

Global Jacobian and Γ -convergence in a two-dimensional Ginzburg-Landau model for boundary vortices

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December 14, 2020

Abstract

In the theory of $2D$ Ginzburg-Landau vortices, the Jacobian plays a crucial role for the detection of topological singularities. We introduce a related distributional quantity, called the global Jacobian that can detect both interior and boundary vortices for a $2D$ map u . We point out several features of the global Jacobian, in particular, we prove an important stability property. This property allows us to study boundary vortices in a $2D$ Ginzburg-Landau model arising in thin ferromagnetic films, where a weak anchoring boundary energy penalising the normal component of u at the boundary competes with the usual bulk potential energy. We prove an asymptotic expansion by Γ -convergence at the second order for this mixed boundary/interior energy in a regime where boundary vortices are preferred. More precisely, at the first order of the limiting expansion, the energy is quantised and determined by the number of boundary vortices detected by the global Jacobian, while the second order term in the limiting energy expansion accounts for the interaction between the boundary vortices.

AMS classification: Primary: 35Q56, Secondary: 35B25, 49J45

Keywords: Jacobian, stability, compactness, Γ -convergence, Ginzburg-Landau vortices, boundary vortices

Contents

1	Introduction	2
1.1	Global Jacobian	4
1.2	Main results	4
2	Stability of the global Jacobian. Proof of Theorem 1.1	10
3	Approximation by S^1-valued maps	16
4	Second order Γ-convergence in the case of S^1-valued maps	22

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1 Introduction

For two small parameters $\varepsilon > 0$ and $\eta > 0$, we consider the energy functional

$$E_{\varepsilon,\eta}(u) = \int_{\Omega} |\nabla u|^2 dx + \frac{1}{\eta^2} \int_{\Omega} (1 - |u|^2)^2 dx + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (u \cdot \nu)^2 d\mathcal{H}^1, \quad (1)$$

for every 2D map $u \in H^1(\Omega; \mathbb{R}^2)$. The domain $\Omega \subset \mathbb{R}^2$ is a bounded, simply connected and $C^{1,1}$ regular domain (unless explicitly stated otherwise) with ν being the outer unit normal field on $\partial\Omega$. We always consider the following unit tangent field

$$\tau = \nu^\perp = (-\nu_2, \nu_1) \quad \text{on} \quad \partial\Omega,$$

so that (ν, τ) forms an oriented frame on $\partial\Omega$.

From a physical or modelling perspective, the functional (1) has been used as a somewhat ad hoc model for thin ferromagnetic films, for example by Moser [34] and Cantero-Álvarez [13], highlighting an interplay between interior and boundary vortices. In [25], we show explicit bounds that relate (1) to an effective micromagnetic energy in a thin film regime where boundary vortices are preferred. The results of the present article are essential in obtaining the Γ -convergence results for the full micromagnetic energy in that regime. A different regime corresponding to slightly larger films (where the nonlocality plays a more important role) was studied by Moser [35], who obtained convergence results only at the level of minimizers. We refer to [25] for a thorough discussion of the micromagnetic energy and the relevant thin-film regimes.

From a purely mathematical point of view, (1) combines two penalisation terms leading to two well-known singularly perturbed problems that we explain in the following.

Ginzburg-Landau functional for interior vortices. If we formally set $\varepsilon = 0$ in (1), then a finite energy configuration u must be tangential to the boundary $\partial\Omega$. Therefore, the following minimisation problem plays an essential role in our study for small $\eta > 0$:

$$E_\eta^{GL}(u) = \int_{\Omega} |\nabla u|^2 dx + \frac{1}{\eta^2} \int_{\Omega} (1 - |u|^2)^2 dx, \quad u \in H^1(\Omega; \mathbb{R}^2), \quad (2)$$

within the boundary constraint that $u = \tau$ on $\partial\Omega$. As Ω is a bounded simply connected $C^{1,1}$ regular domain, the tangent field τ has winding number 1 on $\partial\Omega$. This situation fits with the setting of the seminal book of Bethuel-Brezis-Hélein [8] who showed in particular that minimizers u_η of (2) with $u = \tau$ on $\partial\Omega$ have an energy of leading order $2\pi|\log \eta| + O(1)$ as $\eta \rightarrow 0$ and converge in various spaces to a singular \mathbb{S}^1 -valued harmonic map having one point-singularity of topological degree 1 (called *interior vortex point*). Moreover, for small $\eta > 0$, minimizers satisfy $|u| \approx 1$ outside a single “bad disc” around the interior vortex point of radius comparable to η , and the precise asymptotic behaviour of the minimal energy at the second order was determined in [8] by introducing a novel notion of renormalized energy governing the location of the interior vortex point.

A weak anchoring energy for \mathbb{S}^1 -valued maps. If we formally set $\eta = 0$ in (1), then a finite energy configuration must satisfy $|u| = 1$ in Ω . Therefore, we are interested in minimising the

following weak anchoring energy for \mathbb{S}^1 -valued maps with small $\varepsilon > 0$:

$$E_\varepsilon^{KS}(u) = \int_\Omega |\nabla u|^2 dx + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (u \cdot \nu)^2 d\mathcal{H}^1, \quad u \in H^1(\Omega; \mathbb{S}^1). \quad (3)$$

This model was first derived by Kohn-Slastikov [30] as a Γ -limit in a certain thin-film regime in micromagnetics. The asymptotic behaviour of the energy E_ε^{KS} at the minimal level when $\varepsilon \rightarrow 0$ was studied by the second author [31]: in particular, the minimizers have an energy of leading order $2\pi |\log \varepsilon| + O(1)$ and converge to a \mathbb{S}^1 -valued harmonic map with two boundary singularities. Each of these singularities can be interpreted as carrying a ‘‘half’’ topological degree. For small $\varepsilon > 0$, minimizers satisfy $u \approx \pm\tau$ outside of two ‘‘bad discs’’ of radius comparable to ε , and again, it is possible to precisely determine the asymptotics of the minimal energy at the second order.

Both (2) and (3) have also been studied from the point of view of Γ -convergence. A difficulty is that the diverging energies typically lead to a lack of compactness for the order parameter u . To overcome this problem, it was observed that instead of the map u , other quantities have much better compactness properties. In the case of the Ginzburg-Landau functional E_η^{GL} for interior vortices, the natural quantity is the Jacobian determinant $\text{jac}(u) = \det \nabla u$. It was shown that for families $(u_\eta)_\eta$ with $E_\eta^{GL}(u_\eta) = O(|\log \eta|)$ as $\eta \rightarrow 0$, the Jacobians $(\text{jac}(u_\eta))_\eta$ are relatively compact in $(W_0^{1,\infty}(\Omega))^*$ and other dual spaces of functions that are zero on the boundary, see Jerrard-Soner [29] or Sandier-Serfaty [39]. The limits of the Jacobians are of the form $\pi \sum_k d_k \delta_{a_k}$ for some distinct points $a_k \in \Omega$, corresponding to interior vortex points, carrying the topological degrees $d_k \in \mathbb{Z}$. As the Jacobian is controlled only in the dual space of functions that are zero on the boundary, there is no control over vortices escaping to the boundary.

For the weak anchoring energy E_ε^{KS} over \mathbb{S}^1 -valued maps, the problem is slightly easier: as a map u_ε with finite energy possesses a global lifting $u_\varepsilon = e^{i\phi_\varepsilon}$ with $\phi_\varepsilon \in H^1(\Omega)$, then every family $(u_\varepsilon)_\varepsilon$ of \mathbb{S}^1 -valued maps with $E_\varepsilon^{KS}(u_\varepsilon) = O(|\log \varepsilon|)$ has the liftings $(\phi_\varepsilon)_\varepsilon$ (up to an additive constant) relatively compact in $L^p(\partial\Omega)$ for every $p \in [1, \infty)$ with limits ϕ_* on $\partial\Omega$ such that $\partial_\tau \phi_* - \varkappa$ is a multiple of a sum of Dirac masses on $\partial\Omega$ (see [32]). Here, ∂_τ denotes tangential differentiation and \varkappa the curvature on $\partial\Omega$. This approach relies strongly on the constraint $|u_\varepsilon| = 1$ in Ω .

The energy (1) allows for both types of topological phenomena (boundary and interior vortex), so we need a tool that captures these singularities and does not require the existence of a global lifting. The natural tool is the notion of global Jacobian that we discuss in the next section. The Γ -convergence results for the energy (1) are proved in this paper in the regime

$$|\log \varepsilon| \ll |\log \eta|, \quad (4)$$

i.e., interior vortices cost more energy than boundary vortices.

Notation. We always denote by $a_\varepsilon \ll b_\varepsilon$ or $a_\varepsilon = o(b_\varepsilon)$ if $\frac{a_\varepsilon}{b_\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$, and similar $a_\varepsilon \lesssim b_\varepsilon$ or $a_\varepsilon = O(b_\varepsilon)$ if there exists $C > 0$ such that $a_\varepsilon \leq Cb_\varepsilon$ for all small $\varepsilon > 0$. Also, $a = o_b(1)$ if $a \rightarrow 0$ as $b \rightarrow 0$. In the following, $\varepsilon \rightarrow 0$ can mean both a sequence $\varepsilon_k \rightarrow 0$ as well as the continuous parameter $\varepsilon \rightarrow 0$. More precisely, in our Γ -convergence results, the limits $\varepsilon \rightarrow 0$ in \liminf (and \limsup) are understood in both cases sequence / family of the parameter ε ; only for the compactness result, we start with a fixed sequence $\varepsilon_k \rightarrow 0$ and then we take further subsequences of this sequence.

1.1 Global Jacobian

For a two-dimensional map $u \in H^1(\Omega; \mathbb{R}^2)$ defined in a Lipschitz bounded domain $\Omega \subset \mathbb{R}^2$, we call *global Jacobian* of u the following linear functional $\mathcal{J}(u) : W^{1,\infty}(\Omega) \rightarrow \mathbb{R}$ acting on Lipschitz test functions:

$$\langle \mathcal{J}(u), \zeta \rangle := - \int_{\Omega} u \times \nabla u \cdot \nabla^{\perp} \zeta \, dx, \quad \text{for every Lipschitz function } \zeta : \Omega \rightarrow \mathbb{R}. \quad (5)$$

Here $a \times b = a_1 b_2 - a_2 b_1$ for $a, b \in \mathbb{R}^2$, $u \times \nabla u = (u \times \partial_{x_1} u, u \times \partial_{x_2} u)$ that belongs to $L^1(\Omega; \mathbb{R}^2)$ for $u \in H^1(\Omega; \mathbb{R}^2)$, $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ and $\langle \cdot, \cdot \rangle$ stands for the (algebraic) dual pairing between $(W^{1,\infty}(\Omega))^*$ and $W^{1,\infty}(\Omega)$. In particular, the global Jacobian has zero average, i.e.,

$$\langle \mathcal{J}(u), 1 \rangle = 0. \quad (6)$$

Relation with the interior and boundary Jacobian. On the one hand, when applied to test functions $\zeta \in W^{1,\infty}(\Omega)$ vanishing at the boundary $\partial\Omega$, the global Jacobian $\mathcal{J}(u)$ reduces to twice the *interior Jacobian* $\text{jac}(u) = \partial_{x_1} u \times \partial_{x_2} u \in L^1(\Omega)$ for $u \in H^1(\Omega; \mathbb{R}^2)$; indeed, integrating by parts, it follows

$$\langle \mathcal{J}(u), \zeta \rangle = \int_{\Omega} 2 \text{jac}(u) \zeta \, dx \quad \text{if } \zeta = 0 \text{ on } \partial\Omega.$$

Therefore, the global Jacobian carries the topological information in the interior Ω and detects the interior vortices.

On the other hand, the global Jacobian also carries the topological information at the boundary $\partial\Omega$ and enables us to detect boundary vortices; more precisely, we define the *boundary Jacobian* of $u \in H^1(\Omega; \mathbb{R}^2)$ to be the linear functional $\mathcal{J}_{bd}(u) : W^{1,\infty}(\Omega) \rightarrow \mathbb{R}$ given by

$$\langle \mathcal{J}_{bd}(u), \zeta \rangle := \langle \mathcal{J}(u), \zeta \rangle - \int_{\Omega} 2 \text{jac}(u) \zeta \, dx, \quad \text{for every Lipschitz function } \zeta : \Omega \rightarrow \mathbb{R}. \quad (7)$$

In fact, the functional $\mathcal{J}_{bd}(u)$ acts only on the boundary $\partial\Omega$ (see Proposition 2.2 below): in particular, if $u \in C^2(\bar{\Omega}; \mathbb{R}^2)$, then integration by parts yields

$$\langle \mathcal{J}_{bd}(u), \zeta \rangle = - \int_{\partial\Omega} u \times \partial_{\tau} u \zeta \, d\mathcal{H}^1 \quad \text{for every Lipschitz function } \zeta : \Omega \rightarrow \mathbb{R},$$

i.e., $\mathcal{J}_{bd}(u) = -u \times \partial_{\tau} u \mathcal{H}^1 \llcorner \partial\Omega$. While ζ is a priori only defined in Ω , it has a unique Lipschitz extension to $\bar{\Omega}$, and we will tacitly use this extension in the following. In addition, for a \mathbb{S}^1 -valued map u given through a smooth lifting $\varphi \in C^2(\bar{\Omega}; \mathbb{R})$, i.e., $u = (\cos \varphi, \sin \varphi)$ in Ω , then the interior Jacobian $\text{jac}(u)$ vanishes in Ω so that the whole topological information is carried by the tangential derivative of φ at the boundary, i.e.,

$$\text{jac}(u) = 0, \quad \mathcal{J}(u) = \mathcal{J}_{bd}(u) = -\partial_{\tau} \varphi \mathcal{H}^1 \llcorner \partial\Omega \quad \text{and} \quad \langle \mathcal{J}_{bd}(u), 1 \rangle = 0 \quad \text{if } u = e^{i\varphi} \text{ in } \Omega. \quad (8)$$

1.2 Main results

We show the following stability result for the global Jacobian. This is the generalisation of the well known estimate for the interior Jacobian (see the very nice paper of Brezis-Nguyen [10], or Proposition 2.1 below).

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a $C^{1,1}$ bounded domain and $u, v \in H^1(\Omega; \mathbb{R}^2)$ such that*

$$|v| \leq 1 \quad \text{a.e. in } \Omega.$$

Then for every $\zeta \in W^{1,\infty}(\Omega)$, we have

$$|\langle \mathcal{J}(u) - \mathcal{J}(v), \zeta \rangle| \leq f \left(\|u - v\|_{L^2(\Omega)} (\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}) \right) \|\nabla \zeta\|_{L^\infty(\Omega)}$$

where the function f is given by $f(t) = t + C\sqrt{t}$ with $C > 0$ depending only on the geometry of Ω .

Note that the above inequality can be interpreted as a stability property of the global Jacobian $\mathcal{J}(\cdot)$ in the strong L^2 -topology of maps under a certain control on their H^1 -seminorm (that eventually could blow up). This allows us to make perturbations of u that are small in L^2 , but possibly large in H^1 without changing the global Jacobian much.

Theorem 1.1 is an important tool in proving the compactness of the global Jacobian for configurations u_ε satisfying the energetic bound $E_{\varepsilon,\eta}(u_\varepsilon) \leq C|\log \varepsilon|$ in the regime (4), see Theorem 1.2 below. In addition, we prove the compactness of the trace $u_\varepsilon|_{\partial\Omega}$ in the strong $L^p(\partial\Omega)$ -topology for every $p \geq 1$. This compactness result of $u_\varepsilon|_{\partial\Omega}$ is **very surprising** in the context of Ginzburg-Landau type functionals where in general, no compactness of configurations u_ε is expected to occur. Moreover, under a more restrictive energetic regime, we prove that strong $L^p(\Omega)$ -convergence of u_ε inside Ω does also occur, see Theorem 1.4. The role of Theorem 1.2 consists also in proving a lower bound of the energy at the first order that is quantised by the number of boundary vortices detected by the global Jacobian.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected, $C^{1,1}$ regular domain and \varkappa be the curvature on $\partial\Omega$. If $\varepsilon \rightarrow 0$ and $\eta = \eta(\varepsilon) \rightarrow 0$ satisfy $|\log \varepsilon| \ll |\log \eta|$, then the following holds: Assume $u_\varepsilon \in H^1(\Omega; \mathbb{R}^2)$ satisfy*

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} E_{\varepsilon,\eta}(u_\varepsilon) < \infty. \quad (9)$$

*i) **Compactness of global Jacobians and $L^p(\partial\Omega)$ -compactness of $u_\varepsilon|_{\partial\Omega}$.** For a subsequence, the global Jacobians $\mathcal{J}(u_\varepsilon)$ converge to a measure $J \in \mathcal{M}(\overline{\Omega})$ on the closure $\overline{\Omega}$, in the sense that¹*

$$\sup_{|\nabla \zeta| \leq 1 \text{ in } \Omega} |\langle \mathcal{J}(u_\varepsilon) - J, \zeta \rangle| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (10)$$

J is supported on $\partial\Omega$ and has the form

$$J = -\varkappa \mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{j=1}^N d_j \delta_{a_j} \quad \text{with} \quad \sum_{j=1}^N d_j = 2 \quad (11)$$

for N distinct boundary vortices $a_j \in \partial\Omega$ carrying the non-zero multiplicities² $d_j \in \mathbb{Z} \setminus \{0\}$. Moreover, for a subsequence, the trace $u_\varepsilon|_{\partial\Omega}$ converges as $\varepsilon \rightarrow 0$ in $L^p(\partial\Omega)$ (for every

¹Using the notation of Section 2 below, this means $\|\mathcal{J}(u_\varepsilon) - J\|_{(\text{Lip}(\Omega))^*} \rightarrow 0$ as $\varepsilon \rightarrow 0$. This quantity is stronger than the usual $W^{-1,1}(\Omega)$ -norm because our test functions are not necessarily zero on the boundary $\partial\Omega$.

²We use integer ‘‘multiplicities’’ instead of half-integer ‘‘degrees’’ for the boundary vortices in this article.

$p \in [1, \infty)$) to $e^{i\phi_0} \in BV(\partial\Omega; \{\pm\tau\})$ for a BV lifting ϕ_0 of the tangent field $\pm\tau$ on $\partial\Omega$ determined (up to a constant in $\pi\mathbb{Z}$) by

$$\partial_\tau \phi_0 = \varkappa - \pi \sum_{j=1}^N d_j \delta_{a_j} \quad \text{on } \partial\Omega.$$

ii) **Energy lower bound at the first order.** If (u_ε) satisfies the convergence assumption in point i) as the sequence / family $\varepsilon \rightarrow 0$, then the energy lower bound at the first order is the total mass of the measure $J + \varkappa \mathcal{H}^1 \llcorner \partial\Omega$ on $\partial\Omega$:³

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} E_{\varepsilon, \eta}(u_\varepsilon) \geq \pi \sum_{j=1}^N |d_j| = |J + \varkappa \mathcal{H}^1 \llcorner \partial\Omega|(\partial\Omega).$$

Note that the limit Jacobian measure (11) lives on the boundary $\partial\Omega$, having a diffuse part carried by the curvature \varkappa and a singular part carried by (multiples of) Dirac masses at the boundary vortices a_j . The convergence (10) is discussed in Section 2. In particular, by (6) and (10), we have $\langle J, 1 \rangle = 0$; thus, combined with the Gauß-Bonnet formula, we have that

$$\pi \sum_{j=1}^N d_j = \int_{\partial\Omega} \varkappa d\mathcal{H}^1 = 2\pi.$$

This explains the constraint (11) on the sum of the multiplicities $(d_j)_j$. The BV lifting ϕ_0 on $\partial\Omega$ is determined by $\partial_\tau \phi_0 = -J$ up to an additive constant that a-priori is arbitrary in \mathbb{R} ; however, the restriction that the limit $e^{i\phi_0}$ is parallel with τ fixes this constant to be a multiple of π .

Theorem 1.2 is carried out in the regime (4), so that the formation of boundary singularities is preferred over interior singularities. In particular, we have that $\eta \ll \varepsilon$, so the typical core size of an interior vortex is much smaller than the length scale of a boundary transition from a parallel to an antiparallel tangent direction $\pm\tau$ at $\partial\Omega$. In this context, as interior vortices of non-zero winding number are expected to be absent, we prove in Theorem 3.1 below that u_ε can be replaced by an \mathbb{S}^1 -valued map without raising the energy by much and without affecting the convergence and limit of the global Jacobians (thanks to Theorem 1.1). The \mathbb{S}^1 -valued problem is studied in Section 4 (in particular Theorem 4.2), and we improve results in the literature [31, 32, 33] by giving simpler, more direct proofs and obtaining new and significantly stronger results for the second order energy expansion. In particular, we adapt a co-area argument of Sandier [38] in the nonlocal context of (3) (see the rewriting (55) below) to show a new single multiplicity result and use arguments inspired by Colliander-Jerrard [14] to obtain lower bounds using purely energy methods. Owing to our approximation Theorem 3.1, these results can then be transferred to the study of (1).

For the analysis of the asymptotic expansion at the second order, we need to introduce a renormalized energy similar to that of Bethuel-Brezis-Hélein [8] that consists in eliminating the “infinite” energy carried asymptotically in small “half” disks around the boundary vortices.

³Recall that \varkappa is the curvature of the boundary and $J + \varkappa \mathcal{H}^1 \llcorner \partial\Omega = \pi \sum_{j=1}^N d_j \delta_{a_j}$.

Definition 1.3 Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected, $C^{1,1}$ regular domain and \varkappa be the curvature on $\partial\Omega$. Consider $\phi_0 : \partial\Omega \rightarrow \mathbb{R}$ to be a BV function such that $e^{i\phi_0} \cdot \nu = 0$ on $\partial\Omega \setminus \{a_1, \dots, a_N\}$ and

$$\partial_\tau \phi_0 = \varkappa - \pi \sum_{j=1}^N d_j \delta_{a_j} \quad \text{on } \partial\Omega \quad \text{with } d_j \in \{\pm 1\} \text{ and } \sum_{j=1}^N d_j = 2$$

for N distinct points $a_j \in \partial\Omega$ carrying the multiplicities $d_j \in \{\pm 1\}$. If ϕ_* is the harmonic extension to Ω of ϕ_0 , then the **renormalized energy** of $\{(a_j, d_j)\}$ is defined as

$$W_\Omega(\{(a_j, d_j)\}) = \lim_{\rho \rightarrow 0} \left(\int_{\Omega \setminus \bigcup_{j=1}^N B_\rho(a_j)} |\nabla \phi_*|^2 dx - N\pi \log \frac{1}{\rho} \right), \quad (12)$$

where $B_\rho(a_j)$ is the disk of radius ρ centered at a_j .

In Definition 1.3, ϕ_0 is uniquely determined (up to an additive constant in $\pi\mathbb{Z}$) and stands for a BV lifting of some tangent unit vector field $\pm\tau$ on $\partial\Omega$ with prescribed jumps at a_j (see e.g., [20] for more details on BV liftings). The difference with respect to the lifting in Theorem 1.2, point *i*) consists in allowing here only jumps of height $\pm\pi$ at the boundary vortices a_j . Note also that the limit in (12) exists, see [31, Prop. 7.1]. The renormalized energy $W_\Omega(\{(a_j, d_j)\})$ can be computed in a $C^{1,1}$ -domain Ω , in particular, it depends on $\log|a_j - a_k|$ for every $j \neq k$ and on the curvature \varkappa of $\partial\Omega$, see [25] for details. In a disk $\Omega = B_R$, the renormalized energy has a particularly simple form, see [25]:

$$W_{B_R}(\{(a_j, d_j)\}) = -2\pi \sum_{1 \leq k < j \leq N} d_k d_j \log|a_k - a_j|, \quad a_j \in \partial B_R, d_j \in \{\pm 1\} \text{ and } \sum_{j=1}^N d_j = 2. \quad (13)$$

In particular, if $N = 2$, then $d_1 = d_2 = 1$ and the renormalized energy achieves the minimum value $-2\pi \log 2R$ for diametrically opposed singularities and has no other critical points than this rotationally symmetric family of minimizers.

We have the following refinement of Theorem 1.2 at the second order using the renormalized energy (12):

Theorem 1.4 Under the hypothesis of Theorem 1.2, we assume that the sequence / family (u_ε) satisfies the convergence at point *i*) in Theorem 1.2 as $\varepsilon \rightarrow 0$. In addition, we assume the following more precise bound than (9):

$$\limsup_{\varepsilon \rightarrow 0} (E_{\varepsilon, \eta}(u_\varepsilon) - \pi \sum_{j=1}^N |d_j| |\log \varepsilon|) < \infty. \quad (14)$$

Then the following results hold:

i) Single multiplicity and second order lower bound. The multiplicities satisfy $d_j = \pm 1$ for $1 \leq j \leq N$, so $\sum_{j=1}^N |d_j| = N$ and there holds the finer energy lower bound:

$$\liminf_{\varepsilon \rightarrow 0} (E_{\varepsilon, \eta}(u_\varepsilon) - \pi N |\log \varepsilon|) \geq W_\Omega(\{(a_j, d_j)\}) + \gamma_0 N, \quad (15)$$

with $\gamma_0 = \pi \log \frac{e}{4\pi}$ a universal constant and W_Ω the renormalized energy defined in (12).

ii) **Penalty bound.** The penalty (potential) terms in the energy are of order $O(1)$, i.e.,

$$\limsup_{\varepsilon \rightarrow 0} \left(\frac{1}{\eta^2} \int_{\Omega} (1 - |u_{\varepsilon}|^2)^2 dx + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (u_{\varepsilon} \cdot \nu)^2 d\mathcal{H}^1 \right) < \infty. \quad (16)$$

iii) **Lower bound for the energy near the boundary vortex core.** There exist $\rho_0 > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that the energy of u_{ε} near the singularities $\{a_j\}$ satisfies for all the $0 < \varepsilon < \varepsilon_0$ in the sequence / family and for all $\rho \in (0, \rho_0)$:

$$\left(\int_{\Omega \cap \bigcup_{j=1}^N B_{\rho}(a_j)} |\nabla u_{\varepsilon}|^2 dx - \pi N \log \frac{\rho}{\varepsilon} \right) > -C, \quad (17)$$

iv) **$L^p(\Omega)$ -compactness of maps u_{ε} .** For any $q \in [1, 2)$, the sequence /family $(u_{\varepsilon})_{\varepsilon}$ is uniformly bounded in $W^{1,q}(\Omega; \mathbb{R}^2)$. Moreover, for a subsequence, u_{ε} converges as $\varepsilon \rightarrow 0$ strongly in $L^p(\Omega; \mathbb{R}^2)$ for any $p \in [1, \infty)$ to $e^{i\hat{\phi}_0}$, where $\hat{\phi}_0 \in W^{1,q}(\Omega)$ is an extension (not necessarily harmonic) to Ω of the lifting $\phi_0 \in BV(\partial\Omega)$ determined in Theorem 1.2, point i).

Finally, we have a matching upper bound that complements Theorems 1.2 and 1.4 to yield a full asymptotic expansion by Γ -convergence at the second order for the energy (1).

Theorem 1.5 Let $\{a_j\}_{1 \leq j \leq N} \subset \partial\Omega$ be N distinct points and $d_j \in \mathbb{Z} \setminus \{0\}$ corresponding multiplicities that satisfy the constraint $\sum_{j=1}^N d_j = 2$. For every $\varepsilon, \eta = \eta(\varepsilon) \in (0, \frac{1}{2})$ such that $|\log \varepsilon| \ll |\log \eta|$ as $\varepsilon \rightarrow 0$, we can construct $u_{\varepsilon} \in H^1(\Omega; \mathbb{S}^1)$ such that the global Jacobians $\mathcal{J}(u_{\varepsilon})$ converge to $J = -\varkappa \mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{j=1}^N d_j \delta_{a_j}$ as in (10). Furthermore, u_{ε} converges strongly to $e^{i\hat{\phi}}$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$ for all $p \in [1, \infty)$, where $\hat{\phi}$ is the harmonic extension in Ω of a boundary lifting ϕ_0 satisfying $e^{i\phi_0} \cdot \nu = 0$ and $\partial_{\tau} \phi_0 = \varkappa - \pi \sum_{j=1}^N d_j \delta_{a_j}$ on $\partial\Omega$; the energies of u_{ε} satisfy

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} E_{\varepsilon, \eta}(u_{\varepsilon}) = \pi \sum_{j=1}^N |d_j|.$$

If furthermore $|d_j| = 1$ for all $j = 1, \dots, N$, then u_{ε} can be chosen such that

$$\lim_{\varepsilon \rightarrow 0} (E_{\varepsilon, \eta}(u_{\varepsilon}) - \pi N |\log \varepsilon|) = W_{\Omega}(\{(a_j, d_j)\}) + N\gamma_0.$$

Remark 1.1 In fact, a slightly stronger statement than the lower bound of Theorem 1.4 part i) can be proved: Considering the limit functions $\hat{\phi}_0$ found in Theorem 1.4 part iv),⁴ we define

$$\mathcal{W}(\hat{\phi}_0; \{(a_j, d_j)\}) = \limsup_{\rho \rightarrow 0} \left(\int_{\Omega \setminus \bigcup_{j=1}^N B_{\rho}(a_j)} |\nabla \hat{\phi}_0|^2 dx - N\pi \log \frac{1}{\rho} \right). \quad (18)$$

Then our proof of Theorem 1.4 shows that $\mathcal{W}(\hat{\phi}_0; \{(a_j, d_j)\}) < \infty$ and $\mathcal{W}(\hat{\phi}_0; \{(a_j, d_j)\}) \geq W_{\Omega}(\{(a_j, d_j)\})$, with equality if and only if $\hat{\phi}_0$ is harmonic in Ω ; moreover, the estimate (15) of Theorem 1.4 part i) is then strengthened to

$$\liminf_{\varepsilon \rightarrow 0} (E_{\varepsilon, \eta}(u_{\varepsilon}) - \pi N |\log \varepsilon|) \geq \mathcal{W}(\hat{\phi}_0; \{(a_j, d_j)\}) + \gamma_0 N. \quad (19)$$

⁴The difference to the extension ϕ_* used in (12) is that $\hat{\phi}_0$ is not necessarily harmonic.

We expect that the corresponding upper bound holds for these general limit configurations $\hat{\phi}_0$, slightly generalizing Theorem 1.5.

As a consequence of Theorems 1.4 and 1.5, we deduce the following asymptotic behaviour for minimizers u_ε of $E_{\varepsilon,\eta}$:

Corollary 1.6 *For $\varepsilon, \eta = \eta(\varepsilon) \in (0, \frac{1}{2})$ such that $|\log \varepsilon| \ll |\log \eta|$ as $\varepsilon \rightarrow 0$, let u_ε be a minimizer of $E_{\varepsilon,\eta}$ over the set $H^1(\Omega; \mathbb{R}^2)$. Then there exists a sequence $\varepsilon \rightarrow 0$ such that*

$$\begin{cases} \mathcal{J}(u_\varepsilon) \rightarrow J = -\varkappa \mathcal{H}^1 \llcorner \partial\Omega + \pi(\delta_{a_1} + \delta_{a_2}) \text{ as in (10),} \\ u_\varepsilon \rightharpoonup e^{i\phi_*} \text{ weakly in } W^{1,q}(\Omega; \mathbb{R}^2) \text{ for every } q \in [1, 2), \end{cases} \quad \text{as } \varepsilon \rightarrow 0,$$

where a_1 and a_2 are two distinct points in $\partial\Omega$ that minimize the renormalized energy (for the multiplicities $d_1 = d_2 = 1$):

$$W_\Omega(\{(a_1, 1), (a_2, 1)\}) = \min \left\{ W_\Omega(\{(\tilde{a}_1, 1), (\tilde{a}_2, 1)\}) : \tilde{a}_1, \tilde{a}_2 \in \partial\Omega \text{ distinct points} \right\}$$

and ϕ_* is the harmonic extension of $\phi_0 \in BV(\partial\Omega)$ satisfying $\partial_\tau \phi_0 = \varkappa - \pi(\delta_{a_1} + \delta_{a_2})$ on $\partial\Omega$ and $e^{i\phi_0(P)} \in \{\pm\tau(P)\}$ for some point $P \in \Omega \setminus \{a_1, a_2\}$.⁵ Moreover, we have the following second order energy expansion:

$$E_{\varepsilon,\eta}(u_\varepsilon) = 2\pi \log \frac{1}{\varepsilon} + W_\Omega(\{(a_1, 1), (a_2, 1)\}) + 2\gamma_0 + o(1), \text{ as } \varepsilon \rightarrow 0.$$

Combining Corollary 1.6 with (13), in the case of a disk domain Ω , we deduce that the two (limit) boundary vortices a_1 and a_2 in Corollary 1.6 are diametrically opposite to each other and are unique up to a rotation. This existence of infinitely many minimizers shows that the convergence of the Jacobians (and of the maps u_ε) in Corollary 1.6 only needs to hold true up to subsequences.

Remark 1.2 It is possible to replace the normal vector ν on $\partial\Omega$ in the boundary potential term of the energy (1) by an arbitrary Lipschitz vector field $V : \partial\Omega \rightarrow \mathbb{S}^1$ of winding number $\deg(V) \in \mathbb{Z}$. Partial results in this direction have been obtained in [31, 32] in the setting of the reduced functional E_ε^{KS} . The main changes in our results for the case of a general vector field V are that \varkappa is replaced by the scalar function $V \times \partial_\tau V$ on $\partial\Omega$ and that the constraint $\sum_j d_j = 2$ is replaced by

$$\sum_j d_j = \frac{1}{\pi} \int_{\partial\Omega} V \times \partial_\tau V d\mathcal{H}^1 = 2 \deg(V).$$

For the proofs, note that our approximation result in Theorem 3.1 only requires $|V| \leq 1$ on $\partial\Omega$. The subsequent analysis of the \mathbb{S}^1 -valued model when ν is replaced by V can be performed with only minor changes.

⁵Note that the vector field $e^{i\phi_0}$ is uniquely determined (up to a sign) by the points a_1 and a_2 on $\partial\Omega$, yielding the uniqueness (up to a sign) of the limit vector field $e^{i\phi_*}$ inside Ω .

Some of our results were announced in [22, Section 11]. We expect that our results can be extended to situations where both interior and boundary vortices are present in the limit $\varepsilon, \eta \rightarrow 0$ as long as sufficiently tight energy bounds hold, and they generalise the results for minimizers of Moser [34] (see also the case of boundary “boojums” in a liquid crystal model studied by Alama-Bronsard-Golovaty [1]). Even if boundary singularities are favourable compared to interior ones in the regime (4), certain configurations with interior vortices are still conjectured to be local minimizers (see [13] for partial results). However, an extension of our method will require an approximation result that can be used in the presence of interior vortices, see Ignat-Otto [27]. We also expect that our results extend to 2-dimensional Riemannian manifolds with boundary, by following the approach of Ignat-Jerrard [23, 24].

2 Stability of the global Jacobian. Proof of Theorem 1.1

In this section we discuss some properties of the global Jacobian $\mathcal{J}(u)$ introduced in (5) for a two-dimensional map $u \in H^1(\Omega; \mathbb{R}^2)$ defined on a Lipschitz bounded domain $\Omega \subset \mathbb{R}^2$; in particular, we prove Theorem 1.1. We recall that the global Jacobian $\mathcal{J}(u)$ is an element of the (algebraic) dual $(W^{1,\infty}(\Omega))^*$ of $W^{1,\infty}(\Omega)$. In order to speak about the continuity of this linear functional, some natural seminorms are considered on the space of Lipschitz functions $W^{1,\infty}(\Omega)$ and the subspace

$$W_0^{1,\infty}(\Omega) = \{\zeta \in W^{1,\infty}(\Omega) : \zeta = 0 \text{ on } \partial\Omega\}.$$

These seminorms lead to the following dual quantities that measure the global and the interior Jacobian: if $A \in (W^{1,\infty}(\Omega))^*$, we define

$$\begin{aligned} \|A\|_{(\text{Lip}(\Omega))^*} &= \sup \{ \langle A, \zeta \rangle : \zeta \in W^{1,\infty}(\Omega), |\nabla\zeta| \leq 1 \}, \\ \|A\|_{(W^{1,\infty}(\Omega))^*} &= \sup \{ \langle A, \zeta \rangle : \zeta \in W^{1,\infty}(\Omega), |\zeta| + |\nabla\zeta| \leq 1 \}, \\ \|A\|_{(W_0^{1,\infty}(\Omega))^*} &= \sup \{ \langle A, \zeta \rangle : \zeta \in W_0^{1,\infty}(\Omega), |\zeta| + |\nabla\zeta| \leq 1 \}. \end{aligned}$$

We write $\|A\|$ as a shorthand for $\|A\|_{(\text{Lip}(\Omega))^*}$ and is the quantity we use in the next sections. Note that by homogeneity,

$$\|A\| < \infty \text{ implies } \langle A, 1 \rangle = 0. \tag{20}$$

Clearly, we have for all $A \in (W^{1,\infty}(\Omega))^*$:

$$\|A\| \geq \|A\|_{(W^{1,\infty}(\Omega))^*} \geq \|A\|_{(W_0^{1,\infty}(\Omega))^*}.$$

In particular,

$$\|\mathcal{J}(u)\| \geq \|\mathcal{J}(u)\|_{(W_0^{1,\infty}(\Omega))^*} = 2\|\text{jac}(u)\|_{(W_0^{1,\infty}(\Omega))^*}.$$

Identifying \mathbb{R}^2 with the complex plane, both operators $\text{jac}(\cdot)$ and $\mathcal{J}(\cdot)$ are invariant under (complex) multiplication with a fixed unit length vector $a \in \mathbb{S}^1$ on $H^1(\Omega; \mathbb{R}^2)$. While $\text{jac}(\cdot)$ is invariant under addition of a fixed vector $a \in \mathbb{R}^2$, $\mathcal{J}(\cdot)$ is not.⁶ Therefore, when estimating $\text{jac}(u)$ for

⁶For example, consider u of the form $u = e^{i\varphi}$ with a smooth lifting φ in $\bar{\Omega}$. Then (8) implies $\mathcal{J}(u) = \mathcal{J}_{bd}(u) = -\partial_\tau\varphi \mathcal{H}^1 \llcorner \partial\Omega$ and $\mathcal{J}(u+1) = \mathcal{J}_{bd}(u+1) = -\partial_\tau(\varphi + \sin\varphi) \mathcal{H}^1 \llcorner \partial\Omega$ (because $\text{jac}(u+1) = 0$ where 1 is identified with $(1, 0) \in \mathbb{R}^2 \simeq \mathbb{C}$); therefore, $\mathcal{J}(u) \neq \mathcal{J}(u+1)$ provided that $\partial_\tau(\sin\varphi) \neq 0$ at some point on $\partial\Omega$.

$u \in H^1(\Omega; \mathbb{R}^2)$ on a bounded domain Ω , we may replace u by

$$\tilde{u} := u - \frac{1}{|\Omega|} \int_{\Omega} u. \quad (21)$$

We start by recalling the following stability inequality of the interior Jacobian⁷ (see the nice paper of Brezis-Nguyen [10]) that represents a weaker form of Theorem 1.1:

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain and let $u, v \in H^1(\Omega; \mathbb{R}^2)$. Then*

$$\|\text{jac}(u) - \text{jac}(v)\|_{(W_0^{1,\infty}(\Omega))^*} \leq \frac{1}{2} \|u - v\|_{L^2(\Omega)} (\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}).$$

The above estimate can be improved by using $\|\tilde{u} - \tilde{v}\|_{L^2(\Omega)} = \min_{a \in \mathbb{R}^2} \|u - v - a\|_{L^2(\Omega)}$ defined in (21) instead of $\|u - v\|_{L^2(\Omega)}$.

Proof. First assume that u and v are smooth maps in Ω . Note that

$$2\text{jac}(u) = \nabla \times (u \times \nabla u) \quad \text{in } \Omega. \quad (22)$$

If $\zeta \in C_c^\infty(\Omega)$, then integration by parts yields:

$$\begin{aligned} 2 \left| \int_{\Omega} (\text{jac}(u) - \text{jac}(v)) \zeta \, dx \right| &= \left| \int_{\Omega} (u \times \nabla u - v \times \nabla v) \cdot \nabla^\perp \zeta \, dx \right| \\ &= \left| \int_{\Omega} \left\{ (u - v) \times (\nabla u + \nabla v) \cdot \nabla^\perp \zeta + \nabla(v \times u) \cdot \nabla^\perp \zeta \right\} dx \right| \\ &\leq \|u - v\|_{L^2(\Omega)} (\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}) \|\nabla \zeta\|_{L^\infty(\Omega)}. \end{aligned} \quad (23)$$

The general case follows by a density argument: every test function $\zeta \in W^{1,\infty}(\Omega)$ with $\zeta = 0$ on $\partial\Omega$ is approximated in $W^{1,1}(\Omega)$ by $\zeta_n \in C_c^\infty(\Omega)$ such that $\|\nabla \zeta_n\|_{L^\infty(\Omega)} \leq \|\nabla \zeta\|_{L^\infty(\Omega)}$ (in particular $(\zeta_n)_n$ is uniformly bounded in $L^\infty(\Omega)$), while the maps $u, v \in H^1(\Omega; \mathbb{R}^2)$ are approximated in $H^1(\Omega)$ by smooth maps u_n and v_n in Ω implying in particular that $\text{jac}(u_n) \rightarrow \text{jac}(u)$ and $\text{jac}(v_n) \rightarrow \text{jac}(v)$ in $L^1(\Omega)$. Finally, passing at the limit $n \rightarrow \infty$ in the above inequality for (u_n, v_n, ζ_n) , the conclusion is proved. Note that for $u \in H^1$, (22) holds true in the distribution sense by the same density argument since $u \times \nabla u \in L^1$ (so, $\nabla \times (u \times \nabla u) \in (W_0^{1,\infty}(\Omega))^*$) and $\text{jac}(u) \in L^1$. The last statement of Proposition 2.1 follows from the invariance of $\text{jac}(\cdot)$ under addition of a fixed vector $a \in \mathbb{R}^2$. \square

In order to prove Theorem 1.1, we need to investigate the stability properties of the boundary Jacobian defined in (7). The following lemma proves that the boundary Jacobian is indeed a quantity living on the boundary $\partial\Omega$ and we obtain a stability inequality for the boundary Jacobian in the strong $H^{1/2}(\partial\Omega)$ -topology, i.e., endowed by the norm $\|u\|_{H^{1/2}(\partial\Omega)} := \|u\|_{L^2(\partial\Omega)} + \|u\|_{\dot{H}^{1/2}(\partial\Omega)}$ with

$$\|u\|_{\dot{H}^{1/2}(\partial\Omega)}^2 := \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^2} \, dx dy.$$

⁷For the case of BV maps, we refer the reader to the paper [21].

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain. Then for every $u \in H^1(\Omega; \mathbb{R}^2)$, the boundary Jacobian $\mathcal{J}_{bd}(u)$ of u defined in (7) can be identified with the following linear functional acting on Lipschitz functions $W^{1,\infty}(\partial\Omega)$ at the boundary $\partial\Omega$:*

$$\zeta \in W^{1,\infty}(\partial\Omega) \mapsto -(\zeta u \times \partial_\tau u)_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} \quad (24)$$

where the right hand side is interpreted as a dual (cross) product⁸ between $\partial_\tau u \in H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$ and $\zeta u \in H^{1/2}(\partial\Omega)$. Moreover, for every $u, v \in H^1(\Omega, \mathbb{R}^2)$ and $\zeta \in W^{1,\infty}(\Omega)$, we have

$$|\langle (\mathcal{J}_{bd}(u) - \mathcal{J}_{bd}(v)), \zeta \rangle| \leq C \|u - v\|_{H^{1/2}(\partial\Omega)} (\|u\|_{H^{1/2}(\partial\Omega)} + \|v\|_{H^{1/2}(\partial\Omega)}) \|\zeta\|_{W^{1,\infty}(\partial\Omega)}, \quad (25)$$

where $C > 0$ is a constant depending only on Ω and $\|\zeta\|_{W^{1,\infty}(\partial\Omega)} := \|\zeta\|_{L^\infty(\partial\Omega)} + \|\partial_\tau \zeta\|_{L^\infty(\partial\Omega)}$.

Proof. First, we prove that for any $u \in H^1(\Omega; \mathbb{R}^2)$ the linear functional (24) is continuous on $W^{1,\infty}(\partial\Omega)$ endowed with the norm $\|\cdot\|_{W^{1,\infty}(\partial\Omega)}$. Indeed, we have for $\zeta \in W^{1,\infty}(\partial\Omega)$:

$$\left| (\zeta u \times \partial_\tau u)_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} \right| \leq \|\zeta u\|_{H^{1/2}(\partial\Omega)} \|\partial_\tau u\|_{H^{-1/2}(\partial\Omega)} \leq C \|u\|_{H^{1/2}(\partial\Omega)}^2 \|\zeta\|_{W^{1,\infty}(\partial\Omega)}$$

because $\|\partial_\tau u\|_{H^{-1/2}(\partial\Omega)} \leq \|u\|_{H^{1/2}(\partial\Omega)}$, $\|\zeta u\|_{L^2(\partial\Omega)} \leq \|\zeta\|_{L^\infty(\partial\Omega)} \|u\|_{L^2(\partial\Omega)}$ and

$$\begin{aligned} \|\zeta u\|_{H^{1/2}(\partial\Omega)}^2 &= \int_{\partial\Omega} \int_{\partial\Omega} \frac{|\zeta(x)u(x) - \zeta(y)u(y)|^2}{|x-y|^2} dx dy \\ &\leq C \|\zeta\|_{L^\infty(\partial\Omega)}^2 \|u\|_{H^{1/2}(\partial\Omega)}^2 + C \|\partial_\tau \zeta\|_{L^\infty(\partial\Omega)}^2 \|u\|_{L^2(\partial\Omega)}^2 \\ &\leq C \|u\|_{H^{1/2}(\partial\Omega)}^2 \|\zeta\|_{W^{1,\infty}(\partial\Omega)}^2 \end{aligned}$$

where $C > 0$ depends only on the geometry of Ω . Now let us check that the boundary Jacobian $\mathcal{J}_{bd}(u)$ coincides with (24) for a map $u \in H^1(\Omega; \mathbb{R}^2)$ and a test function $\zeta \in W^{1,\infty}(\Omega)$. Indeed, if $u \in C^2(\bar{\Omega})$, then integration by parts yields:

$$-\langle \mathcal{J}_{bd}(u), \zeta \rangle \stackrel{(7)}{=} \int_{\Omega} u \times \nabla u \cdot \nabla^\perp \zeta dx + 2 \int_{\Omega} \text{jac}(u) \zeta dx \stackrel{(22)}{=} \int_{\partial\Omega} \zeta u \times \partial_\tau u d\mathcal{H}^1.$$

Since the trace operator is continuous from $H^1(\Omega; \mathbb{R}^2)$ to $H^{1/2}(\partial\Omega; \mathbb{R}^2)$ as well as the operator mapping $u \in H^{1/2}(\partial\Omega; \mathbb{R}^2) \mapsto \partial_\tau u \in H^{-1/2}(\partial\Omega; \mathbb{R}^2)$, by the density of $C^2(\bar{\Omega}; \mathbb{R}^2)$ maps into $H^1(\Omega; \mathbb{R}^2)$, we conclude that the last identity also holds for general $u \in H^1(\Omega; \mathbb{R}^2)$ within the duality $(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. Finally, we prove (25) for $u, v \in H^1(\Omega; \mathbb{R}^2)$ and $\zeta \in W^{1,\infty}(\Omega)$. Indeed, using the same estimates as above, there exists a constant $C > 0$ depending on Ω such that

$$\begin{aligned} |\langle (\mathcal{J}_{bd}(u) - \mathcal{J}_{bd}(v)), \zeta \rangle| &= \left| (\zeta u \times \partial_\tau u - \zeta v \times \partial_\tau v)_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} \right| \\ &= \left| \underbrace{(\zeta(u-v) \times \partial_\tau(u+v))}_{:=I} - \underbrace{(\zeta u \times \partial_\tau v + \zeta v \times \partial_\tau u)}_{:=II} \right|_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} \end{aligned}$$

⁸Using a Lipschitz arc-length parametrisation $\{\gamma(\theta)\}_{\theta \in \mathbb{S}^1}$ of $\partial\Omega$, the right hand side of (24) becomes up to sign

$$((\zeta u) \times \partial_\tau u)_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} = (\tilde{\zeta} \bar{u} \times \partial_\theta \bar{u})_{H^{1/2}(\mathbb{S}^1), H^{-1/2}(\mathbb{S}^1)}$$

where $\bar{u}(\theta) = u(\gamma(\theta)) \in H^{1/2}(\mathbb{S}^1)$ and $\tilde{\zeta}(\theta) = \zeta(\gamma(\theta)) \in W^{1,\infty}(\mathbb{S}^1)$.

with

$$\begin{aligned} |I| &\leq \|\zeta(u-v)\|_{H^{1/2}(\partial\Omega)} \|u+v\|_{H^{1/2}(\partial\Omega)} \\ &\leq C \|u-v\|_{H^{1/2}(\partial\Omega)} \left(\|u\|_{H^{1/2}(\partial\Omega)} + \|v\|_{H^{1/2}(\partial\Omega)} \right) \|\zeta\|_{W^{1,\infty}(\partial\Omega)} \end{aligned}$$

and II is interpreted as a duality between $\partial_\tau(v \times u) \in (W^{1,\infty}(\partial\Omega))^*$ and $\zeta \in W^{1,\infty}(\partial\Omega)$ which combined with $v \times u = v \times (u-v) \in L^1(\partial\Omega)$ leads to

$$\begin{aligned} |II| &= \left| \int_{\partial\Omega} v \times u \partial_\tau \zeta \, d\mathcal{H}^1 \right| \leq \|u-v\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \|\zeta\|_{W^{1,\infty}(\partial\Omega)} \\ &\leq C \|u-v\|_{H^{1/2}(\partial\Omega)} \|v\|_{H^{1/2}(\partial\Omega)} \|\zeta\|_{W^{1,\infty}(\partial\Omega)}. \end{aligned}$$

Summing up, we conclude with (25) which implies in particular for every $\zeta \in W^{1,\infty}(\Omega)$:

$$|\langle \mathcal{J}_{bd}(u) - \mathcal{J}_{bd}(v), \zeta \rangle| \leq C \|u-v\|_{H^1(\Omega)} (\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}) \|\zeta\|_{W^{1,\infty}(\Omega)}. \quad (26)$$

□

Remark 2.1 i) Note that the above inequality (26) is weaker than the estimate in Theorem 1.1 because it represents a stability inequality for the boundary Jacobian in the strong $H^1(\Omega)$ -topology, while in Theorem 1.1 only $L^2(\Omega)$ closeness is required together with a slight control of the $\dot{H}^1(\Omega)$ -seminorm that may blow-up.

ii) For $u \in H^1(\Omega; \mathbb{R}^2)$, while the interior Jacobian $\text{jac}(u)$ is a measure in Ω , the global Jacobian is not in general a measure on $\bar{\Omega}$ because \mathcal{J}_{bd} is not in general a measure on $\partial\Omega$. Indeed, if $|u| = 1$ in a smooth simply connected domain Ω , then $\text{jac}(u) = 0$ in Ω (because $\partial_{x_1}u$ and $\partial_{x_2}u$ are parallel vectors, both being orthogonal to u) and by the Bethuel-Zheng theorem in [9], we know that $u = e^{i\varphi}$ with a lifting $\varphi \in H^1(\Omega)$ so that $\varphi \in H^{1/2}(\partial\Omega)$. Then it follows by Proposition 2.2 that for every $\zeta \in W^{1,\infty}(\Omega)$, $\langle \mathcal{J}_{bd}(u), \zeta \rangle = -(\zeta \partial_\tau \varphi)_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)}$ where $\partial_\tau \varphi$ belongs to $H^{-1/2}(\partial\Omega)$ which clearly can be chosen not to be a measure on $\partial\Omega$.

Proof of Theorem 1.1. Since Ω is $C^{1,1}$, there exists $r_1 := r_1(\Omega) > 0$ such that every point $x \in \Omega$ with $\text{dist}(x, \partial\Omega) < r_1$ has a unique orthogonal projection on the boundary $\partial\Omega$, i.e., the crossing of two normal directions on $\partial\Omega$ in the interior of Ω happens at a distance larger than r_1 from the boundary.

Assume for the moment that u, v are smooth maps in $\bar{\Omega}$. Note that the inequality is trivial if u and v are equal or if they are both constant maps. Therefore, in the following, we can assume that

$$\delta = \frac{\|u-v\|_{L^2(\Omega)}}{\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}} \in (0, \infty).$$

Let ζ be smooth in $\bar{\Omega}$. In the following, **we denote by $C > 0$ a constant depending only on the geometry of Ω that can change from line to line.**

Case 1. Suppose that $\delta \geq r_1/4$. In this case, we have

$$\begin{aligned} & \left| \langle \mathcal{J}(u) - \mathcal{J}(v), \zeta \rangle \right| = \left| \int_{\Omega} (u \times \nabla u - v \times \nabla v) \cdot \nabla^{\perp} \zeta \, dx \right| \\ & = \left| \int_{\Omega} \left\{ (u - v) \times \nabla u \cdot \nabla^{\perp} \zeta + v \times (\nabla u - \nabla v) \cdot \nabla^{\perp} \zeta \right\} dx \right| \\ & \leq \|v - u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla \zeta\|_{L^\infty(\Omega)} + \|v\|_{L^2(\Omega)} (\|\nabla v\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) \|\nabla \zeta\|_{L^\infty(\Omega)}. \end{aligned}$$

The conclusion follows by

$$\|v\|_{L^2(\Omega)} \leq \mathcal{H}^2(\Omega)^{1/2} \leq Cr_1^{1/2} \leq C\delta^{1/2},$$

where we used the hypothesis $|v| \leq 1$ in Ω and the assumption $\delta \geq r_1/4$.

Case 2. Suppose that $\delta \leq r_1/4$. In this case, we denote by

$$\Omega_R = \{x \in \Omega : \text{dist}(x, \partial\Omega) < R\}$$

the region around the boundary $\partial\Omega$ at a distance less than R . Then by averaging on the interval $(\delta, 2\delta)$, the co-area formula yields the existence of some $R \in (\delta, 2\delta)$ such that:

$$\begin{aligned} \delta \int_{\partial\Omega_R \cap \Omega} |v \times u| \, d\mathcal{H}^1 &= \int_{\delta}^{2\delta} dr \int_{\partial\Omega_r \cap \Omega} |v \times u| \, d\mathcal{H}^1 = \int_{\Omega_{2\delta} \setminus \Omega_{\delta}} |v \times u| \, dx \\ &\leq C\delta^{1/2} \|v - u\|_{L^2(\Omega)} \end{aligned} \quad (27)$$

because $v \times u = v \times (u - v)$ and $|v| \leq 1$. We estimate the desired quantity on $\Omega \setminus \Omega_R$:

$$\begin{aligned} I &:= \left| \int_{\Omega \setminus \Omega_R} (u \times \nabla u - v \times \nabla v) \cdot \nabla^{\perp} \zeta \, dx \right| \\ &\stackrel{(23)}{\leq} \|\nabla \zeta\|_{L^\infty(\Omega)} \int_{\Omega \setminus \Omega_R} |u - v| (|\nabla u| + |\nabla v|) \, dx + \left| \int_{\Omega \setminus \Omega_R} \nabla(v \times u) \cdot \nabla^{\perp} \zeta \, dx \right| \end{aligned}$$

where the integration by parts leads to

$$\begin{aligned} \left| \int_{\Omega \setminus \Omega_R} \nabla(v \times u) \cdot \nabla^{\perp} \zeta \, dx \right| &= \left| \int_{\partial\Omega_R \cap \Omega} v \times u \, \partial_{\tau} \zeta \, d\mathcal{H}^1 \right| \\ &\leq \|\nabla \zeta\|_{L^\infty(\Omega)} \int_{\partial\Omega_R \cap \Omega} |v \times u| \, d\mathcal{H}^1 \\ &\stackrel{(27)}{\leq} C \|v - u\|_{L^2(\Omega)}^{1/2} (\|\nabla v\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)})^{1/2} \|\nabla \zeta\|_{L^\infty(\Omega)}. \end{aligned}$$

Next we estimate the desired quantity on Ω_R :

$$\begin{aligned} II &:= \left| \int_{\Omega_R} (u \times \nabla u - v \times \nabla v) \cdot \nabla^{\perp} \zeta \, dx \right| \\ &= \left| \int_{\Omega_R} \left\{ (u - v) \times \nabla u \cdot \nabla^{\perp} \zeta + v \times (\nabla u - \nabla v) \cdot \nabla^{\perp} \zeta \right\} dx \right| \\ &\leq \|\nabla \zeta\|_{L^\infty(\Omega)} \int_{\Omega_R} |v - u| |\nabla u| \, dx + \mathcal{H}^2(\Omega_R)^{1/2} (\|\nabla v\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) \|\nabla \zeta\|_{L^\infty(\Omega)}, \end{aligned}$$

with $\mathcal{H}^2(\Omega_R)^{1/2} \leq C\delta^{1/2}$. Adding *I* and *II* we obtain the desired inequality. By a standard density argument (as in the proof of Proposition 2.1), the inequality holds for general H^1 -maps u and v and general Lipschitz test function ζ . \square

As a consequence of Theorem 1.1 and Proposition 2.1, we have the following stability result for the global and interior Jacobian:

Corollary 2.3 *Let $\Omega \subset \mathbb{R}^2$ be a $C^{1,1}$ bounded domain and let $u_\varepsilon, v_\varepsilon : \Omega \rightarrow \mathbb{R}^2$ be two sequences / families of H^1 -maps such that $|v_\varepsilon| = 1$ in Ω and*

$$\|u_\varepsilon - v_\varepsilon\|_{L^2(\Omega)} (\|\nabla u_\varepsilon\|_{L^2(\Omega)} + \|\nabla v_\varepsilon\|_{L^2(\Omega)}) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (28)$$

Then

1. (Stability of the interior Jacobian) We have $\text{jac}(v_\varepsilon) = 0$ and $\|\text{jac}(u_\varepsilon)\|_{(W_0^{1,\infty}(B_1))^*} \xrightarrow{\varepsilon \rightarrow 0} 0$;
2. (Stability of the global Jacobian) We have $\|\mathcal{J}(u_\varepsilon) - \mathcal{J}(v_\varepsilon)\| = \|\mathcal{J}(u_\varepsilon) - \mathcal{J}_{bd}(v_\varepsilon)\| \xrightarrow{\varepsilon \rightarrow 0} 0$.

Let us show now that the boundary Jacobian \mathcal{J}_{bd} is *not* stable under the condition (28) (recall that \mathcal{J}_{bd} is stable in the strong $H^1(\Omega)$ -topology, see (26)).

Proposition 2.4 *Let $P = (0, -\frac{3}{4})$, $\Omega = B_{1/4}(P)$ be the disk of center P and radius $1/4$. Then for every small $\varepsilon > 0$, there exists a map $u_\varepsilon \in H^1(\Omega; \mathbb{R}^2)$ such that (28) holds for $v_\varepsilon = 1$ in Ω , but*

$$\langle \mathcal{J}_{bd}(u_\varepsilon) - \mathcal{J}_{bd}(v_\varepsilon), 1 \rangle \not\rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

In particular, $\|\mathcal{J}_{bd}(u_\varepsilon) - \mathcal{J}_{bd}(v_\varepsilon)\| \not\rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. Set $P_\pm = (0, \pm\frac{1}{2}) \in \mathbb{R}^2$ and B^2 be the unit disk of \mathbb{R}^2 . Note that $P_- \in \partial\Omega$.

Step 1. *Construction of a function $U : B^2 \rightarrow \overline{B^2}$.* First, we set

$$G = B^2 \setminus \left(\overline{B_{1/4}(P_-)} \cup \overline{B_{1/4}(P_+)} \right)$$

and we define $U : \partial G \rightarrow \mathbb{S}^1$ as follows: $U = 1$ on ∂B^2 , $U(P_\pm + \frac{1}{4}e^{i\theta}) = e^{\pm i\theta}$ for $\theta \in [0, 2\pi)$. Note that the total topological degree of this smooth boundary data U over ∂G is zero. Therefore, we can smoothly extend $U : \overline{G} \rightarrow \mathbb{S}^1$ to the closure of the set G (see Bethuel-Brezis-Hélein [8] or Struwe [40]). Finally, we extend U to the whole disk B^2 by setting $U(P_\pm + re^{i\theta}) = 4re^{\pm i\theta}$ for $0 \leq r \leq \frac{1}{4}$ and $\theta \in [0, 2\pi)$. Then U is continuous, $U \in H^1(B^2; \mathbb{R}^2)$ and U has degree ± 1 on the circles $\partial B_{\frac{1}{4}}(P_\pm)$.

Step 2. *Construction of u_ε and v_ε on Ω .* For every $0 < \varepsilon < \frac{1}{4}$, set $v_\varepsilon \equiv 1$ in Ω and $u_\varepsilon : \Omega \rightarrow \overline{B^2}$ is defined as follows: $u_\varepsilon(x) = U(\frac{x-P_-}{\varepsilon})$ in $B_\varepsilon(P_-) \cap \Omega$ and $u_\varepsilon = 1$ in $\Omega \setminus B_\varepsilon(P_-)$. In other words, u_ε has one interior vortex point going to the boundary $\partial\Omega$ as $\varepsilon \rightarrow 0$. Then

$$\int_{\Omega} (|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \leq \int_{B^2} |\nabla U|^2 dx.$$

Since $\|u_\varepsilon\|_{L^\infty} = \|v_\varepsilon\|_{L^\infty} = 1$ and $\mathcal{L}^2(\{u_\varepsilon \neq v_\varepsilon\}) \leq \pi\varepsilon^2$, we deduce that

$$\int_{\Omega} |u_\varepsilon - v_\varepsilon|^2 dx \leq C\varepsilon^2;$$

thus, (28) holds in Ω as $\varepsilon \rightarrow 0$. We finally calculate the boundary Jacobians in the disk Ω with boundary $\partial\Omega$. Clearly $\mathcal{J}_{bd}(v_\varepsilon) = 0$ and by Proposition 2.2,

$$\langle \mathcal{J}_{bd}(u_\varepsilon), 1 \rangle = - \int_{\partial\Omega} u_\varepsilon \times \partial_\tau u_\varepsilon d\mathcal{H}^1.$$

For ε sufficiently small, we have $|u_\varepsilon| = 1$ on $\partial\Omega$, so we obtain

$$\langle \mathcal{J}_{bd}(u_\varepsilon), 1 \rangle = -2\pi \deg(u_\varepsilon; \partial\Omega) = 2\pi,$$

which clearly does not tend to zero as $\varepsilon \rightarrow 0$. In particular, $\|\mathcal{J}_{bd}(u_\varepsilon) - \mathcal{J}_{bd}(v_\varepsilon)\|_{(W^{1,\infty}(\Omega))^*} \not\rightarrow 0$ as $\varepsilon \rightarrow 0$ (note that $\|\mathcal{J}_{bd}(u_\varepsilon) - \mathcal{J}_{bd}(v_\varepsilon)\|_{(W^{1,\infty}(\Omega))^*} < \infty$ by (26)), while $\|\mathcal{J}_{bd}(u_\varepsilon) - \mathcal{J}_{bd}(v_\varepsilon)\| = \infty$ due to (20). \square

3 Approximation by \mathbb{S}^1 -valued maps

In this section we show that maps $u : \Omega \rightarrow \mathbb{R}^2$ with energy of order $E_{\varepsilon,\eta}(u) \leq C|\log \varepsilon|$ can be approximated by suitable \mathbb{S}^1 -valued maps $U : \Omega \rightarrow \mathbb{S}^1$ in the regime $|\log \varepsilon| \ll |\log \eta|$. The approximation is realised such that u and U are close energetically, and also in $L^2(\Omega)$ and in $L^2(\partial\Omega)$, and such that their global Jacobians are close to each other. This is an essential step in the reduction of our model to the study of a simpler problem for \mathbb{S}^1 -valued maps. Our result is based on some ideas introduced by Ignat-Otto [27] (see also Côte-Ignat-Miot [15]) where the approximation argument was done locally; here the improvement consists in developing a global analysis of the configurations u , in particular at the boundary $\partial\Omega$.

Notation: If $G \subset \Omega$ and $u : \Omega \rightarrow \mathbb{R}^2$, we denote

$$E_{\varepsilon,\eta}(u; G) = \int_G (|\nabla u|^2 + \frac{1}{\eta^2}(1 - |u|^2)^2) dx + \frac{1}{2\pi\varepsilon} \int_{G \cap \partial\Omega} (u \cdot \nu)^2 d\mathcal{H}^1.$$

Theorem 3.1 *Let $\beta \in (\frac{1}{2}, 1)$, $C > 0$ and $\Omega \subset \mathbb{R}^2$ be a simply connected $C^{1,1}$ bounded domain. We consider the sequence / family $\varepsilon > 0$ and $\eta = \eta(\varepsilon) > 0$ satisfying $\eta(\varepsilon) \rightarrow 0$ and $|\log \varepsilon| \ll |\log \eta|$ as $\varepsilon \rightarrow 0$. Then there exist $\varepsilon_0, c_0, \tilde{C} > 0$ depending only on β, C , the function $\varepsilon \mapsto \eta(\varepsilon)$ and Ω and $0 < \tilde{\beta} < \frac{1-\beta}{6}$ such that for $\varepsilon \leq \varepsilon_0$ in the sequence / family and every $u = u_\varepsilon : \Omega \rightarrow \mathbb{R}^2$ with $E_{\varepsilon,\eta}(u) \leq C|\log \varepsilon|$, we can construct a unit-length map $U = U_\varepsilon : \Omega \rightarrow \mathbb{S}^1$ such that*⁹

$$\int_{\Omega} |U - u|^2 dx \lesssim \eta^{2\beta} E_{\varepsilon,\eta}(u), \quad \int_{\Omega} (|\nabla U|^2 + |\nabla u|^2) dx \lesssim E_{\varepsilon,\eta}(u), \quad (29)$$

$$\int_{\partial\Omega} |U - u|^2 d\mathcal{H}^1 \lesssim \eta^\beta E_{\varepsilon,\eta}(u) \quad (30)$$

and

$$E_{\varepsilon,\eta}(U) \leq E_{\varepsilon,c_0\eta}(u) + \tilde{C}\eta^{\tilde{\beta}} \left(E_{\varepsilon,c_0\eta}(u) + \sqrt{E_{\varepsilon,c_0\eta}(u)} \right). \quad (31)$$

⁹In the following, \lesssim denotes an upper bound with a constant depending only on β, C and Ω .

As consequence, for every $p \geq 1$, $\|U - u\|_{L^p(\partial\Omega)} \rightarrow 0$, $\|U - u\|_{L^p(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$,

$$\|\text{jac}(u)\|_{(W_0^{1,\infty})^*(\Omega)} \lesssim \eta^\beta E_{\varepsilon,\eta}(u) \quad \text{and} \quad \|\mathcal{J}(U) - \mathcal{J}(u)\|_{(\text{Lip}(\Omega))^*} \lesssim \sqrt{\eta^\beta E_{\varepsilon,\eta}(u)}.$$

The map U also satisfies the following local estimates: for any open set $G \subset \Omega$ independent of ε , there exists a constant $\tilde{C}_G > 0$ such that

$$E_{\varepsilon,\eta}(U; G_\eta) \leq E_{\varepsilon,c_0\eta}(u; G) + \tilde{C}_G \eta^{\tilde{\beta}} \left(E_{\varepsilon,c_0\eta}(u; G) + \sqrt{E_{\varepsilon,c_0\eta}(u; G)} \right) \quad (32)$$

where ¹⁰

$$G_\eta = \{x \in G : \text{dist}(x, \partial G \cap \Omega) > 3\eta^\beta\}.$$

Proof. We start by proving the result in the case of the unit disk $\Omega = B^2$ and then we treat the general case of a simply connected $C^{1,1}$ domain Ω .

Step 1. *Construction of a polar grid \mathcal{R} in B^2 .* We use the polar coordinates $(r, \theta) \in (0, 1) \times [0, 2\pi)$ corresponding to $x = (x_1, x_2) \in B^2$. For each (radial) shift $R \in (0, \eta^\beta)$, write

$$V_R := \{x \in B^2 : r = |x| \in (\eta^\beta, 1), r \equiv R \pmod{\eta^\beta}\}$$

for the net of concentric circles at a distance η^β in B^2 . By the mean value theorem, there exists $R \in (0, \eta^\beta)$ such that

$$\int_{V_R} e_\eta(u) d\mathcal{H}^1 \leq \frac{1}{\eta^\beta} \int_{B^2} e_\eta(u) dx,$$

where $e_\eta(\cdot)$ is the Ginzburg-Landau energy density:

$$e_\eta(u) = |\nabla u|^2 + \frac{1}{\eta^2}(1 - |u|^2)^2. \quad (33)$$

If one repeats the above argument for the net of radii lines at an angular distance η^β in B^2 , we obtain for some angle $\Theta \in (0, \eta^\beta)$ a net

$$\tilde{V}_\Theta := \{x \in B^2 : \theta = \arg x \in (\eta^\beta, 2\pi), \theta \equiv \Theta \pmod{\eta^\beta}\}$$

such that

$$\int_{\tilde{V}_\Theta} e_\eta(u(x)) d\mathcal{H}^1(x) = \int_{\{r < 1, \theta \equiv \Theta\}} e_\eta(u(r, \theta)) r dr \leq \frac{1}{\eta^\beta} \int_{B^2} e_\eta(u) dx.$$

Therefore, we obtain a polar grid $\mathcal{R} = V_R \cup \tilde{V}_\Theta$ of size at most η^β such that

$$\int_{\mathcal{R}} e_\eta(u) d\mathcal{H}^1 \leq \frac{2}{\eta^\beta} \int_{B^2} e_\eta(u) dx \lesssim \frac{E_{\varepsilon,\eta}(u)}{\eta^\beta}. \quad (34)$$

We regroup the cells¹¹ of \mathcal{R} in order that each new cell has approximatively the same area of order $\sim \eta^{2\beta}$: the first new cell has the interior given by the disk $B(0, R + \eta^\beta)$ (by regrouping all the sectors of \mathcal{R} of radius less than $R + \eta^\beta$ and containing the origin). Then for each annulus of \mathcal{R} of the form $B(0, R + (k+1)\eta^\beta) \setminus B(0, R + k\eta^\beta)$ with $k \geq 1$, we regroup the neighbouring cells of

¹⁰The typical example is $G = B(a, r) \cap \Omega$ for some $a \in \partial\Omega$ and $r > 0$, so that $G_\eta = B(a, r - 3\eta^\beta) \cap \Omega$.

¹¹Our terminology here is slightly non-standard: a ‘‘cell’’ for us is a union of one-dimensional (straight or circular) segments of \mathcal{R} , not the corresponding two-dimensional interior (which a topologist would call a 2-cell).

the angular sectors $(\Theta + j\eta^\beta, \Theta + (j+1)\eta^\beta)$ ($j \geq 1$) so that the length of the angular arc gets of order $\sim \eta^\beta$ and their area become of order $\sim \eta^{2\beta}$. (In the annuli close to the origin, many cells are regrouped, while in the annuli far away from the origin, no regrouping is needed). Therefore, from now on, we can assume that all cells of \mathcal{R} (excepting the first one $B(0, R + \eta^\beta)$) are rather identical (all the four sides of the cell having the length of order $\sim \eta^\beta$). For any cell $\mathcal{C} \subset \mathcal{R}$ (which is one dimensional as a union of straight and circular segments we denote by $\text{int}(\mathcal{C})$ the $2D$ region enclosed by \mathcal{C} and let

$$\text{int}(\mathcal{R}) = \cup_{\mathcal{C} \subset \mathcal{R}} \text{int}(\mathcal{C}).$$

Therefore, we have that the closure $\overline{\text{int}(\mathcal{R})}$ of $\text{int}(\mathcal{R})$ is a disk strictly included in B^2 at a distance less than η^β from the boundary ∂B^2 . The cells we have constructed all satisfy uniform conditions on their geometry so we can apply Proposition 3.2 with uniform constants.

Step 2. An approximating \mathbb{S}^1 -valued map \hat{U} for u inside $\text{int}(\mathcal{R})$. In the interior $\text{int}(\mathcal{C})$ of a polar cell \mathcal{C} of \mathcal{R} having each side of length $\sim \eta^\beta$, we define $w = w_\varepsilon \in H^1(\text{int}(\mathcal{C}), \mathbb{R}^2)$ (depending on ε through $\eta = \eta(\varepsilon)$) be a minimizer of

$$\min_{w=u \text{ on } \mathcal{C}} \int_{\text{int}(\mathcal{C})} e_\eta(w) dx. \quad (35)$$

Putting together all the cells, w is now defined in the whole $\text{int}(\mathcal{R})$. We apply Proposition 3.2 below (for $\kappa := C|\log \varepsilon| \ll |\log \eta|$): Since (34) holds (in particular, (44) holds for e_η on the domain \mathcal{D}_η and $\partial \mathcal{D}_\eta$), we have the existence of $0 < \tilde{\beta} < \frac{1-\beta}{6}$ such that for some $\hat{C} > 0$:

$$\sup_{\text{int}(\mathcal{R})} \| |w|^2 - 1 \| \leq \hat{C} \eta^{\tilde{\beta}} =: \delta \ll 1. \quad (36)$$

In particular, $|u| \geq 1/2$ on \mathcal{R} and u has vanishing degree on each cell, i.e., $\deg(u, \mathcal{C}) = 0$. The same conclusion holds for the central cell of interior $B(0, R + \eta^\beta)$. Therefore, we can define

$$\hat{U} := \frac{w}{|w|} \quad \text{in } \text{int}(\mathcal{R}).$$

Then $|w|^2 |\nabla \hat{U}|^2 \leq |\nabla w|^2$ and we deduce for small $\varepsilon > 0$:

$$\begin{aligned} \int_{\text{int}(\mathcal{R})} |\nabla \hat{U}|^2 dx &\stackrel{(36)}{\leq} (1+2\delta) \int_{\text{int}(\mathcal{R})} |\nabla w|^2 dx \leq (1+2\delta) \int_{\text{int}(\mathcal{R})} e_\eta(w) dx \\ &\stackrel{(35)}{\leq} (1+2\delta) \int_{\text{int}(\mathcal{R})} e_\eta(u) dx \leq (1+2\delta) \int_{B^2} e_\eta(u) dx. \end{aligned} \quad (37)$$

For the local estimates inside a set $G \subset B^2$, we set $\hat{\mathcal{R}}$ be the union of cells $\mathcal{C} \subset \mathcal{R}$ such that $\text{int}(\mathcal{C}) \subset G$ and by the same notation as above, we call $\text{int}(\hat{\mathcal{R}}) = \cup_{\mathcal{C} \subset \hat{\mathcal{R}}} \text{int}(\mathcal{C})$. Then we have $\text{int}(\hat{\mathcal{R}}) \subset G$ and we conclude as above

$$\int_{\text{int}(\hat{\mathcal{R}})} |\nabla \hat{U}|^2 dx \leq (1+2\delta) \int_{\text{int}(\hat{\mathcal{R}})} |\nabla w|^2 dx \leq (1+2\delta) \int_G e_\eta(u) dx.$$

Step 3. Our approximating \mathbb{S}^1 -valued map U of u in B^2 . We have defined \hat{U} in $\text{int}(\mathcal{R}) \subset\subset B^2$. However, we have that $B^2 = (1 + O(\eta^\beta)) \text{int}(\mathcal{R})$. For simplicity of notation, we assume in the following that

$$\overline{B^2} = (1 + \eta^\beta) \overline{\text{int}(\mathcal{R})} \quad \text{and} \quad U(\tilde{x}) := \hat{U}(x), \quad \tilde{x} = (1 + \eta^\beta)x \quad \text{for every } x \in \text{int}(\mathcal{R}) \quad (38)$$

and our goal is to prove that $U : B^2 \rightarrow \mathbb{S}^1$ is indeed the desired approximating map of the given u . We also set $\tilde{u} : (1 + \eta^\beta)B^2 \rightarrow \mathbb{R}^2$ by $\tilde{u}(\tilde{x}) = u(x)$ for every $x \in B^2$.

Step 4. *Estimate the L^2 -norm of gradients.* We have that

$$\int_{B^2} |\nabla U|^2 d\tilde{x} = \int_{\text{int}(\mathcal{R})} |\nabla \hat{U}|^2 dx \stackrel{(37)}{\leq} (1 + 2\delta) \int_{B^2} e_\eta(u) dx.$$

Combined with $\|\nabla u\|_{L^2(\Omega)}^2 \leq \int_{B^2} e_\eta(u) dx \leq E_{\varepsilon, \eta}(u)$, the second estimate in (29) follows. For the local estimate, we have $G_\eta \subset (1 + \eta^\beta)\text{int}(\hat{\mathcal{R}}) \subset B^2$ by convention (38) so that by Step 2 it follows

$$\|\nabla U\|_{L^2(G_\eta)}^2 \leq \|\nabla \hat{U}\|_{L^2(\text{int}(\hat{\mathcal{R}}))}^2 \leq (1 + 2\delta) \int_G e_\eta(u) dx.$$

Step 5. *Estimate $\|\hat{U} - u\|_{L^2(\text{int}\mathcal{R})}$.* By Poincaré's inequality, we have for each cell $\mathcal{C} \subset \mathcal{R}$:

$$\int_{\text{int}(\mathcal{C})} \left| \hat{U} - \fint_{\mathcal{C}} \hat{U} \right|^2 dx \lesssim \eta^{2\beta} \int_{\text{int}(\mathcal{C})} |\nabla \hat{U}|^2 dx \quad (39)$$

and

$$\int_{\text{int}(\mathcal{C})} \left| u - \fint_{\mathcal{C}} u \right|^2 dx \lesssim \eta^{2\beta} \int_{\text{int}(\mathcal{C})} |\nabla u|^2 dx, \quad (40)$$

where $\fint_{\mathcal{C}} = \frac{1}{\mathcal{H}^1(\mathcal{C})} \int_{\mathcal{C}}$ is the average on the cell \mathcal{C} . As $\rho := |u| \geq \frac{1}{2}$ on \mathcal{R} , we can set $v = \frac{u}{\rho}$ on \mathcal{R} with $|v| = 1$; therefore, we have $v = \hat{U}$ on \mathcal{R} and by Jensen's inequality, we estimate

$$\begin{aligned} \int_{\text{int}(\mathcal{C})} \left| \fint_{\mathcal{C}} (\hat{U} - u) d\mathcal{H}^1 \right|^2 dx &= \int_{\text{int}(\mathcal{C})} \left| \fint_{\mathcal{C}} (v - \rho v) d\mathcal{H}^1 \right|^2 dx \\ &\lesssim \eta^{2\beta} \int_{\mathcal{C}} (1 - \rho)^2 d\mathcal{H}^1 \lesssim \eta^\beta \int_{\mathcal{C}} (1 - \rho^2)^2 d\mathcal{H}^1 \lesssim \eta^{\beta+2} \int_{\mathcal{C}} e_\eta(u) d\mathcal{H}^1. \end{aligned} \quad (41)$$

Summing (39), (40) and (41) over all the cells \mathcal{C} of the grid \mathcal{R} , by (34) and (37), we obtain that

$$\int_{\text{int}(\mathcal{R})} |\hat{U} - u|^2 dx \lesssim \eta^{2\beta} \int_{B^2} e_\eta(u) dx.$$

Step 6. *The L^2 -estimate of $U - u$ in B^2 .* From (38) and the previous step, we clearly have that

$$\|U - \tilde{u}\|_{L^2(B^2)}^2 \lesssim \eta^{2\beta} \int_{B^2} e_\eta(u) dx.$$

Hence it remains to show that the L^2 norm of $u - \tilde{u}$ satisfies the same estimate. We compute

$$\int_{B^2} |u(x) - \tilde{u}(x)|^2 dx = \int_{B^2} \left| u(x) - u\left(\frac{x}{1 + \eta^\beta}\right) \right|^2 dx.$$

We set $\lambda(t) = (1 - t) + \frac{t}{1 + \eta^\beta}$ for $t \in [0, 1]$. Then $\frac{1}{1 + \eta^\beta} \leq \lambda(t) \leq 1$, $|\lambda'(t)| = 1 - \frac{1}{1 + \eta^\beta} = O(\eta^\beta)$ and

$$|u(x) - \tilde{u}(x)| = \left| \int_0^1 \lambda'(t)x \cdot \nabla u(\lambda(t)x) dt \right|$$

so integrating on B^2 , we obtain

$$\int_{B^2} |u(x) - \tilde{u}(x)|^2 dx \lesssim \eta^{2\beta} \int_0^1 \int_{B^2} |x \cdot \nabla u(\lambda(t)x)|^2 dt dx.$$

Changing variables $y = \lambda(t)x$ and using Fubini, we see that

$$\int_{B^2} |u(x) - \tilde{u}(x)|^2 dx \lesssim \eta^{2\beta} \int_0^1 \frac{1}{|\lambda(t)|^4} \int_{B^2} |y \cdot \nabla u(y)|^2 dy dt \lesssim \eta^{2\beta} \int_{B^2} e_\eta(u) dx$$

as claimed. This proves the first inequality in (29).

Step 7. *The L^2 -estimate of $U - u$ at the boundary ∂B^2 and $\overline{G_\eta} \cap \partial B^2$.* Let $R_0 \in (0, 1)$ be the largest radius such that $\partial B(0, R_0) \subset \mathcal{R}$. By the convention (38), we have chosen

$$R_0 = \frac{1}{1 + \eta^\beta}$$

and we have defined U in terms of \hat{U} . Since $v = \hat{U}$, $|v| = 1$ and $u = \rho v$ on \mathcal{R} , we have

$$\int_{\partial B(0, R_0)} |\hat{U} - u|^2 d\mathcal{H}^1 = \int_{\partial B(0, R_0)} (1 - \rho)^2 d\mathcal{H}^1 \leq \eta^2 \int_{\mathcal{R}} e_\eta(u) d\mathcal{H}^1 \stackrel{(34)}{\lesssim} \eta^{2-\beta} \int_{B^2} e_\eta(u) dx,$$

$$\begin{aligned} \int_{\partial B(0, R_0)} |u(x) - u(\frac{x}{R_0})|^2 d\mathcal{H}^1(x) &\stackrel{(38)}{=} R_0 \int_{\partial B^2} |u(R_0 \tilde{x}) - u(\tilde{x})|^2 d\mathcal{H}^1(\tilde{x}) \\ &\leq R_0 \int_0^{2\pi} \left(\int_{R_0}^1 |\partial_r u|(r e^{i\theta}) dr \right)^2 d\theta \\ &\lesssim \eta^\beta \int_{B^2 \setminus B(0, R_0)} |\nabla u|^2 dx \lesssim \eta^\beta \int_{B^2} e_\eta(u) dx. \end{aligned}$$

Combining these inequalities, we conclude

$$\int_{\partial B^2} |u(\tilde{x}) - U(\tilde{x})|^2 d\mathcal{H}^1(\tilde{x}) = (1 + \eta^\beta) \int_{\partial B(0, R_0)} |u(\frac{x}{R_0}) - \hat{U}(x)|^2 d\mathcal{H}^1(x) \lesssim \eta^\beta \int_{B^2} e_\eta(u) dx.$$

For the local estimate at the boundary $\overline{G_\eta} \cap \partial B^2$, we have as before for ε small:

$$\int_{G_\eta \cap \partial B(0, R_0)} |u(x) - u(\frac{x}{R_0})|^2 d\mathcal{H}^1(x) \lesssim \eta^\beta \int_G |\nabla u|^2 dx$$

because by the definition of G_η , we know that for every $x \in G_\eta \cap \partial B(0, R_0)$, the open segment $(x, \frac{x}{R_0}) \subset G$. It remains to prove that

$$\int_{G_\eta \cap \partial B(0, R_0)} |\hat{U} - u|^2 d\mathcal{H}^1 = \int_{G_\eta \cap \partial B(0, R_0)} (1 - \rho)^2 d\mathcal{H}^1 \lesssim \eta \int_G e_\eta(u) dx.$$

Indeed, we consider the covering $G_\eta \cap \partial B(0, R_0) \subset \mathcal{U} \subset G$ and for each cell \mathcal{C} we consider the function w constructed at Step 2. For simplicity of notation, we write such a cell \mathcal{C} to be the sector $(R_0 - \eta^\beta, R_0) \times (\Theta, \Theta + \eta^\beta)$ in the polar coordinates. By averaging in the radial coordinates, one can find an arc $\mathcal{L}_* = \{r_*\} \times (\Theta, \Theta + \eta^\beta)$ with $r_* \in (R_0 - \eta^\beta, R_0)$ such that

$$\int_{\mathcal{L}_*} (1 - |w|)^2 d\mathcal{H}^1 \leq \frac{1}{\eta^\beta} \int_{\text{int}(\mathcal{C})} (1 - |w|)^2 dx \leq \eta^{2-\beta} \int_{\text{int}(\mathcal{C})} e_\eta(w) dx. \quad (42)$$

Then

$$\begin{aligned} \int_{\mathcal{C} \cap \partial B(0, R_0)} (1 - |w|)^2 d\mathcal{H}^1 &\leq \int_{\mathcal{L}_*} (1 - |w|)^2 d\mathcal{H}^1 + 2 \int_{\text{int}(\mathcal{C})} (1 - |w|) |\partial_r w| dx \\ &\stackrel{(42)}{\lesssim} \eta \int_{\text{int}(\mathcal{C})} e_\eta(w) dx \lesssim \eta \int_{\text{int}(\mathcal{C})} e_\eta(u) dx. \end{aligned}$$

Summing up over cells \mathcal{C} covering $G_\eta \cap \partial B(0, R_0)$, we conclude that

$$\int_{G_\eta \cap \partial B^2} |u(\tilde{x}) - U(\tilde{x})|^2 d\mathcal{H}^1 \lesssim \eta^\beta \int_G e_\eta(u) dx.$$

Step 8. *Estimate of the global / interior Jacobian and L^p -estimates of $U - u$ in Ω and $\partial\Omega$.* The estimates of the global / interior are consequences of (29), Theorem 1.1 and Proposition 2.1. For the L^p -estimates of $U - u$ in Ω , we use (29), the Gagliardo-Nirenberg interpolation inequality for $p > 2$ ¹² or simply, the Hölder inequality for $p \leq 2$ as well as $\eta \leq \varepsilon$ (due to the regime (4)), in particular, $\eta^\sigma |\log \varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0$ for every $\sigma > 0$. The same argument applies for the estimate $L^p(\partial\Omega)$ of $U - u$.

Step 9. *Estimate of the energy of U in B^2 and G_η .* Using that $a^2 \leq b^2 + 2|a - b|$ for every $a \in [-1, 1]$ and $b \in \mathbb{R}$, then Step 7 and Cauchy-Schwarz yield

$$\begin{aligned} \frac{1}{2\pi\varepsilon} \int_{\partial B^2} (U \cdot \nu)^2 d\mathcal{H}^1 &\leq \frac{1}{2\pi\varepsilon} \int_{\partial B^2} (u \cdot \nu)^2 d\mathcal{H}^1 + \frac{1}{\pi\varepsilon} \int_{\partial B^2} |(U - u) \cdot \nu| d\mathcal{H}^1 \\ &\leq \frac{1}{2\pi\varepsilon} \int_{\partial B^2} (u \cdot \nu)^2 d\mathcal{H}^1 + \frac{c}{\varepsilon} \|U - u\|_{L^2(\partial B^2)} \\ &\leq \frac{1}{2\pi\varepsilon} \int_{\partial B^2} (u \cdot \nu)^2 d\mathcal{H}^1 + \frac{c\eta^{\beta/2}}{\varepsilon} \sqrt{\int_{B^2} e_\eta(u) dx}, \end{aligned} \quad (43)$$

for some $c > 0$. Since $|\log \varepsilon| \ll |\log \eta|$, we can choose $\varepsilon_0 > 0$ (depending on β) such that $\frac{\eta^{\beta/2}}{\varepsilon} \leq \delta$ for every $\varepsilon \leq \varepsilon_0$ where δ is defined in (36). (Here, the assumption $\beta > \frac{1}{2}$ is essential.)

By Step 4, we obtain

$$E_{\varepsilon, \eta}(U) \leq E_{\varepsilon, \eta}(u) + \tilde{C}\delta \left(\sqrt{\int_{B^2} e_\eta(u) dx} + \int_{B^2} e_\eta(u) dx \right),$$

for some constant $\tilde{C} > 0$. The local estimate (32) (with $c_0 = 1$) follows by the same argument, the constant c in (43) depending only on the length of $\partial G \cap \partial B^2$.

Step 10. *The general case of a simply connected $C^{1,1}$ domain Ω .* By the Kellogg-Warschawski theorem (see Pommerenke [37, Theorem 3.5]), there exists a conformal map $\Psi \in C^{1,\alpha}(\bar{\Omega}; \bar{B}^2)$ that transforms Ω and $\partial\Omega$ in B^2 and ∂B^2 respectively, for every $\alpha \in (0, 1)$. Since the Jacobian $\text{jac}(\Psi)$ is bounded above and below by some positive constants, the corresponding energy on B^2 is bounded (above and below) by $E_{\tilde{\varepsilon}, \tilde{\eta}}$ where $\varepsilon \sim \tilde{\varepsilon}$ and $\eta \sim \tilde{\eta}$. Therefore, (29) and (30) (as well as the estimates for the interior / global Jacobian) follow immediately because the prefactor in those inequalities

¹²For every $p \in (2, \infty)$, there exists $C > 0$ such that $\|f\|_{L^p} \leq C(\|\nabla f\|_{L^2} + \|f\|_{L^2})^{1-2/p} \|f\|_{L^2}^{2/p}$ for every $f \in H^1(\Omega)$.

is not essential. However, as the prefactor is essential for the global / local estimates (31) and (32), we note that our argument in Steps 1-9 is based only on the control of the Ginzburg-Landau density e_η and therefore, the estimates (31) and (32) hold true by changing η by $\tilde{\eta} = c_0\eta$. \square

In the previous proof, we used the following global uniform estimate for solutions of the standard Ginzburg-Landau equation in a cell, which was obtained in [26] (with slightly different notation, using ε instead of η). Let $\mathcal{D} \subset \mathbb{R}^2$ be a Lipschitz bounded domain. For a sequence / family $\eta \rightarrow 0$ and $\beta \in (0, 1)$, we consider $\mathcal{D}_\eta := \eta^\beta \mathcal{D}$ a cell of size η^β , τ be a unit tangent vector field a.e. on $\partial\mathcal{D}_\eta$ and a boundary data $g_\eta \in H^1(\partial\mathcal{D}_\eta; \mathbb{R}^2)$. For every $u \in H^1(\mathcal{D}_\eta; \mathbb{R}^2)$, we recall the Ginzburg-Landau energy density $e_\eta(u)$ defined in (33).

Proposition 3.2 ([26], Corollary 2) *For a sequence / family $\eta \rightarrow 0$, let $u_\eta \in H^1(\mathcal{D}_\eta; \mathbb{R}^2)$ be a minimizer of*

$$\min_{u=g_\eta} \int_{\mathcal{D}_\eta} e_\eta(u) dx.$$

Let $\kappa = \kappa(\eta) \ll |\log \eta|$ as $\eta \rightarrow 0$. Assume that

$$\int_{\partial\mathcal{D}_\eta} |\partial_\tau g_\eta|^2 + \frac{1}{\eta^2} (1 - |g_\eta|^2)^2 d\mathcal{H}^1 \leq \frac{\kappa}{\eta^\beta} \quad \text{and} \quad \int_{\mathcal{D}_\eta} e_\eta(u_\eta) dx \leq \kappa. \quad (44)$$

Then there exists $0 < \tilde{\beta} < \frac{1-\beta}{6}$ such that for the members of the sequence / family with $0 < \eta \leq \eta_0$,

$$\sup_{\mathcal{D}_\eta} \left| |u_\eta|^2 - 1 \right| \leq C\eta^{\tilde{\beta}},$$

where $C > 0$ and $\eta_0 > 0$ depend only on the geometry of \mathcal{D} . In particular, $\deg(g_\eta; \partial\mathcal{D}_\eta) = 0$.

4 Second order Γ -convergence in the case of \mathbb{S}^1 -valued maps

In this section we start with the setting of \mathbb{S}^1 -valued maps motivated by the previous section, and perform a Γ -development at second order of $E_{\varepsilon, \eta}$ restricted to such \mathbb{S}^1 -valued maps. The main benefit is seen in the following lifting argument, which simplifies the analysis and geometry of the problem:

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ regular domain. If $u \in H^1(\Omega; \mathbb{S}^1)$ then there exists a lifting $\phi \in H^1(\Omega; \mathbb{R})$ with $u = e^{i\phi}$ and ϕ is unique up to an additive constant in $2\pi\mathbb{Z}$. Furthermore, for every small $\varepsilon > 0$ and $\eta > 0$,*

$$E_{\varepsilon, \eta}(u) \stackrel{(3)}{=} E_\varepsilon^{KS}(u) = \int_\Omega |\nabla \phi|^2 dx + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} \sin^2(\phi - g) d\mathcal{H}^1 =: \mathcal{G}_\varepsilon(\phi), \quad (45)$$

where g is a lifting of the unit tangent vector field τ at $\partial\Omega$, i.e.,

$$e^{ig} = \tau = i\nu \quad \text{on} \quad \partial\Omega \quad (46)$$

and g is continuous except at a point of $\partial\Omega$.

Proof. For the existence and uniqueness of the lifting ϕ of u in Ω , we refer to Bethuel-Zheng [9]. For the existence of g , we note that τ has winding number 1 around $\partial\Omega$ as Ω is simply connected, and hence no continuous $g : \partial\Omega \rightarrow \mathbb{R}$ with $e^{ig} = \tau$ can exist. However, if $\partial\Omega$ is $C^{1,1}$, we can choose g to be locally Lipschitz except at one point of $\partial\Omega$ where it jumps by -2π (see e.g. Ignat [20] for the theory of BV liftings). Clearly, the curvature \varkappa of $\partial\Omega$ is given by the absolutely continuous part of the derivative of g (as a BV function), i.e.,

$$\varkappa = (\partial_\tau g)_{ac} \quad \text{and} \quad \int_{\partial\Omega} \varkappa d\mathcal{H}^1 = 2\pi$$

which is in fact the Gauß-Bonnet formula for the boundary of a simply connected domain. As $|\nabla u| = |\nabla \phi|$ in Ω and $u \cdot \nu = \sin(g - \phi)$ on $\partial\Omega$, the equality of $E_{\varepsilon,\eta}(u)$ and $\mathcal{G}_\varepsilon(\phi)$ is straightforward. \square

The functional \mathcal{G}_ε has been studied before: compactness and a first order Γ -convergence result were established by the second author in [32], while the second order lower bound was shown for the restricted case of minimizers in [31] and under a stronger a priori single multiplicity assumption in [33] (which is true for critical points, see [6]). We use a different approach here that leads to new and significantly improved results and proofs: For the first order compactness, unlike the proof in [32], our new approach incorporates ideas of Garroni-Müller [17] so that it does not require the fairly elaborate rearrangement inequality for *functions* from Garsia-Rodemich [18], but instead uses a much more straightforward rearrangement inequality for *sets* from Alberti-Bouchitté-Seppecher [2].

For our more precise second order results, we employ a new method, adapting a co-area argument of Sandier [38], see Proposition 4.10 below. We can avoid the use of a “ball construction” by directly working with the one-dimensional nonlocal energy (see (55) below), and directly obtain some single multiplicity results from the energy bounds. A further central new step is Proposition 4.16, a comparison argument inspired by Colliander-Jerrard [14] that yields the second order lower bounds by purely energy methods. We can thus completely avoid the PDE arguments used in [33] or [11]. We also find new strong compactness results on the level of the functions (in Ω) that are in addition to the typical compactness of Jacobians for Ginzburg-Landau theory. These results are essential to show compactness of the magnetisation in a dimension reduction argument in our work [25].

We now state the main compactness and Γ -convergence results for \mathcal{G}_ε defined at (45). The proof requires several steps and is completed at the end of this section. Recall that for our compactness results, we often label sequences with a continuous parameter ε , which means that we start with a fixed sequence $\varepsilon_k \rightarrow 0$ and then take further subsequences of this sequence.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ regular domain.*

1. $L^p(\partial\Omega)$ -compactness and first order lower bound. *Let $(\phi_\varepsilon)_\varepsilon$ be a sequence / family in $H^1(\Omega; \mathbb{R})$ such that*

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \mathcal{G}_\varepsilon(\phi_\varepsilon) < \infty.$$

*Then there is a sequence / family $(z_\varepsilon)_\varepsilon$ of integers such that $(\phi_\varepsilon - \pi z_\varepsilon)_\varepsilon$ converges (up to a subsequence) **strongly** in $L^p(\partial\Omega)$ to a limit ϕ_0 such that $\phi_0 - g \in BV(\partial\Omega; \pi\mathbb{Z})$ with g given in (46) and*

$$\partial_\tau \phi_0 = \varkappa - \pi \sum_{j=1}^N d_j \delta_{a_j}, \quad a_j \in \partial\Omega \text{ distinct points, } d_j \in \mathbb{Z} \setminus \{0\} \quad \text{with} \quad \sum_{j=1}^N d_j = 2$$

and $\partial_\tau \phi_\varepsilon \rightarrow \partial_\tau \phi_0$ in $W^{-1,p}(\partial\Omega)$ for every $1 \leq p < \infty$. Furthermore, we have the following first order lower bound

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \mathcal{G}_\varepsilon(\phi_\varepsilon) \geq |\partial_\tau \phi_0 - \varkappa|(\partial\Omega) = \pi \sum_{j=1}^N |d_j|.$$

2. $W^{1,q}(\Omega)$ weak compactness and second order lower bound. Let $(\phi_\varepsilon)_\varepsilon$ be a sequence / family in $H^1(\Omega; \mathbb{R})$ satisfying the convergence at point 1. with the limit ϕ_0 on $\partial\Omega$ as $\varepsilon \rightarrow 0$. If additionally we assume that

$$\limsup_{\varepsilon \rightarrow 0} \left(\mathcal{G}_\varepsilon(\phi_\varepsilon) - \pi |\log \varepsilon| \sum_{j=1}^N |d_j| \right) < \infty, \quad (47)$$

then $d_j \in \{\pm 1\}$ for all $j = 1, \dots, N$, $(\nabla \phi_\varepsilon)_\varepsilon$ converges weakly (for a subsequence) in $L^q(\Omega; \mathbb{R}^2)$ for any $q \in [1, 2)$ to $\nabla \hat{\phi}_0$, where $\hat{\phi}_0 \in W^{1,q}(\Omega)$ is an extension (not necessarily harmonic) of ϕ_0 to Ω . The following second order lower bound holds for the sequence / family $\varepsilon \rightarrow 0$:

$$\liminf_{\varepsilon \rightarrow 0} (\mathcal{G}_\varepsilon(\phi_\varepsilon) - \pi N |\log \varepsilon|) \geq W_\Omega(\{(a_j, d_j)\}) + N\gamma_0, \quad (48)$$

where W_Ω is the renormalized energy defined in (12) and $\gamma_0 = \pi \log \frac{e}{4\pi}$.

3. Upper bound construction: Let $\phi_0 : \partial\Omega \rightarrow \mathbb{R}$ be such that $\partial_\tau \phi_0 = \varkappa - \pi \sum_{j=1}^N d_j \delta_{a_j}$, $d_j \in \mathbb{Z} \setminus \{0\}$ with $\sum_j d_j = 2$, $e^{i\phi_0} \cdot \nu = 0$ in $\partial\Omega \setminus \{a_1, \dots, a_N\}$. Then for every $\varepsilon > 0$ small, there exists $\hat{\phi}_\varepsilon \in H^1(\Omega; \mathbb{R})$ such that $\hat{\phi}_\varepsilon \rightarrow \phi_0$ in $L^p(\partial\Omega)$ and $\hat{\phi}_\varepsilon \rightarrow \phi_*$ in $L^p(\Omega)$ for every $p \in [1, \infty)$ where ϕ_* is the harmonic extension of ϕ_0 given in Definition 1.3 and

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \mathcal{G}_\varepsilon(\hat{\phi}_\varepsilon) = \pi \sum_{j=1}^N |d_j|. \quad (49)$$

If in addition $d_j = \pm 1$ for all j , then we have additionally

$$\limsup_{\varepsilon \rightarrow 0} \left(\mathcal{G}_\varepsilon(\hat{\phi}_\varepsilon) - N\pi \log \frac{1}{\varepsilon} \right) = W_\Omega(\{(a_j, d_j)\}) + N\gamma_0. \quad (50)$$

Our first steps towards the analysis of \mathcal{G}_ε are flattening the boundary and getting rid of the effect of g . For the first order in the energy expansion, this can be done as in Alberti-Bouchitté-Seppecher [3], by locally flattening the boundary with maps of small isometry defect, requiring only C^1 smoothness of the boundary. In order to obtain slightly more precise estimates, we use a locally conformal flattening, requiring $C^{1,1}$ smoothness. We introduce the following notation for half disks and intervals centred at the origin, and use it throughout this section:

$$B_r^+ = \{(x_1, x_2) \in \mathbb{R}^2 : |x| < r, x_2 > 0\} \quad \text{and} \quad I_r = (-r, r), \quad r > 0,$$

where I_r is the straight part of the boundary of B_r^+ . We also denote by

$$\mathbb{R}_+^2 = \mathbb{R} \times (0, \infty).$$

The localisation lemma is proved in the following:

Lemma 4.3 *Let $\Omega \subset \mathbb{R}^2$ be a simply connected $C^{1,1}$ domain. There exist constants $c_1 = c_1(\Omega) > 0$ and $r_0 = r_0(\Omega) \in (0, 1)$ such that for any $a \in \partial\Omega$, we can find a C^1 map $\Psi_a : \overline{B_{2r_0}^+} \rightarrow \overline{\Omega}$ with the following properties:*

- (a) $\Psi_a : \overline{B_{r_0(1+c_1r_0 \log \frac{1}{r_0})}^+} \rightarrow \overline{\Psi_a(B_{r_0(1+c_1r_0 \log \frac{1}{r_0})}^+)}$ is a conformal diffeomorphism with $\Psi_a(0) = a$;
- (b) For any $\phi \in H^1(\Omega; \mathbb{R})$, setting $\psi = \phi \circ \Psi_a$, we have for any $r < r_0$:

$$\int_{B_{r(1-c_1r \log \frac{1}{r})}^+} |\nabla \psi|^2 dx \leq \int_{B_r(a) \cap \Omega} |\nabla \phi|^2 dx \leq \int_{B_{r(1+c_1r \log \frac{1}{r})}^+} |\nabla \psi|^2 dx$$

and

$$(1-c_1r \log \frac{1}{r}) \int_{I_{r(1-c_1r \log \frac{1}{r})}} \sin^2 \psi d\mathcal{H}^1 \leq \int_{\partial\Omega \cap B_r(a)} \sin^2 \phi d\mathcal{H}^1 \leq (1+c_1r \log \frac{1}{r}) \int_{I_{r(1+c_1r \log \frac{1}{r})}} \sin^2 \psi d\mathcal{H}^1.$$

Proof. For a point $a \in \partial\Omega$ with the unit tangent vector τ_a at $\partial\Omega$, the Riemann mapping theorem yields existence of a conformal map $\Psi_a : \mathbb{R}_+^2 \rightarrow \Omega$ such that $\Psi_a(0) = a$ and $\Psi'_a(0) = \tau_a$ where Ψ'_a denotes the complex differential of Ψ_a . By the Kellogg-Warschawski theorem (see Pommerenke [37, Theorem 3.5]), it follows that Ψ'_a extends to a Dini continuous map up to the boundary $\partial\mathbb{R}_+^2 = \mathbb{R} \times \{0\}$. Near the origin, it has a modulus of continuity $\omega(\delta) = C\delta \log \frac{1}{\delta}$ for $\delta > 0$ small, where $C > 0$ denotes here and in the following a constant depending only on Ω that can change from line to line. In particular,

$$|\Psi'_a(z) - \tau_a| \leq C|z| \log \frac{1}{|z|}, \quad \text{for } |z| \text{ small.}$$

By complex integration, we deduce $|\Psi_a(z) - a - \tau_a z| \leq C|z|^2 \log \frac{1}{|z|}$ for $|z|$ small. This implies that for $r < r_0$ sufficiently small,

$$\Psi_a(B_{r(1-c_1r \log \frac{1}{r})}^+) \subset B_r(a) \cap \Omega \subset \Psi_a(B_{r(1+c_1r \log \frac{1}{r})}^+).$$

Together with conformal invariance of the Dirichlet integral this implies the first part of claim (b). The second part follows from the same inclusion together with our bounds on $|\Psi'_a(z) - \tau_a|$. \square

For an open set $G \subset \mathbb{R}_+^2$ and $\psi : G \rightarrow \mathbb{R}$, we define the localised functionals

$$\hat{F}_\varepsilon^{(g)}(\psi; G) := \int_G |\nabla \psi|^2 dx + \frac{1}{2\pi\varepsilon} \int_{\overline{G} \cap (\mathbb{R} \times \{0\})} \sin^2(\psi(\cdot, 0) - g) dx_1, \quad (51)$$

where g stands here for a function defined on $\overline{G} \cap (\mathbb{R} \times \{0\})$. Usually, this is the lifting of the tangent vector field defined in (46), composed with the change of variables in Lemma 4.3.

Usually, we integrate over sets of the type $G = B_r^+$ or $G = B_r^+ \setminus B_s^+$, where the corresponding boundary integral is over one or two intervals. We can compare $\hat{F}_\varepsilon^{(g)}$ and the special case $\hat{F}_\varepsilon^{(0)}$ of zero boundary g by subtracting a suitable harmonic extension:

Lemma 4.4 *Let g be a Lipschitz function in $C^{0,1}((-1, 1))$ and $\psi : B_1^+ \rightarrow \mathbb{R}$. For every $r \in (0, 1)$, we define $\tilde{g}_r : \mathbb{R} \rightarrow \mathbb{R}$ by*

$$\tilde{g}_r(x_1) = \begin{cases} g(x_1) & \text{if } |x_1| \leq r, \\ g(\frac{x_1^2}{r}) & \text{if } |x_1| > r \end{cases}$$

and let $\hat{g}_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be the unique bounded harmonic extension of \tilde{g}_r to \mathbb{R}_+^2 . Then we have for every $1 \leq s < \infty$ and $r \in (0, 1)$, with a constant C depending only on s and the Lipschitz constant $\|g'\|_{L^\infty}$ of g :

- (i) $\|\nabla \hat{g}_r\|_{L^s(B_r^+)} \leq Cr^{\frac{2}{s}}$;
- (ii) $\|\partial_{x_2} \hat{g}_r(\cdot, 0)\|_{L^s(I_r)} \leq Cr^{\frac{1}{s}}$;
- (iii) $\nu \cdot \nabla \hat{g}_r = 0$ on $\partial B_r(0) \cap \mathbb{R}_+^2$ and ν is the unit outer normal vector to $\partial B_r(0) \cap \mathbb{R}_+^2$.

If we set

$$A(\psi; r) = \left| \hat{F}_\varepsilon^{(g)}(\psi; B_r^+) - \hat{F}_\varepsilon^{(0)}(\psi - \hat{g}_r; B_r^+) \right|, \quad r \in (0, 1),$$

then for every $p \in (1, \infty)$ and $r \in (0, 1)$, we have, with constants C depending on p and the Lipschitz constant of g ,

$$A(\psi; r) \leq C \|\nabla \psi\|_{L^p(B_r^+)} r^{2 - \frac{2}{p}} + Cr^2; \quad (52)$$

in particular, for $p = 2$,

$$A(\psi; r) \leq Cr \left(1 + \sqrt{\hat{F}_\varepsilon^{(g)}(\psi; B_r^+)} \right). \quad (53)$$

Furthermore,

$$A(\psi; r) \leq C \|\psi(\cdot, 0)\|_{L^p(I_r)} r^{1 - \frac{1}{p}} + Cr^2, \quad p \in (1, \infty). \quad (54)$$

Proof. In order to prove (i) and (ii), we start by noting that $\|\tilde{g}'_r\|_{L^s(\mathbb{R})} \leq Cr^{\frac{1}{s}} \|g'\|_{L^\infty(-1, 1)}$ for every $s \in [1, \infty]$ and $r \in (0, 1)$ for some universal constant $C > 0$ (with the convention that $1/\infty = 0$). It is known that $x_1 \mapsto \partial_{x_2} \hat{g}_r(x_1, 0)$ represents the Dirichlet-to-Neumann operator applied to \tilde{g}_r that is given by the Hilbert transform H of the derivative \tilde{g}'_r . As $H : L^s(\mathbb{R}) \rightarrow L^s(\mathbb{R})$ is a bounded linear operator for $s \in (1, \infty)$, the estimates on $\|\tilde{g}'_r\|_{L^s(\mathbb{R})}$ yield (ii); for the case $s = 1$ we use the Hölder inequality and the embedding $L^2(I_r) \subset L^1(I_r)$. As $\partial_{x_j} \hat{g}_r$ is harmonic in \mathbb{R}_+^2 for $j = 1, 2$, the standard theory of harmonic functions, see e.g. Axler et al. [5, Theorem 7.6], implies (also for $s = 1$) that $\|\partial_{x_j} \hat{g}_r(\cdot, x_2)\|_{L^s(\mathbb{R})} \leq C \|\partial_{x_j} \hat{g}_r(\cdot, 0)\|_{L^s(\mathbb{R})}$ for every $x_2 > 0$. Integrating on the strip $\mathbb{R} \times (0, r)$, we deduce the desired estimate in (i). For proving (iii), note that $\hat{g}_r(x) = \hat{g}_r\left(\frac{xr^2}{|x|^2}\right)$ in \mathbb{R}_+^2 because uniqueness of the bounded harmonic extension implies invariance under the inversion at the circle $\partial B_r(0)$ (satisfied by the boundary data \tilde{g}_r). Then differentiating in radial direction and comparing both sides on the circle $\partial B_r(0)$ yield the claim (iii).

For the claims on $A(\psi; r)$ note that

$$A(\psi; r) = \left| \int_{B_r^+} (|\nabla \psi|^2 - |\nabla(\psi - \hat{g}_r)|^2) dx \right| = \left| \int_{B_r^+} 2\nabla \psi \cdot \nabla \hat{g}_r - |\nabla \hat{g}_r|^2 dx \right|$$

Now $\int_{B_r^+} |\nabla \hat{g}_r|^2 dx = O(r^2)$ by (i). Furthermore, we have by Hölder's inequality and (i) applied with $\frac{1}{s} = 1 - \frac{1}{p}$

$$\left| \int_{B_r^+} 2\nabla \psi \cdot \nabla \hat{g}_r dx \right| \leq Cr^{2 - \frac{2}{p}} \left(\int_{B_r^+} |\nabla \psi|^p dx \right)^{\frac{1}{p}},$$

which yields (52). For the final claim (54), integration by parts and Hölder's inequality applied with $\frac{1}{s} = 1 - \frac{1}{p}$, combined with (ii) and (iii) imply

$$\left| \int_{B_r^+} \nabla \psi \cdot \nabla \hat{g}_r dx \right| = \left| \int_{I_r} \psi \partial_{x_2} \hat{g}_r d\mathcal{H}^1 \right| \leq C \left(\int_{I_r} |\psi|^p d\mathcal{H}^1 \right)^{\frac{1}{p}} r^{1 - \frac{1}{p}}.$$

□

After reducing the study of $\hat{F}_\varepsilon^{(g)}$ defined at (51) to the special case $\hat{F}_\varepsilon^{(0)}$ thanks to the above lemma, we further reduce the analysis of the two-dimensional energy functional $\hat{F}_\varepsilon^{(0)}$ to a one-dimensional (nonlocal) functional defined for functions $\varphi : I \rightarrow \mathbb{R}$ for an interval $I \subset \mathbb{R}$:

$$F_\varepsilon(\varphi; I) = \frac{1}{2\pi} \int_{I \times I} \left| \frac{\varphi(s) - \varphi(t)}{s - t} \right|^2 ds dt + \frac{1}{2\pi\varepsilon} \int_I \sin^2 \varphi dt. \quad (55)$$

Lemma 4.5 *If $\psi : B_r^+ \rightarrow \mathbb{R}$ is an H^1 function in B_r^+ for some $r > 0$, then*

$$\hat{F}_\varepsilon^{(0)}(\psi; B_r^+) \geq F_\varepsilon(\psi(\cdot, 0); I_r)$$

where the right hand side is given by the trace $\psi(\cdot, 0)$ of ψ on the interval $I_r = \partial B_r^+ \cap (\mathbb{R} \times \{0\})$.

Proof. For the half-space \mathbb{R}_+^2 (corresponding to $r = \infty$), the Dirichlet integral in $\hat{F}_\varepsilon^{(0)}$ and the nonlocal functional in F_ε can be compared using a standard Fourier space argument:

$$\int_{\mathbb{R}_+^2} |\nabla \psi|^2 dx \geq \|\psi(\cdot, 0)\|_{H^{1/2}(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} \left| \frac{\psi(x_1, 0) - \psi(\tilde{x}_1, 0)}{x_1 - \tilde{x}_1} \right|^2 dx_1 d\tilde{x}_1.$$

The bounded domain version in B_r^+ can be deduced by inversion at ∂B_r as in Lemma 4.4 (see Alberti-Bouchitté-Seppecher [3, Corollary 6.4] for details). The constant 1 in the above inequality is optimal (see e.g. [3, Remark 6.5]). □

The following rearrangement inequality is essential in the proof of the compactness result for the functional F_ε :

Lemma 4.6 *Let $I \subset \mathbb{R}$ be a bounded interval and $A, B \subset I$ be measurable sets of positive measure with $A \cap B = \emptyset$. Set $P = I \setminus (A \cup B)$. Then*

$$\int_A \int_B \frac{1}{|s - t|^2} ds dt \geq \log \frac{(|I| - |A|)(|I| - |B|)}{|I|(|I| - |A| - |B|)} \geq \log \frac{|B|}{|I|} + \log \frac{|A|}{|I|} - \log \frac{|I| - |A| - |B|}{|I|}. \quad (56)$$

If additionally $|B| \geq c|I|$ for some $c \in (0, 1)$, we have

$$\int_A \int_B \frac{1}{|s - t|^2} ds dt \geq \log \left(1 + \frac{c|A|}{|P|} \right). \quad (57)$$

Proof. By a simple rearrangement lemma (see [2, Lemme 2]),

$$\int_A \int_B \frac{1}{|s - t|^2} ds dt \geq \int_0^{|A|} \int_{|I| - |B|}^{|I|} \frac{1}{|s - t|^2} ds dt = \log \frac{(|I| - |A|)(|I| - |B|)}{|I|(|I| - |A| - |B|)}$$

and the last part of (56) follows using that $|I| - |A| \geq |B|$ and $|I| - |B| \geq |A|$. We note that

$$\frac{(|I| - |A|)(|I| - |B|)}{|I|(|I| - |A| - |B|)} = 1 + \frac{|B|}{|I|} \frac{|A|}{|I| - |A| - |B|} \geq 1 + c \frac{|A|}{|P|}$$

so (57) now follows by the monotonicity of the logarithm. □

Now we prove a first compactness result for functional F_ε in (55) in the weak L^p topology:

Proposition 4.7 *Let $I \subset \mathbb{R}$ be a bounded open interval and $M > 0$. Then there exists $\varepsilon_0 > 0$ such that for every sequence / family $(\varphi_\varepsilon)_\varepsilon$ of functions such that the functional F_ε defined in (55) satisfies $F_\varepsilon(\varphi_\varepsilon; I) \leq M|\log \varepsilon|$, there exists a sequence / family $(k_\varepsilon)_\varepsilon$ of integers such that $(\varphi_\varepsilon - \pi k_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ is bounded in $L^p(I)$ for every $p \in [1, \infty)$.*

Proof. We assume without loss of generality that $|I| = 1$ (otherwise, one rescales by the length of the interval I which implies only a change of the parameter ε in the functional F_ε as the nonlocal part of F_ε is scaling invariant). We denote

$$a \wedge b = \min(a, b) \quad \text{and} \quad a \vee b = \max(a, b), \quad a, b \in \mathbb{R}.$$

A particular case. We assume that $|\{\varphi_\varepsilon < 0\}| > \frac{1}{4}$ for every small $\varepsilon > 0$. We want to prove that the positive part $(\varphi_\varepsilon)_+ = \varphi_\varepsilon \vee 0$ is uniformly bounded in $L^p(I)$ for every $p \in [1, \infty)$ as $\varepsilon \rightarrow 0$. For every ε , we use the truncations of φ_ε between $k\pi$ and $(k+1)\pi$ given by

$$T_k \varphi_\varepsilon = (\varphi_\varepsilon \wedge (k+1)\pi) \vee k\pi, \quad \text{for every } k \in \mathbb{Z}. \quad (58)$$

Fix a small $\gamma > 0$. We consider the following sets

$$A_k^\varepsilon = \{T_k \varphi_\varepsilon > (k+1)\pi - \gamma\}, \quad B_k^\varepsilon = \{T_k \varphi_\varepsilon < k\pi + \gamma\}, \quad k \geq 0$$

together with

$$\alpha_k^\varepsilon = |A_k^\varepsilon| \quad \text{and} \quad \rho_k^\varepsilon = 1 - |A_k^\varepsilon \cup B_k^\varepsilon|.$$

Note that $\{\varphi_\varepsilon < 0\} \subset B_k^\varepsilon$ for $k \geq 0$ so that $|B_k^\varepsilon|/|I| > 1/4$ (by the assumption of this case); also $(\alpha_k^\varepsilon)_{k \geq 0}$ is a non-increasing sequence and we have the estimate

$$M|\log \varepsilon| \geq F_\varepsilon(\varphi_\varepsilon; I) \geq \frac{1}{2\pi\varepsilon} \int_{I \setminus (A_k^\varepsilon \cup B_k^\varepsilon)} \sin^2 \varphi_\varepsilon \, dt \geq \frac{\rho_k^\varepsilon}{C(\gamma)\varepsilon}, \quad (59)$$

i.e., $\rho_k^\varepsilon \leq C(\gamma)M\varepsilon|\log \varepsilon|$. Moreover, if $\alpha_k^\varepsilon > 0$, then $\rho_k^\varepsilon > 0$ (otherwise, we would have $|A_k^\varepsilon| + |B_k^\varepsilon| = 1$ for the nontrivial partition $A_k^\varepsilon \neq \emptyset$ and $B_k^\varepsilon \neq \emptyset$ of I which leads to a contradiction with the fact that $H^{1/2}$ -functions have no jump discontinuities) and by Lemma 4.6, we obtain:

$$M \geq \frac{1}{|\log \varepsilon|} F_\varepsilon(T_k \varphi_\varepsilon; I) \geq \frac{2}{2\pi|\log \varepsilon|} \int_{A_k^\varepsilon} \int_{B_k^\varepsilon} \frac{|\varphi_\varepsilon(s) - \varphi_\varepsilon(t)|^2}{|s-t|^2} \, ds dt \geq \frac{(\pi - 2\gamma)^2 \log(1 + \frac{\alpha_k^\varepsilon}{4\rho_k^\varepsilon})}{\pi|\log \varepsilon|}. \quad (60)$$

Now we decompose the set of non-negative integers:

$$\mathbb{N} = \mathcal{K}_\varepsilon \cup \mathcal{N}_\varepsilon, \quad \mathcal{K}_\varepsilon := \{k \geq 0 : \alpha_k^\varepsilon < \varepsilon^{\frac{1}{3}}\}, \quad \mathcal{N}_\varepsilon := \{k \geq 0 : \alpha_k^\varepsilon \geq \varepsilon^{\frac{1}{3}}\}.$$

Subcase i). Assume that $\mathcal{N}_\varepsilon \neq \emptyset$. Note that for $\varepsilon \leq \varepsilon(M, \gamma)$, we have for every $k \in \mathcal{N}_\varepsilon$ that $\log(1 + \frac{\alpha_k^\varepsilon}{4\rho_k^\varepsilon}) > \frac{1}{3}|\log \varepsilon|$ because $0 < \rho_k^\varepsilon \leq C(\gamma)M\varepsilon|\log \varepsilon|$. Let $k_0^\varepsilon = \sup \mathcal{N}_\varepsilon \in \mathbb{N} \cup \{+\infty\}$. We claim that $(k_0^\varepsilon)_\varepsilon$ is uniformly bounded in ε . Indeed, as $(\alpha_k^\varepsilon)_{k \geq 0}$ is non-increasing, we know that $\mathcal{N}_\varepsilon = \{0, 1, \dots, k_0^\varepsilon\}$, i.e., $\alpha_k^\varepsilon \geq \varepsilon^{\frac{1}{3}}$ for every $0 \leq k \leq k_0^\varepsilon$, so that

$$M \geq \frac{1}{|\log \varepsilon|} \sum_{k=0}^{k_0^\varepsilon} F_\varepsilon(T_k \varphi_\varepsilon; I) \stackrel{(60)}{\geq} \sum_{k=0}^{k_0^\varepsilon} \frac{(\pi - 2\gamma)^2 \log(1 + \frac{\alpha_k^\varepsilon}{4\rho_k^\varepsilon})}{\pi|\log \varepsilon|} \geq \frac{k_0^\varepsilon}{3}$$

which proves our claim. Let $k_0 = \limsup_{\varepsilon \in (0, \varepsilon_0]} k_0^\varepsilon < \infty$. In particular, for $\varepsilon \leq \varepsilon_0$, $\alpha_{k_0+1}^\varepsilon < \varepsilon^{1/3}$. Now the one-dimensional Moser-Trudinger inequality (see Taylor [41, Proposition 4.2]; compare [32, Lemma 2.10]) implies the existence of constants $c_1, c_2 > 0$ such that

$$\int_{\{\varphi_\varepsilon > (k_0+2)\pi - \gamma\}} \exp \frac{c_1(\varphi_\varepsilon - (k_0+2)\pi + \gamma)^2}{M|\log \varepsilon|} dt \leq c_2 \alpha_{k_0+1}^\varepsilon (\leq c_2 \varepsilon^{1/3}),$$

so that for every $k \geq k_0 + 1$, by definition of α_k^ε it follows

$$\alpha_k^\varepsilon \exp \left(c_1 \pi^2 \frac{(k - k_0 - 1)^2}{M|\log \varepsilon|} \right) \leq \int_{\{\varphi_\varepsilon > (k_0+2)\pi - \gamma\}} \exp \frac{c_1(\varphi_\varepsilon - (k_0+2)\pi + \gamma)^2}{M|\log \varepsilon|} dt \leq c_2 \exp \left(-\frac{1}{3} |\log \varepsilon| \right),$$

yielding for $k \geq k_0 + 1$

$$\alpha_k^\varepsilon \leq c_2 \exp \left(-\frac{1}{3} |\log \varepsilon| - c_1 \pi^2 \frac{(k - k_0 - 1)^2}{M|\log \varepsilon|} \right) \leq c_2 \exp \left(-\frac{2\pi\sqrt{c_1}(k - k_0 - 1)}{\sqrt{3M}} \right), \quad (61)$$

where we used $a^2 + b^2 \geq 2ab$ in the argument of the exponential. Therefore, we obtain for the positive part of φ_ε and $p \in [1, \infty)$:

$$\begin{aligned} \int_I |(\varphi_\varepsilon)_+|^p dt &= \sum_{k \geq 0} \int_{\{k\pi < \varphi_\varepsilon < (k+1)\pi\}} |T_k \varphi_\varepsilon|^p dt \leq \pi^p + \sum_{k \geq 1} \int_{A_{k-1}^\varepsilon} |T_k \varphi_\varepsilon|^p dt \\ &\leq \pi^p + C \sum_{k=0}^{\infty} (k+1)^p \alpha_k^\varepsilon \\ &\stackrel{(61)}{\leq} \tilde{C} \sum_{k=0}^{k_0} (k+1)^p + \tilde{C} \sum_{k \geq k_0+1} (k+1)^p \exp \left(-\frac{2\pi\sqrt{c_1}(k - k_0 - 1)}{\sqrt{3M}} \right), \end{aligned}$$

which is bounded independently of ε , yielding the claimed L^p bound of $(\varphi_\varepsilon)_+$.

Subcase ii). Assume that $\mathcal{N}_\varepsilon = \emptyset$, i.e., $\alpha_k^\varepsilon < \varepsilon^{\frac{1}{3}}$ for every $k \geq 0$. Then by (61), we deduce that α_k^ε satisfies an exponential decay for every $k \geq 1$ and the same argument as in Subcase i) yields the L^p bound of $(\varphi_\varepsilon)_+$.

The general case. For a measurable function $\varphi_\varepsilon : I \rightarrow \mathbb{R}$ with $|I| = 1$, there exists an integer k_ε such¹³ that $|\{\varphi_\varepsilon < k_\varepsilon \pi\}| > \frac{1}{4}$ and $|\{\varphi_\varepsilon > (k_\varepsilon - 1)\pi\}| > \frac{1}{4}$. By considering $\tilde{\varphi}_\varepsilon := \varphi_\varepsilon - k_\varepsilon \pi$, we deduce that $F_\varepsilon(\tilde{\varphi}_\varepsilon; I) = F_\varepsilon(\varphi_\varepsilon; I)$ and by the particular case discussed before, we have that the positive parts of the sequence / family $(\tilde{\varphi}_\varepsilon)_\varepsilon$ are bounded in L^p . The same argument yields that the sequence / family of positive parts of $-(\pi + \tilde{\varphi}_\varepsilon)$, i.e., $(-\pi - \tilde{\varphi}_\varepsilon)_+ = (\tilde{\varphi}_\varepsilon + \pi)_-$, is also bounded in L^p . Together these bounds yield the L^p bound of $(\tilde{\varphi}_\varepsilon)_\varepsilon$. \square

Remark 4.1 From (61) we can actually deduce a bound not just in L^p , but in a certain Orlicz space. The type of Orlicz space (e^{cL} with a constant of order $\frac{1}{\sqrt{M}}$) is essentially optimal by an example presented in [32].

We can improve now the result in Proposition 4.7 by showing the compactness in **strong** L^p topology and derive a first order lower bound for the functional F_ε defined in (55).

¹³One can consider the smallest $k_\varepsilon \in \mathbb{Z}$ such that $|\{\varphi_\varepsilon < k_\varepsilon \pi\}| > \frac{1}{4}$.

Proposition 4.8 *Let $I \subset \mathbb{R}$ be a bounded open interval and let $(\varphi_\varepsilon)_\varepsilon$ be a sequence / family of functions such that $F_\varepsilon(\varphi_\varepsilon; I) \leq M|\log \varepsilon|$ as $\varepsilon \rightarrow 0$ for some fixed $M > 0$. Then for a subsequence $\varepsilon \rightarrow 0$ (still denoted (φ_ε)), there exists a sequence $(k_\varepsilon)_\varepsilon$ of integers such that $\varphi_\varepsilon - k_\varepsilon \pi \rightarrow \varphi$ **strongly** in $L^p(I)$ for every $p \in [1, \infty)$, where φ is a piecewise constant function in $BV(I; \pi\mathbb{Z})$. Furthermore, every sequence / family (φ_ε) satisfying the above convergence as $\varepsilon \rightarrow 0$ yields the following energy lower bound at first order:¹⁴*

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_\varepsilon(\varphi_\varepsilon; I) \geq \|\varphi\|_{BV} = \sum_{t \in S(\varphi)} |\varphi(t+) - \varphi(t-)|, \quad (62)$$

where $S(\varphi)$ denotes the finite set of jumps of φ and $\varphi(t_\pm) \in \pi\mathbb{Z}$ the traces of φ at a jump t .

Proof. We may assume that $I = (0, 1)$ (by the same argument as in the proof of Proposition 4.7). We start by treating a particular case and then we prove the general case.

A particular case. Assume that φ_ε takes values into $[0, \pi]$ for every ε . We can then follow the argument of Alberti-Bouchitté-Seppecher [2]: Since (φ_ε) is uniformly bounded, then for a subsequence, we can assume that φ_ε is weakly* convergent in $L^\infty(I)$ to a function $\varphi : I \rightarrow [0, \pi]$. By the fundamental theorem of Young measures (see Ball [7] or Müller [36]), there exists a family of probability measures $\{\mu_t\}_{t \in I}$ (depending measurably on $t \in I$) over the range $[0, \pi]$ such that for any continuous test function $\zeta \in C^0([0, \pi] \times [0, 1])$,

$$\int_0^1 \zeta(\varphi_\varepsilon(t), t) dt \rightarrow \int_0^1 \int_0^\pi \zeta(z, t) d\mu_t(z) dt \quad \text{as } \varepsilon \rightarrow 0.$$

Choosing $\zeta(z, t) = \sin^2 z$ for every $z \in [0, \pi]$ and $t \in I$, since $F_\varepsilon(\varphi_\varepsilon; I) \leq M|\log \varepsilon|$, it follows that

$$0 = \lim_{\varepsilon \rightarrow 0} \int_0^1 \sin^2 \varphi_\varepsilon(t) dt = \int_0^1 \int_0^\pi \sin^2 z d\mu_t(z) dt,$$

and since $\sin^2 z > 0$ for $z \in (0, \pi)$, it follows that $\text{supp } \mu_t \subset \{0, \pi\}$ for almost every t , and we can write $\mu_t = \theta(t)\delta_0 + (1 - \theta(t))\delta_\pi$ for some measurable function $\theta : I \rightarrow [0, 1]$.

Claim: For a.e. $t \in I$, $\theta(t) \in \{0, 1\}$, i.e., μ_t is a Dirac measure.

To prove the claim, we first set

$$S = I \setminus \left\{ t_0 \in I : \lim_{r \rightarrow 0} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \theta(t) dt \text{ exists and belongs to } \{0, 1\} \right\}.$$

Setting $I_r(t) := (t - r, t + r) \subset I$ for $t \in I$ and small $r > 0$, the above definition implies for $t_0 \in S$ that there exist $\delta > 0$ and a decreasing sequence $r_k \rightarrow 0$ such that for all k ,

$$\frac{1}{2r_k} \int_{I_{r_k}(t_0)} \theta(s) ds \in (\delta, 1 - \delta). \quad (63)$$

Indeed, the function $r \mapsto \frac{1}{2r} \int_{I_r(t_0)} \theta(s) ds$ is continuous for small $r > 0$, which implies that $J_{t_0} := \left[\liminf_{r \rightarrow 0} \frac{1}{2r} \int_{I_r(t_0)} \theta(s) ds, \limsup_{r \rightarrow 0} \frac{1}{2r} \int_{I_r(t_0)} \theta(s) ds \right]$ is a closed interval $\subset [0, 1]$ that is not

¹⁴In the following, we use the BV -seminorm of a function $\varphi : \Omega \rightarrow \mathbb{R}$: $\|\varphi\|_{BV(\Omega)} = |D\varphi|(\Omega)$.

reduced to $\{0\}$ or $\{1\}$ for $t_0 \in S$. Therefore, there exists $\delta > 0$ such that $J_{t_0} \cap (\delta, 1 - \delta) \neq \emptyset$ which yields (63).

Step 1. We show that for every $t_0 \in S$ and any $\gamma \in (0, \pi)$, we have that there exists a decreasing sequence $r_k \rightarrow 0$ such that

$$\liminf_{k \rightarrow \infty} \liminf_{\varepsilon \rightarrow 0} \frac{|I_{r_k}(t_0) \cap \{\varphi_\varepsilon < \gamma\}|}{|I_{r_k}(t_0)|} > 0 \quad (64)$$

and

$$\liminf_{k \rightarrow \infty} \liminf_{\varepsilon \rightarrow 0} \frac{|I_{r_k}(t_0) \cap \{\varphi_\varepsilon > \pi - \gamma\}|}{|I_{r_k}(t_0)|} > 0. \quad (65)$$

Indeed, let $t_0 \in S$ with $\delta > 0$ and $r_k \rightarrow 0$ satisfying (63). We choose γ_1 and γ_2 such that $0 < \gamma_1 < \gamma_2 < \pi$ and we consider a test function $\zeta = \zeta(z)$ such that ζ is continuous on $[0, 1]$, $\zeta = 1$ on $[0, \gamma_1]$, $\zeta = 0$ on $[\gamma_2, \pi]$ and $0 < \zeta < 1$ on (γ_1, γ_2) . Then

$$\int_{I_r(t_0)} \zeta(\varphi_\varepsilon) dt \rightarrow \int_{I_r(t_0)} \int_0^\pi \zeta(z) d\mu_t(z) dt = \int_{I_r(t_0)} \theta(t) dt \quad \text{as } \varepsilon \rightarrow 0,$$

because $\{\mu_t\}_{t \in I_r(t_0)}$ is also the family of Young measures of the restriction $(\varphi_\varepsilon|_{I_r(t_0)})_\varepsilon$ for every small r . As $\int_{I_r(t_0)} \zeta(\varphi_\varepsilon) dt \leq |I_r(t_0) \cap \{\varphi_\varepsilon < \gamma_2\}|$ we deduce that

$$\int_{I_r(t_0)} \theta(t) dt \leq \liminf_{\varepsilon \rightarrow 0} |I_r(t_0) \cap \{\varphi_\varepsilon < \gamma_2\}|.$$

Setting $r := r_k$ and $\gamma := \gamma_2$, after dividing by $2r_k$ and passing to \liminf as $k \rightarrow \infty$, the desired inequality (64) holds true. The proof of (65) is analogous.

Step 2. We show that the set S is finite. For that, let (I_j) , $1 \leq j \leq J$ be a finite family of disjoint open intervals inside I such that $I_j \cap S \neq \emptyset$ for every j . For some $\gamma \in (0, \frac{\pi}{2})$, we consider the sets $A_j^\varepsilon = I_j \cap \{\varphi_\varepsilon < \gamma\}$ and $B_j^\varepsilon = I_j \cap \{\varphi_\varepsilon > \pi - \gamma\}$. For every j , there exists $t_j \in I_j \cap S$ so that by (64) and (65), there exist $r_j, \tilde{r}_j > 0$ small satisfying $I_{r_j}(t_j), I_{\tilde{r}_j}(t_j) \subset I_j$ and

$$\liminf_{\varepsilon \rightarrow 0} |A_j^\varepsilon| \geq \liminf_{\varepsilon \rightarrow 0} |I_{r_j}(t_j) \cap \{\varphi_\varepsilon < \gamma\}| > 0, \quad \liminf_{\varepsilon \rightarrow 0} |B_j^\varepsilon| \geq \liminf_{\varepsilon \rightarrow 0} |I_{\tilde{r}_j}(t_j) \cap \{\varphi_\varepsilon > \pi - \gamma\}| > 0.$$

Furthermore, since $\sin^2 z \geq c(\gamma) > 0$ for every $z \in (\gamma, \pi - \gamma)$, we deduce by (59) that $|I_j \setminus (A_j^\varepsilon \cup B_j^\varepsilon)| \leq C(\gamma, M)\varepsilon |\log \varepsilon|$. Applying the rearrangement result (56) with $|A_j^\varepsilon| \leq |I_j| - |B_j^\varepsilon|$, we obtain

$$\int_{A_j^\varepsilon} \int_{B_j^\varepsilon} \frac{1}{|t-s|^2} dt ds \geq \log \frac{|B_j^\varepsilon| |A_j^\varepsilon|}{|I_j| |I_j|} - \log \frac{|I_j| - |A_j^\varepsilon| - |B_j^\varepsilon|}{|I_j|} = \log(a_j^\varepsilon b_j^\varepsilon) - \log \rho_j^\varepsilon,$$

within the notation $a_j^\varepsilon := \frac{|A_j^\varepsilon|}{|I_j|}$, $b_j^\varepsilon := \frac{|B_j^\varepsilon|}{|I_j|}$ and $\rho_j^\varepsilon := \frac{|I_j \setminus (A_j^\varepsilon \cup B_j^\varepsilon)|}{|I_j|} = 1 - (a_j^\varepsilon + b_j^\varepsilon)$. Combined with the argument in (59) and (60), we obtain

$$\frac{1}{|\log \varepsilon|} F_\varepsilon(\varphi_\varepsilon; I_j) \geq \frac{(\pi - 2\gamma)^2}{\pi |\log \varepsilon|} \left(\log(a_j^\varepsilon b_j^\varepsilon) - \log \rho_j^\varepsilon + \frac{|I_j| c(\gamma)}{2(\pi - 2\gamma)^2 \varepsilon} \rho_j^\varepsilon \right).$$

Using $\liminf_{\varepsilon \rightarrow 0} a_j^\varepsilon b_j^\varepsilon > 0$ and¹⁵ $-\log \rho_j^\varepsilon + K \rho_j^\varepsilon \geq \log K + 1$ for $K = \frac{|I_j| c(\gamma)}{2(\pi - 2\gamma)^2 \varepsilon} \gg 1$, then summing over j we conclude that

$$M \geq \sum_{j=1}^J \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_\varepsilon(\varphi_\varepsilon; I_j) \geq J \liminf_{\varepsilon \rightarrow 0} \frac{(\pi - 2\gamma)^2}{\pi |\log \varepsilon|} (|\log \varepsilon| + \tilde{c}) = \frac{(\pi - 2\gamma)^2}{\pi} J \quad (66)$$

¹⁵If $K > 1$ and $f(\rho) = K\rho - \log \rho$ for $\rho \in (0, 1)$, then the minimum of f is achieved at $\rho_K = 1/K$ and $f(\rho) \geq f(\rho_K) = \log K + 1$.

for a constant \tilde{c} depending on γ and the product $\Pi_j |I_j|$. Therefore, J is bounded by M (up to a constant), hence S must be a finite set.

Proof of Claim: By the above considerations, we deduce that we can choose a representative θ defined on I such that for every $t \in I \setminus S$,

$$\theta(t) = \lim_{r \rightarrow 0} \frac{1}{2r} \int_{t-r}^{t+r} \theta(s) ds \in \{0, 1\}.$$

If $t_1 < t_2$ are two consecutive points in the (finite) set S , then θ satisfies the above condition for every $t \in (t_1, t_2)$ which implies that either $\theta \equiv 0$, or $\theta \equiv 1$ in the interval (t_1, t_2) . In other words, θ is a piecewise constant function with values into $\{0, 1\}$ whose jump points belong to S (i.e., θ is a characteristic function of a finite union of disjoint open intervals, so $\theta \in BV$). In particular, this shows that μ_t is a Dirac measure for almost every t , finishing the proof of the Claim.

It now follows that $\varphi_\varepsilon \rightarrow \varphi$ in $L^1(I)$ by a well known property of Young measures (see Valadier [43, Theorem 9]). Moreover, since $\varphi(t) = \int_0^\pi z d\mu_t(z) = (1 - \theta(t))\pi$ for a.e. $t \in I$, we find a representative φ that is piecewise constant with values into $\{0, \pi\}$ almost everywhere, and the jump points of φ are those of θ , hence included in S . By the finiteness of S we obtain $\varphi \in BV(I; \{0, \pi\})$. Since both φ and all φ_ε are bounded, we obtain the convergence $\varphi_\varepsilon \rightarrow \varphi$ in $L^p(I)$ for $1 \leq p < \infty$.

From (66) we find

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_\varepsilon(\varphi_\varepsilon; I) \geq \frac{(\pi - 2\gamma)^2}{\pi} \mathcal{H}^0(S),$$

and letting $\gamma \rightarrow 0$ we find that $\liminf_{\gamma \rightarrow 0} \frac{(\pi - 2\gamma)^2}{\pi} \mathcal{H}^0(S) \geq \|\varphi\|_{BV(I)}$, finishing the proof in our assumed particular case.

General case. To recover the general case, we use a truncation argument similar to that of Garroni-Müller [17]. From Proposition 4.7, we find integers k_ε such that $\varphi_\varepsilon - k_\varepsilon\pi$ is bounded in $L^p(I)$ for every $p \geq 1$. We may assume that $k_\varepsilon = 0$ and (choosing a subsequence) $\varphi_\varepsilon \rightarrow \varphi$ in $L^p(I)$ for any $p \geq 1$. For every $k \in \mathbb{Z}$, the particular case above applied to the truncations $T_k\varphi_\varepsilon$ defined at (58) yields for further subsequences, $T_k\varphi_\varepsilon \rightarrow f_k$ in L^1 for some $f_k \in BV(I; \{k\pi, (k+1)\pi\})$ and

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_\varepsilon(T_k\varphi_\varepsilon; I) \geq \|f_k\|_{BV}.$$

For any positive integer M , we now consider another truncation operator:

$$T^M\psi = (\psi \vee (-M\pi)) \wedge M\pi.$$

Adding up the above pieces $T_k\varphi_\varepsilon$ in the set $\{k\pi \leq \varphi_\varepsilon \leq (k+1)\pi\}$ for $k = -M, \dots, M-1$, we obtain the existence of $\varphi^M \in BV(I; \pi\mathbb{Z})$ such that

$$T^M\varphi_\varepsilon \rightarrow \varphi^M \quad \text{in } L^1(I).$$

As in [17], we note that $|\varphi_\varepsilon| \leq \frac{|\varphi_\varepsilon|^2}{M}$ on $\{|\varphi_\varepsilon| \geq M\}$. The uniform L^2 -bound and weak lower semicontinuity of the L^1 norm then yield for every $M > 0$:

$$\|\varphi^M - \varphi\|_{L^1} \leq \liminf_{\varepsilon \rightarrow 0} \|T^M\varphi_\varepsilon - \varphi_\varepsilon\|_{L^1} = \liminf_{\varepsilon \rightarrow 0} \int_{\{|\varphi_\varepsilon| > M\}} (|\varphi_\varepsilon| - M) dx \leq \frac{C}{M}.$$

Splitting $\varphi_\varepsilon - \varphi = \varphi_\varepsilon - T^M \varphi_\varepsilon + T^M \varphi_\varepsilon - \varphi^M + \varphi^M - \varphi$, we find that

$$\|\varphi_\varepsilon - \varphi\|_{L^1} \leq \frac{C}{M} + \|T^M \varphi_\varepsilon - \varphi^M\|_{L^1} + \frac{C}{M}.$$

As $T^M \varphi_\varepsilon \rightarrow \varphi^M$ in L^1 and M is arbitrary, we obtain $\varphi_\varepsilon \rightarrow \varphi$ in $L^1(I)$. As (φ_ε) is uniformly bounded in any L^p , by interpolation, we obtain $\varphi_\varepsilon \rightarrow \varphi$ in $L^p(I)$ for all $p \in [1, \infty)$. We also obtain that $T_k \varphi = f_k$. By super-additivity in F_ε , Fatou's lemma and the lower bound from the particular case

$$\infty > \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_\varepsilon(\varphi_\varepsilon; I) \geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \sum_k F_\varepsilon(T_k \varphi_\varepsilon; I) \geq \sum_k \|f_k\|_{BV}.$$

As f_k are piecewise constant with the only possible jumps of size π , we find that $\|f_k\|_{BV} = 0$ for all but finitely many k . Since $\varphi = \sum_k T_k \varphi = \sum_k f_k$, we deduce that $\varphi \in BV(I; \pi\mathbb{Z})$, and using additivity of the BV seminorm of f_k taking the values $\{k\pi, (k+1)\pi\}$, we finally obtain (62). \square

In Proposition 4.8, the lower bounds for F_ε are accurate up to $o(|\log \varepsilon|)$. In the following, we improve the error to $O(1)$ by means of a co-area argument inspired by the work of Sandier [38] on the Ginzburg-Landau energy (a different method was found by Jerrard [28]). To this end, we need to compare the nonlocal energy of a (scalar) function to that of a $\mathbb{S}^0 \approx \{0, \pi\}$ -valued variant of the same function (the corresponding step in Sandier's argument compares the Dirichlet energy of a complex valued function u with a \mathbb{S}^1 -valued variant given by $\frac{u}{|u|}$).

Lemma 4.9 *Let I be a bounded interval and $\varphi \in H^{1/2}(I)$ with $0 \leq \varphi \leq \pi$ and define $\hat{\varphi} : I \rightarrow \{0, \pi\}$ by*

$$\hat{\varphi} = \begin{cases} 0 & \varphi < \frac{\pi}{2}, \\ \pi & \varphi \geq \frac{\pi}{2}. \end{cases}$$

For $0 \leq \gamma \leq \frac{\pi}{2}$ we let $E_\gamma = \{s \in I : |\varphi(s) - \frac{\pi}{2}| > \frac{\pi}{2} - \gamma\}$. Let

$$\Theta_\varphi(\gamma) = \int_{E_\gamma} \int_{E_\gamma} \left| \frac{\hat{\varphi}(s) - \hat{\varphi}(t)}{s - t} \right|^2 ds dt, \quad \gamma \in [0, \frac{\pi}{2}],$$

then Θ_φ is a nondecreasing function and

$$\int_I \int_I \left| \frac{\varphi(s) - \varphi(t)}{s - t} \right|^2 ds dt \geq \int_0^{\frac{\pi}{2}} (1 - \frac{2\gamma}{\pi})^2 d\Theta_\varphi(\gamma) \quad (67)$$

where $d\Theta_\varphi$ denotes the measure corresponding to the (distributional) derivative of Θ_φ .

Proof. Let

$$\tilde{\Theta}_\varphi(\gamma) = \int_{E_\gamma} \int_{E_\gamma} \left| \frac{\varphi(s) - \varphi(t)}{s - t} \right|^2 ds dt, \quad \gamma \in [0, \frac{\pi}{2}].$$

As $(E_\gamma)_\gamma$ is nondecreasing in γ (with respect to inclusion), then Θ_φ and $\tilde{\Theta}_\varphi$ are nondecreasing functions. For $\pi \geq \gamma > \tilde{\gamma} \geq 0$, we have that

$$\tilde{\Theta}_\varphi(\gamma) - \tilde{\Theta}_\varphi(\tilde{\gamma}) \geq \left(\frac{\pi - 2\gamma}{\pi} \right)^2 (\Theta_\varphi(\gamma) - \Theta_\varphi(\tilde{\gamma})).$$

Letting $\tilde{\gamma} \rightarrow \gamma$, we see that the distributional derivatives satisfy (as measures on $[0, \frac{\pi}{2}]$)

$$d\tilde{\Theta}_\varphi(\gamma) \geq \frac{(\pi - 2\gamma)^2}{\pi^2} d\Theta_\varphi(\gamma)$$

and since $\tilde{\Theta}_\varphi(0) = 0$, we obtain by integrating over $\gamma \in (0, \frac{\pi}{2})$:

$$\int_I \int_I \left| \frac{\varphi(s) - \varphi(t)}{s - t} \right|^2 ds dt \geq \tilde{\Theta}_\varphi\left(\frac{\pi}{2}\right) \geq \int_0^{\frac{\pi}{2}} \left(1 - \frac{2\gamma}{\pi}\right)^2 d\Theta_\varphi(\gamma).$$

□

Now we show a more precise lower bound of F_ε up to an error $O(1)$:

Proposition 4.10 *There is a universal constant $M_0 > 0$ such that the following holds. Assume $\ell \in \mathbb{Z}$ and $\varphi_\varepsilon \rightarrow \ell\varphi_*$ in $L^1((-1, 1))$ for a sequence / family $\varepsilon \rightarrow 0$, where $\varphi_*(x) = \pi$ for $x \in (-1, 0)$ and $\varphi_*(x) = 0$ for $x \in (0, 1)$. Then for every $r \in (0, 1)$ we have*

$$\liminf_{\varepsilon \rightarrow 0} \left(F_\varepsilon(\varphi_\varepsilon; (-r, r)) - \pi|\ell| \log \frac{r}{\varepsilon} \right) \geq -|\ell|M_0. \quad (68)$$

Proof. Without loss of generality, we can assume $\ell \neq 0$ and

$$F_\varepsilon(\varphi_\varepsilon; (-r, r)) \leq \pi|\ell| \log \frac{r}{\varepsilon} \quad \text{for every } r \in (0, 1) \text{ and } \varepsilon < r \text{ small}$$

(otherwise the conclusion is obvious). We consider first the case of a single limit transition layer (i.e., $\ell = 1$) and then we deduce the general case.

The particular case of $\ell = 1$. Without loss of generality we may assume $0 \leq \varphi_\varepsilon \leq \pi$, by replacing φ_ε with $(\varphi_\varepsilon \vee 0) \wedge \pi$ that keeps the same limit and decreases the energy functional. By Lemma 4.9, using the notation of $\Theta_{\varphi_\varepsilon}$ for φ_ε inside the interval $I_r = (-r, r)$, we estimate

$$F_\varepsilon(\varphi_\varepsilon; I_r) \geq \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \left(1 - \frac{2\gamma}{\pi}\right)^2 d\Theta_{\varphi_\varepsilon}(\gamma) + \frac{1}{2\pi\varepsilon} \int_{I_r} \sin^2 \varphi_\varepsilon dt, \quad r \in (0, 1).$$

Now for every $\gamma \in [0, \frac{\pi}{2}]$ we have

$$\int_{I_r} \sin^2 \varphi_\varepsilon dt \geq \int_{\{\gamma \leq \varphi_\varepsilon \leq \pi - \gamma\}} \sin^2 \varphi_\varepsilon dx \geq |\{\gamma \leq \varphi_\varepsilon \leq \pi - \gamma\}| \sin^2 \gamma,$$

where the sets are understood as intersected with I_r . Averaging over $\gamma \in [0, \frac{\pi}{2}]$ this yields

$$\int_{I_r} \sin^2 \varphi_\varepsilon dt \geq \frac{2}{\pi} \int_0^{\frac{\pi}{2}} |\{\gamma \leq \varphi_\varepsilon \leq \pi - \gamma\}| \sin^2 \gamma d\gamma.$$

Integrating by parts, as $\Theta_{\varphi_\varepsilon}(0) = 0$, we have that

$$\int_0^{\frac{\pi}{2}} \left(1 - \frac{2\gamma}{\pi}\right)^2 d\Theta_{\varphi_\varepsilon}(\gamma) = \int_0^{\frac{\pi}{2}} \frac{4}{\pi} \left(1 - \frac{2\gamma}{\pi}\right) \Theta_{\varphi_\varepsilon}(\gamma) d\gamma,$$

so we obtain

$$F_\varepsilon(\varphi_\varepsilon; I_r) \geq \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \left(\frac{4}{\pi} \left(1 - \frac{2\gamma}{\pi}\right) \Theta_{\varphi_\varepsilon}(\gamma) + \frac{2}{\pi\varepsilon} \sin^2 \gamma |\{\gamma \leq \varphi_\varepsilon \leq \pi - \gamma\}| \right) d\gamma, \quad r \in (0, 1). \quad (69)$$

We set for $\gamma \in (0, \frac{\pi}{2})$:

$$a_\gamma^\varepsilon = \frac{|\{t \in I_r : \varphi_\varepsilon(t) < \gamma\}|}{2r}, \quad b_\gamma^\varepsilon = \frac{|\{t \in I_r : \varphi_\varepsilon(t) > \pi - \gamma\}|}{2r}, \quad c_\gamma^\varepsilon = \frac{|\{t \in I_r : \gamma \leq \varphi_\varepsilon(t) \leq \pi - \gamma\}|}{2r}.$$

Since the integrand on the right hand side of (69) is nonnegative, we use in the following only the restriction to $\gamma \in (\varepsilon^{1/3}, \frac{\pi}{2})$ (which is enough to deduce the desired lower bound for $F_\varepsilon(\varphi_\varepsilon; I_r)$). This choice is motivated by the fact that $c_{\varepsilon^{1/3}}^\varepsilon \leq \frac{C\varepsilon |\log \varepsilon|}{2r \sin^2(\varepsilon^{1/3})} \rightarrow 0$ as $\varepsilon \rightarrow 0$ (following from (59)); combined with the assumption $\varphi_\varepsilon \rightarrow \varphi_*$ in $L^1(I_r)$ and the fact that $a_{\varepsilon^{1/3}}^\varepsilon + b_{\varepsilon^{1/3}}^\varepsilon + c_{\varepsilon^{1/3}}^\varepsilon = 1$, we deduce that $a_{\varepsilon^{1/3}}^\varepsilon \rightarrow \frac{1}{2}$ and $b_{\varepsilon^{1/3}}^\varepsilon \rightarrow \frac{1}{2}$ as $\varepsilon \rightarrow 0$. Using (56), we have for every $\gamma \in (\varepsilon^{1/3}, \frac{\pi}{2})$:

$$\Theta_{\varphi_\varepsilon}(\gamma) \geq 2\pi^2 \int_{\{\varphi_\varepsilon < \gamma\}} \int_{\{\varphi_\varepsilon > \pi - \gamma\}} \frac{1}{|s-t|^2} ds dt \geq 2\pi^2 (\log a_\gamma^\varepsilon + \log b_\gamma^\varepsilon - \log c_\gamma^\varepsilon)$$

so

$$F_\varepsilon(\varphi_\varepsilon; I_r) \geq 2 \int_{\varepsilon^{1/3}}^{\frac{\pi}{2}} \left(2\left(1 - \frac{2\gamma}{\pi}\right) (\log a_\gamma^\varepsilon + \log b_\gamma^\varepsilon - \log c_\gamma^\varepsilon) + \frac{r}{\pi^2 \varepsilon} c_\gamma^\varepsilon \sin^2 \gamma \right) d\gamma.$$

For every fixed $\gamma \in (0, \frac{\pi}{2})$, as $a_\gamma^\varepsilon \geq a_{\varepsilon^{1/3}}^\varepsilon$ for $\varepsilon \leq \varepsilon_\gamma$, we deduce that $\liminf_{\varepsilon \rightarrow 0} a_\gamma^\varepsilon \geq \frac{1}{2}$; similarly, $\liminf_{\varepsilon \rightarrow 0} b_\gamma^\varepsilon \geq \frac{1}{2}$ for every $\gamma \in (0, \frac{\pi}{2})$. Using footnote 15, for every $\gamma \in (\varepsilon^{1/3}, \frac{\pi}{2})$ and $\varepsilon \leq \varepsilon_r$, we consider $K_\gamma = \frac{r \sin^2 \gamma}{2\pi \varepsilon (\pi - 2\gamma)} > 1$ and we obtain that $-\log c_\gamma^\varepsilon + K_\gamma c_\gamma^\varepsilon \geq \log K_\gamma + 1$ yielding for $\varepsilon > 0$ small enough

$$F_\varepsilon(\varphi_\varepsilon; I_r) \geq -C + 2 \int_{\varepsilon^{1/3}}^{\frac{\pi}{2}} \left(2\left(1 - \frac{2\gamma}{\pi}\right) + 2\left(1 - \frac{2\gamma}{\pi}\right) \log K_\gamma \right) d\gamma.$$

As

$$2 \int_0^{\frac{\pi}{2}} 2\left(1 - \frac{2\gamma}{\pi}\right) \log \frac{r}{\varepsilon} d\gamma = \pi \log \frac{r}{\varepsilon},$$

$\varepsilon^{1/3} \log \frac{r}{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$ and

$$\int_0^{\frac{\pi}{2}} \left(1 - \frac{2\gamma}{\pi}\right) \log \frac{\sin^2 \gamma}{2\pi(\pi - 2\gamma)} d\gamma < \infty,$$

we conclude to the existence of $M_0 > 0$ with

$$\liminf_{\varepsilon \rightarrow 0} \left(F_\varepsilon(\varphi_\varepsilon; I_r) - \pi \log \frac{r}{\varepsilon} \right) \geq -M_0.$$

The general case of $\ell \in \mathbb{Z}$. For the higher-multiplicity statement, we may assume $\ell > 0$ (otherwise, replace φ_ε with $-\varphi_\varepsilon$) and decompose $\varphi_\varepsilon = \sum_{j=0}^{\ell-1} \varphi_\varepsilon^{(j)}$, where

$$\varphi_\varepsilon^{(j)} = (\varphi_\varepsilon \vee j\pi) \wedge (j+1)\pi - j\pi.$$

Using $(\varphi_\varepsilon^{(j)}(t) - \varphi_\varepsilon^{(j)}(s))(\varphi_\varepsilon^{(k)}(t) - \varphi_\varepsilon^{(k)}(s)) \geq 0$ for every $t, s \in I_r$ and the π -periodicity of \sin^2 , we easily deduce that

$$F_\varepsilon(\varphi_\varepsilon; I_r) \geq \sum_{j=0}^{\ell-1} F_\varepsilon(\varphi_\varepsilon^{(j)}; I_r).$$

As $\varphi_\varepsilon^{(j)} \rightarrow \varphi_*$ in $L^1(I_r)$ as $\varepsilon \rightarrow 0$ for $0 \leq j \leq \ell - 1$, we can use the case $\ell = 1$ on every $\varphi_\varepsilon^{(j)}$ and conclude with (68) for ℓ general. \square

In the following two corollaries, we show that (68) holds without the \liminf , for sufficiently small r and ε .

Corollary 4.11 *Under the assumptions of Proposition 4.10, consider sequences $r = r_k \rightarrow 0$, $\varepsilon = \varepsilon_k \rightarrow 0$ with $\frac{r_k}{\varepsilon_k} \rightarrow \infty$ and $\varphi_k = \varphi_{\varepsilon_k}$. Then*

$$\liminf_{k \rightarrow \infty} (F_{\varepsilon_k}(\varphi_k; I_{r_k}) - \pi|\ell| \log \frac{r_k}{\varepsilon_k}) \geq -|\ell|M_0.$$

Proof. Set $\hat{\varepsilon}_k = \frac{\varepsilon_k}{r_k}$ and $\hat{\varphi}_k(x) = \varphi_k(r_k x)$. Then $F_{\hat{\varepsilon}_k}(\hat{\varphi}_k; I_1) = F_{\varepsilon_k}(\varphi_k; I_{r_k})$ and $\hat{\varphi}_k \rightarrow \ell\varphi_*$ in $L^1(I_1)$. By Proposition 4.10, it follows that

$$\liminf_{k \rightarrow \infty} (F_{\varepsilon_k}(\varphi_k; I_{r_k}) - \pi|\ell| \log \frac{r_k}{\varepsilon_k}) = \liminf_{k \rightarrow \infty} (F_{\hat{\varepsilon}_k}(\hat{\varphi}_k; I_1) - \pi|\ell| \log \frac{1}{\hat{\varepsilon}_k}) \geq -|\ell|M_0.$$

□

Corollary 4.12 *There exist constants $M_2 > 0$, $\varepsilon_0 > 0$, $r_0 \in (0, 1)$ such that for every sequence / family $(\varphi_\varepsilon)_{\varepsilon \rightarrow 0}$ converging to φ_* as in Proposition 4.10 and for all $r, \varepsilon > 0$ with $\varepsilon < \varepsilon_0$, $r < r_0$, the following holds:*

$$F_\varepsilon(\varphi_\varepsilon; (-r, r)) - \pi|\ell| \log \frac{r}{\varepsilon} \geq -M_2|\ell|. \quad (70)$$

Proof. First, note that it is enough to show the existence of a universal constant $K > 0$ such that the conclusion holds true in the restricted case $r > K\varepsilon$. Indeed, the other case $r \leq K\varepsilon$ follows because then $\log \frac{r}{K\varepsilon} \leq 0$ and hence

$$F_\varepsilon(\varphi_\varepsilon; (-r, r)) \geq 0 \geq \pi|\ell| \log \frac{r}{K\varepsilon} = \pi|\ell| \log \frac{r}{\varepsilon} - \pi|\ell| \log K,$$

so (70) is true up to replacing M_2 with $\max(M_2, \log K)$.

For the existence of the constant K , we argue by contradiction. Assume that for $M_2 = n$, $K = n$, $\varepsilon_0 = \frac{1}{n^3}$ and $r_0 = \frac{1}{n}$ there exist a sequence $(\varphi_{\varepsilon_n})_{n \rightarrow \infty}$ converging to φ_* as in Proposition 4.10 and $\varepsilon_n < \frac{1}{n^3}$ and $r_n \in (K\varepsilon_n, \frac{1}{n})$ with

$$F_{\varepsilon_n}(\varphi_{\varepsilon_n}; I_{r_n}) - \pi|\ell| \log \frac{r_n}{\varepsilon_n} < -n|\ell|,$$

then $\frac{r_n}{\varepsilon_n} \rightarrow \infty$ but

$$\liminf_{n \rightarrow \infty} \left(F_{\varepsilon_n}(\varphi_{\varepsilon_n}; I_{r_n}) - \pi|\ell| \log \frac{r_n}{\varepsilon_n} \right) = -\infty$$

in contradiction to Corollary 4.11. □

We also need the following simple but powerful lemma, a variant of an observation by del Pino and Felmer [16].

Lemma 4.13 *For every $M_3 > 0$, there is $M_4 := 2(M_2 + M_3 + \pi \log 2) > 0$ (with M_2 given in (70)) such that for every sequence / family $\varphi_\varepsilon \rightarrow \varphi_*$ in $L^1((-1, 1))$ as $\varepsilon \rightarrow 0$, where $\varphi_*(x) = \pi$ for $x \in (-1, 0)$ and $\varphi_*(x) = 0$ for $x \in (0, 1)$, that satisfies*

$$F_\varepsilon(\varphi_\varepsilon; (-r, r)) \leq \pi \log \frac{r}{\varepsilon} + M_3 \quad \text{for every } r \in (0, 1) \text{ and } \varepsilon \text{ small,}$$

then

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{2\pi\varepsilon} \int_{-r}^r \sin^2 \varphi_\varepsilon d\mathcal{H}^1 \leq M_4 \quad \text{for every } r \in (0, 1).$$

Proof. Let $\varphi_\varepsilon \rightarrow \varphi_*$ in $L^1((-1, 1))$. Denoting $\tilde{\varphi}_{2\varepsilon} := \varphi_\varepsilon$, we have that $\tilde{\varphi}_{2\varepsilon} \rightarrow \varphi_*$ in $L^1((-1, 1))$. Hence for small ε , we apply Corollary 4.12 for $\tilde{\varphi}_{2\varepsilon}$ on I_r :

$$F_{2\varepsilon}(\varphi_\varepsilon; I_r) = F_{2\varepsilon}(\tilde{\varphi}_{2\varepsilon}; I_r) \geq \pi \log \frac{r}{2\varepsilon} - M_2$$

so

$$\frac{1}{4\pi\varepsilon} \int_{I_r} \sin^2 \varphi_\varepsilon d\mathcal{H}^1 = F_\varepsilon(\varphi_\varepsilon; I_r) - F_{2\varepsilon}(\varphi_\varepsilon; I_r) \leq \pi \log \frac{r}{\varepsilon} + M_3 - \pi \log \frac{r}{2\varepsilon} + M_2,$$

for every $r \in (0, 1)$ and ε small. \square

For the second order lower bound of the two-dimensional functional $\hat{F}_\varepsilon^{(0)}$ defined at (51), we need the following result comparing some optimal profile problems. To simplify notation we **skip** $^{(0)}$ in $\hat{F}_\varepsilon^{(0)}$, i.e., we denote for an open set $G \subset \mathbb{R}_+^2$ and $\psi : G \rightarrow \mathbb{R}$ the localised functional

$$\hat{F}_\varepsilon(\psi; G) := \int_G |\nabla \psi|^2 dx dy + \frac{1}{2\pi\varepsilon} \int_{G \cap (\mathbb{R} \times \{0\})} \sin^2(\psi(\cdot, 0)) dx. \quad (71)$$

Lemma 4.14 *We set $\phi^*(x, y) = \arg(x + iy)$ and $\phi_\varepsilon^*(x, y) = \arg(x + i(y + 2\pi\varepsilon))$ for $(x, y) \in \mathbb{R}_+^2$. Setting $I_r = (-r, r)$ for $r > 0$,*

$$\gamma_1 = \liminf_{\varepsilon \rightarrow 0} \left(\inf_{\psi = \phi^* \text{ on } \partial B_r^+ \setminus I_r} \hat{F}_\varepsilon(\psi; B_r^+) - \pi \log \frac{r}{\varepsilon} \right)$$

and

$$\gamma_2 = \lim_{r \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \left(\inf_{\psi = \phi_\varepsilon^* \text{ on } \partial B_r^+ \setminus I_r} \hat{F}_\varepsilon(\psi; B_r^+) - \pi \log \frac{r}{\varepsilon} \right),$$

then these limits are equal (in particular, γ_1 is independent of r), and moreover,

$$\gamma_1 = \gamma_2 = \gamma_0 := \pi + \pi \log \frac{1}{4\pi} = \pi \log \frac{e}{4\pi}.$$

Proof. We remark that in the definition of γ_1 , we can scale out r if we replace r by 1 and ε by ε/r without changing the result, so the limit is in fact independent of r , i.e., γ_1 is independent of r . The harmonic function ϕ_ε^* is Peierls' solution of the Euler-Lagrange equations for \hat{F}_ε (see Toland [42]).

Step 1. *We show that $\gamma_1 = \gamma_2$.* For that, we construct comparison functions ϕ_ε on $B_{r(1+r)}^+ \setminus B_r$ for some $r > 0$ that satisfy $\phi_\varepsilon = \phi_\varepsilon^*$ on the half-circle $\partial B_{r(1+r)}^+ \setminus I_{r(1+r)}$ and $\phi_\varepsilon = \phi^*$ on $\partial B_r^+ \setminus I_r$. For example, we can choose an interpolation function such as

$$\phi_\varepsilon(x, y) = \arg\left(x + i\left(y + 2\pi\varepsilon \frac{\sqrt{x^2 + y^2} - r}{r^2}\right)\right), \quad (x, y) \in B_{r(1+r)}^+ \setminus B_r.$$

As both the argument function and the function multiplied by ε are smooth away from 0, it is straightforward to see that

$$\lim_{\varepsilon \rightarrow 0} \int_{B_{r(1+r)}^+ \setminus B_r} |\nabla \phi_\varepsilon|^2 dx dy = \int_{B_{r(1+r)}^+ \setminus B_r} |\nabla \arg(x + iy)|^2 dx dy = \int_0^\pi \int_r^{r(r+1)} \frac{1}{s} ds d\theta = \pi \log(1+r)$$

and $\sin^2 \phi_\varepsilon(x, 0) \leq \sin^2 \phi_\varepsilon(r(r+1), 0) \leq C(\frac{\varepsilon}{r})^2$ for $x \in I_{r(1+r)} \setminus I_r$, so letting first $\varepsilon \rightarrow 0$ and then $r \rightarrow 0$ it follows that $\gamma_2 \leq \gamma_1$. The opposite inequality follows from a similar interpolation argument.

Step 2. We compute that $\gamma_1 = \gamma_2 = \pi \log \frac{r}{4\pi}$. To identify the limit, we use a result of Cabré and Solà-Morales [12, Lemma 3.1] that states that ϕ_ε^* is not only a critical point of \hat{F}_ε , but actually the minimizer of \hat{F}_ε with respect to its own boundary conditions, i.e., ϕ_ε^* is the minimizer inside the limit γ_2 . Therefore, we compute explicitly the energy of ϕ_ε^* . First, note that by rescaling $\psi(z) := \phi_\varepsilon^*(2\pi\varepsilon z)$ for $z = (x, y) \in \mathbb{R}_+^2$, we have that $\psi(x, y) = \frac{\pi}{2} - \arctan \frac{x}{y+1}$. For $R = \frac{r}{2\pi\varepsilon}$ we then have

$$\int_{B_R^+} |\nabla \psi|^2 dx dy = \int_{B_r^+} |\nabla \phi_\varepsilon^*|^2 dx dy$$

and

$$\int_{-R}^R \sin^2 \psi dx = \frac{1}{2\pi\varepsilon} \int_{-r}^r \sin^2 \phi_\varepsilon^* dx.$$

By direct calculation, $|\nabla \psi(x, y)|^2 = \frac{1}{x^2 + (y+1)^2}$ and changing variables we obtain

$$\int_{B_R^+} \frac{1}{x^2 + (y+1)^2} dx dy = \int_{B_R(0,1) \cap \{y > 1\}} \frac{1}{x^2 + y^2} dx dy.$$

Setting $A_R = B_R^+ \cap \{y > 1\} = \{(x, y) : x^2 + y^2 < R^2, y > 1\}$, we clearly have for $R > 1$:

$$A_R \subset B_R(0,1) \cap \{y > 1\} \subset A_{R+1}.$$

Using polar coordinates $x = s \cos \theta$, $y = s \sin \theta$ in A_R , we have that $y > 1$ corresponds to $\sin \theta > \frac{1}{s}$ and $s > 1$ (as $s > y$) so

$$\int_{A_R} \frac{1}{x^2 + y^2} dx dy = \int_1^R \int_{\arcsin \frac{1}{s}}^{\pi - \arcsin \frac{1}{s}} \frac{1}{s} d\theta ds.$$

Evaluating the θ -integral and changing variables $s = \frac{1}{\sin t}$ we see

$$\int_{A_R} \frac{1}{x^2 + y^2} dx dy = \int_1^R \left(\frac{\pi}{s} - \frac{2 \arcsin \frac{1}{s}}{s} \right) ds = \pi \log R - 2 \int_{\arcsin \frac{1}{R}}^{\frac{\pi}{2}} t \cot t dt.$$

We note that $\int_0^{\arcsin \frac{1}{R}} t \cot t dt = O(\frac{1}{R})$ as R is large and integrate by parts:

$$\int_0^{\frac{\pi}{2}} t \cot t dt = \int_0^{\frac{\pi}{2}} t \frac{d}{dt} (\log \sin t) dt = - \int_0^{\frac{\pi}{2}} \log \sin t dt = \frac{\pi}{2} \log 2,$$

where the final equality is a standard integral, see Gradshteyn-Ryzhik [19, 3.747].

We thus have $\int_{A_R} |\nabla \psi|^2 dx dy = \pi \log R - \pi \log 2 - O(\frac{1}{R})$ and so, using $\log(R+1) - \log R = O(\frac{1}{R})$ as R is large that

$$\int_{B_R^+} |\nabla \psi|^2 dx dy = \pi \log R - \pi \log 2 - O(\frac{1}{R}).$$

For the boundary term, we calculate

$$\int_{-R}^R \sin^2 \psi dx = \int_{-R}^R \frac{1}{1+x^2} dx = 2 \arctan R = \pi - O(\frac{1}{R})$$

as $R \rightarrow \infty$. Putting everything together we see that

$$\hat{F}_\varepsilon(\phi_\varepsilon^*; B_r^+) = \pi \log \frac{r}{2\pi\varepsilon} - \pi \log 2 + \pi - O(\frac{\varepsilon}{r}) = \pi \log \frac{r}{\varepsilon} + \pi + \pi \log \frac{1}{4\pi} - O(\frac{\varepsilon}{r}),$$

and passing to the limit $\varepsilon \rightarrow 0$ and then $r \rightarrow 0$ we obtain that $\gamma_2 = \gamma_0$ as claimed. \square

Lemma 4.14 clearly applies to boundary vortices of multiplicities ± 1 by suitable sign change. For higher multiplicity transitions, we have the following result.

Lemma 4.15 *Let $d > 0$ be an integer and set $\phi_d^*(x, y) = d \arg(x + iy)$ for every $(x, y) \in \mathbb{R}^2$. For every small $r > 0$ and $\varepsilon < e^{-1/r^2}$, there exists $\phi_{d,\varepsilon} : B_r^+ \rightarrow \mathbb{R}$ such that $\phi_{d,\varepsilon} = \phi_d^*$ on ∂B_r and*

$$\hat{F}_\varepsilon(\phi_{d,\varepsilon}; B_r^+) \leq \pi d \log \frac{r}{\varepsilon} + Cd^2(1 + |\log r| + \log |\log \varepsilon|)$$

where $C > 0$ is independent of r and ε .

Proof. The idea of the proof is to replace a near-jump of $d\pi$ at 0 by d near-jumps of height π at points x_ε^j that all converge to 0 and to estimate their interaction energy. Set $a_\varepsilon = \frac{1}{|\log \varepsilon|}$ and $x_\varepsilon^j = ja_\varepsilon$, $j = 1, \dots, d$. With the interpolation function

$$f(x, y) = \begin{cases} 1 & \text{if } \sqrt{x^2 + y^2} < r(1-r) \\ \frac{r - \sqrt{x^2 + y^2}}{r^2} & \text{if } r(1-r) \leq \sqrt{x^2 + y^2} \leq r \\ 0 & \text{if } \sqrt{x^2 + y^2} > r \end{cases}$$

we set

$$\phi_{d,\varepsilon} = \sum_{j=1}^d \arg(x - f(x, y)x_\varepsilon^j + i(y + 2\pi\varepsilon f(x, y))).$$

As in the proof of Lemma 4.14, the interpolation function does not contribute much to the energy, in fact

$$\int_{B_r^+ \setminus B_{r(1-r)}} |\nabla \phi_{d,\varepsilon}|^2 dx dy + \frac{1}{\varepsilon} \int_{I_r \setminus I_{r(1-r)}} \sin^2 \phi_{d,\varepsilon} dx \leq Cd^2 \left(|\log(1-r)| + \frac{\varepsilon}{r^2} \right) \leq Cd^2.$$

It suffices to compute the energy of $\phi_{d,\varepsilon}$ in B_ρ^+ for $\rho = r(1-r)$, where $f \equiv 1$. For that, we note that $\phi(x, y) = \arg(x + i(y + 2\pi\varepsilon))$ and $\psi(x, y) = \log|x + i(y + 2\pi\varepsilon)|$ are (up to sign) harmonic conjugates. Then

$$|\nabla \phi_{d,\varepsilon}|^2 = \sum_{j=1}^d |\nabla \psi(\cdot - x_\varepsilon^j)|^2 + \sum_{j \neq k} \nabla \psi(\cdot - x_\varepsilon^j) \cdot \nabla \psi(\cdot - x_\varepsilon^k).$$

The integral over B_ρ^+ of the first sum is bounded by

$$d\pi \log \frac{\rho}{\varepsilon} + O(1),$$

while for the second part we compute for $j \neq k$

$$\int_{B_\rho^+} \nabla \psi(\cdot - x_\varepsilon^j) \cdot \nabla \psi(\cdot - x_\varepsilon^k) dx = \int_{\partial B_\rho^+} \psi(\cdot - x_\varepsilon^j) \frac{\partial}{\partial \nu} \psi(\cdot - x_\varepsilon^k).$$

The integrals over $\partial B_\rho \cap \mathbb{R}_+^2$ are estimated by $\pi |\log \rho| + O(a_\varepsilon^2)/\rho^2 \leq \pi |\log \rho| + C$, while the integrals over the straight part are of the form

$$\int_{-\rho}^{\rho} \frac{1}{2} \log[(x - x_\varepsilon^j)^2 + (2\pi\varepsilon)^2] \frac{2\pi\varepsilon}{(x - x_\varepsilon^k)^2 + (2\pi\varepsilon)^2} dx.$$

Extending the integration interval to $(-\infty, \infty)$ provides an upper bound (up to the contribution of the region where the logarithm is negative, which is bounded by $C|\log \rho|$ since $\varepsilon < \rho^2$). The remaining integral,

$$\int_{-\infty}^{\infty} \frac{1}{2} \log [(x - x_\varepsilon^j)^2 + (2\pi\varepsilon)^2] \frac{2\pi\varepsilon}{(x - x_\varepsilon^k)^2 + (2\pi\varepsilon)^2} dx$$

can be evaluated using the residue theorem: The function can be extended to the upper half plane as

$$(\log |z - (x_\varepsilon^j - i2\pi\varepsilon)| + i \arg(z - (x_\varepsilon^j - i2\pi\varepsilon))) \frac{2\pi\varepsilon}{(z - x_\varepsilon^k)^2 + (2\pi\varepsilon)^2}$$

for a branch of the argument that is smooth on the upper half plane. Integrating over ∂B_R^+ and letting $s \rightarrow 0$ and $R \rightarrow \infty$, we find that the only singularity in the contour is a simple pole at $z = x_\varepsilon^k + i2\pi\varepsilon$, and using the residue theorem we obtain after taking real parts

$$\int_{-\infty}^{\infty} \frac{1}{2} \log [(x - x_\varepsilon^j)^2 + (2\pi\varepsilon)^2] \frac{2\pi\varepsilon}{(x - x_\varepsilon^k)^2 + (2\pi\varepsilon)^2} dx = 2\pi \log ((j - k)^2 a_\varepsilon^2 + (2\pi\varepsilon)^2).$$

From $\sin^2(x + y) \leq 2(\sin^2 x + \sin^2 y)$ we see that $\sin^2(\phi_{d,\varepsilon}) \leq Cd \sum_{j=1}^d \sin^2(\phi(\cdot - x_\varepsilon^j))$ and using the calculation in Step 2 of the previous lemma, the boundary term contributes only by a constant, and adding up we arrive at the conclusion of the lemma since $|\log \rho - \log r| = |\log(1 - r)| \leq C$.¹⁶ \square

Now we show a precise estimate which is the central step in the Γ -expansion beyond the leading (logarithmic) order proved at (62). This is based on an argument that is new in the context of boundary vortices, inspired by the work of Colliander-Jerrard [14] for interior vortices. A different proof of the same result (due to Alicandro-Ponsiglione [4]) uses a dyadic decomposition argument; we expect that such an approach can also be used here.

Proposition 4.16 *Let $\rho > 0$ and $\phi_\varepsilon \in H^1(B_\rho^+)$ be a sequence / family with $\phi_\varepsilon(x, 0) \rightarrow \phi^*(x, 0) = \pi \mathbf{1}_{\{x < 0\}}(x)$ in $L^1(I_\rho)$. For the functional (71), we have the following second order lower bound:*

$$\liminf_{\varepsilon \rightarrow 0} \left(\hat{F}_\varepsilon(\phi_\varepsilon; B_\rho^+) - \pi \log \frac{\rho}{\varepsilon} \right) \geq \gamma_0,$$

where $\gamma_0 = \pi \log \frac{e}{4\pi}$.

Proof. As our statement is about the \liminf , it is enough to consider sequences in the following. First note the invariance of the desired estimate with respect to rescaling in ρ . Therefore, it suffices to consider the case $\rho = 1$. Let $\delta \in (0, \pi)$. We let C_j denote generic positive constants **independent of ε , ϕ_ε and δ** . We may assume that ϕ_ε are C^1 smooth in $\overline{B_1^+}$, since for any $\eta > 0$ and $\phi_\varepsilon \in H^1(B_1^+)$ there exists $\hat{\phi}_\varepsilon \in C^1(\overline{B_1^+})$ with $\left| \hat{F}_\varepsilon(\hat{\phi}_\varepsilon; B_1^+) - \hat{F}_\varepsilon(\phi_\varepsilon; B_1^+) \right| \leq \eta$.

Second, we may assume that there is $C_1 > 0$ such that

$$\hat{F}_\varepsilon(\phi_\varepsilon; B_1^+) \leq \pi \log \frac{1}{\varepsilon} + \gamma_0 + C_1$$

(otherwise the desired estimate is trivially satisfied).

¹⁶The proof sketched above is fully local. A nonlocal proof of a less precise estimate is given in [32].

Step 1. Finding a radius $\rho_* = \rho_*(\delta) \in (0, \frac{1}{2})$ such that for all ε along a sequence, $\phi_\varepsilon(\rho_* e^{i\theta})$ has similar properties on $\partial B_{\rho_*}^+$ to that of the limit function $\phi^*(\rho_* e^{i\theta}) = \theta$ where θ is the polar angle. For that, we start by recalling from Proposition 4.10 that for every $r_0 \in (0, \frac{1}{2})$:

$$\liminf_{\varepsilon \rightarrow 0} \left(\hat{F}_\varepsilon(\phi_\varepsilon; B_{r_0}^+) - \pi \log \frac{r_0}{\varepsilon} \right) \geq -M_0.$$

Combining the two estimates, we obtain for a constant $C_2 > 0$ independent of r_0 :

$$\limsup_{\varepsilon \rightarrow 0} \hat{F}_\varepsilon(\phi_\varepsilon; B_1^+ \setminus B_{r_0}) \leq \pi \log \frac{1}{r_0} + C_2,$$

and reducing the domain of integration and setting $C_3 = C_2 - \pi \log \frac{1}{2} > 0$ we can write

$$\limsup_{\varepsilon \rightarrow 0} \hat{F}_\varepsilon(\phi_\varepsilon; B_{1/2}^+ \setminus B_{r_0}^+) \leq \pi \log \frac{1}{2r_0} + C_3. \quad (72)$$

For $s \in (0, \frac{1}{2})$, we introduce

$$f_\varepsilon(s) := \int_{\partial B_s^+ \setminus I_s} |\nabla \phi_\varepsilon|^2 d\mathcal{H}^1 + \frac{1}{2\pi\varepsilon} \int_{\partial I_s} \sin^2 \phi_\varepsilon(\cdot, 0) d\mathcal{H}^0, \quad (73)$$

so that $\hat{F}_\varepsilon(\phi_\varepsilon; B_r^+) = \int_0^r f_\varepsilon(s) ds$, as well as the sets

$$A_\varepsilon = \left\{ s \in (0, \frac{1}{2}) : f_\varepsilon(s) \leq \frac{\pi + \delta}{s} \right\} \quad \text{and} \quad G_\varepsilon = \left\{ s \in (0, 1) : |\phi_\varepsilon(s, 0)| + |\phi_\varepsilon(-s, 0) - \pi| < \frac{1}{4} \right\},$$

We fix r_0 such that

$$r_0 = r_0(\delta) \leq \frac{1}{2} \exp\left(-\frac{2C_3}{\delta}\right)$$

and

$$\text{from now on, } \varepsilon \text{ is small, i.e., } \varepsilon < r_0. \quad (74)$$

Thus, we have $\delta \log \frac{1}{2r_0} \geq 2C_3$. The aim of this step is to show that

$$[r_0, \frac{1}{2}] \cap A_\varepsilon \cap G_\varepsilon \neq \emptyset$$

(any point ρ_* in this intersection can be used as the desired radius in the claim of Step 1). To do so, we estimate $a_\varepsilon = |[r_0, \frac{1}{2}] \cap A_\varepsilon|$ as follows: as $s \mapsto \frac{1}{s}$ is decreasing in $(0, \frac{1}{2})$, we may estimate

$$\pi \log \frac{1}{2r_0} + C_3 \stackrel{(72)}{\geq} \int_{r_0}^{\frac{1}{2}} f_\varepsilon(s) ds \geq (\pi + \delta) \int_{r_0 + a_\varepsilon}^{1/2} \frac{1}{s} ds = (\pi + \delta) \log \frac{1}{2(r_0 + a_\varepsilon)}.$$

Using our choice of r_0 , it follows that

$$-C_3 \geq C_3 - \delta \log \frac{1}{2r_0} \geq (\pi + \delta) \log \frac{r_0}{r_0 + a_\varepsilon},$$

so for every $0 < \delta < \pi$, we can estimate

$$a_\varepsilon \geq r_0 \left(e^{\frac{C_3}{\pi + \delta}} - 1 \right) \geq r_0 C_5, \quad C_5 := e^{\frac{C_3}{2\pi}} - 1 > 0.$$

Choosing a sequence $\varepsilon_n \rightarrow 0$, we have $|G_{\varepsilon_n} \cap [r_0, \frac{1}{2}]| \rightarrow \frac{1}{2} - r_0$ and hence (using Fatou's lemma) that $|[r_0, \frac{1}{2}] \cap \limsup_{n \rightarrow \infty} (A_{\varepsilon_n} \cap G_{\varepsilon_n})| > 0$. In particular there is a radius $\rho_* = \rho_*(\delta) \in [r_0, \frac{1}{2}]$ that lies in infinitely many sets $A_{\varepsilon_n} \cap G_{\varepsilon_n}$. In particular, $\rho_* > \varepsilon$.

Step 2. We show that $\phi_\varepsilon(\rho_* e^{i\theta})$ is close to $\phi^*(\rho_* e^{i\theta}) = \theta$ in $L^2(\partial B_{\rho_*}^+)$. Indeed, setting

$$w_\varepsilon(\theta) := \phi_\varepsilon(\rho_* e^{i\theta}) - \theta,$$

where θ is the polar angle, we have $|w_\varepsilon(\theta = 0)|, |w_\varepsilon(\theta = \pi)| < \frac{1}{4}$ (since $\rho_* \in G_\varepsilon$). Since $x/\sin x$ is increasing on $(0, \frac{1}{4})$, there exists $C_7 > 0$ such that

$$|w_\varepsilon| \leq \frac{1/4}{\sin \frac{1}{4}} |\sin w_\varepsilon| \leq \frac{1/4}{\sin \frac{1}{4}} \sqrt{\frac{4\pi^2 \varepsilon}{\rho_*}} =: C_7 \sqrt{\frac{\varepsilon}{\rho_*}} \quad \text{at } \theta \in \{0, \pi\} \quad (75)$$

(since $\rho_* \in A_\varepsilon$) so

$$\begin{aligned} \int_0^\pi |\partial_\theta w_\varepsilon(\theta)|^2 d\theta &= \int_0^\pi (|\partial_\theta \phi_\varepsilon(\rho_* e^{i\theta})|^2 + 2\partial_\theta \phi_\varepsilon(\rho_* e^{i\theta}) + 1) d\theta \\ &= \int_0^\pi (|\partial_\theta \phi_\varepsilon(\rho_* e^{i\theta})|^2 - 1 + 2\partial_\theta w_\varepsilon) d\theta \\ &\leq \delta + 2 \int_0^\pi \partial_\theta w_\varepsilon d\theta = \delta + 2(w_\varepsilon(\pi) - w_\varepsilon(0)) \\ &\leq \delta + 4C_7 \sqrt{\frac{\varepsilon}{\rho_*}}. \end{aligned} \quad (76)$$

In particular, for a suitably chosen $C_8 > 0$, we obtain thanks to (74) and (75) (in particular, $\varepsilon < \rho_*$):

$$\int_0^\pi |w_\varepsilon|^2 d\theta \leq \int_0^\pi \left(w_\varepsilon(0) + \int_0^\theta \partial_\theta w_\varepsilon(y) dy \right)^2 d\theta \leq C_8 \left(\delta + \sqrt{\frac{\varepsilon}{\rho_*}} \right). \quad (77)$$

Step 3. We prove that

$$\liminf_{\varepsilon \rightarrow 0} (\hat{F}_\varepsilon(\phi_\varepsilon; B_{\rho_*}^+) - \pi \log \frac{\rho_*}{\varepsilon}) \geq \gamma_0 - o_\delta(1). \quad (78)$$

The idea is to estimate the energy of the interpolation between ϕ_ε and $\phi^* = \theta$ in a small annulus around $\partial B_{\rho_*}^+$. In the small annulus $B_{\rho_* + \eta}^+ \setminus B_{\rho_*}$ with η to be chosen later (see (79)), we set the interpolation function between $\phi_\varepsilon(\rho_* e^{i\theta})$ and $\phi^*((\rho_* + \eta)e^{i\theta}) = \theta$:

$$\hat{\phi}_\varepsilon(r, \theta) = \theta + \frac{\rho_* + \eta - r}{\eta} w_\varepsilon(\theta), \quad r \in (\rho_*, \rho_* + \eta), \theta \in (0, \pi).$$

Then we estimate the energy of $\hat{\phi}_\varepsilon$:

$$\begin{aligned} \hat{F}_\varepsilon(\hat{\phi}_\varepsilon; B_{\rho_* + \eta}^+ \setminus B_{\rho_*}) &= \int_{\rho_*}^{\rho_* + \eta} \left(\int_0^\pi \frac{1}{r} \left(1 + \frac{\rho_* + \eta - r}{\eta} \partial_\theta w_\varepsilon \right)^2 + \frac{r}{\eta^2} |w_\varepsilon|^2 d\theta + \right. \\ &\quad \left. + \frac{1}{2\pi\varepsilon} \left(\sin^2 \left(\frac{\rho_* + \eta - r}{\eta} w_\varepsilon(0) \right) + \sin^2 \left(\frac{\rho_* + \eta - r}{\eta} w_\varepsilon(\pi) \right) \right) \right) dr \end{aligned}$$

For the first term on the right hand side, we use (75) and (76) to estimate:

$$\begin{aligned} &\int_{\rho_*}^{\rho_* + \eta} \int_0^\pi \left(\frac{1}{r} + \frac{2(\rho_* + \eta - r)}{r\eta} \partial_\theta w_\varepsilon + \frac{1}{r} (\partial_\theta w_\varepsilon)^2 \right) d\theta dr \\ &\leq \pi \log \left(1 + \frac{\eta}{\rho_*} \right) + \int_{\rho_*}^{\rho_* + \eta} \frac{2}{r} |w_\varepsilon(\pi) - w_\varepsilon(0)| + \frac{1}{r} (\delta + 4C_7 \sqrt{\frac{\varepsilon}{\rho_*}}) dr \\ &\leq \log \left(1 + \frac{\eta}{\rho_*} \right) \left(\pi + 8C_7 \sqrt{\frac{\varepsilon}{\rho_*}} + \delta \right). \end{aligned}$$

Since $\rho_* \in A_\varepsilon$, this estimate combined with (77) yield

$$\hat{F}_\varepsilon(\hat{\phi}_\varepsilon; B_{\rho_*+\eta}^+ \setminus B_{\rho_*}) \leq \log\left(1 + \frac{\eta}{\rho_*}\right) \left(\pi + 8C_7\sqrt{\frac{\varepsilon}{\rho_*}} + \delta\right) + C_8\left(\delta + \sqrt{\frac{\varepsilon}{\rho_*}}\right)\left(\frac{\rho_*}{\eta} + 1\right) + \frac{2\pi\eta}{\rho_*}.$$

Letting $\varepsilon \rightarrow 0$ and setting

$$\eta = \delta^{1/4}\rho_*, \quad (79)$$

we obtain that

$$\limsup_{\varepsilon \rightarrow 0} \hat{F}_\varepsilon(\hat{\phi}_\varepsilon; B_{\rho_*+\eta}^+ \setminus B_{\rho_*}) \leq (\pi + \delta)\log(1 + \delta^{1/4}) + C_8(\delta^{3/4} + \delta) + 2\pi\delta^{1/4}, \quad (80)$$

which tends to 0 as $\delta \rightarrow 0$. If we extend $\hat{\phi}_\varepsilon$ in the ball $B_{\rho_*+\eta=\rho_*(1+\delta^{1/4})}^+$ by setting $\hat{\phi}_\varepsilon := \phi_\varepsilon$ in $B_{\rho_*}^+$, we can now use the lower bounds from the definition of γ_1 in Lemma 4.14 (because $\hat{\phi}_\varepsilon = \phi^*$ on $\partial B_{\rho_*+\eta}^+ \setminus I_{\rho_*+\eta}$), giving us

$$\liminf_{\varepsilon \rightarrow 0} (\hat{F}_\varepsilon(\hat{\phi}_\varepsilon; B_{\rho_*(1+\delta^{1/4})}^+) - \pi \log \frac{\rho_*}{\varepsilon}) \geq \gamma_0 - o_\delta(1).$$

Since $\phi_\varepsilon = \hat{\phi}_\varepsilon$ on $B_{\rho_*}^+$, we can use (80) and obtain (78) (recall that ρ_* depends on δ , that's why the last term $o_\delta(1)$ is needed in (78)).

Step 4. *We prove the optimal lower bound in the outer annulus*

$$\liminf_{\varepsilon \rightarrow 0} \hat{F}_\varepsilon(\phi_\varepsilon; B_1^+ \setminus B_{\rho_*}) \geq \pi \log \frac{1}{\rho_*}.$$

In fact, we prove the following more general case that is needed in the proof of Theorem 4.2:

Claim: If $\ell \in \mathbb{Z}$ and $\phi_\varepsilon \in H^1(B_1^+)$ with $\phi_\varepsilon(x, 0) \rightarrow \ell\phi^*(x, 0)$ in $L^1((-1, 1))$ as $\varepsilon \rightarrow 0$, then

$$\liminf_{\varepsilon \rightarrow 0} \hat{F}_\varepsilon(\phi_\varepsilon; B_1^+ \setminus B_{\rho_*}) \geq \pi\ell^2 \log \frac{1}{\rho_*}, \quad \text{for every } \rho_* \in (0, 1). \quad (81)$$

For that, we start by fixing $\rho_* \in (0, 1)$ and focusing on the set

$$S_\varepsilon = \left\{ s \in (\rho_*, 1) : |\phi_\varepsilon(-s, 0) - \ell\pi| + |\phi_\varepsilon(s, 0)| < \frac{1}{4} \right\}.$$

It is clear that $|S_\varepsilon| \rightarrow 1 - \rho_*$ as $\varepsilon \rightarrow 0$, since $\phi_\varepsilon \rightarrow \ell\phi^*$ in $L^1(I_1)$. By Hölder's inequality,

$$|\phi_\varepsilon(-r, 0) - \phi_\varepsilon(r, 0)| \leq \int_0^\pi |\partial_\theta \phi_\varepsilon(re^{i\theta})| d\theta \leq \left(\int_0^\pi |\partial_\theta \phi_\varepsilon(re^{i\theta})|^2 d\theta \right)^{1/2} \pi^{1/2}, \quad r \in (0, 1),$$

so using f_ε defined in (73), we estimate

$$f_\varepsilon(r) \geq \frac{1}{\pi r} (\phi_\varepsilon(r, 0) - \phi_\varepsilon(-r, 0))^2 + \frac{1}{2\pi\varepsilon} (\sin^2 \phi_\varepsilon(r, 0) + \sin^2 \phi_\varepsilon(-r, 0)), \quad r \in (0, 1).$$

If we restrict to $r \in S_\varepsilon$, there is a constant $C_9 > 0$ such that

$$\begin{aligned} \sin^2 \phi_\varepsilon(r, 0) + \sin^2 \phi_\varepsilon(-r, 0) &\geq 2C_9((\ell\pi - \phi_\varepsilon(-r, 0))^2 + \phi_\varepsilon(r, 0)^2) \\ &\geq C_9(\ell\pi - \phi_\varepsilon(-r, 0) + \phi_\varepsilon(r, 0))^2, \quad r \in (\rho_*, 1) \end{aligned}$$

so

$$f_\varepsilon(r) \geq \inf_{s \in \mathbb{R}} \left(\frac{s^2}{\pi r} + C_9 \frac{(\ell\pi - s)^2}{2\pi\varepsilon} \right), \quad r \in S_\varepsilon.$$

Optimising over s , we obtain for a constant $C_{10} > 0$:¹⁷

$$f_\varepsilon(r) \geq \frac{\pi\ell^2}{r + C_{10}\varepsilon}, \quad r \in S_\varepsilon$$

yielding

$$\begin{aligned} \hat{F}_\varepsilon(\phi_\varepsilon; B_1^+ \setminus B_{\rho_*}) &\geq \int_{S_\varepsilon} \frac{\pi\ell^2}{r + C_{10}\varepsilon} dr \\ &\geq \int_{1-|S_\varepsilon|}^1 \frac{\pi\ell^2}{r + C_{10}\varepsilon} dr = \pi\ell^2 \log \frac{1 + C_{10}\varepsilon}{1 - |S_\varepsilon| + C_{10}\varepsilon}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, as $|S_\varepsilon| \rightarrow 1 - \rho_*$, this yields (81) and proves the claim.

Combining (81) and (78) (in the case $\ell = 1$), we obtain

$$\liminf_{\varepsilon \rightarrow 0} \left(\hat{F}_\varepsilon(\phi_\varepsilon; B_1^+) - \pi \log \frac{1}{\varepsilon} \right) \geq \gamma_0 - o_\delta(1),$$

so letting $\delta \rightarrow 0$ we obtain the desired conclusion. \square

We need the following estimate, which is closely related to a result from Struwe [40].

Lemma 4.17 *Let f be a measurable function on the unit half disk $B_1^+ \subset \mathbb{R}^2$ with the following property: There exists $r_0 < 1$ and $A > 0$ such that for every $0 < r \leq r_0$,*

$$\|f\|_{L^2(B_1^+ \setminus B_r)}^2 \leq A(1 + \log \frac{1}{r}).$$

Then for $1 \leq q < 2$ we have

$$\|f\|_{L^q(B_1^+)} \leq C(A, q, r_0) < \infty,$$

where $C(A, q, r_0)$ is independent of f .

Proof. Let $1 \leq q < 2$. Using Hölder's inequality (as $q < 2$), we calculate for $r_j = 2^{-j}r_0$:

$$\begin{aligned} \int_{B_{r_0}^+} |f|^q dx &= \sum_{j=0}^{\infty} \int_{B_{r_j}^+ \setminus B_{r_{j+1}}} (|f|^2)^{\frac{q}{2}} dx \\ &\leq \sum_{j=0}^{\infty} \left(\int_{B_{r_j}^+ \setminus B_{r_{j+1}}} |f|^2 dx \right)^{\frac{q}{2}} |B_{r_j}^+ \setminus B_{r_{j+1}}|^{1-\frac{q}{2}} \\ &\leq \sum_{j=0}^{\infty} \left(\int_{B_1^+ \setminus B_{r_{j+1}}} |f|^2 dx \right)^{\frac{q}{2}} \left(\frac{\pi}{2} \right)^{1-\frac{q}{2}} (2^{-j}r_0)^{2-q} \\ &\leq C \sum_{j=0}^{\infty} (1 + j \log 2 - \log r_0)^{\frac{q}{2}} 2^{-(2-q)j}. \end{aligned}$$

¹⁷The function $h(s) = \frac{s^2}{\pi r} + C_9 \frac{(\ell\pi - s)^2}{2\pi\varepsilon}$ is a parabola having the minimum s_* satisfying $s_* = C_9 r (\ell\pi - s_*) / (2\varepsilon)$ which yields $h(s_*) = \ell s_* / r = \pi\ell^2 / (r + \frac{2\varepsilon}{C_9})$.

The sum converges by the root test so $\|f\|_{L^q(B_{r_0}^+)} \leq C(A, q, r_0)$. We also clearly have that $\|f\|_{L^q(B_1^+ \setminus B_{r_0})} \leq C(A, q, r_0)$. \square

Proof of Theorem 4.2. We divide the proof in several steps:

Step 1. *Proof of point 1.* For small $r > 0$, using Lemma 4.3 (and the notation therein), we can cover a neighbourhood of $\partial\Omega$ with a finite number of patches $A_j = \Psi_{p_j}(B_{r(1-c_1r \log \frac{1}{r})}^+ \cup I_{r(1-c_1r \log \frac{1}{r})}) \subset B_r(p_j) \cap \overline{\Omega}$ for a finite set of points $p_j \in \partial\Omega$, such that $\cup A_j$ is relatively open in $\overline{\Omega}$ and the functions $\psi_\varepsilon^{(j)} = \phi_\varepsilon \circ \Psi_{p_j}$ satisfy the energy estimate for the functionals (51):

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \hat{F}_\varepsilon^{(g^{(j)})}(\psi_\varepsilon^{(j)}; B_\rho^+) < \infty$$

where we have denoted $g^{(j)} = g \circ \Psi_{p_j}$ for g a lifting as given in (46) and $\rho = r(1 - c_1r \log \frac{1}{r})$. On each patch, we arrange $g^{(j)}$ to be continuous. From (53) in Lemma 4.4, we find that the functions $w_\varepsilon^{(j)} = \psi_\varepsilon^{(j)} - \widehat{g^{(j)}}_\rho$ then satisfy

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \hat{F}_\varepsilon^{(0)}(w_\varepsilon^{(j)}; B_\rho^+) < \infty.$$

We can now use Lemma 4.5 to reduce $\hat{F}_\varepsilon^{(0)}$ to F_ε defined at (55) and apply Proposition 4.7 to see that for a sequence / family $z_\varepsilon^{(j)} \in \mathbb{Z}$, $w_\varepsilon^{(j)}(\cdot, 0) - \pi z_\varepsilon^{(j)}$ are bounded in $L^p(I_\rho)$ for every $p \in [1, \infty)$. By Proposition 4.8, we have up to a subsequence the L^p convergence $w_\varepsilon^{(j)}(\cdot, 0) - \pi z_\varepsilon^{(j)} \rightarrow w_0^{(j)} \in BV(I_\rho; \pi\mathbb{Z})$. Changing variables, we obtain convergence for $\phi_\varepsilon - \pi z_\varepsilon^{(j)}$ in $L^p(A_j \cap \partial\Omega)$. If $\partial\Omega \cap A_j \cap A_k \neq \emptyset$, it follows that $z_\varepsilon^{(j)} - z_\varepsilon^{(k)} (\in \mathbb{Z})$ converges as $\varepsilon \rightarrow 0$, i.e., it is constant for small ε ; in particular, we may choose a subsequence $z_\varepsilon \in \mathbb{Z}$ that works for all of the patches A_j . Adding up the results on the patches, it follows that $(\phi_\varepsilon - \pi z_\varepsilon)_\varepsilon$ is bounded and converges as claimed in $L^p(\partial\Omega)$ for every $p \in [1, \infty)$ to a limit function ϕ_0 on $\partial\Omega$ that satisfies $\phi_0 - g \in BV(\partial\Omega; \pi\mathbb{Z})$. Furthermore, $\partial_\tau \phi_\varepsilon \rightarrow \partial_\tau \phi_0$ in $W^{-1,p}$. With $\varkappa = [\partial_\tau g]_{ac}$ we obtain that $\partial_\tau \phi_0 - \varkappa = -\pi \sum_{j=1}^N d_j \delta_{a_j}$, where the a_j can be chosen mutually distinct. The measures $\partial_\tau \phi_\varepsilon$ all average to zero, so $\partial_\tau \phi_0$ does as well, and we must have that $\sum d_j = 2$ (due to the Gauß-Bonnet theorem in the proof of Lemma 4.1). To show the lower bound, we consider for small $r > 0$ disjoint patches A_j as above, centred at a_j . Defining $w_\varepsilon^{(j)}$ as above and setting $\rho = r(1 - c_1r \log \frac{1}{r})$ and $\tilde{\varepsilon} = \frac{\varepsilon}{1 - c_1r \log \frac{1}{r}}$, the results of Lemma 4.3, the convergence of ϕ_ε in $L^2(\partial\Omega)$ and (54) imply

$$\int_{B_r(a_j) \cap \Omega} |\nabla \phi_\varepsilon|^2 dx + \frac{1}{2\pi\varepsilon} \int_{B_r(a_j) \cap \partial\Omega} \sin^2(\phi_\varepsilon - g) d\mathcal{H}^1 \geq \hat{F}_{\tilde{\varepsilon}}^{(0)}(w_\varepsilon^{(j)}; B_\rho^+) - Cr^{\frac{1}{2}}. \quad (82)$$

In I_ρ , we have $w_\varepsilon^{(j)}(\cdot, 0) \rightarrow w_*^{(j)}$, where $w_*^{(j)}$ is locally constant except for a single jump of height $d_j\pi$. Subtracting a suitable constant, we can apply Corollary 4.12 and obtain

$$\hat{F}_{\tilde{\varepsilon}}^{(0)}(w_\varepsilon^{(j)}; B_\rho^+) \geq \pi |d_j| \log \frac{\rho}{\tilde{\varepsilon}} - M_2 |d_j| = \pi |d_j| \log \frac{r}{\tilde{\varepsilon}} - M_2 |d_j| + 2\pi |d_j| \log(1 - c_1r \log \frac{1}{r}). \quad (83)$$

Combining the results on each of the disjoint patches and dividing by $|\log \varepsilon|$, it follows that

$$\frac{1}{|\log \varepsilon|} \mathcal{G}_\varepsilon(\phi_\varepsilon) \geq \pi \sum_j |d_j| - \frac{1}{|\log \varepsilon|} (\pi \sum_j |d_j| (|\log r| + M_2) + Cr^{\frac{1}{2}}),$$

and letting $\varepsilon \rightarrow 0$ we obtain the first order lower bound as claimed.

Step 2. Proof of point 2. Assume now the stronger condition (47). For small $r > 0$, let $\rho = r(1 - c_1 r \log \frac{1}{r})$. From (82), we then must have

$$\sum_{j=1}^N \hat{F}_\varepsilon^{(0)}(w_\varepsilon^{(j)}; B_\rho^+) \leq \sum_{j=1}^N \pi |d_j| \log \frac{\rho}{\varepsilon} + K_0,$$

where $K_0 = K_0(\rho)$ is independent of ε .

For $\sigma < \rho$, we use Lemma 4.5 and Corollary 4.12, which shows for $\varepsilon < \varepsilon_0$

$$\sum_{j=1}^N \hat{F}_\varepsilon^{(0)}(w_\varepsilon^{(j)}; B_\sigma^+) \geq \sum_{j=1}^N \pi |d_j| \log \frac{\sigma}{\varepsilon} - \sum_{j=1}^N |d_j| M_2$$

so in $B_\rho^+ \setminus B_\sigma$, we obtain

$$\limsup_{\varepsilon \rightarrow 0} \sum_{j=1}^N \hat{F}_\varepsilon^{(0)}(w_\varepsilon^{(j)}; B_\rho^+ \setminus B_\sigma) \leq \sum_{j=1}^N \pi |d_j| \log \frac{\rho}{\sigma} + K_0 + \sum_{j=1}^N |d_j| M_2. \quad (84)$$

However, as $w_\varepsilon^{(j)} \rightarrow w_*^{(j)}$ in $L^1(I_\rho)$ where $w_*^{(j)}$ is locally constant away from one jump point of size $d_j \pi$, by (81), we get

$$\liminf_{\varepsilon \rightarrow 0} \sum_{j=1}^N \hat{F}_\varepsilon^{(0)}(w_\varepsilon^{(j)}; B_\rho^+ \setminus B_\sigma) \geq \sum_{j=1}^N \pi d_j^2 \log \frac{\rho}{\sigma},$$

so

$$\sum_{j=1}^N \pi (d_j^2 - |d_j|) \log \frac{\rho}{\sigma} \leq K_0 + \sum_{j=1}^N |d_j| M_2.$$

Letting $\sigma \rightarrow 0$, we obtain that this is only possible if $\sum_{j=1}^N (d_j^2 - |d_j|) \leq 0$, so $d_j = \pm 1$ as claimed.

From (82), (83) and (47), we find the existence of a constant K_1 such that for every small $\varepsilon > 0$ and $r > 0$,

$$\int_{\Omega_r} |\nabla \phi_\varepsilon|^2 dx \leq N \pi \log \frac{1}{r} + K_1,$$

where

$$\Omega_r = \Omega \setminus \bigcup_j B_r(a_j).$$

We conclude using Lemma 4.17 that $\nabla \phi_\varepsilon$ are uniformly bounded in $L^q(\Omega)$ for every $q < 2$.

It follows that there exists $\hat{\phi}_0 \in W^{1,q}(\Omega)$ such that for a subsequence, $\phi_\varepsilon \rightharpoonup \hat{\phi}_0$ weakly in $W^{1,q}(\Omega)$ and weakly in $H^1(O)$ for any open O with $\bar{O} \subset \bar{\Omega} \setminus \{a_1, \dots, a_N\}$. By the trace theorem, we deduce that $\hat{\phi}_0$ is an extension (in Ω) of the boundary limit ϕ_0 found at point 1. By lower semicontinuity of the Dirichlet integral, we find that

$$\int_{\Omega_r} |\nabla \hat{\phi}_0|^2 dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_r} |\nabla \phi_\varepsilon|^2 dx.$$

In order to prove the second order lower bound (48) for ϕ_ε , we replace ϕ_ε by ϕ_ε^* which is the harmonic extension of $\phi_\varepsilon|_{\partial\Omega}$ to Ω , i.e., ϕ_ε^* is the minimizer of the Dirichlet energy in Ω under the Dirichlet boundary condition $\phi_\varepsilon|_{\partial\Omega}$. Therefore, $\mathcal{G}_\varepsilon(\phi_\varepsilon) \geq \mathcal{G}_\varepsilon(\phi_\varepsilon^*)$ and it is sufficient to prove (48) for ϕ_ε^* . By the above argument for the convergence of ϕ_ε , replacing ϕ_ε by ϕ_ε^* , we know that ϕ_ε^*

converges weakly in $W^{1,p}(\Omega)$ and weakly in $H^1(O)$ for any open O with $\overline{O} \subset \overline{\Omega} \setminus \{a_1, \dots, a_N\}$ to the harmonic extension ϕ_* to Ω of $\phi_0 : \partial\Omega \rightarrow \mathbb{R}$. Using lower semicontinuity of the Dirichlet integral, we find by letting $\varepsilon \rightarrow 0$:

$$\int_{\Omega_r} |\nabla \phi_*|^2 dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_r} |\nabla \phi_\varepsilon|^2 dx.$$

By definition of W_Ω we know that

$$\int_{\Omega_r} |\nabla \phi_*|^2 dx = \pi N \log \frac{1}{r} + W_\Omega(\{(a_j, d_j)\}) + o_r(1).$$

From Proposition 4.16 and (82), we find

$$\liminf_{\varepsilon \rightarrow 0} \left(\int_{\bigcup_j B_r(a_j) \cap \Omega} |\nabla \phi_\varepsilon|^2 dx + \frac{1}{2\pi\varepsilon} \int_{\bigcup_j B_r(a_j) \cap \partial\Omega} \sin^2(\phi_\varepsilon - g) d\mathcal{H}^1 - N(\pi \log \frac{r}{\varepsilon} + \gamma_0) \right) \geq -CNr^{\frac{1}{2}}. \quad (85)$$

Combining the last three relations, we see that

$$\liminf_{\varepsilon \rightarrow 0} \left(\mathcal{G}_\varepsilon(\phi_\varepsilon) - \pi N \log \frac{1}{\varepsilon} - N\gamma_0 - W_\Omega(\{(a_j, d_j)\}) \right) \geq o_r(1)$$

Letting $r \rightarrow 0$, we conclude (48). Note that if we do not replace ϕ_ε by ϕ_ε^* , the same argument shows that, for a sequence $r \rightarrow 0$ that realizes the lim sup in (18),

$$\liminf_{\varepsilon \rightarrow 0} \left(\mathcal{G}_\varepsilon(\phi_\varepsilon) - \pi N \log \frac{1}{\varepsilon} - N\gamma_0 - \mathcal{W}(\hat{\phi}_0; \{(a_j, d_j)\}) \right) \geq o_r(1),$$

which proves the lower bound (19).

Step 3. Proof of point 3. Let ϕ_* be the harmonic extension of ϕ_0 given in Definition 1.3, and let $r > 0$ be a small radius. For each j we use Lemma 4.3 and find $\Psi_{a_j} : \overline{B_{2r}^+} \rightarrow \overline{\Omega}$ as there.

Close to a_j , for a suitable choice of the argument function and arguing as in Lemma 4.4, $\phi_* = \pm d_j \arg(z - a_j) + h(z)$, for $h \in W^{1,p}$ in a neighbourhood of a_j for all $p \in [1, \infty)$, with bounds depending only on p , $\partial\Omega$ and the choice of $\{(a_j, d_j)\}$ (since g is Lipschitz). Clearly $\sin(h - g) = 0$ on $\partial\Omega$. It follows using the Dini regularity of Ψ_{a_j} that $\hat{\phi}_* = \phi_* \circ \Psi_{a_j}$ can be written as

$$\hat{\phi}_*(z) = \pm d_j \arg(z) + \hat{h}(z)$$

in a neighborhood of 0 with $\hat{h} = h \circ \Psi_{a_j}$ bounded in $W^{1,p}$ around the origin as above. We now define

$$\hat{\psi}_\varepsilon = \begin{cases} \phi_*(z) & \text{if } |\Psi_{a_j}^{-1}(z)| > r \\ h(z) \pm \phi_\varepsilon(\Psi_{a_j}^{-1}(z)) & \text{elsewhere,} \end{cases}$$

where ϕ_ε is the function as defined in Step 1 of the proof of Lemma 4.14 for $d_j = \pm 1$ and $\phi_\varepsilon = \phi_{d_j, \varepsilon}$ as in the proof of Lemma 4.15 for $|d_j| > 1$. Then $\hat{\psi}_\varepsilon$ is continuous in $\overline{\Omega}$. From our construction, it is clear that $\hat{\psi}_\varepsilon \rightarrow \phi_0$ in all $L^p(\partial\Omega)$.

Using Lemma 4.3 and the definition of ϕ_* , denoting $E_r = \bigcup_{j=1}^N (B_r(a_j) \Delta \Psi_{a_j}(B_r^+))$, we have

$$\int_{E_r} |\nabla \hat{\psi}_\varepsilon|^2 dx \leq C \int_{r(1-c_1r \log \frac{1}{r})}^{r(1+c_1r \log \frac{1}{r})} \frac{1}{s} ds = O(r \log \frac{1}{r}).$$

Inside $\Psi_{a_j}(B_r^+)$, we compute the energy of $\hat{\psi}_\varepsilon$. By conformal mapping, we can compute it in B_r^+ . Note that

$$\int_{B_r^+} |\nabla\phi_\varepsilon + \nabla\hat{h}|^2 dx = \int_{B_r^+} |\nabla\phi_\varepsilon|^2 dx + 2 \int_{B_r^+} \nabla\phi_\varepsilon \cdot \nabla\hat{h} dx + \int_{B_r^+} |\nabla\hat{h}|^2 dx.$$

For $d_j = \pm 1$, by Lemma 4.14,

$$\hat{F}_\varepsilon(\phi_\varepsilon; B_r^+) \leq \pi \log \frac{r}{\varepsilon} + \gamma_0 + o_r(1).$$

Using $\int_{B_r \cap \mathbb{R}} \sin^2 \phi_\varepsilon d\mathcal{H}^1 \leq C$, we find using Lemma 4.3 that

$$\int_{B_r(a_j) \cap \partial\Omega} \sin^2(\hat{\psi}_\varepsilon - g) d\mathcal{H}^1 = \int_{B_r \cap \mathbb{R}} \sin^2 \phi_\varepsilon d\mathcal{H}^1 + O(r \log \frac{1}{r}).$$

As $\hat{h} \in W^{1,p}$ and $|\nabla\phi_\varepsilon(x, y)| \leq \frac{1}{|(x, y)|}$, we can estimate using Hölder's inequality

$$\left| \int_{B_r^+} \nabla\phi_\varepsilon \cdot \nabla\hat{h} dx \right| \leq \left(\int_{B_r^+} |\nabla\phi_\varepsilon|^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \left(\int_{B_r^+} |\nabla\hat{h}|^3 dx \right)^{\frac{1}{3}} \leq o_r(1) \quad \text{and} \quad \int_{B_r^+} |\nabla\hat{h}|^2 dx \leq o_r(1).$$

As the Dirichlet energy of ϕ_* in $\Omega \setminus \bigcup_{j=1}^N B_r(a_j)$ is $W_\Omega(\{(a_j, d_j)\}) + N\pi|\log r| + o_r(1)$, we can thus establish that the upper bound (50) holds for $\hat{\psi}_\varepsilon$ with an error $o_r(1)$. Choosing r sufficiently small, we see that (50) must hold. Replacing $\hat{\psi}_\varepsilon$ by the harmonic function $\hat{\psi}_\varepsilon^*$ with the same boundary conditions on $\partial\Omega$, the energy does not increase. As harmonic functions satisfy $\|f\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}$, we obtain that $\hat{\psi}_\varepsilon^* \rightarrow \phi_*$ in $L^2(\Omega)$ and using the maximum principle and Hölder's inequality we obtain $\hat{\psi}_\varepsilon^* \rightarrow \phi_*$ in $L^p(\Omega)$ for every $p \in [1, \infty)$. For $|d_j| > 1$, applying the result of Lemma 4.15 similarly leads to (49).¹⁸

□

5 Second order Γ -convergence for the full energy. Proof of Theorems 1.2, 1.4 and 1.5

In this section we prove Theorems 1.2, 1.4 and 1.5.

Proof of Theorem 1.2. Let (u_ε) be a sequence / family of maps with $E_{\varepsilon, \eta}(u_\varepsilon) \leq C|\log \varepsilon|$. Then we can use Theorem 3.1 to construct a sequence / family U_ε with the following properties:

- $U_\varepsilon \in H^1(\Omega; \mathbb{S}^1)$;
- $\|U_\varepsilon - u_\varepsilon\|_{L^p(\Omega)} \rightarrow 0$, $\|U_\varepsilon - u_\varepsilon\|_{L^p(\partial\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$ for every $p \in [1, \infty)$;
- $E_{\varepsilon, \eta}(U_\varepsilon) \leq E_{\varepsilon, \eta}(u_\varepsilon) + o_\varepsilon(1)$
- $\mathcal{J}(U_\varepsilon) - \mathcal{J}(u_\varepsilon) \rightarrow 0$ in $(\text{Lip}(\Omega))^*$ as $\varepsilon \rightarrow 0$.

¹⁸For a different proof of the upper bound construction in the case $g = 0$, we refer to [11].

By Lemma 4.1, we find ϕ_ε such that $U_\varepsilon = e^{i\phi_\varepsilon}$ and $E_{\varepsilon,\eta}(U_\varepsilon) = \mathcal{G}_\varepsilon(\phi_\varepsilon)$, with \mathcal{G}_ε defined in (45). The global Jacobian of U_ε is given by (8) as $\mathcal{J}(U_\varepsilon) = \mathcal{J}_{bd}(U_\varepsilon) = -\partial_\tau \phi_\varepsilon|_{\partial\Omega}$ as a distribution in $H^{-1/2}(\partial\Omega)$ (see Proposition 2.2). By Theorem 4.2, for a subsequence, there exist integers $z_\varepsilon \in \mathbb{Z}$ (either all of them are even, or all are odd) such that $\phi_\varepsilon - \pi z_\varepsilon$ converges to a limit ϕ_0 in $L^p(\partial\Omega)$ for every $p \in [1, \infty)$ where $\phi_0 - g \in BV(\partial\Omega; \pi\mathbb{Z})$ with g given in (46). As $|e^{is} - e^{it}| \leq \frac{\pi}{2}|s - t|$ for every $s, t \in \mathbb{R}$, we deduce that $U_\varepsilon \rightarrow \pm e^{i\phi_0}$ in any $L^p(\partial\Omega)$. Changing ϕ_0 in $\phi_0 - \pi$ (in the case where z_ε are all odd), we obtain the desired convergence $u_\varepsilon \rightarrow e^{i\phi_0}$ in $L^p(\partial\Omega)$ for every $p \in [1, \infty)$. Moreover, the convergence $\partial_\tau \phi_\varepsilon \rightarrow \partial_\tau \phi_0$ in $W^{-1,p}(\partial\Omega)$ for any $p \in (1, \infty)$ directly induces the convergence of $\mathcal{J}(U_\varepsilon) \rightarrow J$ as $\varepsilon \rightarrow 0$ as claimed in $(\text{Lip}(\Omega))^*$. As $\|\mathcal{J}(U_\varepsilon) - \mathcal{J}(u_\varepsilon)\| \rightarrow 0$ in $(\text{Lip}(\Omega))^*$, we obtain that $\mathcal{J}(u_\varepsilon)$ tends to the same limit. Since $E_{\varepsilon,\eta}(u_\varepsilon) \geq E_{\varepsilon,\eta}(U_\varepsilon) - o(1) = \mathcal{G}_\varepsilon(\phi_\varepsilon) - o(1)$, the lower bounds for $\mathcal{G}_\varepsilon(\phi_\varepsilon)$ directly translate into the claimed lower bound for $E_{\varepsilon,\eta}(u_\varepsilon)$ at the first order. \square

Proof of Theorem 1.4. Continuing as in the previous proof (with the same notation), we note that the estimate (14) transfers to $\mathcal{G}_\varepsilon(\phi_\varepsilon)$ so that Theorem 4.2 yields the claims about $|d_i| = 1$ and again, $E_{\varepsilon,\eta}(u_\varepsilon) \geq E_{\varepsilon,\eta}(U_\varepsilon) - o(1) = \mathcal{G}_\varepsilon(\phi_\varepsilon) - o(1)$ implies the desired lower bound for $E_{\varepsilon,\eta}$ at the second order.

To show the $L^q(\Omega)$ bound for ∇u_ε for every $q < 2$, we proceed as follows: Using the boundary vortices a_j with their multiplicities d_j from Theorem 4.2 coming from the lifting ϕ_ε of U_ε , we have by (85) that

$$\liminf_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \left(\int_{\bigcup B_\rho(a_j) \cap \Omega} |\nabla U_\varepsilon|^2 dx + \frac{1}{2\pi\varepsilon} \int_{\bigcup B_\rho(a_j) \cap \partial\Omega} (U_\varepsilon \cdot \nu)^2 d\mathcal{H}^1 - N(\pi \log \frac{\rho}{\varepsilon} + \gamma_0) \right) \geq 0.$$

From (32) applied to $G = B_{\rho(1+\rho)}(a_j) \cap \Omega$ so that $G_\eta \supset B_\rho(a_j) \cap \Omega$, we now deduce (since $\log \frac{\rho(1+\rho)}{\rho} \rightarrow 0$ as $\rho \rightarrow 0$)

$$\liminf_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \left(\int_{\bigcup B_\rho(a_j) \cap \Omega} |\nabla u_\varepsilon|^2 + \frac{1}{\eta^2} (1 - |u_\varepsilon|^2)^2 dx + \frac{1}{2\pi\varepsilon} \int_{\bigcup B_\rho(a_j) \cap \partial\Omega} (u_\varepsilon \cdot \nu)^2 d\mathcal{H}^1 - N(\pi \log \frac{\rho}{\varepsilon} + \gamma_0) \right) \geq 0. \quad (86)$$

Using Corollary 4.12 and (32), we also find C such that for $\rho < \rho_0$, $\varepsilon < \varepsilon_0$:

$$\int_{\bigcup B_\rho(a_j) \cap \Omega} |\nabla u_\varepsilon|^2 + \frac{1}{\eta^2} (1 - |u_\varepsilon|^2)^2 dx + \frac{1}{2\pi\varepsilon} \int_{\bigcup B_\rho(a_j) \cap \partial\Omega} (u_\varepsilon \cdot \nu)^2 d\mathcal{H}^1 \geq N\pi \log \frac{\rho}{\varepsilon} - C, \quad (87)$$

hence, by (14),

$$\int_{\Omega \setminus \bigcup B_\rho(a_j)} |\nabla u_\varepsilon|^2 dx \leq N\pi \log \frac{1}{\rho} + C. \quad (88)$$

so Lemma 4.17 applies and shows that $\limsup_{\varepsilon \rightarrow 0} \|\nabla u_\varepsilon\|_{L^q(\Omega)} < \infty$ for every $q < 2$.

Finally, we need to show (16), which clearly implies (17) via (87). However, (16) follows from the exact same argument as used in Lemma 4.13: Let $\tilde{u}_{2\varepsilon} := u_\varepsilon$, $\tilde{U}_{2\varepsilon} := U_\varepsilon$ and as in (87), apply Corollary 4.12 for $(\tilde{u}_{2\varepsilon})$ to get for ε sufficiently small,

$$E_{2\varepsilon, 2\eta}(u_\varepsilon) = E_{2\varepsilon, 2\eta}(\tilde{u}_{2\varepsilon}) \geq N\pi \log \frac{1}{2\varepsilon} - C,$$

while by the upper bound for some fixed $\rho > 0$:

$$E_{\varepsilon, \eta}(u_\varepsilon) \leq N\pi \log \frac{1}{\varepsilon} + C,$$

so

$$\frac{3}{4\eta^2} \int_{\Omega} (1 - |u_{\varepsilon}|^2)^2 dx + \frac{1}{4\pi\varepsilon} \int_{\partial\Omega} (u_{\varepsilon} \cdot \nu)^2 d\mathcal{H}^1 = E_{\varepsilon,\eta}(u_{\varepsilon}) - E_{2\varepsilon,2\eta}(u_{\varepsilon}) \leq \tilde{C},$$

which clearly implies (16).

For point *iv*), by Theorem 4.2, we know that up to a subsequence and an additive constant, $\phi_{\varepsilon} \rightarrow \phi_0$ a.e. in $\partial\Omega$ which by dominated convergence theorem implies that $U_{\varepsilon} = e^{i\phi_{\varepsilon}} \rightarrow e^{i\phi_0}$ in every $L^p(\partial\Omega)$ for every $p \in [1, \infty)$. By (30), we know that $u_{\varepsilon} - U_{\varepsilon} \rightarrow 0$ in $L^p(\partial\Omega)$, therefore $u_{\varepsilon} - e^{i\phi_0} \rightarrow 0$ in $L^p(\partial\Omega)$. As (u_{ε}) is bounded in $W^{1,q}(\Omega)$ for every $q \in [1, 2)$ and $\|U_{\varepsilon} - u_{\varepsilon}\|_{L^p(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$ for every $p \in [1, \infty)$, by the trace theorem and Theorem 4.2 point 2), for a subsequence, u_{ε} converges weakly in $W^{1,q}(\Omega)$ and strongly in $L^p(\Omega)$ for every $p \geq 1$ to the limit of $U_{\varepsilon} = e^{i\phi_{\varepsilon}}$ which is an \mathbb{S}^1 -valued extension $e^{i\hat{\phi}_0}$ of $e^{i\phi_0}$ in Ω . \square

Proof of Theorem 1.5. The upper bound construction is a direct consequence of the corresponding construction for $\mathcal{G}_{\varepsilon}$ in Theorem 4.2. With $\hat{\psi}_{\varepsilon}^*$ as constructed there, we set $u_{\varepsilon} = e^{i\hat{\psi}_{\varepsilon}^*}$, then $|u_{\varepsilon}| = 1$ and $\mathcal{J}(u_{\varepsilon}) = -\partial_{\tau}\hat{\psi}_{\varepsilon}^* \mathcal{H}^1 \llcorner \partial\Omega$, and then the convergence and energy bound results follow directly using $E_{\varepsilon,\eta}(u_{\varepsilon}) = \mathcal{G}_{\varepsilon}(\hat{\psi}_{\varepsilon}^*)$. \square

Proof of Corollary 1.6. The existence of minimizers u_{ε} of $E_{\varepsilon,\eta}$ is a consequence of the direct method in the calculus of variations. Let a_1^* and a_2^* be two distinct points in $\partial\Omega$ that minimize the renormalized energy (for the multiplicities $d_1^* = d_2^* = 1$):

$$W_{\Omega}(\{(a_1^*, 1), (a_2^*, 1)\}) = \min \left\{ W_{\Omega}(\{(\tilde{a}_1, 1), (\tilde{a}_2, 1)\}) : \tilde{a}_1, \tilde{a}_2 \in \partial\Omega \text{ distinct points} \right\} \quad (89)$$

(such pair (a_1^*, a_2^*) exists, compare [25]). First, by the upper bound in Theorem 1.5 applied to $\{(a_1^*, 1), (a_2^*, 1)\}$, we deduce that every minimizer u_{ε} of $E_{\varepsilon,\eta}$ has to satisfy

$$E_{\varepsilon,\eta}(u_{\varepsilon}) \leq 2\pi |\log \varepsilon| + W_{\Omega}(\{(a_1^*, 1), (a_2^*, 1)\}) + 2\gamma_0 + o(1) \quad \text{as } \varepsilon \rightarrow 0. \quad (90)$$

By the compactness result in Theorem 1.2 point *i*), we have for a subsequence that $\mathcal{J}(u_{\varepsilon}) \rightarrow -\varkappa \mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{k=1}^N d_k \delta_{a_k}$ as in (10) for some distinct points $a_k \in \partial\Omega$ and nonzero integers d_k satisfying $\sum_k d_k = 2$. Moreover, the lower bound in Theorem 1.2 point *ii*) combined with (90) yields $\sum_k |d_k| \leq 2$. It entails that $N = 2$ and $d_1 = d_2 = 1$. Applying the sharper lower bound in Theorem 1.4 point *i*), we deduce that $E_{\varepsilon,\eta}(u_{\varepsilon}) \geq 2\pi |\log \varepsilon| + W_{\Omega}(\{(a_1, 1), (a_2, 1)\}) + 2\gamma_0 + o(1)$. Then (90) leads as $\varepsilon \rightarrow 0$ to $W_{\Omega}(\{(a_1, 1), (a_2, 1)\}) \leq W_{\Omega}(\{(a_1^*, 1), (a_2^*, 1)\})$. In other words, the pair (a_1, a_2) is a minimizer of the the renormalized energy in (89), so $W_{\Omega}(\{(a_1, 1), (a_2, 1)\}) = W_{\Omega}(\{(a_1^*, 1), (a_2^*, 1)\})$ yielding the desired second order expansion of $E_{\varepsilon,\eta}(u_{\varepsilon})$. Also, point *iv*) in Theorem 4.2 implies that for a subsequence, u_{ε} converges weakly in $W^{1,q}(\Omega)$ and strongly in $L^p(\Omega)$ for every $q \in [1, 2)$ and $p \geq 1$ to an \mathbb{S}^1 -valued extension $e^{i\hat{\phi}_0}$ of $e^{i\phi_0}$ in Ω . Moreover, by (88), we for every small $\rho > 0$:

$$\int_{\Omega \setminus (B_{\rho}(a_1) \cup B_{\rho}(a_2))} |\nabla \hat{\phi}_0|^2 dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega \setminus (B_{\rho}(a_1) \cup B_{\rho}(a_2))} |\nabla u_{\varepsilon}|^2 dx \leq 2\pi \log \frac{1}{\rho} + C.$$

It remains to prove that the extension $\hat{\phi}_0$ is harmonic in Ω . Assume towards a contradiction that $\hat{\phi}_0$ is not equal (up to a constant) to the harmonic extension ϕ_* of ϕ_0 . This implies the positivity of

$$\delta := \liminf_{\rho \rightarrow 0} \int_{\Omega \setminus (B_{\rho}(a_1) \cup B_{\rho}(a_2))} (|\nabla \hat{\phi}_0|^2 - |\nabla \phi_*|^2) dx > 0.$$

Indeed, we have $|\nabla\hat{\phi}_0|^2 - |\nabla\phi_*|^2 = |\nabla(\hat{\phi}_0 - \phi_*)|^2 + 2\nabla\phi_* \cdot \nabla(\hat{\phi}_0 - \phi_*)$. Since $\hat{\phi}_0 - \phi_* \in W_0^{1,q}(\Omega)$ for every $q \in [1, 2)$ and ϕ_* behaves (up to the conformal map in Lemma 4.3) as the sum of an angular function around a_1 and a_2 and a harmonic function $h \in W^{1,p}(\Omega)$ for every $p \in [1, \infty)$ (see details in Step 3 of the proof of Theorem 4.2), integration by parts yields

$$\begin{aligned} \left| \int_{\Omega \setminus (B_\rho(a_1) \cup B_\rho(a_2))} \nabla\phi_* \cdot \nabla(\hat{\phi}_0 - \phi_*) dx \right| &= \left| \int_{\Omega \cap (\partial B_\rho(a_1) \cup \partial B_\rho(a_2))} \partial_\nu \phi_* (\hat{\phi}_0 - \phi_*) d\mathcal{H}^1 \right| \\ &= \left| \int_{(B_\rho(a_1) \cup B_\rho(a_2)) \cap \Omega} \nabla h \cdot \nabla(\hat{\phi}_0 - \phi_*) dx \right| \\ &\leq \|\nabla h\|_{L^3((B_\rho(a_1) \cup B_\rho(a_2)) \cap \Omega)} \|\nabla(\hat{\phi}_0 - \phi_*)\|_{L^{3/2}((B_\rho(a_1) \cup B_\rho(a_2)) \cap \Omega)} \rightarrow 0 \quad \text{as } \rho \rightarrow 0 \end{aligned}$$

where ν is the normal vector at $\partial B_\rho(a_1)$ and $\partial B_\rho(a_2)$; thus, $\delta > 0$. By (86) we obtain

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \left(E_{\varepsilon, \eta}(u_\varepsilon) - 2(\pi \log \frac{1}{\varepsilon} + \gamma_0) \right) &\geq \liminf_{\rho \rightarrow 0} \left(\int_{\Omega \setminus (B_\rho(a_1) \cup B_\rho(a_2))} |\nabla\hat{\phi}_0|^2 dx - 2\pi \log \frac{1}{\rho} \right) \\ &\geq \delta + W_\Omega(\{(a_1, 1), (a_2, 1)\}) \end{aligned}$$

which contradicts the upper bound satisfied by $E_{\varepsilon, \eta}(u_\varepsilon)$. \square

Acknowledgment. R.I. acknowledges partial support by the ANR project ANR-14-CE25-0009-01.

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