{R}}\left|\frac{d\chi}{dx_1}\right|^2\right)^{1/2}\left(\int_\omega |\nabla m_\varepsilon|^2\right)^{1/2}\\ &\to 0 \quad \text{as} \quad k\to\infty.\end{aligned}$$

For the first term of the LHS in (71), integration by parts implies that

$$\begin{aligned}\int_\omega \nabla\cdot\{\Phi(m_\varepsilon)\}\,\chi_k(x_1)\,dx &= -\int_\omega \Phi(m_\varepsilon(kx_1,x_2))\cdot\boldsymbol{e}_1\frac{d\chi}{dx_1}\,dx\\ &\to \Big(\Phi(m^+) - \Phi(m^-)\Big)\cdot\boldsymbol{e}_1 \overset{(26)}{=} 2 \quad \text{as} \quad k\to\infty,\end{aligned}$$

since $m_\varepsilon \in M$. For the second term of the LHS in (71), we have that

$$\begin{aligned}\left|\int_\omega \alpha(m_\varepsilon)\nabla\cdot m_\varepsilon'\chi_k(x_1)\,dx\right| &\le \|\nabla\cdot m_\varepsilon'\|_{\dot H^{-1}(\omega)}\|\nabla[\alpha(m_\varepsilon)\chi_k]\|_{L^2(\omega)}\\ &\le \|\nabla\cdot(m_\varepsilon'\mathbf{1}_\omega)\|_{\dot H^{-1}(\mathbf{R}^2)}\Big(\|\nabla\alpha\|_{L^\infty}\|\nabla m_\varepsilon\|_{L^2(\omega)} + \|\alpha\|_{L^\infty}\|\nabla\chi_k\|_{L^2(\omega)}\Big)\\ &\le \|\nabla\alpha\|_{L^\infty}\left(\frac{2\beta}{\varepsilon}\right)^{1/2}E_{\varepsilon,\beta}(m_\varepsilon) + \|\alpha\|_{L^\infty}\left(\frac{2\beta}{k}E_{\varepsilon,\beta}(m_\varepsilon)\right)^{1/2},\end{aligned}$$

which means

$$\limsup_{k\to\infty}\left|\int_\omega \alpha(m_\varepsilon)\nabla\cdot m_\varepsilon'\chi_k(x_1)\,dx\right| \le \|\nabla\alpha\|_{L^\infty}\left(\frac{2\beta}{\varepsilon}\right)^{1/2}E_{\varepsilon,\beta}(m_\varepsilon).$$

Finally, summing the above relations and passing to lim inf as $\varepsilon\downarrow 0$, (71) leads to (25) in the regime (3) and Proposition 2 is proved.

Finally we prove Proposition 3. We assume $0 < \overline{m}_2 < 1$ and for $m\in S_{\overline{m}_2}$, we set

$$\Phi_{\overline{m}_2}(m) = (\varphi_{\overline{m}_2}(m), \psi_{\overline{m}_2}(m)) \;:=\; \frac{1}{\overline{m}_2}\Phi(m) + \frac{\overline{m}_1}{2\overline{m}_2}(0, m_3^2) \quad \text{and} \quad \alpha_{\overline{m}_2}(m) \;:=\; \frac{\alpha(m)}{\overline{m}_2}.$$

Again (26) holds. From (70), we have that for every $m \in M$,

$$\nabla \cdot \{\Phi_{\overline{m}_2}(m)\} + \alpha_{\overline{m}_2}(m)\nabla \cdot m' \quad = \quad m_3 \frac{1}{\overline{m}_2} \begin{pmatrix} \overline{m}_1 - m_1 \\ -m_2 \\ m_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_2 m_3 \\ \partial_1 m_3 \\ \partial_2 m_1 + \partial_1 m_2 \end{pmatrix}.$$

Now observe that for $m \in S_{\overline{m}_2}$, we have

$$\left| \frac{1}{\overline{m}_2} \begin{pmatrix} \overline{m}_1 - m_1 \\ -m_2 \\ m_3 \end{pmatrix} \right|^2 \quad = \quad \frac{1 - 2m_1\overline{m}_1 + \overline{m}_1^{\,2}}{\overline{m}_2^{\,2}} \leq \frac{1 - \overline{m}_1^{\,2}}{\overline{m}_2^{\,2}} = 1.$$

We deduce again by Young's inequality, that for $m \in M$

$$\nabla \cdot \{\Phi_{\overline{m}_2}(m)\} + \alpha_{\overline{m}_2}(m)\nabla \cdot m' \quad \leq \quad \frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon} m_3^2 + \varepsilon \nabla \cdot \begin{pmatrix} m_2 \partial_2 m_1 \\ -m_2 \partial_1 m_1 \end{pmatrix}.$$

So (27) holds for every $\varepsilon > 0$ and the same smooth a_ε as in Proposition 2. As above, the same argument yields (25) which concludes Proposition 3. ∎

6.3 Proof of Lemma 1

In this section we prove the pointwise bounds of Lemma 1. These bounds are the key ingredients leading to the contradiction establishing Theorem 6

Proof of Lemma 1. We define the following operator $L = (L_1, L_2)$: for every $m \in S^2$ and $(v_1, v_2) \in (T_m S^2)^2$,

$$\langle L(m)\,;\,(v_1, v_2)\rangle \quad := \quad \Big((\nabla\varphi(m) + \alpha(m)\Pi_m \mathbf{e}_1, \nabla\psi(m) + \alpha(m)\Pi_m \mathbf{e}_2) \; ; \; (v_1, v_2) \Big),$$

where $(\cdot, \cdot)$ denotes the scalar product in the Euclidian space $\mathbf{R}^3 \times \mathbf{R}^3$. Then for every smooth map $m \in C^\infty(\omega, S^2)$, (27) writes as in (70):

$$\begin{aligned} \langle L(m)\,;\,(\partial_{x_1} m, \partial_{x_2} m)\rangle &= \nabla \cdot \{\Phi(m)\} + \alpha(m)\nabla \cdot m' \\ &\leq \frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon} m_3^2 + \nabla \cdot \{a_\varepsilon(m)\nabla m\}, \qquad \text{for a.e. } x \in \omega. \end{aligned} \tag{72}$$

Now let $\tilde{x} \in \omega$ be fixed and $\tilde{m} \in S^2$ be a Lebesgue point of $\nabla\Phi$ and ∇a_ε.

For simplicity, we transpose our problem from S^2 to $\mathbf{R}^2$. Let R be an isomorphism between $\mathbf{R}^2$ and the tangent plane $T_{\tilde{m}} S^2$. We consider the following parameterization of S^2 in the neighborhood of $\tilde{m}$:

$$\Lambda \; : \; \mathbf{R}^2 \longrightarrow S^2, \quad n \quad \longmapsto \quad \frac{\tilde{m} + Rn}{|\tilde{m} + Rn|}.$$

Through the map Λ, we will associate to every map $n \in C^\infty(\omega, \mathbf{R}^2)$ the following map $m := \Lambda \circ n \in C^\infty(\omega, S^2)$. Moreover, the operator a_ε can be written via the following Lipschitz operator $\tilde{a}_\varepsilon \; : \mathbf{R}^2 \to \mathcal{L}(\mathbf{R}^2 \times \mathbf{R}^2, \mathbf{R}^2)$ defined by

$$\tilde{a}_\varepsilon(n)\nabla n \quad = \quad a_\varepsilon(m)\nabla m, \qquad \text{for a.e. } x \in \omega.$$

We prove that the operator a_ε has the following property:

Claim: *Provided that $m(\tilde{x}) = \tilde{m}$, there exists $b_\varepsilon \in \mathbf{R}$ such that*

$$\nabla \cdot \{a_\varepsilon(m)\nabla m\}(\tilde{x}) = b_\varepsilon \left[\partial_1 m(\tilde{x})\,;\,\partial_2 m(\tilde{x})\,;\,\tilde{m}\right],$$

where $[\cdot\,;\,\cdot\,;\,\cdot]$ stands for the scalar triple product.

Proof of Claim. We compute

$$\begin{aligned}\nabla\cdot\{a_\varepsilon(m)\nabla m\} &= \sum_{i,j,k=1,2}\partial_i(\tilde a^i_{\varepsilon\, j,k}(n)\,\partial_j n_k)\\ &= \sum_{i,j,k,l=1,2}\partial_l\tilde a^i_{\varepsilon\, j,k}(n)\partial_i n_l\,\partial_j n_k+\sum_{i,j,k=1,2}\tilde a^i_{\varepsilon\, j,k}(n)\partial_i\partial_j n_k \quad \text{a.e. in } \omega. \quad (73)\end{aligned}$$

(Here, we identified the linear operator $\tilde a_\varepsilon(n)\in\mathcal{L}(\mathbf{R}^2\times\mathbf{R}^2,\mathbf{R}^2)$ by the corresponding tensor $(\tilde a^i_{\varepsilon\, j,k}(n))_{i,j,k}\in\mathbf{R}^8$.) Let $y\in\mathbf{R}^2$ be such that $\Lambda(y)$ is a Lebesgue point of $\nabla\Phi$ and ∇a_ε. For every vector $(v_{i,j,k})_{i,j,k=1,2}\in\mathbf{R}^8$ satisfying $v_{i,j,k}=v_{j,i,k}$, we choose $n\in C^\infty(\omega,\mathbf{R}^2)$ such that $n(\tilde x)=y$, $\nabla n(\tilde x)=0$ and $\partial_i\partial_j n_k(\tilde x)=v_{i,j,k}$, for $i,j,k=1,2$. Then $m(\tilde x)=\Lambda(y)$, $\nabla m(\tilde x)=0$ and we deduce via (72) and (73) applied at $\tilde x$:

$$0 \;\le\; \frac{1}{2\varepsilon}\Lambda_3(y)^2+\sum_{i,j,k=1,2}\tilde a^i_{\varepsilon\, j,k}(y)v_{i,j,k}.$$

Since $(v_{i,j,k})$ was arbitrarily chosen such that $v_{i,j,k}=v_{j,i,k}$, we easily deduce that

$$\tilde a^i_{\varepsilon\, i,k}(y) \;=\; 0 \qquad\text{and}\qquad \tilde a^1_{\varepsilon\, 2,k}(y)+\tilde a^2_{\varepsilon\, 1,k}(y) \;=\; 0,\ \text{for } i,k=1,2. \quad (74)$$

Since y is an arbitrary point in a dense set of ω and $\tilde a_\varepsilon$ is continuous, it implies that the above identities hold true in $\mathbf{R}^2$.

Now we consider maps n such that $n(\tilde x)=0$. Since $\tilde m=\Lambda(0)$ is a Lebesgue point of $\nabla\Phi$ and ∇a_ε, by (73) applied at $\tilde x$ and (74), we conclude:

$$\begin{aligned}\nabla\cdot\{a_\varepsilon(m)\nabla m\}(\tilde x) &= \sum_{i,j,k,l=1,2}\partial_l\tilde a^i_{\varepsilon\, j,k}(0)\partial_i n_l\,\partial_j n_k\\ &= \sum_{k,l=1,2}\partial_l\tilde a^1_{\varepsilon\, 2,k}(0)(\partial_1 n_l\partial_2 n_k-\partial_1 n_k\partial_2 n_l)\\ &= \Big(\partial_1\tilde a^1_{\varepsilon\, 2,2}(0)-\partial_2\tilde a^1_{\varepsilon\, 2,1}(0)\Big)(\partial_1 n_1\partial_2 n_2-\partial_1 n_2\partial_2 n_1)\\ &=: b_\varepsilon \det\big(\partial_1 n(\tilde x)\,,\,\partial_2 n(\tilde x)\big)=b_\varepsilon\,[\partial_1 m(\tilde x)\,;\,\partial_2 m(\tilde x)\,;\,\tilde m]\,.\end{aligned}$$

(Here, we used that $\partial_j m(\tilde x)=R\partial_j n(\tilde x)$, j=1,2.) ∎

Applying our claim for a smooth map m such that $m(\tilde x)=\tilde m$, (72) at $\tilde x$ reads

$$\langle (L_1,L_2)(\tilde m)\,;\,(\partial_1 m,\partial_2 m)\rangle \;\le\; \frac{\varepsilon}{2}|\nabla m|^2+\frac{1}{2\varepsilon}\tilde m_3^2+b_\varepsilon\,[\partial_1 m\,;\,\partial_2 m\,;\,\tilde m]\,.$$

Finally, for every vector $v\in T_{\tilde m}S^2$ such that $|v|=|\tilde m_3|/\varepsilon$ we choose succesively two maps m such that $(\partial_1 m,\partial_2 m)(\tilde m):=(v,0)$ and $(\partial_1 m,\partial_2 m)(\tilde m):=(0,v)$, respectively. We get that $\left\langle L_j(\tilde m)\,;\,\frac{v}{|v|}\right\rangle\le|\tilde m_3|$ for $j=1,2$ and since $\tilde m$ is an arbitrary point in a dense set of S^2, we conclude with (29) and (30). ∎

6.4 Proof of Theorem 6

We now prove Theorem 6. In fact we prove the following stronger result which together with Lemma 1 yields the Theorem.

Proposition 5. *There exists $\varepsilon>0$ such that for $0<\overline{m}_2<\varepsilon$, $\overline{m}_1:=\sqrt{1-\overline{m}_2{}^2}$, $m_1^\pm:=(\overline{m}_1,\pm\overline{m}_2,0)$ there is no triplet $(\varphi,\psi,\alpha)\in Lip(S^2,\mathbf{R}^2)$ such that (26), (29) and (30) hold.*

Proof of Proposition 5. Assume by contradiction that there exists a sequence $\{\rho_k\} \subset (0,1)$ converging to 0 and a sequence of triplets $(\varphi_k, \psi_k, \alpha_k) \subset Lip(S^2, \mathbf{R}^3)$ adapted to the jumps (m_k^-, m_k^+) with $m_k^\pm := (\overline{m}_{1k}, \pm\overline{m}_{2k})$, $\overline{m}_{2k} := \rho_k$ and $\overline{m}_{1k} := \sqrt{1-\rho_k^2}$. Then for $k \geq 0$, we have

$$\varphi_k(m_k^+) - \varphi_k(m_k^-) \;=\; 2\rho_k^2 \tag{75}$$

and from Lemma 1, we have for $k \geq 0$ and for almost every $m \in S^2$:

$$|\nabla\varphi_k(m) + \alpha_k(m)\Pi_m \mathbf{e}_1| \;\leq\; |m_3| \quad \text{and} \quad |\nabla\psi_k(m) + \alpha_k(m)\Pi_m \mathbf{e}_2| \;\leq\; |m_3|. \tag{76}$$

Let us denote by I the symmetry transform with respect to the plane $\{m_2 = 0\}$. Replacing if necessary $(\varphi_k, \psi_k, \alpha_k)$ by

$$\left(\frac{\varphi_k - \varphi_k\circ I}{2}, \frac{\psi_k + \psi_k\circ I}{2}, \frac{\alpha_k - \alpha_k\circ I}{2}\right),$$

we may assume whitout loss of generality that the following properties

$$\varphi_k = -\varphi_k\circ I, \qquad \psi_k = \psi_k\circ I, \qquad \alpha_k = -\alpha_k\circ I, \qquad \text{for every } k \geq 0, \tag{77}$$

hold and that (75) & (76) are still true.

We want to perform a blow up around $m = (1,0,0)$ as k tends to infinity. For this reason let us transport the problem from S^2 to $\mathbf{R}^2$ (similarly as in Lemma 1). We introduce the map

$$P \;:\; \mathbf{R}^2 \;\to\; S^2, \quad n = (n_2, n_3) \;\longmapsto\; m = (m_1, m_2, m_3) = \frac{1}{1+|n|^2/4}(1-|n|^2/4, n_2, n_3),$$

Notice that the inverse P^{-1} of this map is the stereographic projection of vertex $(-1,0,0)$ on the tangent plane to the sphere at $(1,0,0)$.

For $k \geq 0$, we set $\tilde\varphi_k = \varphi_k\circ P, \tilde\psi_k = \psi_k\circ P$ and $\tilde\alpha_k = \alpha_k\circ P$. With these notations, (75) reads

$$\tilde\varphi_k\left(\frac{2\rho_k}{1+\sqrt{1-\rho_k^2}}, 0\right) - \tilde\varphi_k\left(-\frac{2\rho_k}{1+\sqrt{1-\rho_k^2}}, 0\right) \quad = \quad 2\rho_k^2 \tag{78}$$

In order to translate the pointwise bounds (76) in stereographic coordinates, we write $\nabla P_i(n) = \Pi_{P(n)}\mathbf{e}_i \cdot DP(n)$ for $i = 1,2$ where $DP(n)$ is the differential of P at n. Since the stereographic projection is a conformal map, $DP(n)$ is the product of a rotation and a dilation of factor $q(n)$, i.e., $DP(n)\cdot {}^tDP(n) = q^2(n)Id$ with $q(n) = \left(1+|n|^2/4\right)^{-1}$. Then (76) reads: for almost every $n = (n_2, n_3) \in \mathbf{R}^2$,

$$|\nabla\tilde\varphi_k + \tilde\alpha_k\nabla P_1|(n) \;\leq q(n)|P_3(n)| \quad \text{and} \quad \left|\nabla\tilde\psi_k + \tilde\alpha_k\nabla P_2\right|(n) \;\leq\; q(n)|P_3(n)|.$$

A straightforward computation leads to: for almost every $(n_2, n_3) \in \mathbf{R}^2$,

$$\left|\nabla\tilde\varphi_k - \frac{|n|\tilde\alpha_k}{(1+|n|^2/4)^2}\mathbf{e}_r\right| \;\leq\; |n_3|\,, \qquad \left|\nabla\tilde\psi_k + \frac{\sqrt{(1+|n|^2/4)^2 - n_2^2}\,\tilde\alpha_k}{(1+|n|^2/4)^2}\mathbf{f}_2\right| \;\leq\; |n_3|, \tag{79}$$

where we have introduced the unit vectors $\mathbf{e}_r := n/|n|$ and

$$\mathbf{f}_2 := \frac{\nabla P_2}{|\nabla P_2|} = \frac{1}{\sqrt{(1+|n|^2/4)^2 - n_2^2}}\left(1 + \frac{n_3^2 - n_2^2}{4}, -\frac{n_2 n_3}{2}\right).$$

Now we rescale the problem in order to pass to the limit as k goes to ∞ and reach the desired contradiction. Namely we set

$$\overline{\varphi}_k(n) \;:=\; \frac{1}{\rho_k^2}\tilde\varphi_k(\rho_k n), \quad \overline{\psi}_k(n) \;:=\; \frac{1}{\rho_k}\tilde\psi_k(\rho_k n) \quad \text{and} \quad \overline{\alpha}_k(n) \;:=\; \tilde\alpha_k(\rho_k n).$$

The conditions (78) & (79) imply that there exist a sequence of positive real numbers $\{\delta_{0,k}\}$, two sequences of positive functions $\{\delta_{1,k}\}$ and $\{\delta_{2,k}\}$ and a sequence of maps $\{\mathbf{f}_{2,k}\}$ such that

$$\delta_{0,k} \overset{k\uparrow\infty}{\longrightarrow} 1, \quad \delta_{1,k}, \delta_{2,k} \overset{k\uparrow\infty}{\longrightarrow} 1 \ \text{ in } C^1_{loc}(\mathbf{R}^2), \qquad |\mathbf{f}_{2,k}| = 1, \quad \mathbf{f}_{2,k} \overset{k\uparrow\infty}{\longrightarrow} \mathbf{e}_2 \ \text{ in } C^2_{loc}(\mathbf{R}^2, S^1) \tag{80}$$

and

$$\overline{\varphi}_k(\delta_{0,k}, 0) - \overline{\varphi}_k(-\delta_{0,k}, 0) \;=\; 2, \tag{81}$$

$$|\nabla\overline{\varphi}_k(n) - \delta_{1,k}(n)|n|\overline{\alpha}_k(n)\mathbf{e}_r| \;\le\; |n_3| \qquad \text{for a.e. } n = (n_2, n_3) \in \mathbf{R}^2, \tag{82}$$

$$\left|\nabla\overline{\psi}_k(n) + \delta_{2,k}(n)\overline{\alpha}_k(n)\mathbf{f}_{2,k}(n)\right| \;\le\; \rho_k|n_3| \qquad \text{for a.e. } n = (n_2, n_3) \in \mathbf{R}^2. \tag{83}$$

Moreover, from (77), we have

$$\overline{\varphi}_k(0, n_3) \;=\; 0, \qquad \text{for every} \qquad k \ge 0 \text{ and } n_3 \in \mathbf{R}. \tag{84}$$

In order to pass to the limit $k \uparrow \infty$, we prove the following Lemma.

Lemma 7. *The sequence of Lipschitz maps $\{(\overline{\varphi}_k, \overline{\psi}_k)\}_{k\in\mathbf{N}}$ is locally uniformly equicontinuous.*

Proof. We will use several families of local orthonormal basis in the vertical plane $(n_2, n_3) \in \mathbf{R}^2$: $(\mathbf{e}_2, \mathbf{e}_3)$, $(\mathbf{e}_r(n), \mathbf{e}_\theta(n))$ and $\{(\mathbf{f}_{2,k}(n), \mathbf{f}_{3,k}(n))\}_{k\in\mathbf{N}}$ with $\mathbf{e}_\theta = \mathbf{e}_r^\perp$ and $\mathbf{f}_{3,k} = \mathbf{f}_{2,k}^\perp$ for every $k \in \mathbf{N}$.

Let B be an arbitrary closed ball in $\mathbf{R}^2$. Along the proof C denotes a (possibly changing) positive constant only depending on B. In the sequel, $\{a_{1,k}\}, \{a_{2,k}\}, \cdots$ will denote bounded sequences in $L^\infty(B)$ and $\{b_{1,k}\}, \{b_{2,k}\}, \cdots$ will denote bounded sequences in $C^1(B)$ such that $|b_{i,k}| \ge 1/C$ holds uniformly.

We set

$$a_{1,k} := (\mathbf{e}_\theta \cdot \nabla)\overline{\varphi}_k \quad \text{ and } \quad a_{2,k} := (\mathbf{f}_{3,k} \cdot \nabla)\overline{\psi}_k, \qquad \text{for every} \qquad k \ge 0. \tag{85}$$

Then inequalities (82) and (83) imply that the sequences $\{a_{1,k}\}$ and $\{a_{2,k}\}$ are uniformly bounded in $L^\infty(B)$ (as requested above). Together with (84), it leads in particular to:

$$\|\overline{\varphi}_k\|_{L^\infty(B)} \;\le\; C. \tag{86}$$

Now combining (82) and (83) in order to eliminate $\overline{\alpha}_k$, we obtain two sequences $\{a_{3,k}\}$ and $\{b_{1,k}\}$ defined by

$$a_{3,k} \;:=\; (\mathbf{e}_r \cdot \nabla)\overline{\varphi}_k(n) + |n| b_{1,k}(\mathbf{f}_{2,k} \cdot \nabla)\overline{\psi}_k(n) \qquad \text{for } k \ge 0, \tag{87}$$

that satisfy the required conditions.

Now our goal is to establish that $\overline{\varphi}_k$ solves a uniformly elliptic second order PDE on B with a sufficiently integrable RHS in order to deduce some uniform regularity on $\{\overline{\varphi}_k\}$. For this, we now assume that the closed ball B is away from the n_2-axis, i.e., $B \cap \{(n_2, 0) : n_2 \in \mathbf{R}\} = \emptyset$. Since $\{\mathbf{f}_{3,k}\}$ converges uniformly to $\mathbf{e}_3$ on B, this assumption implies that for k large enough, we have

$$1 \ge |\mathbf{e}_r \cdot \mathbf{f}_{3,k}|(n) = |\mathbf{e}_\theta \cdot \mathbf{f}_{2,k}|(n) \ge 1/C \qquad \text{for every } n \in B. \tag{88}$$

In particular, there exists a sequence of angle functions $\{x_k\}$ bounded in $C^1(B)$ such that for k large enough, we have $\mathbf{f}_{3,k} = (\cos x_k)\mathbf{e}_r + (\sin x_k)\mathbf{e}_\theta$ with $|\cos x_k| \ge 1/C$ in B. Plugging this identity in (87), we get for k large enough,

$$(\mathbf{f}_{3,k} \cdot \nabla)\overline{\varphi}_k \;=\; (\cos x_k)a_{3,k} + (\sin x_k)(\mathbf{e}_\theta \cdot \nabla)\overline{\varphi}_k - (\cos x_k)|n| b_{2,k}(\mathbf{f}_{2,k} \cdot \nabla)\overline{\psi}_k, \qquad \text{on } B.$$

The first term in the right hand side is uniformly bounded and by (85), the second term is also uniformly bounded. For the last term, we notice that the coefficient $(-\cos x_k)|n| b_{2,k}$ required the desired properties so that we may rewrite the last equation as

$$(\mathbf{f}_{3,k} \cdot \nabla)\overline{\varphi}_k \;=\; a_{4,k} + b_{3,k}(\mathbf{f}_{2,k} \cdot \nabla)\overline{\psi}_k. \tag{89}$$

We now prove

$$\nabla \cdot \{(\mathbf{f}_{3,k} \otimes \mathbf{f}_{3,k}) \nabla \overline{\varphi}_k\} \;=\; a_{5,k} + \partial_{n_2} a_{6,k} + \partial_{n_3} a_{7,k}. \tag{90}$$

Multiplying (89) by $\mathbf{f}_{3,k}$ and applying the divergence operator, we obtain

$$\nabla \cdot \{(\mathbf{f}_{3,k} \otimes \mathbf{f}_{3,k}) \nabla \overline{\varphi}_k\} \;=\; \partial_{n_2} a_{8,k} + \partial_{n_3} a_{9,k} + \nabla \cdot \{b_{3,k} \mathbf{f}_{3,k}\} \left\{(\mathbf{f}_{2,k} \cdot \nabla) \overline{\psi}_k\right\} + b_{3,k} \{(\mathbf{f}_{3,k} \cdot \nabla) \mathbf{f}_{2,k}\} \cdot \nabla \overline{\psi}_k + b_{3,k} D^2 \overline{\psi}_k (\mathbf{f}_{3,k}, \mathbf{f}_{2,k}). \tag{91}$$

Since $\{b_{3,k}\}$ and $\{\mathbf{f}_{3,k}\}$ are bounded in $C^1(B)$, we obtain that the third term in the right hand side has the form $a_{10,k} (\mathbf{f}_{2,k} \cdot \nabla) \overline{\psi}_k$. Now since $|b_{3,k}|$ is uniformly bounded from below, we deduce from (89) that $(\mathbf{f}_{2,k} \cdot \nabla) \overline{\psi}_k = a_{11,k} + b_{4,k} (\mathbf{f}_{3,k} \cdot \nabla) \overline{\varphi}_k$. Finally by (86) we obtain that the third term in the RHS of (91) has the form

$$\nabla \cdot \{b_{3,k} \mathbf{f}_{3,k}\} \left\{(\mathbf{f}_{2,k} \cdot \nabla) \overline{\psi}_k\right\} \;=\; a_{12,k} + \partial_{n_2} a_{13,k} + \partial_{n_3} a_{14,k}.$$

For the fourth term in the RHS of (91), since $\mathbf{f}_{2,k}$ is a unit C^1 vector field $(\mathbf{f}_{3,k} \cdot \nabla) \mathbf{f}_{2,k}$, has the form $a_{14,k} \mathbf{f}_{3,k}$ and from (85), we deduce

$$b_{3,k} \{(\mathbf{f}_{3,k} \cdot \nabla) \mathbf{f}_{2,k}\} \cdot \nabla \overline{\psi}_k \;=\; a_{15,k}.$$

For the last term, we write $D^2 \overline{\psi}_k (\mathbf{f}_{3,k}, \mathbf{f}_{2,k}) \;=\; (\mathbf{f}_{2,k} \cdot \nabla) \{(\mathbf{f}_{3,k} \cdot \nabla) \overline{\psi}_k\} - \{(\mathbf{f}_{2,k} \cdot \nabla) \mathbf{f}_{3,k}\} \cdot \nabla \overline{\psi}_k$. Using (85) and the fact that $\mathbf{f}_{3,k}$ is a unit C^1 vector field, we have $(\mathbf{f}_{2,k} \cdot \nabla) \mathbf{f}_{3,k} = \tilde{a}_k \mathbf{f}_{2,k}$ with $\tilde{a}_k$ uniformly bounded in $C^1(B)$. Therefore

$$b_{2,k} D^2 \overline{\psi}_k (\mathbf{f}_{3,k}, \mathbf{f}_{2,k}) \;=\; b_{2,k} (\mathbf{f}_{2,k} \cdot \nabla) a_{2,k} + b_{2,k} \tilde{a}_k (\mathbf{f}_{2,k} \cdot \nabla) \overline{\psi}_k.$$

As above, using (89) and (86), we deduce that the last term in the RHS of (91) has the desired form. We conclude that (90) holds.

Similarly, multiplying the first part of (85) by $\mathbf{e}_\theta$ and taking the divergence, we obtain that $\nabla \cdot \{(\mathbf{e}_\theta \otimes \mathbf{e}_\theta) \nabla \overline{\varphi}_k\}$ has the same form. Adding this result and (90), we conclude that $\overline{\varphi}_k$ solves a second order PDE on divergence form:

$$\nabla \cdot \{(\mathbf{f}_{3,k} \otimes \mathbf{f}_{3,k} + \mathbf{e}_\theta \otimes \mathbf{e}_\theta) \nabla \overline{\varphi}_k\} = a_{16,k} + \partial_{n_2} a_{17,k} + \partial_{n_3} a_{18,k}. \tag{92}$$

By (88) the family of matrices

$$\{\mathbf{f}_{3,k} \otimes \mathbf{f}_{3,k} + \mathbf{e}_\theta \otimes \mathbf{e}_\theta\}$$

is uniformly elliptic on B, uniformly in k. Using (86) and (92), classical elliptic theory implies that $\{\overline{\varphi}_k\}$ is bounded in $H^1(B')$ for every closed ball B' in the interior of B. By (85) and (87) we deduce that $\{\nabla \overline{\psi}_k\}$ is also bounded in $L^2(B')$. By Lemma 9 (seeAppendix), these L^2-bounds on the gradients together with the one-direction L^∞-bounds (85) imply that $\{\overline{\varphi}_k\}$ and $\{\overline{\psi}_k\}$ are uniformly 1/3-Hölder continuous on B'.

Finally we deduce from (85) and the fact that $\{\mathbf{f}_{3,k}\}$ tends to $\mathbf{e}_3 = (0,1)$ in L^∞_{loc} that $\{\overline{\varphi}_k\}$ and $\{\overline{\psi}_k\}$ are also equicontinuous on bounded sets intersecting $\mathbf{R} \times \{0\}$. ■

By Lemma 7 and (84), Ascoli's theorem implies the existence of $(\varphi, \psi) \in C(\mathbf{R}^2, \mathbf{R}^2)$ and constants $\{p_k\}_k \subset \mathbf{R}$ such that (up to a subsequence) $\{(\overline{\varphi}_k, \overline{\psi}_k - p_k)\}_k$ converges to (φ, ψ) uniformly on every compact of $\mathbf{R}^2$. In the sequel, we identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ and we use both cartesian and polar coordinates $(n_2, n_3) = re^{i\theta}$ with $r \geq 0$ and $\theta \in \mathbf{R}$. Passing to the limit $k \uparrow \infty$ in (81), we obtain

$$\varphi(1,0) - \varphi(-1,0) \;=\; 2. \tag{93}$$

From (80), we have $(\mathbf{f}_{2,k}^\perp \cdot \nabla) \overline{\psi}_k \to (e_3 \cdot \nabla) \psi$ in $\mathcal{D}'(\mathbf{R}^2)$. So (83) imply $(\mathbf{e}_3 \cdot \nabla) \psi = 0$ and ψ only depends on n_2 as well as the distribution defined by $\alpha := -(\mathbf{e}_2 \cdot \nabla) \psi$. Next using again (80) and (83), we obtain $\overline{\alpha}_k \to \alpha$ in $\mathcal{D}'(\mathbf{R}^2)$. Finally, passing to the limit in (82), we obtain that

$$\nabla \varphi - |n| \alpha(n_2) \mathbf{e}_r \in L^\infty_{loc}(\mathbf{R}^2, \mathbf{R}^2) \quad \text{and} \quad |\nabla \varphi - |n| \alpha(n_2) \mathbf{e}_r| \;\leq\; |n_3| \quad \text{for a.e. } n \in \mathbf{R}^2. \tag{94}$$

The couple (φ, α) inherits the symmetries (77) of the sequence $(\overline{\varphi}_k, \overline{\alpha}_k)$, so we have

$$\varphi \;=\; -\varphi \circ I, \qquad \alpha \;=\; -\alpha \circ I \tag{95}$$

where we recall that I is the symmetry with respect to $\{m_2 = 0\}$ on S^2 that turns through the stereographic projection into the symmetry with respect to the axis $\{n_2 = 0\}$. In particular φ vanishes on the axis $\{n_2 = 0\}$.

It turns out that the conditions (93), (94) and (95) are not compatible so we will obtain a contradiction, which proves Proposition 5. Namely:

Lemma 8. *There is no couple $(\varphi, \alpha) \in C(\mathbf{R}^2) \times \mathcal{D}'(\mathbf{R})$ satisfying* (93),(94) *and* (95).

The end of the paper is dedicated to the proof of Lemma 8.

A simple case: For convenience of the reader, we first prove Lemma 8 in the simple case of a C^2 function φ. The idea of the proof in the general case will be the same but some technical issues are to be detailed. Assume that φ is of class C^2 and denote the angular derivative by $\frac{1}{r}\partial_\theta\varphi(re^{i\theta}) = e_\theta \cdot \nabla\varphi(re^{i\theta})$. By (94), $\alpha \in L^\infty_{loc}(\mathbf{R}^2)$. First, applying (94) on the unit circle $r = 1$, we obtain that

$$|\partial_\theta\varphi(e^{i\theta})| \;\leq\; \sin\theta, \qquad 0 < \theta < \pi.$$

Now from (93), we have

$$\int_0^\pi -\partial_\theta\varphi(e^{i\theta})\, d\theta \;=\; \int_0^\pi \sin\theta\, d\theta,$$

thus we have $\partial_\theta\varphi(e^{i\theta}) = -\sin\theta$ for $0 < \theta < \pi$. Then (95) leads to $\varphi(e^{i\theta}) = \cos\theta$ for $0 < \theta < \pi$. Therefore, (94) (where the equality holds for $r = 1$) yields

$$\alpha(\cos\theta) \;=\; \partial_r\varphi(e^{i\theta}), \qquad 0 < \theta < \pi. \tag{96}$$

Moreover, using again (94), it results that

$$\left(\frac{1}{r}\partial_\theta\varphi(re^{i\theta})\right)^2 \;\leq\; r^2\sin^2\theta \quad \text{for} \quad 0 < \theta < \pi \text{ and } r > 0. \tag{97}$$

This inequality is an equality for $r = 1$, so the derivatives with respect to r of the left and right hand sides of (97) must be equal in $r = 1$. That means:

$$\partial_r\partial_\theta\varphi(e^{i\theta}) \;=\; -2\sin\theta, \qquad 0 < \theta < \pi.$$

Combining with the symmetry (95) of α and (96), we obtain

$$\alpha(n_2) \;=\; 2n_2, \qquad -1 < n_2 < 1. \tag{98}$$

To end the proof, we write (94) at $n_3 = 0$ and using (98), it implies $\partial_{n_2}\varphi(n_2, 0) \;=\; n_2\alpha(n_2) \;=\; 2n_2^2$. This is not compatible with (93) since

$$2 \overset{(93)}{=} \int_{-1}^1 \partial_{n_2}\varphi(n_2)\, dn_2 \;=\; \int_{-1}^1 2n_2^2\, dn_2 \;=\; \frac{4}{3}.$$

The general case: Here we only assume $\varphi \in C(\mathbf{R}^2)$. We begin by improving the regularity of φ.

Claim 1. *The function φ is locally Lipschitz and α is locally bounded.*

Proof. Using the polar coordinates, we write $\mathbf{e}_3 = \cos\theta \mathbf{e}_\theta + \sin\theta \mathbf{e}_r$ and

$$\begin{aligned}(\mathbf{e}_3\cdot\nabla)\varphi &= \cos\theta(\mathbf{e}_\theta\cdot\nabla)\varphi + \sin\theta(\mathbf{e}_r\cdot\nabla)\varphi \\ &= \cos\theta(\mathbf{e}_\theta\cdot\nabla)\varphi + \sin\theta\left\{(\mathbf{e}_r\cdot\nabla)\varphi - 2\alpha(n_2)|n|\right\} + 2\alpha(n_2)n_3.\end{aligned}$$

Denoting $f := 2\alpha(n_2)n_3 - (\mathbf{e}_3\cdot\nabla)\varphi$, by (94), we know that $f \in L^\infty_{loc}(\mathbf{R}^2)$, $|f| \le |n_3|$. Regularizing with symmetric mollifiers in n_2 and n_3 the following distribution

$$2\alpha(n_2)n_3 \;=\; (\mathbf{e}_3\cdot\nabla)\varphi + f,$$

then integrating in n_3 on $[0,1]$ and letting the mollifiers to converge to Dirac masses, one proves that $\alpha \in L^\infty_{loc}(\mathbf{R}^2)$ and satisfies

$$|\alpha(n_2)| \;\le\; |\varphi(n_2,1) - \varphi(n_2,0)| + \frac{1}{2} \quad \text{for a.e. } n_2 \in \mathbf{R}.$$

Combining with (94), we deduce that φ is locally Lipschitz. ∎

Claim 2. *For every $0 < \theta < \pi$, we have $\varphi(e^{i\theta}) = \cos\theta$. Moreover the map*

$$(0,+\infty) \longrightarrow L^1(0,\pi), \quad r \longmapsto \frac{1}{r}\partial_\theta\varphi(r\exp(i\cdot))$$

is continuous at $r = 1$.

Proof. Since φ is Lipschitz, for every $r > 0$ the function $\varphi_r : \theta \mapsto \varphi(re^{i\theta})$ is absolutely continuous with derivative $\frac{1}{r}\partial_\theta\varphi$ (defined for $\mathcal{H}^1-$almost every θ). From (94), we have that for almost every $r > 0$:

$$r\sin\theta + \frac{1}{r}\partial_\theta\varphi(re^{i\theta}) \;\ge\; 0 \qquad \text{for a.e. } \theta \in (0,\pi). \tag{99}$$

Since the map $r \mapsto \varphi_r$ with values in $\mathcal{D}'(\mathbf{R}/2\pi\mathbf{Z})$ is continuous, (99) holds for every $r > 0$. In particular, if $r = 1$, integrating for $\theta \in (0,\pi)$, one has by (93):

$$\int_0^\pi \left(\sin\theta + \partial_\theta\varphi(e^{i\theta})\right) d\theta = 0,$$

which implies by (99) that $\partial_\theta\varphi(e^{i\theta}) = -\sin\theta$ a.e. in $(0,\pi)$ and using (95) we obtain $\varphi(e^{i\theta}) = \cos\theta$ for $\theta \in (0,\pi)$. Finally, by (93) and the continuity of φ, we get

$$\int_{-\pi}^{\pi} \left(r\sin\theta + \frac{1}{r}\partial_\theta\varphi(re^{i\theta})\right) d\theta \;\stackrel{r\to 1}{\longrightarrow}\; 0,$$

which means that the map $\theta \mapsto \frac{1}{r}\partial_\theta\varphi$ converges to $\theta \mapsto -\sin\theta$ in $L^1(0,\pi)$ as r tends to 1 which establishes Claim 2. ∎

Remark 6.1. *In general if $F : [0,1]^2 \to \mathbf{R}$ is Lipschitz, then the map $[0,1] \to L^1(0,1)$, $y \mapsto \partial_x F(x,y)$ is not continuous. As a counterexample, let us set $F(x,\frac{1}{n}) := \frac{1}{2^n}\sin(2^n x)$ for $n \ge 1$ and $x \in [0,1]$. Then we extend $F(x,\cdot)$ as an affine function on $\left[\frac{1}{2^{n+1}},\frac{1}{2^n}\right]$. It is easy to check that $\partial_x F(\cdot,y)$ converges only weakly to 0 as y tends to 0.*

Claim 3. *We have $\alpha(n_2) = 2n_2$ for a.e. $-1 < n_2 < 1$.*

Proof. By (94), for almost every $r > 0$ and $\theta \in (0,\pi)$,

$$|\partial_r\varphi(re^{i\theta}) - \alpha(r\cos\theta)r|^2 \;\le\; r^2\sin^2\theta - \left|\frac{1}{r}\partial_\theta\varphi(re^{i\theta})\right|^2,$$

By Claim 2, the above RHS (as a function of θ) tends to 0 in $L^1(0,\pi)$ as $r \to 1$. Hence,

$$\int_0^\pi |\partial_r\varphi(re^{i\theta}) - \alpha(r\cos\theta)|^2\, d\theta \quad \overset{r\to 1}{\longrightarrow} \quad 0.$$

(Here, we used that $(r-1)\alpha(r\cos\theta) \overset{r\to 1}{\longrightarrow} 0$ in $L^2(0,\pi)$ because α is locally bounded.) Averaging for radii s between 1 and r (r can be less than 1 or larger than 1) and using the identity $\varphi(e^{i\theta}) = \cos\theta$, we obtain

$$\varphi(re^{i\theta}) \;=\; \cos\theta + (r-1)\fint_1^r \alpha(s\cos\theta)\, ds + (r-1)R(re^{i\theta}) \qquad \text{for a.e. } \theta\in(0,\pi), \tag{100}$$

with $\int_0^\pi |R(re^{i\theta})|^2\, d\theta \overset{r\to 1}{\longrightarrow} 0$. On the other hand, by (95), we have for all $\theta_0 \in (0,\pi)$ that

$$\varphi(re^{i\theta_0}) \;=\; r\int_{\pi/2}^{\theta_0} \frac{1}{r}\partial_\theta\varphi(re^{i\theta})\, d\theta \;=\; r^2\cos\theta_0 + r\int_{\pi/2}^{\theta_0}\left\{\frac{1}{r}\partial_\theta\varphi(re^{i\theta}) + r\sin\theta\right\} d\theta.$$

Plugging this equality in (100), we obtain

$$r\int_{\pi/2}^{\theta_0}\left\{\partial_\theta\varphi(re^{i\theta}) + r\sin\theta\right\} d\theta \;=\; (r-1)\left\{\fint_1^r \alpha(s\cos\theta_0)\, ds - (r+1)\cos\theta_0 - R(re^{i\theta_0})\right\}.$$

Finally, by (99), the integrand in the above LHS is non-negative which implies (dividing by $r-1$ and letting $r\downarrow 1$ and $r\uparrow 1$, respectively):

$$\alpha(\cos\theta_0) \;=\; 2\cos\theta_0 \qquad \text{for a.e } 0<\theta_0<\pi,$$

which proves Claim 3. (Here, we used that for a.e. θ_0, $\cos\theta_0$ is a Lebesgue point of α, i.e., $\lim_{r\to 1}\fint_1^r \alpha(s\cos\theta_0)\, ds = \alpha(\cos\theta_0)$.) ∎

Finally, we prove that (93),(94) and (95) lead to a contradiction. For that, we use (94) in the neighborhood of $\{n_3 = 0\}$. For almost every $\theta\in(0,\pi)$, we have

$$-r\sin\theta \le \partial_r\varphi - 2r^2\cos\theta \;\le\; r\sin\theta \qquad \text{for a.e. } r>0.$$

The continuity of φ implies that the above expression holds true for every $\theta\in(0,\pi)$. Integrating in r on $(0,1)$ and letting $\theta\downarrow 0$ and $\theta\uparrow\pi$, since $\varphi(0,0)=0$ by (95), we obtain

$$\varphi(1,0) - \varphi(-1,0) \;=\; \frac{4}{3},$$

which contradicts (93). That concludes the proof of Proposition 5. ∎

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7 Appendix

We used the following embedding theorem:

Lemma 9. *Let $B_2 \subset \mathbf{R}^2$ be the disk of radius 2. Every function $\varphi \in H^1(B_2)$ satisfying $\partial_1\varphi \in L^\infty(B_2)$ is $\frac{1}{3}-$Hölder continuous on the unit disk, i.e., $\varphi \in C^{0,\frac{1}{3}}(B_1)$ and*

$$|\varphi|_{C^{0,\frac{1}{3}}(B_1)} \le 2\Big(\|\partial_1\varphi\|_{L^\infty(B_2)} + \|\partial_2\varphi\|_{L^2(B_2)}\Big). \tag{101}$$

Proof. We first show that (101) holds for $\varphi \in C^1(B_2)$. For that, let $(x, y) \in B_1$ and we will estimate $\varphi(x, y) - \varphi(0, 0)$. First, we have that

$$|\varphi(x, y) - \varphi(0, y)| \leq \|\partial_1 \varphi\|_{L^\infty(B_2)} |x| \leq \|\partial_1 \varphi\|_{L^\infty(B_2)} |x|^{1/3}.$$

For $\delta := \left(\frac{|y|}{2}\right)^{1/3} \in (0, 1)$, we compute

$$\begin{aligned}
|\varphi(0, y) - \varphi(0, 0)| &\leq \left|\varphi(0, y) - \fint_{-\delta}^{\delta} \varphi(x', y)\, dx'\right| + \fint_{-\delta}^{\delta} |\varphi(x', y) - \varphi(x', 0)|\, dx' \\
&\qquad + \left|\varphi(0, 0) - \fint_{-\delta}^{\delta} \varphi(x', 0)\, dx'\right| \\
&\leq \delta \|\partial_1 \varphi\|_{L^\infty(B_2)} + \fint_{-\delta}^{\delta} \left| \int_0^y \partial_2 \varphi(x', y')\, dy' \right| dx' \\
&\leq \delta \|\partial_1 \varphi\|_{L^\infty(B_2)} + \left(\frac{|y|}{2\varepsilon}\right)^{1/2} \|\partial_2 \varphi\|_{L^2(B_2)} \\
&\leq |y|^{1/3} \Big(\|\partial_1 \varphi\|_{L^\infty(B_2)} + \|\partial_2 \varphi\|_{L^2(B_2)} \Big).
\end{aligned}$$

Therefore,

$$|\varphi(x, y) - \varphi(0, 0)| \leq 2|(x, y)|^{1/3} \Big(\|\partial_1 \varphi\|_{L^\infty(B_2)} + \|\partial_2 \varphi\|_{L^2(B_2)} \Big)$$

and (101) holds. For a general function φ, one can use a density argument (by regularizing φ with mollifiers in direction x and y) and conclude by passing to the limit in (101). ∎

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