Singularities in some variational problems

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To my parents
and my sister, Claudia.

Contents

Introduction

This book represents the PhD thesis of the author that was carried out under the supervision of Haïm Brezis at the University Paris 6 between 2003-2006. This PhD thesis was refereed by Francois Alouges and Peter Sternberg and was defended in December 11, 2006 in front of the Jury composed by Amandine Aftalion, Francois Alouges, Fabrice Béthuel, Haïm Brezis, Mariano Giaquinta, Robert Jerrard and Felix Otto. For this PhD thesis, the author received the Arconati-Visconti Prize in Sciences (Pure and Applied Mathematics) awarded by the Chancellery of Universities of Paris in 2007.

The topic of this PhD thesis concerns the structure of singularities that appear in several variational problems. These singularities play the role of vortex lines for superconductors, vortices in Bose-Einstein condensates or domain walls in micromagnetics. Since these problems are an interplay between analysis and geometry, the study of singularities lies on analytical and geometrical tools based on variational methods, geometric measure theory as well as regularity theory for partial differential equations.

The book is divided in three parts. A first one, formed by Chapters 1-5, deals with functions with values into the unit circle S^1, in particular, the study of the (optimal) lifting of functions of bounded variation. A second part, including Chapters 6 and 7, is devoted to the asymptotic behavior of vortices in a two-dimensional rotating Bose-Einstein condensate. The third part (Chapter 8) concerns the optimality of Néel walls in thin-film micromagnetics.

In the following, we present the main features of each chapter.

1 Lifting of functions with values into the unit circle S^1

The study of functions with values into S^1 and having a certain regularity (for example, belonging to some Sobolev space) is motivated by the theory of Ginzburg-Landau equation and the degree theory. In this context, the main questions concern the regularity of the lifting of such functions and the analysis of their topological singularities. The goal of this part is to answer to these questions in the case of BV functions with values into S^1.

This part is a collection of several works published by the author during his PhD thesis (cf. [37, 51, 52, 53, 58]). Several changes and additional details have been introduced here with respect to the published version, together with a work in progress (cf. [59]). Each chapter can be considered self-contained.

1.1 Lifting of BV functions with values into S^1 (joint work with J. Dávila)

Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to S^1$ be a measurable function. We call lifting of u, every measurable function $\varphi : \Omega \to \mathbb{R}$ satisfying

$$u(x) = e^{i\varphi(x)}$$

for almost every $x \in \Omega$. A natural question concerns the existence of a lifting φ that preserves the regularity of the function u. For example, if Ω is simply connected and u is continuous, then there exists a continuous lifting φ of u that is unique up to an additional $2\pi\mathbb{Z}$ constant.

The first existence result for the lifting in the case of Sobolev spaces has been proved by Béthuel and Zheng [19]: if Ω is a bounded simply connected domain in $\mathbb{R}^N$ and $u \in W^{1,p}(\Omega, S^1)$ with $p \geq 2$ then u has a lifting $\varphi \in W^{1,p}(\Omega, \mathbb{R})$ that is unique up to a constant. Otherwise, if $1 \leq p < 2$ and $N \geq 2$, then there exist some functions $u \in W^{1,p}(\Omega, S^1)$ that cannot be lifted in $W^{1,p}$; the standard example (in the case of $N = 2$ and $0 \in \Omega$) is given by

$$u(x) = \frac{x}{|x|}. \tag{1}$$

Later, Bourgain, Brezis and Mironescu [20] studied the existence of lifting in the general case of Sobolev spaces $W^{s,p}(\Omega, S^1)$, $0 < s < \infty$ and $1 < p < \infty$. A complete description depending on N, s and p is given for the cases where a lifting with the same regularity exists and for the other cases where one can construct functions $u \in W^{s,p}(\Omega, S^1)$ with no lifting belonging to $W^{s,p}$.

In the case of BMO functions, we recall the work of Coifman and Meyer [35]. For the one-dimensional case, they have showed that if $u : \mathbb{R} \to S^1$ belongs to BMO and $|u|_{BMO} < \gamma$ (where $\gamma > 0$ is a specific constant) then u has a BMO lifting φ with a certain control on the BMO seminorm of φ. Later, Brezis and Nirenberg [31] extended this result to the case of general domains $\Omega \subset \mathbb{R}^N$; moreover, they also proved that every function $u \in VMO(\Omega, S^1)$ has a VMO lifting that is unique up an additional constant.

In this chapter, we study the case of functions of bounded variation with values into the unit circle S^1, i.e., $u = (u_1, u_2) \in L^1_{loc}(\Omega, \mathbb{R}^2)$, $|u(x)| = 1$ for almost every $x \in \Omega$ and the $BV-$seminorm is finite:

$$|u|_{BV} = \sup\left\{ \int_\Omega \sum_{k=1}^{2} u_k \operatorname{div} \zeta_k \, dx \; : \; \zeta_k \in C_0^\infty(\Omega, \mathbb{R}^N), \; \sum_{k=1}^{2} |\zeta_k|^2 \leq 1 \text{ in } \Omega \right\} < \infty,$$

where $|\cdot|$ denotes the euclidian norm over $\mathbb{R}^N$. Our main result shows that u always has a BV lifting with an optimal control on the total variation:

Theorem 0.1 *([37]) Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in BV(\Omega, S^1)$. Then there exists a lifting $\varphi \in BV \cap L^\infty(\Omega, \mathbb{R})$ of u such that*

$$|\varphi|_{BV} \leq 2|u|_{BV}. \tag{2}$$

If $N \geq 2$ and Ω is a bounded open set in $\mathbb{R}^N$, the constant 2 in inequality (2) is optimal; it can be checked for the standard example (1). In dimension $N = 1$ (i.e., Ω is an interval), every function $u \in BV(\Omega, S^1)$ has a BV lifting φ such that $|\varphi|_{BV} \leq \frac{\pi}{2}|u|_{BV}$ and the constant $\frac{\pi}{2}$ is optimal.

The idea of the proof of Theorem 0.1 is to consider the argument function $L : S^1 \to \mathbb{R}$,

$$L(e^{i\theta}) = \theta, \quad \forall -\pi \leq \theta < \pi.$$

Then $\varphi = L(u)$ is a (measurable) lifting of u, as well as every function $L(e^{i\alpha}u) - \alpha$ with $\alpha \in \mathbb{R}$. Next we prove a co-area type inequality

$$\int_0^{2\pi} |L(e^{i\alpha}u)|_{BV}\, d\alpha \leq 4\pi |u|_{BV} \tag{3}$$

which leads to our result. In particular, for almost every $\alpha \in \mathbb{R}$, the lifting $L(e^{i\alpha}u) - \alpha$ is of bounded variation. The main tool to prove (3) resides in the chain rule for BV functions; a new proof of (3) without using the chain rule was later found by Merlet [69].

Remark 0.1 *(a)* If $u \in W^{1,1}(\Omega, S^1)$ and $\Omega \subset \mathbb{R}^2$ is a bounded smooth simply connected open set, Brezis and Mironescu showed that u has a lifting $\varphi \in BV(\Omega, \mathbb{R})$ satisfying (2). The idea consists in applying the density result of Béthuel and Zheng [19] so that the proof reduces to the case of functions that are smooth away from a finite set of singular points: for such a function u, one can construct a $BV-$lifting that has the jump set concentrated on the minimal connection between the point singularities of u (taking into account their topological degree) and the boundary of Ω. This lifting satisfies condition (2).

(b) The existence of a BV lifting of $u \in BV(\Omega, S^1)$ (when Ω is a bounded smooth simply connected open set) was proved before by Giaquinta, Modica and Soucek [47], but without an optimal control on the $BV-$seminorm of a lifting.

1.2 Optimal lifting for $BV(S^1, S^1)$

Let $g \in BV(S^1, S^1)$, i.e. $g \in BV(S^1, \mathbb{R}^2)$ and $|g(y)| = 1$ for almost every $y \in S^1$. The aim of this chapter is to compute the total variation of an *optimal* BV lifting of g:

$$E(g) = \inf\left\{ \int_{S^1} |\dot{\varphi}| \,:\, \varphi \in BV(S^1, \mathbb{R}), e^{i\varphi} = g \text{ a.e. in } S^1 \right\} \tag{4}$$

where "$\dot{}$" stands for the tangential derivative. The above infimum is achieved and equal to the relaxed energy

$$E_{\mathrm{rel}}(g) = \inf\left\{ \liminf_{n\to\infty} \int_{S^1} |\dot{g}_n| \mathrm{d}\mathcal{H}^1 \,:\, g_n \in C^\infty(S^1, S^1), \deg g_n = 0,\ g_n \to g \text{ a.e. in } S^1 \right\}.$$

In the sequel, we identify g to the precise representative, that is a Borel function defined as

$$g(y) = \frac{g(y+) + g(y-)}{2},\ \forall y \in S^1,$$

where $g(y\pm)$ are the left and right limits of g at y with respect to the counterclockwise orientation of S^1. The vector measure $\dot{g}$ is decomposed as

$$\dot{g} = (\dot{g})^a + (\dot{g})^c + (\dot{g})^j,$$
$$\text{where} \quad (\dot{g})^j = \sum_{y \in S(g)} (g(y+) - g(y-))\delta_y.$$

Here, $(\dot{g})^a, (\dot{g})^c$ and $(\dot{g})^j$ correspond to the absolutely continuous part, to the Cantor part and to the jump part of the measure $\dot{g}$, respectively. The (most) countable set

$$S(g) = \{y \in S^1 : \dot{g}(\{y\}) \neq 0\}$$

corresponds to the jump set of g. For every jump point $y \in S(g)$, we consider the signed jump size $d_y(g) \in (-\pi, \pi] \setminus \{0\}$ defined as

$$e^{i\, d_y(g)} = \frac{g(y+)}{g(y-)},$$

so that the modulus $|d_y(g)| = d_{S^1}(g(y+), g(y-))$ coincides with the geodesic distance on S^1.

We want to study the structure of an (optimal) lifting of g. So, let $\varphi \in BV(S^1, \mathbb{R})$ be a lifting of g (such a function exists thanks to Theorem 0.1). As above, we decompose the finite measure $\dot{\varphi}$ into three terms:

$$\dot{\varphi} = (\dot{\varphi})^a + (\dot{\varphi})^c + \sum_{z \in S(\varphi)} (\varphi(z+) - \varphi(z-))\delta_z.$$

We deduce that the absolutely continuous part and the Cantor part of $\dot{\varphi}$ are completely determined by the following equations:

$$(\dot{\varphi})^a = g \wedge (\dot{g})^a \quad \text{and} \quad (\dot{\varphi})^c = g \wedge (\dot{g})^c \quad \text{in} \quad S^1.$$

For the jump part, we only know that

$$S(g) \subset S(\varphi) \;\text{ and }\; \begin{cases} \varphi(y+) - \varphi(y-) \equiv d_y(g) \,(\mathrm{mod}\, 2\pi) & \text{if } y \in S(g), \\ \varphi(y+) - \varphi(y-) \equiv 0 \,(\mathrm{mod}\, 2\pi) & \text{if } y \in S(\varphi) \setminus S(g). \end{cases}$$

In order to compute (4), it is sufficient to determine the minimal total variation of the jump part of φ. If g was defined on an interval of $\mathbb{R}$ instead of S^1, then the jump part of an optimal lifting would have the total variation given by

$$\sum_{y \in S(g)} |d_y(g)|. \tag{5}$$

In our framework, since g is defined on S^1, a new constraint needs to be added for a lifting φ:

$$\int_{S^1} \dot{\varphi} = 0. \tag{6}$$

Because of this (topological) constraint, the minimal total variation of a lifting is in general larger than (5). For example, if $g = Id : S^1 \to S^1$ is the indentity, i.e., $g(z) = z$ for $z \in S^1$, then g has a topological degree equal to 1, so that any $BV-$lifting of g cannot be continuous and thus, has a jump of size larger than 2π. In fact, in the case of continuous functions $g \in BV(S^1, S^1)$, Bourgain, Brezis and Mironescu [22] proved the following formula:

$$E(g) = \int_{S^1} |\dot{g}| + 2\pi \deg g.$$

In the general case of an arbitrary function $g \in BV(S^1, S^1)$, the presence of jump points turns the analysis more delicate. A jump of g cannot be always lifted to have the size of the jump equal to the geodesic distance on S^1 (as in the case of an interval of $\mathbb{R}$); therefore, we often have $|\varphi(y+) - \varphi(y-)| > |d_y(g)|$. The idea is to define a new quantity $m(g) \in \mathbb{Z}$ (a "pseudo-degree" of g) corresponding to the number of jumps where the above inequality holds. Our main result is the following:

Theorem 0.2 *([52]) If $S(g) \neq \varnothing$, then*

$$E(g) = \int_{S^1} \Big(|(\dot{g})^a| + |(\dot{g})^c| \Big) + \min_{\substack{\alpha_y \in \mathbb{Z},\, y \in S(g) \\ \#\{y \in S(g) :\, \alpha_y \neq 0\} < \infty \\ \sum_{y \in S(g)} \alpha_y = m(g)}} \sum_{y \in S(g)} |d_y(g) - 2\pi\alpha_y|.$$

Next we construct a minimal configuration $\{\alpha_y\}_{y \in S(g)}$ that allows us to define an optimal lifting. From this formula, we can give a different proof of (2), i.e., $E(g) \leq 2 \int_{S^1} |\dot{g}|$.

1.3 The space $BV(S^2, S^1)$: minimal connection and optimal lifting

The concept of minimal connection associated to a function defined in $\mathbb{R}^3$ with values into the unit sphere S^2 has been introduced by Brezis, Coron and Lieb [27]. That problem was motivated by the theory of liquid crystals. Later, this notion has been used by Bourgain, Brezis and Mironescu [22] in the case of a three-dimensional model for the Ginzburg-Landau equation: the vortex lines correspond to minimal connection between the point singularities of a given boundary data. Recently, Brezis, Mironescu and Ponce [30] studied the topological singularities of functions $g \in W^{1,1}(S^2, S^1)$. They show that the Jacobian of g (in the sense of distributions) detects the position and the degree of the topological singularities of g. More precisely, let $T(g) \in \mathcal{D}'(S^2, \mathbb{R})$ be the distribution defined on S^2 by

$$T(g) = 2\det(\nabla g) = -(g \wedge g_x)_y + (g \wedge g_y)_x;$$

then there exist two sequences of points $(p_k), (n_k)$ on S^2 such that

$$\sum_k |p_k - n_k| < \infty \quad \text{and} \quad T(g) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}).$$

The distribution $T(g)$ in general is not a finite measure and it always has a infinite number of representations as a sum of dipoles. The length of a minimal connection of $T(g)$ is defined as:

$$\|T(g)\| = \sup_{\substack{\zeta \in C^1(S^2) \\ |\nabla \zeta| \leq 1}} \langle T(g), \zeta \rangle.$$

For example, if $T(g) = 2\pi \sum_{k=1}^{m} (\delta_{p_k} - \delta_{n_k})$ is a finite sum of dipoles, Brezis, Coron and Lieb [27] have proved that

$$\|T(g)\| = 2\pi \min_{\sigma \in S_m} \sum_{k=1}^{m} d_{S^2}(p_k, n_{\sigma(k)}),$$

where S_m denotes the group of permutations of $\{1, 2, \dots, m\}$ and d_{S^2} stands for the geodesic distance on S^2. For a countable sum of dipoles, Bourgain, Brezis and Mironescu [22] have generalized the above result by showing that $T(g)$ can be characterized as:

$$\|T(g)\| = \inf_{(p_k),(n_k)} \left\{ 2\pi \sum_k d_{S^2}(p_k, n_k) \; : \; T(g) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}) \text{ and } \sum_k |p_k - n_k| < \infty \right\}. \tag{7}$$

The aim of this chapter is to generalize these notions for functions $u \in BV(S^2, S^1)$. In this case, the difficulty of the analysis comes from the existence of two types of singularities: on one hand, topological point singularities (carrying a degree), on the other hand, jump singularities concentrated on curves. In the sequel, we will always identify u with the precise representative; the 2×2 matrix measure Du is decomposed into three terms

$$Du = D^a u + D^c u + (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^1 \llcorner S(u),$$

where $D^a u, D^c u$ and $D^j u$ correspond to the absolutely continuous part, to the Cantor part and to the jump part of Du. The jump set $S(u)$ is an $\mathcal{H}^1$-rectifiable set on S^2, oriented by the unit normal vector $\nu_u : S(u) \to S^1$. The Borel functions $u^+, u^- : S(u) \to S^1$ represent the traces of u on the jump set $S(u)$ with respect to the orientation ν_u.

We introduce the distribution $T(u) \in \mathcal{D}'(S^2, \mathbb{R})$ as

$$\langle T(u), \zeta \rangle = \int_{S^2} \nabla^\perp \zeta \cdot (u \wedge (D^a u + D^c u)) + \int_{S(u)} \rho(u^+, u^-) \, \nu_u \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1, \; \forall \zeta \in C^1(S^2, \mathbb{R}). \tag{8}$$

Here, $\nabla^\perp \zeta = (\zeta_y, -\zeta_x)$ and the antisymmetric application $\rho(\cdot, \cdot) : S^1 \times S^1 \to [-\pi, \pi]$ corresponds to a signed geodesic distance on S^1:

$$\rho(\omega_1, \omega_2) = \begin{cases} \mathrm{Arg}\left(\frac{\omega_1}{\omega_2}\right) & \text{if } \frac{\omega_1}{\omega_2} \neq -1 \\ \mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2) & \text{if } \frac{\omega_1}{\omega_2} = -1 \end{cases}, \; \forall \omega_1, \omega_2 \in S^1,$$

where $\mathrm{Arg}\,(\omega) \in (-\pi, \pi]$ stands for the argument of a unit complex number $\omega \in S^1$.

Our first result shows that $T(u)$ is a countable sum of dipoles. It is a generalization to the case of BV functions of the result mentioned above for $W^{1,1}$ functions in [30].

Theorem 0.3 *([53]) If $u \in BV(S^2, S^1)$, then there exist two sequences of points $(p_k), (n_k)$ in S^2 such that*

$$\sum_k |p_k - n_k| < \infty \quad \textit{and} \quad T(u) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}). \tag{9}$$

The proof relies on the fact that the derivative (in the sense of distributions) of the characteristic function of a bounded measurable set in $\mathbb{R}$ can be written as a sum of countable dipoles. This property allows us to introduce the set of functions defined on curves of S^2 and taking values into $2\pi\mathbb{Z}$ so that their tangential derivative is given by $T(u)$:

$$\mathcal{J}(T(u)) = \left\{ (f, S, \nu) \, : \, \begin{array}{l} S \text{ is countably } \mathcal{H}^1\text{- rectifiable in } S^2, \ \nu \text{ is an orientation on } S, \\ f \in L^1(S, 2\pi\mathbb{Z}) \text{ so that } \displaystyle\int_S f\, \nu \cdot \nabla^{\perp}\zeta \, \mathrm{d}\mathcal{H}^1 = \langle T(u), \zeta \rangle, \ \forall \zeta \in C^1(S^2) \end{array} \right\}.$$

Then we deduce the following version of (7):

$$\|T(u)\| = \min_{(f,S,\nu)\in\mathcal{J}(T(u))} \int_S |f| \, \mathrm{d}\mathcal{H}^1.$$

While the infimum in (7) in general is not achieved, the advantage of the above formula consists in having the minimum always attained. It means that $\|T(u)\|$ corresponds to the minimal mass that a function with values into the discrete set $2\pi\mathbb{Z}$ could carry between the set of dipoles prescribed by $T(u)$.

In the sequel, we deal with the question of lifting of functions $u \in BV(S^2, S^1)$. A BV lifting of u can be characterized via the set $\mathcal{J}(T(u))$; more precisely, any lifting $\varphi \in BV(S^2, \mathbb{R})$ of u corresponds to a triplet $(f, S, \nu) \in \mathcal{J}(T(u))$ so that

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \, \mathcal{H}^1 \llcorner S(u) - f \nu \, \mathcal{H}^1 \llcorner S.$$

As in the case of functions $BV(S^1, S^1)$, we are interested in the minimal total variation of a lifting of $u \in BV(S^2, S^1)$, i.e.,

$$E(u) = \inf \left\{ \int_{S^2} |D\varphi| \, : \, \varphi \in BV(S^2, \mathbb{R}), \, e^{i\varphi} = u \text{ a.e. in } S^2 \right\}, \tag{10}$$

as well as in constructing a minimizer of (10) called optimal lifting. The following result establishes the expression of $E(u)$ via the distribution $T(u)$.

Theorem 0.4 *([53]) If $u \in BV(S^2, S^1)$, then*

$$E(u) = \int_{S^2} (|D^a u| + |D^c u|) + \min_{(f,S,\nu)\in\mathcal{J}(T(u))} \int_{S \cup S(u)} \left| f\nu \, \chi_S - \rho(u^+, u^-)\nu_u \, \chi_{S(u)} \right| \mathrm{d}\mathcal{H}^1.$$

In particular, we recover the result of Brezis, Mironescu and Ponce [30] for the total variation of an optimal BV lifting of functions $g \in W^{1,1}(S^2, S^1)$: the gap between $E(g)$ and the total variation of g corresponds to the length of a minimal connection between the topological singularities of g, i.e.,

$$E(g) = \int_{S^2} |\nabla g| \, \mathrm{d}\mathcal{H}^2 + \|T(g)\|.$$

In the spirit of [30], the length $\|T(u)\|$ has an interpretation as a distance:

$$\|T(u)\| = \min_{\psi \in BV(S^2,\mathbb{R})} \int_{S^2} \Big| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \,\mathcal{H}^1 \llcorner S(u) - D\psi \Big|. \tag{11}$$

Moreover, there exists at least one minimizer $\psi \in BV(S^2, \mathbb{R})$ of (11) that is a lifting of u.

1.4 On the relation between minimizers of a Γ-limit energy and optimal lifting in BV (joint work with A. Poliakovsky)

A natural method to approximate liftings of a function $u \in BV(\Omega, S^1)$ is to consider the following family of functionals $\{F_\varepsilon^{(u,p)}\}_{\varepsilon>0}$ depending on a parameter $0 < p < +\infty$:

$$F_\varepsilon^{(u,p)}(\varphi) = \varepsilon \int_\Omega |\nabla\varphi|^2 + \frac{1}{\varepsilon}\int_\Omega |u - e^{i\varphi}|^p, \quad \forall \varphi \in H^1(\Omega, \mathbb{R}). \tag{12}$$

Due to the penalizing term in (12), sequences of minimizers φ_ε of $F_\varepsilon^{(u,p)}$ are expected to converge to a lifting φ_0 of u as $\varepsilon \to 0$. Since we are interested in the asymptotic behavior of minimizers, the concept of $\Gamma-$convergence appears to be adapted to our context. Indeed, Poliakovsky [70] proved that for $p > 1$ and for bounded domains $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary, any sequence of minimizers $\varphi_\varepsilon \in H^1(\Omega, \mathbb{R})$ of $F_\varepsilon^{(u,p)}$, satisfying $|\int_\Omega \varphi_\varepsilon| \leq C$, converges strongly in L^1 (up to a subsequence) to a lifting $\varphi_0 \in BV(\Omega, \mathbb{R})$ of u as $\varepsilon \to 0$ and φ_0 is a minimizer of the $\Gamma-$limit energy $F_0^{(u,p)} : L^1(\Omega, \mathbb{R}) \to \mathbb{R}$ given by

$$F_0^{(u,p)}(\varphi) = \begin{cases} 2\displaystyle\int_{S(\varphi)} f^{(p)}(|\varphi^+ - \varphi^-|)\, d\mathcal{H}^{N-1} & \text{if } \varphi \text{ is a } BV \text{ lifting of } u, \\ +\infty & \text{otherwise.} \end{cases}$$

Here, $S(\varphi)$ is the jump set of $\varphi \in BV(\Omega, \mathbb{R})$ and φ^-, φ^+ are the traces of φ on each of the sides of the jump set and $f^{(p)} : [0, +\infty) \to \mathbb{R}$ is the function defined by

$$f^{(p)}(\theta) = \inf_{t\in\mathbb{R}} \int_t^{\theta+t} |e^{is} - 1|^{p/2}\, ds, \; \forall \theta \geq 0.$$

Notice that $F_0^{(u,p)}(\varphi) < +\infty$ for a BV lifting φ of u since $f^{(p)}$ is an increasing Lipschitz function. Due to the fact that the energies $\{F_\varepsilon^{(u,p)}\}_{\varepsilon>0}$ and $F_0^{(u,p)}$ are invariant with respect to translations by $2\pi k$, $k \in \mathbb{Z}$, uniqueness of minimizers has a meaning up to additive constants in $2\pi\mathbb{Z}$.

Our goal is to study the question whether the minimizers of $F_0^{(u,p)}$ are necessarily optimal liftings of u, for any p. Surprisingly, this turns out to be the case (in general) only in dimension one, while in dimension $N \geq 2$ this holds only for $p = 4$. Our main result is the following:

Theorem 0.5 *([58]) Let Ω be a bounded domain in $\mathbb{R}^N$.*

(i) If $N = 1$ then for every $u \in BV(\Omega, S^1)$ and $p \in (0, +\infty)$, φ is a minimizer of $F_0^{(u,p)}$ if and only if φ is an optimal lifting of u;

(ii) If $N \geq 2$, the minimizers of $F_0^{(u,p)}$ are optimal BV liftings of u, for every $u \in BV(\Omega, S^1)$ if and only if $p = 4$.

The key point of the proof relies on the construction of counter-examples for the case $p \neq 4$: If $p \in (0,4)$, we construct a piecewise constant function $u \in BV(\Omega, S^1)$ (depending on p) such that $F_0^{(u,p)}$ has a unique minimizer ξ_0, while u has a unique optimal BV lifting ζ_0 and the difference $\xi_0 - \zeta_0$ is not a constant. In the general case $p \neq 4$, we construct a family of functions $\{U_t\}$ among which some elements have a unique optimal BV lifting whose energy $F_0^{(U_t,p)}$ is strictly larger than the minimal value $\min F_0^{(U_t,p)}$. Moreover, for those elements U_t, we prove that $F_0^{(U_t,p)}$ has a unique minimizer.

Finally, we notice that if u belongs to the smaller class $W^{1,1}(\Omega, S^1)$, then a lifting of u is optimal if and only if it is a minimizer of $F_0^{(u,p)}$, for every $p \in (0, +\infty)$.

1.5 On an open problem about how to recognize constant functions

In the theory of Ginzburg-Landau equation, an important issue resides in the problem of existence and uniqueness of lifting in Sobolev spaces. More precisely, if $\Omega \subset \mathbb{R}^N$ is an open set and $u \in W^{s,p}(\Omega, S^1)$, is there a lifting $\varphi \in W^{s,p}(\Omega, \mathbb{R})$ of u (i.e. $u = e^{i\varphi}$ a.e. in Ω)? Is this lifting unique in $W^{s,p}$ (up to $2\pi\mathbb{Z}$ constants)? Here, $0 < s < \infty$ et $1 < p < \infty$. The answer to the question of existence of lifting was given by Bourgain, Brezis and Mironescu (see [20]). Moreover, if $0 < s < \infty$, $p \geq 1$ and $sp \geq 1$, a $W^{s,p}$ lifting is always unique (modulo 2π), i.e., if $u \in W^{s,p}(\Omega, S^1)$ has two liftings $\varphi_1, \varphi_2 \in W^{s,p}(\Omega, \mathbb{R})$ then there exists $k \in \mathbb{Z}$ such that $\varphi_1 - \varphi_2 \equiv 2\pi k$ a.e. in Ω. This is a consequence of the following result of Bourgain, Brezis and Mironescu:

Let Ω be a domain in $\mathbb{R}^N$. If $f : \Omega \to \mathbb{R}$ is a measurable function that satisfies

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \frac{dx\,dy}{|x-y|^N} < +\infty$$

for some fixed $p \geq 1$, then f is a constant in Ω.

The aim of the chapter is to generalize the above result. For that, let us denote by

$$\mathcal{W} = \{\omega \in C(\mathbb{R}_+, \mathbb{R}_+) \,|\, \omega(0) = 0,\, \omega(t) > 0,\, \forall t > 0\}.$$

The following problem now arises:

Problem 1 *Find a necessary and sufficient condition for $\omega \in \mathcal{W}$ so that any measurable function $f : \Omega \to \mathbb{R}$ satisfying*

$$\int_\Omega \int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} < +\infty, \tag{13}$$

is necessarily constant (a.e. in Ω).

Observe that the restriction $\omega \in \mathcal{W}$ is natural. Indeed, the continuity of ω is needed to make the left hand side of (13) well-defined. Also, $\omega(0) = 0$ (since for any constant function f, (13) should hold) and $\omega(t) > 0, \forall t > 0$ (if $\omega(t) = 0$ for some $t > 0$, take $N = 1$ and $f(x) = tx$). Henceforth it is assumed that $\omega \in \mathcal{W}$.

A necessary condition for Problem 1 (in order to avoid jump functions) is the following:

$$\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt = +\infty. \tag{14}$$

Two sufficient conditions for $\omega \in \mathcal{W}$ are also given: A first one consists in assuming $\liminf_{t\to+\infty} \frac{\omega(t)}{t} > 0$. A second one concerns the opposite case $\liminf_{t\to+\infty} \frac{\omega(t)}{t} = 0$; then the answer to Problem 1 is positive if (14) holds, ω is increasing and the function $t \mapsto \frac{\omega(t)}{t}$ is decreasing at infinity. Observe that the two sufficient conditions only concern the behavior of $\omega \in \mathcal{W}$ at infinity without any additional assumption on its behavior at 0. The question whether the necessary condition (14) is also sufficient, remains open.

Next, we deal with the following problem:

Problem 2 *What regularity on f should be assumed so that for any $\omega \in \mathcal{W}$,* (13) *imply f is a constant?*

The motivation is clear: if we don't want any restriction on $\omega \in \mathcal{W}$, we need to impose an additional condition on f in order that (13) yields f to be a constant. We will prove that the condition $f \in W^{1,1}_{loc}(\Omega)$ guarantees that Problem 2 has a positive answer. The other results deal with the question raised by Brezis in [25]: *Is the continuity (or even the $\mathcal{C}^{0,\alpha}_{loc}$ regularity) of f sufficient for Problem 2?* The answer is negative in general: if either $\omega \in \mathcal{W}$ is bounded, or $\omega(t) = t^\theta$ for some $\theta \in (0,1)$, then we construct non-constants $C^{0,\alpha} \cap BV$ functions of Cantor type (for an arbitrary chosen $\alpha \in (0,1)$) that satisfy (13).

We also prove in a joint work with A.-R. Todor that the necessary condition (14) for Problem 1 forbids nontrivial characteristic functions defined on Ω as well as special Cantor type functions.

2 Vortices in a $2d$ rotating Bose-Einstein condensate

The phenomenon of Bose-Einstein condensation has given rise to an intense research since its first realization in alkali gases in 1995. A Bose-Einstein condensate (BEC) is a quantum gas that can be described by a single complex-valued wave function (order parameter). The existence of vortices is a major feature of these systems and they appear as zeroes of the wave function around which there is a circulation of phase. Experimentally, these vortices can be obtained by rotating the harmonic trap that strongly confines the atoms in the direction of the rotation axis (see [1, 65]). For such a model, the wave function decouples and the reduced model becomes two-dimensional (see Castin et Dum [34]). In the case of an asymmetric trap potential, the wave function minimizes the following Gross-Pitaevskii energy

$$\int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} V(x)|u|^2 + \frac{1}{4\varepsilon^2}|u|^4 - \Omega\, x^\perp \cdot (iu, \nabla u) \right\} dx \tag{15}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1, \tag{16}$$

where $\varepsilon > 0$ is a small parameter and $\Omega = \Omega(\varepsilon) \geq 0$ denotes the rotational velocity. Here, the trapping potential is harmonic and given by $V(x) = |x|^2_\Lambda := x_1^2 + \Lambda^2 x_2^2$ where $\Lambda \in (0,1]$ is a fixed parameter. Our aim is to study the number and the location of vortices in function of the angular velocity $\Omega(\varepsilon)$ as $\varepsilon \to 0$. The two chapters included in this part are joint works with V. Millot and have been published in [56, 55, 54].

2.1 The critical velocity for vortex existence in a two dimensional rotating Bose-Einstein condensate (joint work with V. Millot)

We start our analysis by estimating the critical velocity above which the wave function has vortices. According to numerical and theoretical predictions (see [4, 34]), we expect to find the critical speed in the regime $\Omega = \mathcal{O}(|\ln \varepsilon|)$ so that we restrict our study to this situation.

Due to the constraint (16), we may rewrite the energy (15) in the equivalent form

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right] - \Omega x^\perp \cdot (iu, \nabla u) \right\} dx \qquad (17)$$

where $a(x) = a_0 - |x|^2_\Lambda$ and a_0 is determined by $\int_{\mathbb{R}^2} a^+(x) = 1$ so that $a_0 = \sqrt{2\Lambda/\pi}$. Here a^+ and a^- represent respectively the positive and the negative part of a. Then the wave function u_ε is a solution of the variational problem

$$\operatorname{Min}\left\{F_\varepsilon(u) \,:\, u \in \mathcal{H},\, \|u\|_{L^2(\mathbb{R}^2)} = 1\right\} \quad \text{where} \quad \mathcal{H} = \left\{u \in H^1(\mathbb{R}^2, \mathbb{C}) \,:\, \int_{\mathbb{R}^2} |x|^2|u|^2 < +\infty\right\}.$$

In the limit $\varepsilon \to 0$, the minimization of F_ε strongly forces $|u_\varepsilon|^2$ to be close to a^+ which means that the resulting density is asymptotically localized in the ellipsoidal region

$$\mathcal{D} := \left\{x \in \mathbb{R}^2 \,:\, a(x) > 0\right\} = \left\{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1^2 + \Lambda^2 x_2^2 < a_0\right\}.$$

We will also see that $|u_\varepsilon|$ decays exponentially fast outside $\mathcal{D}$. Actually, the domain $\mathcal{D}$ represents the region occupied by the condensate and consequently, vortices will be sought inside $\mathcal{D}$. Here, a vortex corresponds to a small disc whose radius tends to vanish as $\varepsilon \to 0$ and u_ε has a small amplitude and a non-zero degree around the disc.

The main tools for studying vortices were developed by Béthuel, Brezis and Hélein [17] for "Ginzburg-Landau type" problems. We also refer to Sandier [75] and Sandier and Serfaty [76, 77, 78] for complementary techniques. In the case $a(x) \equiv 1$ and for a disc in $\mathbb{R}^2$, Serfaty proved the existence of local minimizers having vortices for different ranges of rotational velocity (see [83]). In [4], Aftalion and Du follow the strategy in [83] for the study of global minimizers of the Gross-Pitaevskii energy (17) where $\mathbb{R}^2$ is replaced by $\mathcal{D}$. In [3], Aftalion, Alama and Bronsard analyze the global minimizers of (17) for potentials of different nature leading to an annular region of confinement. We finally refer to [5, 6, 61] for mathematical studies on 3D models.

We emphasize that we tackle here the problem which corresponds exactly to the physical model. In particular, we minimize F_ε under the unit mass constraint and the admissible configurations are defined in the whole space $\mathbb{R}^2$. Several difficulties arise, especially in the proof of the existence results and the construction of test functions. We point out that we do not assume any implicit bound on the number of vortices. The singular and degenerate behavior of $\sqrt{a^+}$ near $\partial\mathcal{D}$ induces a cost of order $|\ln \varepsilon|$ in the energy and requires specific tools to detect vortices in the boundary region.

We now start to describe our main results. We prove that the critical rotational velocity for the nucleation of a first vortex in $\mathcal{D}$ is asymptotically given by

$$\Omega_1 := \frac{\Lambda^2 + 1}{a_0}|\ln \varepsilon| = \frac{\sqrt{\pi}(\Lambda^2 + 1)}{\sqrt{2\Lambda}}|\ln \varepsilon|.$$

The critical angular velocity Ω_1 coincides with the one found in [4, 34]. We observe that a very stretched condensate, i.e., $\Lambda \ll 1$, yields a very large value of Ω_1 and the smallest Ω_1 is reached for $\Lambda = 1/\sqrt{3}$ (and surprisingly not for the symmetric case, i.e., $\Lambda = 1$). For subcritical velocities, we will see that u_ε behaves as the "vortex-free" profile $\tilde{\eta}_\varepsilon e^{i\Omega S}$ where $\tilde{\eta}_\varepsilon$ is the positive minimizer of

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right] \right\} dx$$

under the constraint (16) and the phase S is given by

$$S(x) = \frac{\Lambda^2 - 1}{\Lambda^2 + 1}\, x_1 x_2\,. \tag{18}$$

For rotational speeds larger than Ω_1, we show the existence of vortices close to the origin. We also give some fundamental energy estimates in the regime $\Omega = \Omega_1 + \mathcal{O}(\ln|\ln\varepsilon|)$ which will allow to study the precise vortex structure of u_ε.

2.2 Energy expansion and vortex location for a two-dimensional rotating Bose-Einstein condensate (joint work with V. Millot)

The goal of this chapter is to compute an asymptotic expansion of the energy $F_\varepsilon(u_\varepsilon)$ and to determine the number and the location of vortices according to the value of the angular speed $\Omega(\varepsilon)$ in the limit $\varepsilon \to 0$. More precisely, we want to estimate the critical velocity Ω_d for which the dth vortex becomes energetically favorable and to derive a reduced energy governing the location of the vortices (the so-called "*renormalized energy*" by analogy with [17, 80, 81]). We prove the following estimate on the critical speed Ω_d for any integer $d \geq 1$ in the asymptotic $\varepsilon \to 0$,

$$\Omega_d = \frac{1+\Lambda^2}{a_0}(|\ln\varepsilon| + (d-1)\ln|\ln\varepsilon|) = \frac{\sqrt{\pi}(1+\Lambda^2)}{\sqrt{2\Lambda}}(|\ln\varepsilon| + (d-1)\ln|\ln\varepsilon|)\,.$$

Then we show that for velocities ranged between Ω_d and Ω_{d+1}, any minimizer has exactly d vortices of degree $+1$ inside $\mathcal{D}$. Establishing an asymptotic expansion of $F_\varepsilon(u_\varepsilon)$ as $\varepsilon \to 0$, we derive the uniform distribution of vortices close to the origin: it is a minimizing configuration of the reduced energy (19) below.

Our main theorem can be stated as follows:

Theorem 0.6 *([56]) Let u_ε be any minimizer of F_ε in $\mathcal{H}$ under the constraint* (16) *and let $0 < \delta \ll 1$ be any small constant.*

(i) If $\Omega \leq \Omega_1 - \delta\ln|\ln\varepsilon|$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_0 = \varepsilon_0(R_0,\delta) > 0$ such that for any $\varepsilon < \varepsilon_0$, u_ε is vortex-free in $B^\Lambda_{R_0} = \left\{x \in \mathbb{R}^2 : |x|^2_\Lambda = x_1^2 + \Lambda^2 x_2^2 < R_0^2\right\}$, i.e., u_ε does not vanish in $B^\Lambda_{R_0}$. In addition,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + o(1).$$

(ii) If $\Omega_d + \delta \ln|\ln\varepsilon| \le \Omega \le \Omega_{d+1} - \delta \ln|\ln\varepsilon|$ *for some integer* $d \ge 1$*, then for any* $R_0 < \sqrt{a_0}$*, there exists* $\varepsilon_1 = \varepsilon_1(R_0, d, \delta) > 0$ *such that for any* $\varepsilon < \varepsilon_1$*,* u_ε *has exactly* d *vortices* $x_1^\varepsilon, \dots, x_d^\varepsilon$ *of degree one in* $B_{R_0}^\Lambda$*. Moreover,*

$$|x_j^\varepsilon| \le C\,\Omega^{-1/2} \quad \text{for any } j = 1, \dots, d\,, \quad \text{and} \quad |x_i^\varepsilon - x_j^\varepsilon| \ge C\,\Omega^{-1/2} \quad \text{for any } i \ne j$$

where $C > 0$ *denotes a constant independent of* ε*. Setting* $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\, x_j^\varepsilon$*, the configuration* $(\tilde{x}_1^\varepsilon, \dots, \tilde{x}_d^\varepsilon)$ *tends to minimize (as* $\varepsilon \to 0$*) the renormalized energy*

$$w(b_1, \dots, b_d) = -\pi a_0 \sum_{i \ne j} \ln|b_i - b_j| + \frac{\pi a_0}{1 + \Lambda^2} \sum_{j=1}^{d} |b_j|_\Lambda^2. \tag{19}$$

In addition,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \frac{\pi a_0^2 d}{1 + \Lambda^2}(\Omega - \Omega_1) + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + \operatorname*{Min}_{b \in \mathbb{R}^{2d}} w(b) + Q_{d,\Lambda} + o(1) \tag{20}$$

where $Q_{d,\Lambda}$ *is a constant depending only on* d *and* Λ*.*

These results are in agreement with the study made by Castin and Dum [34] who have looked for minimizers in a reduced class of functions. More precisely, we find the same critical angular velocities Ω_d as well as a uniform distribution of vortices around the origin at a scale $\Omega^{-1/2}$. The minimizing configurations for the renormalized energy $w(\cdot)$ have been studied in the radial case $\Lambda = 1$ by Gueron and Shafrir in [49]. They prove that for $d \le 6$, regular polygons centered at the origin and *stars* are local minimizers. For larger d, they numerically found minimizers with a shape of concentric polygons and then triangular lattices as d increases. These figures are exactly the ones observed in physical experiments (see [65, 66]).

3 Optimality of the Néel wall (joint work with F. Otto)

Micromagnetics is a nonconvex and nonlocal variational principle whose (local) minimizers correspond to stable states of a ferromagnetic material. One of the most studied issues concerns the analysis of global minimizers. It's because the main features of the steady state are shared by all physical observed local minima. The variational problem contains various asymptotic regimes where singularities at the mesoscopic level of the magnetization represent domain walls (Néel wall, Bloch wall etc.) or vortices (Bloch lines or boundary vortices). The aim of this chapter is to prove compactness of the Néel wall in a $2d$ model of a thin film.

The nondimensionalized magnetization of a ferromagnetic body $\Omega \subset \mathbb{R}^3$ can be described by a unit vector field $m : \Omega \to S^2$. The experimentally observed magnetizations are (local) minimizers of the following energy functional (in the absence of crystalline anisotropy and external magnetic field):

$$E_{3d}(m) = d^2 \int_\Omega |\nabla m|^2\, dx + \int_{\mathbb{R}^3} |\nabla u|^2\, dx.$$

The first term is called exchange energy while the second one represents the stray field or magnetostatic energy. The stray field potential $u : \mathbb{R}^3 \to \mathbb{R}$ is determined by

$$\Delta u = \nabla \cdot \Big(m 1_\Omega \Big), \tag{21}$$

$$\text{i.e.,} \quad \int_{\mathbb{R}^3} \nabla u \cdot \nabla \zeta \, dx = \int_\Omega m \cdot \nabla \zeta \, dx, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^3).$$

It means that u is both generated by the divergence of m inside Ω (volume charges) and by the normal component of m at the boundary of the magnetic body (surface charges). The exchange length d is an intrinsic parameter of the material standing for the relative strength between exchange and magnetostatic energy.

The setting of the following model is determined by our goal to prove the optimality of Néel walls under 2−d variation. We consider the magnetic body as a thin infinitely extended cylinder:

$$\Omega = \Omega' \times (0, t)$$
$$\Omega' = (-1, 1) \times \mathbb{R} \subset \mathbb{R}^2.$$

Here, the thickness t is very small so that the magnetization can be considered invariant in the out-of-plane variable x_3, i.e., $m(x) = m(x')$ and the vertical component of m is strongly penalized, i.e., $m_3(x') = 0$. Therefore, the admissible magnetizations are smooth 2-d unit-length vector fields

$$m' = (m_1, m_2) : \mathbb{R}^2 \to S^1$$

that macroscopically act as an angle wall in Ω', i.e.,

$$m'(x') = \begin{pmatrix} m_{1,\infty} \\ \pm\sqrt{1 - m_{1,\infty}{}^2} \end{pmatrix} \text{ for } \pm x_1 \geq 1,\ x_2 \in \mathbb{R}, \tag{22}$$

where $m_{1,\infty} \in [0, 1)$ is some fixed number and the prime always indicates an in-plane quantity, for example, $x' = (x_1, x_2)$, $x = (x', x_3) \in \mathbb{R}^3$. With these assumptions, in order to write the thin-film energy approximation E_{3d}, we use the following ansatz (see [39, 41]): the equation (21) is equivalent with

$$\Delta(\frac{1}{t}\, u) = \Big(\frac{1}{t} \chi_{(0,t)}(x_3) \Big)\, \nabla' \cdot m',$$

where $\nabla' \cdot m'$ corresponds to the $2d$ divergence of m'. As $t \to 0$, the RHS converges to a distribution concentrated on the horizontal plane $\{x_3 = 0\}$ and we expect that u/t converges to the solution U of the equation

$$\Delta U = \nabla' \cdot m' \, \mathcal{H}^2 \llcorner \{x_3 = 0\}. \tag{23}$$

It explains our choice of considering stray fields $h = (h_1, h_2, h_3) : \mathbb{R}^3 \to \mathbb{R}^3$ related to m' by the following variational formulation:

$$\int_{\mathbb{R}^3} h \cdot \nabla \zeta \, dx = \int_{\mathbb{R}^2} \zeta \, \nabla' \cdot m' \, dx', \quad \forall \zeta \in C_c^\infty(\mathbb{R}^3). \tag{24}$$

To write the energy density of such a configuration, we suppose that

$$m' \text{ and } h \text{ are } L\text{−periodic in the infinite direction } x_2, \tag{25}$$

where L is an arbitrary positive number. After a change of variable, the $2d$ energy functional that we consider in the sequel is given by

$$E_\varepsilon(m', h) = \varepsilon \int_{\mathbb{R}\times[0,L)} |\nabla' \cdot m'|^2 \, dx' + \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} |h|^2 \, dx \tag{26}$$

where

$$\varepsilon := \frac{d^2}{t}$$

is a small parameter and we are interested in the asymptotic behavior as $\varepsilon \to 0$. Remark that we replace the Dirichlet energy of m' by a smaller quantity given by the L^2 norm of the divergence of m'. The equation (24) implies that the minimal stray field energy represents the homogeneous $H^{-1/2}$ norm of $\nabla' \cdot m'$ and the minimal value is achieved for ∇U (where U is the solution of (23)):

$$\min_{h \text{ with } (24)} \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} |h|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}\times[0,L)} \left| |\nabla'|^{-1/2} \nabla' \cdot m' \right|^2 \, dx'.$$

Now we shall informally explain how the principle of pole avoidance leads to the formation of walls. For simplicity, we assume that the mesoscopic transition angle imposed by (22) on the boundary $\partial\Omega'$ is 180°, i.e., $m' \cdot \nu' = 0$ on $\partial\Omega'$. The boundary effects in the tangential direction are excluded by our choice of Ω' which is infinite in x_2−direction. The competition between the exchange and magnetostatic energy will try to enforce the divergence-free condition for m', i.e., $\nabla' \cdot m' = 0$ in Ω'. Therefore, we arrive at

$$|m'| = 1 \text{ and } \nabla' \cdot m' = 0 \text{ in } \Omega', \quad m' \cdot \nu' = 0 \text{ on } \partial\Omega'. \tag{27}$$

This mesoscopic thin-film description has been justified by DeSimone, Kohn, Müller and Otto in [41] using the Γ−convergence method. We notice that the conditions in (27) are too rigid for smooth magnetization m'. This can be seen by writing $m' = \nabla'^{\perp}\psi$ with the help of a "stream function" ψ. Then (27) turns into a Dirichlet problem for the eikonal equation in ψ:

$$|\nabla'^{\perp}\psi| = 1 \text{ in } \Omega', \quad \psi = 0 \text{ on } \partial\Omega'. \tag{28}$$

Using the characteristics method, it follows that there is no smooth solution of the equation (28). On the other hand, there are many continuous solutions that satisfy the first condition of (28) away from a set of vanishing Lebesgue measure. One of them is the "viscosity solution" given by the distance function

$$\psi(x') = \text{dist}\,(x', \partial\Omega')$$

that corresponds to the so-called Landau state for the magnetization m' (see Figure 1). Hence, the divergence-free equation in (27) has to be interpreted in the distribution sense and it is expected to induce line-singularities for solutions m'. These ridges are an idealization of the wall formation in thin-film elements at the mesoscopic level. At the microscopic level, they are

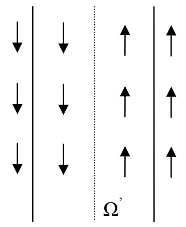


Figure 1: Landau state in Ω'

replaced by smooth transition layers where the magnetization varies very quickly. A final remark is that the normal component of m' does not jump across these discontinuity lines (because of (27)) and therefore, walls are determined by the angle between the mesoscopic levels in the adjacent domains.

In the following we will concentrate on the Néel wall which is the favored wall type in very thin films. It is characterized by a one-dimensional in-plane magnetization:

$$m' = (m_1(x_1), m_2(x_1)), \tag{29}$$

that avoids surface charges, but leads to volume charges (because of (22)), i.e.,

$$\nabla' \cdot m' = \frac{dm_1}{dx_1} \neq 0.$$

The prototype is the 180° Néel wall which corresponds to the boundary condition (22) for $m_{1,\infty} = 0$, i.e.,

$$m'(x_1) = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \text{ for } \pm x_1 \geq 1. \tag{30}$$

Let us now discuss the scaling of the energy of the prototypical Néel wall. For magnetizations (29), the specific energy (26) reduces to

$$E_\varepsilon^{1d}(m') = \varepsilon \int_{\mathbb{R}} |\frac{dm_1}{dx_1}|^2 \, dx_1 + \frac{1}{2} \int_{\mathbb{R}} \left| |\frac{d}{dx_1}|^{1/2} m_1 \right|^2 dx_1. \tag{31}$$

We define the Néel wall as the $1d$ minimizer of (31) under the boundary constraint (30). The Néel wall is a two length scale object: a small core ($|x_1| \lesssim w_{core}$) with fast varying rotation and a logarithmically decaying tail ($w_{core} \lesssim |x_1| \lesssim 1$). The finiteness of Ω' in x_1-direction in our setting serves as the confining mechanism for the Néel wall tail. This two-scale structure permits to the Néel wall to decrease the specific energy by a logarithmic factor.

$$\min_{(29),(30)} E_\varepsilon^{1d}(m') \approx \frac{\pi}{2|\ln \varepsilon|} \quad \text{if } \varepsilon \ll 1;$$

the minimizer m_1 with $m_1(0) = 1$ is symmetric around 0 ($w_{core} \sim \varepsilon$) and satisfies

$$m_1(x_1) \sim \frac{\ln \frac{1}{|x_1|}}{|\ln \varepsilon|} \text{ for } \varepsilon \ll |x_1| \ll 1.$$

The stability of 180° Néel walls under arbitrary $2-d$ modulation was proved by DeSimone, Knüpfer and Otto in [39]:

$$\min_{\substack{m',h \\ m' \text{ with } (8.8)}} E_\varepsilon(m',h) \approx \min_{\substack{m',h \\ m'=m'(x_1) \text{ with } (8.8)}} E_\varepsilon(m',h) \approx \frac{\pi L}{2|\ln\varepsilon|} \quad \text{for } \varepsilon \ll 1.$$

Our first result is a qualitative property of the optimal $1d$ transition layers: We prove that asymptotically, the minimal energy can be assumed *only* by the straight walls. This property holds for general boundary conditions (22). It is based on a compactness result for magnetizations $\{m'_\varepsilon\}$ with energies E_ε close to the minimal energy level: any accumulation limit m' has the singularities concentrated on a vertical line (see FIG. 2).

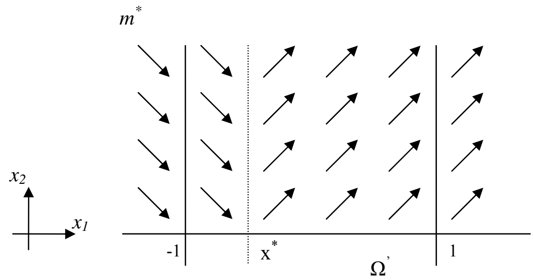


Figure 2: Paroi limite

Theorem 0.7 *([57]) Let $m_{1,\infty} \in [0,1)$ and $L>0$ be given. For any $\delta>0$ there exists $\varepsilon_0>0$ with the following property: given $m':\mathbb{R}^2 \to S^1$ and $h:\mathbb{R}^3\to\mathbb{R}^3$ with*

m' *and* h *are* $L-$*periodic in* x_2*, i.e.,* (25) *holds,*

m' *satisfies the boundary condition* (22),

m' *and* h' *are related by* (24),

$$|\ln\varepsilon| E_\varepsilon(m',h) \le L\frac{\pi}{2}(1-m_{1,\infty})^2 + \varepsilon_0, \quad \textit{for some } 0<\varepsilon\le\varepsilon_0, \tag{32}$$

then we have

$$\int_{\mathbb{R}\times[0,L)} |m'-m^*|\, dx' \le \delta, \tag{33}$$

where m^* is a straight wall given by

$$m^*(x_1,x_2) = \begin{pmatrix} m_{1,\infty} \\ \pm\sqrt{1-{m_{1,\infty}}^2} \end{pmatrix} \quad \textit{for } \pm x_1 > \pm x_1^*, \tag{34}$$

for some $x_1^* \in [-1,1]$.

For that, we investigate the asymptotics as $\varepsilon\to 0$ of families of $2d$ magnetizations when the energy $E_\varepsilon(m'_\varepsilon, h_\varepsilon)$ is placed in the regime $O(\frac{1}{|\ln\varepsilon|})$. One of the issues we discuss here is the question of the L^1_{loc}-compactness of the magnetizations $\{m'_\varepsilon\}_{\varepsilon\downarrow 0}$ in the above energy regime, i.e., whether the topological constraint $|m'_\varepsilon| = 1$ passes to the limit. The difficulty arises from the fact that in general the sequence of divergences $\{\nabla' \cdot m'_\varepsilon\}$ is not uniformly bounded in L^1_{loc}. This was one of the particularities used in the entropy methods for proving compactness results for the Modica-Mortola type problems; we refer to the studies of Jin and Kohn [62], Ambrosio, De Lellis and Mantegazza [10], DeSimone, Kohn, Müller and Otto [40], Rivière and Serfaty [74], Alouges, Rivière and Serfaty [8], Jabin, Otto and Perthame [60]. For our model, the entropy method cannot be applied. Instead, the idea is to use a duality argument in the spirit of [39, 41] based on an ε-perturbation of a logarithmically failing Gagliardo-Nirenberg inequality together with a dynamical system argument. Since the compactness result is a local issue, we state it in the context of the unit ball $B_1 \subset \mathbb{R}^3$ with no imposed boundary conditions:

Theorem 0.8 *([57]) Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$. For $k \in \mathbb{N}$, let $m'_k : B'_1 \to S^1$ and $h_k : B_1 \to \mathbb{R}^3$ be related by*

$$\int_{B_1} h_k \cdot \nabla\zeta \, dx = \int_{B'_1} m'_k \cdot \nabla'\zeta \, dx', \ \forall \zeta \in C_c^\infty(B_1). \tag{35}$$

Suppose that

$$\limsup_{k\to\infty} |\ln \varepsilon_k| \left(\varepsilon_k \int_{B'_1} |\nabla' \cdot m'_k|^2 \, dx' + \int_{B_1} |h_k|^2 \, dx \right) < \infty. \tag{36}$$

Then $\{m'_k\}_{k\uparrow\infty}$ is relatively compact in $L^1(B'_1)$ and any accumulation point $m' : B'_1 \to \mathbb{R}^2$ satisfies

$$|m'| = 1 \text{ a.e. in } B'_1 \quad \text{and} \quad \nabla' \cdot m' = 0 \text{ distributionally in } B'_1. \tag{37}$$

In the case of $1d$ magnetizations, we are able to completely characterize the limit configurations: every accumulation point in L^1_{loc} concentrates on a finite number of limiting walls. However, a sequence of magnetizations is in general not relatively compact in BV.

We also discuss the case of zero-energy states, i.e., m' is an accumulation point of sequences $\{m'_\varepsilon\}_{\varepsilon\downarrow 0}$ such that the limit in (36) vanishes for some stray potentials $\{h_\varepsilon\}$ (in the absence of any boundary condition). The main tool is the principle of characteristics for the eikonal equation. We show that every zero-energy state m' is locally Lipschitz continuous and satisfies the principle of characteristics:

$$m'(x'_0 + tm'(x'_0)^\perp) = m'(x'_0) \text{ for every } t \in \mathbb{R} \text{ where } x'_0 + tm'(x'_0)^\perp \in B'_1.$$

Part I

Lifting of functions with values into the unit circle S^1

Chapter 1

Lifting of BV functions with values into S^1

Abstract

We show that for every $u \in BV(\Omega, S^1)$, there exists a function $\varphi \in BV(\Omega, \mathbb{R})$ such that $u = e^{i\varphi}$ a.e. in Ω and $|\varphi|_{BV} \leq 2|u|_{BV}$. The constant 2 is optimal in dimension $N > 1$.
This chapter is written in collaboration with J. Dávila; the original text is published in C. R. Acad. Sci. Paris, Ser. I **337** (2003), 159–164 (cf. [37]).

1.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to S^1$ a measurable function. A *lifting* of u is a measurable function $\varphi : \Omega \to \mathbb{R}$ such that

$$u(x) = e^{i\varphi(x)}$$

for a.e. $x \in \Omega$. If u has some regularity one may ask whether or not φ can be chosen with some regularity as well. For example, if Ω is simply connected and u is continuous (respectively, $u \in C^k(\Omega, S^1)$), then it is well known that φ can be chosen to be continuous (respectively, $\varphi \in C^k(\Omega, \mathbb{R})$).

Regarding other function spaces there has been recently much research, specially motivated by the study of the Ginzburg-Landau equation. The first result of this type in Sobolev spaces was given by Béthuel and Zheng [19], and it asserts that if Ω is a bounded simply connected domain in $\mathbb{R}^N$ and $u \in W^{1,p}(\Omega, S^1)$ with $p \geq 2$ then $u = e^{i\varphi}$ for some $\varphi \in W^{1,p}(\Omega, \mathbb{R})$. On the other hand, if $N \geq 2$ and $1 \leq p < 2$ then there are functions $u \in W^{1,p}(\Omega, S^1)$ which have no lifting in $W^{1,p}$. One example when $N = 2$ and $0 \in \Omega$ is

$$u(x) = \frac{x}{|x|}.$$

Later Bourgain, Brezis and Mironescu [20] addressed the same question for general Sobolev spaces $W^{s,p}(\Omega, S^1)$, $0 < s < \infty$ and $1 < p < \infty$. They gave a complete description, characterizing in terms of N, s and p all the cases where a lifting is always possible and the cases where there is some $u \in W^{s,p}(\Omega, S^1)$ without lifting in $W^{s,p}$.

Results concerning other spaces include for example the work of Coifman and Meyer [35], who showed among other things that if $u : \mathbb{R} \to S^1$ is BMO and $|u|_{BMO} < \gamma$ (where $\gamma > 0$ is a constant) then u has a lifting in BMO with a certain control of the BMO seminorm of the lifting. Then Brezis and Nirenberg [31] extended this result for general domains Ω and also showed that if $u \in VMO$ then φ can be chosen also in VMO.

We are concerned here with the case when u has bounded variation, and by this we mean that $u = (u_1, u_2) \in L^1_{loc}(\Omega, \mathbb{R}^2)$, $|u(x)| = 1$ for a.e. $x \in \Omega$ and its BV seminorm is finite, i.e.

$$|u|_{BV} = \sup\left\{ \int_\Omega \sum_{k=1}^{2} u_k \operatorname{div} \zeta_k \, dx \, : \, \zeta_k \in C_0^\infty(\Omega, \mathbb{R}^N), \, \sum_{k=1}^{2} |\zeta_k|^2 \le 1 \text{ in } \Omega \right\} < \infty,$$

where the norm in $\mathbb{R}^N$ is the Euclidean norm.

Remark 1.1 Throughout this chapter we will say that $v \in BV(\Omega, \mathbb{R}^m)$ if $v \in L^1_{loc}(\Omega, \mathbb{R}^m)$ and its standard BV seminorm $|v|_{BV}$ is finite. We adopt this convention, because in the case of an open set Ω of infinite Lebesgue measure, the standard definition of BV requires that $v \in L^1(\Omega, \mathbb{R}^m)$ which would not be true for a S^1-valued BV function.

Our main result states the existence of a BV lifting with an optimal control of the BV seminorm:

Theorem 1.1 *Let $u \in BV(\Omega, S^1)$. Then there exists a lifting $\varphi \in BV \cap L^\infty(\Omega, \mathbb{R})$ of u such that*

$$|\varphi|_{BV} \le 2|u|_{BV}. \tag{1.1}$$

The idea for the proof of Theorem 1.1 is to consider the argument type function $L : S^1 \to \mathbb{R}$ defined by

$$L(e^{i\theta}) = \theta \quad \text{for every} \quad -\pi \le \theta < \pi. \tag{1.2}$$

Then $\varphi = L(u)$ is a lifting of u, in the sense that $e^{i\varphi(x)} = u(x)$ for all $x \in \Omega$. We would like to have $|\varphi|_{BV} \le 2|u|_{BV}$, but this is far from true. It may even happen that $L(u)$ does not belong to BV (classical results for composition of functions assert only that if $f : S^1 \to \mathbb{R}$ is Lipschitz then $f(u)$ is BV). There is a way to remedy this situation. Indeed, observe that for fixed $\alpha \in \mathbb{R}$ the function $L(e^{i\alpha}u) - \alpha$ is also a lifting of u. We shall prove

Lemma 1.2 *The function $\alpha \mapsto |L(e^{i\alpha}u)|_{BV}$ is measurable and*

$$\int_0^{2\pi} |L(e^{i\alpha}u)|_{BV} \, d\alpha \le 4\pi |u|_{BV}. \tag{1.3}$$

Remark 1.2 Inequality (1.3) can be viewed as a sort of co-area inequality. In particular it implies that for a.e. $\alpha \in \mathbb{R}$, $L(e^{i\alpha}u) \in BV$. The constant 4π in (1.3) is sharp; see the examples in Section 1.5. The proof of (1.3) is based on the chain rule for BV functions. A new proof of (1.3) without using the chain rule was given by Merlet [69].

Corollary 1.3 *Let $u \in BV(\Omega, S^1)$. Then there exists a sequence $u_k \in C^\infty(\Omega, S^1) \cap BV(\Omega)$ such that $u_k \to u$ a.e. and in L^1_{loc} and*

$$\limsup_{k\to\infty} |u_k|_{BV} \leq 2|u|_{BV}.$$

Remark 1.3 *(1)* If u belongs to the Sobolev space $W^{1,1}(\Omega, S^1)$ and $\Omega \subset \mathbb{R}^2$ is smooth, bounded and simply connected, it was already known that u has a lifting $\varphi \in BV(\Omega, \mathbb{R})$ which satisfies (1.1) (private communication of H. Brezis and P. Mironescu [29]). The idea is to apply the density result of Béthuel and Zheng [19] to reduce the proof to the case where u is smooth except at finitely many points. For such a function u one can construct a lifting whose jump set is precisely the minimal connection between the singularities of u (with respect to their degree) and the boundary of Ω, a notion first introduced by Brezis, Coron and Lieb [27]. This lifting satisfies condition (1.1). None of these tools are available for the case of a function of bounded variation.

(2) The existence of a BV lifting for $u \in BV(\Omega, S^1)$ (when Ω is a bounded simply connected domain of $\mathbb{R}^N$) was first proved by Giaquinta, Modica and Soucek in [47], but the authors did not find the optimal control of the BV seminorm of the lifting.

The control in Theorem 1.1 is optimal for any domain in $\mathbb{R}^N$, $N > 1$. In one-dimensional domains, the best constant is $\pi/2$. The result is stated as follows[1]:

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^N$ be an open set.*

(i) If $N = 1$, the optimal constant is $\frac{\pi}{2}$: if $u \in BV(\Omega, S^1)$ then there exists a lifting $\varphi \in BV(\Omega, \mathbb{R})$ of u such that

$$|\varphi|_{BV} \leq \frac{\pi}{2}|u|_{BV}; \tag{1.4}$$

moreover, if there is a constant $C > 0$ such that any function $u \in BV(\Omega, S^1)$ has a lifting $\varphi \in BV(\Omega, \mathbb{R})$ with the property

$$|\varphi|_{BV} \leq C|u|_{BV}, \tag{1.5}$$

then $C \geq \frac{\pi}{2}$;

(ii) If $N \geq 2$, the constant 2 in (1.1) is optimal, i.e., if there is a constant $C > 0$ such that any function $u \in BV(\Omega, S^1)$ has a lifting $\varphi \in BV(\Omega, \mathbb{R})$ with the property (1.5), then $C \geq 2$.

The outline of the chapter is the following: we start by some preliminaries about functions of bounded variation. In Section 1.3 we prove Theorem 1.1. In Section 1.4, we show that $\frac{\pi}{2}$ is the optimal constant in (1.5) in one-dimensional case. Finally, in Sections 1.5 and 1.6 we prove that 2 is the optimal constant for dimensions $N > 1$.

[1] I added Theorem 1.4 in order to prove the optimality of the constant 2 for liftings in any domain. This result does not appear in the published version of the paper [37].

1.2 Preliminaries about the space BV

The material that we present next is standard and can be found in the book [11] (see also [23, 42]). Let $v \in BV(\Omega, \mathbb{R}^m)$. Its jump set $S(v)$ is defined by the requirement that $x \in \Omega \setminus S(v)$ if and only if there exists $\tilde{v}(x) \in \mathbb{R}^m$ such that $\tilde{v}(x) = \operatorname*{ap-lim}_{y \to x} v(y)$, that is:

$$\lim_{r \to 0} \frac{\mathcal{L}^N\big(B_r(x) \cap \{y \in \Omega \,:\, |v(y) - \tilde{v}(x)| > \varepsilon\}\big)}{\mathcal{L}^N(B_r(x))} = 0, \quad \forall \varepsilon > 0.$$

It can be proved (see [11]) that the set $S(v)$ is a countably $\mathcal{H}^{N-1}$-rectifiable Borel set, i.e., $S(v)$ is σ-finite with respect to the Hausdorff measure $\mathcal{H}^{N-1}$ and there exist countably many $N-1$ dimensional C^1-hypersurfaces $\{S_k\}_{k=1}^{\infty}$ such that $\mathcal{H}^{N-1}\Big(S(v) \setminus \bigcup_{k=1}^{\infty} S_k\Big) = 0$. Moreover, for $\mathcal{H}^{N-1}$-a.e. $x \in S(v)$ there exist $v^+(x), v^-(x) \in \mathbb{R}^m$ and a unit vector $\nu_v(x)$ such that

$$\lim_{r \to 0} \fint_{B_r^+(x, \nu_v(x))} v(y)\, dy = v^+(x), \quad \lim_{r \to 0} \fint_{B_r^-(x, \nu_v(x))} v(y)\, dy = v^-(x), \tag{1.6}$$

where

$$\begin{aligned} B_r^+(x, \nu_v(x)) &= \{y \in B(x,r) \,:\, \langle y - x, \nu_v(x) \rangle > 0\} \\ B_r^-(x, \nu_v(x)) &= \{y \in B(x,r) \,:\, \langle y - x, \nu_v(x) \rangle < 0\}. \end{aligned}$$

The vector field $\nu_v : S(u) \to S^{N-1}$ is called the orientation of the jump set $S(u)$. For a locally bounded function v, (1.6) is equivalent with

$$\operatorname*{ap-lim}_{y \to x,\ \langle y - x, \nu_v(x) \rangle > 0} v(y) = v^+(x), \qquad \operatorname*{ap-lim}_{y \to x,\ \langle y - x, \nu_v(x) \rangle < 0} v(y) = v^-(x).$$

The differential Dv is a matrix valued Radon measure which can be decomposed as

$$Dv = D^a v + D^j v + D^c v,$$

where $D^a v$ is defined as the absolutely continuous part of Dv with respect to the Lebesgue measure, while $D^j v$ and $D^c v$ are defined as

$$D^j v = Dv \llcorner S(v), \quad D^c v = (Dv - D^a v) \llcorner (\Omega \setminus S(v)).$$

$D^j v$ is called the jump part and $D^c v$ the Cantor part of Dv. It can be proved that

$$D^j v = (v^+ - v^-) \otimes \nu_v \, \mathcal{H}^{N-1} \llcorner S(v).$$

Let us consider now the precise representative $v^* : \Omega \to \mathbb{R}^m$ of v, i.e.

$$v^*(x) = \begin{cases} \lim_{r \to 0} \fint_{B_r(x)} v\, dy & \text{if this limit exists} \\ 0 & \text{otherwise} \end{cases}.$$

Remark that if (1.6) holds for some $x \in \Omega$ then

$$v^*(x) = \frac{v^+(x) + v^-(x)}{2}.$$

More generally, $\lim_{r\to 0} \fint_{B_r(x)} v\,dy$ exists for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$; hence, v^* specifies the values of the BV function v except on a $\mathcal{H}^{N-1}$-negligible set and the mollified functions $v \star \rho_\varepsilon$ pointwise converge to v^* in that domain. Since we only use the local behavior of BV functions and we do not need the specific values in each point, henceforth we consider that v coincides with the precise representative v^* in the L^1_{loc}-class.

It is well known that if $v \in BV(\Omega, \mathbb{R}^m)$ and $f : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz then $f \circ v$ belongs to BV, and Ambrosio and Dal Maso [9] proved a chain rule in this context. The following lemma is a slight modification of this chain rule for u in BV with values in S^1 (see Theorem 3.99 in [11] for the case of scalar BV functions):

Lemma 1.5 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in BV(\Omega, S^1)$. Let $f : S^1 \to \mathbb{R}$ be a Lipschitz function. Then $v = f \circ u$ belongs to $BV(\Omega, \mathbb{R})$, f is differentiable at $u(x)$ for $(|D^a u| + |D^c u|)$-a.e. x and*

$$Dv = \vec{f_\tau}(u)(D^a u + D^c u) + (f(u^+) - f(u^-))\nu_u\, \mathcal{H}^{N-1} \llcorner S(u), \tag{1.7}$$

where $\vec{f_\tau}$ denotes the tangential derivative of f.

Proof. Let us consider a Lipschitz extension $\tilde{f}$ of the function f to $\mathbb{R}^2$ such that

$$\tilde{f}(x) = \begin{cases} 0 & \text{if } |x| \le \frac{1}{3}, \\ f(\frac{x}{|x|}) & \text{if } \frac{1}{2} \le |x| \le 2, \\ 0 & \text{if } |x| \ge 3. \end{cases}$$

Denote by F and G the set of Lebesgue points of $\vec{f_\tau} \in \left(L^\infty(S^1)\right)^2$ and $\nabla \tilde{f} \in \left(L^\infty(\mathbb{R}^2)\right)^2$ in S^1 respectively in $\mathbb{R}^2$. Remark that $\mathcal{H}^1(S^1 \setminus F) = 0$, $G \cap S^1 = F$ and $\vec{f_\tau} = \nabla \tilde{f}$ on F. Let (ρ_ε) be the standard mollifiers in $\mathbb{R}^2$; for each $\varepsilon > 0$, consider the functions $\tilde{f}_\varepsilon = \tilde{f} \star \rho_\varepsilon$ and $v_\varepsilon = \tilde{f}_\varepsilon \circ u$. By the chain rule in BV (see Theorem 3.96, [11]), it results that $v_\varepsilon \in BV(\Omega, \mathbb{R})$, $|Dv_\varepsilon|(\Omega) \le ||\nabla \tilde{f}_\varepsilon||_{L^\infty} |Du|(\Omega)$ and

$$Dv_\varepsilon = \nabla \tilde{f}_\varepsilon(u)(D^a u + D^c u) + (\tilde{f}_\varepsilon(u^+) - \tilde{f}_\varepsilon(u^-))\nu_u\, \mathcal{H}^{N-1} \llcorner S(u), \quad \forall \varepsilon > 0. \tag{1.8}$$

Since $v_\varepsilon \to v$ in $L^1_{loc}(\Omega)$, it follows that $v \in BV(\Omega)$ and Dv_ε converge weakly* to Dv.

Remark that $|D^a u|(u^{-1}(S^1 \setminus F)) = |D^c u|(u^{-1}(S^1 \setminus F)) = 0$ (see Proposition 3.92, [11]), so that f is differentiable at $u(x)$ for $(|D^a u| + |D^c u|)$-a.e. x. Therefore the right hand side of (1.7) makes sense and since $\nabla \tilde{f}_\varepsilon(x) \to \nabla \tilde{f}(x)$ for every $x \in G$ as $\varepsilon \to 0$, we deduce that

$$\nabla \tilde{f}_\varepsilon(u) \to \nabla \tilde{f}(u) \quad |D^a u| + |D^c u|\text{-a.e. in } \Omega.$$

By the dominated convergence theorem, the conclusion follows passing to the limit as $\varepsilon \to 0$ in (1.8). $\square$

1.3 Control of a lifting in BV. Proof of Theorem 1.1

We start by proving Lemma 1.2 and as a consequence, we deduce the control of the BV seminorm of a lifting of u.

Proof of Lemma 1.2. Let $u \in BV(\Omega, S^1)$. For the proof of this theorem we consider a sequence of Lipschitz functions that approximate L (defined in (1.2)), and carry out the computations with this approximation. For small $\varepsilon > 0$, let $L_\varepsilon : S^1 \to \mathbb{R}$ denote the following function

$$L_\varepsilon(e^{i\theta}) = \begin{cases} \theta & \text{if } 0 \le \theta \le \pi - \varepsilon, \\ \frac{\pi-\varepsilon}{\varepsilon}(\pi - \theta) & \text{if } \pi - \varepsilon \le \theta \le \pi + \varepsilon, \\ \theta - 2\pi & \text{if } \pi + \varepsilon \le \theta \le 2\pi. \end{cases}$$

Let $\alpha \in \mathbb{R}$ and define $\phi_{\alpha,\varepsilon} : S^1 \to \mathbb{R}$ by

$$\phi_{\alpha,\varepsilon}(e^{i\theta}) = L_\varepsilon(e^{i(\alpha+\theta)}).$$

Then $\phi_{\alpha,\varepsilon}$ is Lipschitz and therefore $\phi_{\alpha,\varepsilon}(u) \in BV$. We use now the chain rule from Lemma 1.5 to compute the derivative of $\phi_{\alpha,\varepsilon}(u)$:

$$D\phi_{\alpha,\varepsilon}(u) = (\vec{\phi}_{\alpha,\varepsilon})_\tau(u)(D^a u + D^c u) + (\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-))\nu_u\, \mathcal{H}^{N-1} \llcorner S(u)$$

where $(\vec{\phi}_{\alpha,\varepsilon})_\tau$ denotes the tangential derivative of $\phi_{\alpha,\varepsilon}$ and is given by

$$(\vec{\phi}_{\alpha,\varepsilon})_\tau(e^{i\theta}) = (\vec{L}_\varepsilon)_\tau(e^{i(\alpha+\theta)}).$$

Hence,

$$D\phi_{\alpha,\varepsilon}(u) = (\vec{L}_\varepsilon)_\tau(e^{i\alpha}u)(D^a u + D^c u) + (\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-))\nu_u\, \mathcal{H}^{N-1} \llcorner S(u).$$

Since the measures in the expression above are mutually singular, for the total variation of the corresponding measures we have

$$|D\phi_{\alpha,\varepsilon}(u)| \le |(\vec{L}_\varepsilon)_\tau(e^{i\alpha}u)|(|D^a u| + |D^c u|) + |\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)|\, \mathcal{H}^{N-1} \llcorner S(u).$$

Integrating this total variation over Ω we get

$$|\phi_{\alpha,\varepsilon}(u)|_{BV} \le \int_\Omega |(\vec{L}_\varepsilon)_\tau(e^{i\alpha}u)|\, d(|D^a u| + |D^c u|) + \int_{S(u)} |\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)|\, d\mathcal{H}^{N-1}. \tag{1.9}$$

Observe that the map $\alpha \to |\phi_{\alpha,\varepsilon}(u)|_{BV}$ is lower semi-continuous because it is the supremum of a family of continuous functions of α:

$$|\phi_{\alpha,\varepsilon}(u)|_{BV} = \sup_{g \in C_0^\infty,\, |g| \le 1} \int_\Omega L_\varepsilon(e^{i\alpha}u)\, \mathrm{div}\, g\, dx.$$

In particular $\alpha \to |\phi_{\alpha,\varepsilon}(u)|_{BV}$ is measurable. Integrating (1.9) with respect to α over $[0, 2\pi]$ we get

$$\int_0^{2\pi} |\phi_{\alpha,\varepsilon}(u)|_{BV}\, d\alpha \le \int_0^{2\pi} \int_\Omega |(\vec{L}_\varepsilon)_\tau(e^{i\alpha}u)|\, d(|D^a u| + |D^c u|)\, d\alpha$$
$$+ \int_0^{2\pi} \int_{S(u)} |\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)|\, d\mathcal{H}^{N-1}\, d\alpha.$$

Let us consider the first term on the right hand side above; by Fubini's theorem

$$\int_0^{2\pi}\int_\Omega |(\vec{L}_\varepsilon)_\tau(e^{i\alpha}u)|\,d(|D^a u|+|D^c u|)\,d\alpha = \int_\Omega\int_0^{2\pi} |(\vec{L}_\varepsilon)_\tau(e^{i\alpha}u)|\,d\alpha\,d(|D^a u|+|D^c u|).$$

But an easy computation shows that for any fixed x

$$\int_0^{2\pi} |(\vec{L}_\varepsilon)_\tau(e^{i\alpha}u(x))|\,d\alpha = 4(\pi-\varepsilon).$$

Therefore

$$\int_0^{2\pi}\int_\Omega |(\vec{L}_\varepsilon)_\tau(e^{i\alpha}u)|\,d(|D^a u|+|D^c u|)\,d\alpha = 4(\pi-\varepsilon)(|D^a u|(\Omega)+|D^c u|(\Omega)). \tag{1.10}$$

Regarding the term

$$\int_0^{2\pi}\int_{S(u)} |\phi_{\alpha,\varepsilon}(u^+)-\phi_{\alpha,\varepsilon}(u^-)|\,d\mathcal{H}^{N-1}\,d\alpha = \int_{S(u)}\int_0^{2\pi} |\phi_{\alpha,\varepsilon}(u^+)-\phi_{\alpha,\varepsilon}(u^-)|\,d\alpha\,d\mathcal{H}^{N-1}$$

we see that we have to estimate

$$\int_0^{2\pi} |L_\varepsilon(e^{i(\alpha+\theta_1)})-L_\varepsilon(e^{i(\alpha+\theta_2)})|\,d\alpha,$$

where θ_1, $\theta_2\in[0,2\pi]$ are fixed. Using the explicit formula for L_ε it is not hard to verify that if $|\theta_1-\theta_2|\le\pi$ then

$$\begin{aligned}\int_0^{2\pi} |L_\varepsilon(e^{i(\alpha+\theta_1)})-L_\varepsilon(e^{i(\alpha+\theta_2)})|\,d\alpha &= 2\frac{\pi-\varepsilon}{\pi}|\theta_1-\theta_2|(2\pi-|\theta_1-\theta_2|)\\ &\le 8(\pi-\varepsilon)\sin(|\theta_1-\theta_2|/2).\end{aligned}$$

Observe that if $e^{i\theta_1}=u^+(x)$ and $e^{i\theta_2}=u^-(x)$ with $|\theta_1-\theta_2|\le\pi$, then $|u^+(x)-u^-(x)| = 2\sin(|\theta_1-\theta_2|/2)$. Hence, for any fixed $x\in S(u)$ we obtain

$$\int_0^{2\pi} |\phi_{\alpha,\varepsilon}(u^+)-\phi_{\alpha,\varepsilon}(u^-)|\,d\alpha \le 4(\pi-\varepsilon)|u^+(x)-u^-(x)|.$$

Therefore, integrating over $S(u)$ we find

$$\int_0^{2\pi}\int_{S(u)} |\phi_{\alpha,\varepsilon}(u^+)-\phi_{\alpha,\varepsilon}(u^-)|\,d\mathcal{H}^{N-1}\,d\alpha \le 4(\pi-\varepsilon)\int_{S(u)} |u^+(x)-u^-(x)|\,d\mathcal{H}^{N-1}. \tag{1.11}$$

Combining (1.10) and (1.11) we establish that

$$\int_0^{2\pi} |\phi_{\alpha,\varepsilon}(u)|_{BV}\,d\alpha \le 4(\pi-\varepsilon)|u|_{BV}. \tag{1.12}$$

To finish the proof note that $\alpha\to|L(e^{i\alpha}u)|_{BV}$ is measurable with values in $[0,\infty]$, because

$$|L(e^{i\alpha}u)|_{BV} = \sup_{g\in C_0^\infty\ |g|\le 1}\int_\Omega L(e^{i\alpha}u)\operatorname{div} g\,dx$$

and for fixed g the map $\alpha \to \int_\Omega L(e^{i\alpha}u)\,\mathrm{div}\,g\,dx$ is measurable.

Also observe that for all except a countable set of $\alpha \in \mathbb{R}$ we have $\mathcal{L}^N(\{y \in \Omega : u(y) = -e^{-i\alpha}\}) = 0$, and for these values of α

$$L_\varepsilon(e^{i\alpha}u) \to L(e^{i\alpha}u) \quad \text{a.e. in } \Omega \text{ as } \varepsilon \to 0.$$

This implies that for a.e. α

$$|L(e^{i\alpha}u)|_{BV} \leq \liminf_{\varepsilon\to 0} |L_\varepsilon(e^{i\alpha}u)|_{BV}.$$

Hence, by Fatou's lemma

$$\begin{aligned}\int_0^{2\pi} |L(e^{i\alpha}u)|_{BV}\,d\alpha &\leq \liminf_{\varepsilon\to 0} \int_0^{2\pi} |L_\varepsilon(e^{i\alpha}u)|_{BV}\,d\alpha \\ &\leq 4\pi|u|_{BV}\end{aligned}$$

by (1.12). □

Proof of Theorem 1.1. Using Lemma 1.2, the mean value theorem yields that there exists α_0 such that

$$|L(e^{i\alpha_0}u)|_{BV} \leq \frac{1}{2\pi}\int_0^{2\pi} |L(e^{i\alpha}u)|_{BV}\,d\alpha \leq 2|u|_{BV}.$$

Therefore, $\varphi = L(e^{i\alpha_0}u) - \alpha_0$ is a lifting of u that satisfies (1.1). □

Remark 1.4 Recall the space of special functions with bounded variation

$$SBV(\Omega, \mathbb{R}^m) = \{u \in BV(\Omega, \mathbb{R}^m) \,|\, D^c u \equiv 0 \text{ in } \Omega\}.$$

We say that $u \in SBV(\Omega, S^1)$ if $u \in SBV(\Omega, \mathbb{R}^2)$ and $|u(x)| = 1$ for a.e. $x \in \Omega$. The previous proof for this case says that for each $u \in SBV(\Omega, S^1)$ there exists a lifting $\varphi \in SBV(\Omega, \mathbb{R})$ of u such that (1.1) holds.

1.4 The constant $\pi/2$ is optimal in $1d$

In this section we prove that the optimal control for the BV seminorm of a lifting is $\pi/2$ in one-dimensional domains.

Proof of *(i)* in Theorem 1.4. Let $\Omega \subset \mathbb{R}$ be an interval and $u \in BV(\Omega, S^1)$. We will construct a lifting φ of u such that (1.4) holds. The derivative $\dot{u}$ decomposes as

$$\dot{u} = (\dot{u})^a + (\dot{u})^c + \sum_{y\in S(u)} (u(y+) - u(y-))\delta_y$$

where $S(u)$ denotes the jump set of u which is at most countable. For any $y \in S(u)$, we denote $d_y(u) = \mathrm{Arg}\,\dfrac{u(y+)}{u(y-)}$ where $\mathrm{Arg}\,\omega \in (-\pi, \pi]$ is the argument of the unit complex number ω. Obviously,

$$|d_y(u)| \leq \frac{\pi}{2}|u(y+) - u(y-)|.$$

Now consider $\varphi^a, \varphi^c, \varphi^j$ the BV functions (unique up to constants) having as derivatives in Ω the finite Radon measures $u \wedge (\dot{u})^a$, $u \wedge (\dot{u})^c$ and $\sum_{y\in S(u)} d_y(u)\delta_y$. Let

$$\varphi = \varphi^a + \varphi^c + \varphi^j \in BV(\Omega, \mathbb{R}).$$

By the chain rule, we have

$$(u\, \dot{e^{-i\varphi}}) = 0 \quad \text{in} \quad \Omega. \tag{1.13}$$

Indeed,

$$\begin{aligned}(\dot{e^{-i\varphi}}) &= -ie^{-i\varphi}\left(\dot{\varphi^a} + \dot{\varphi^c}\right) + \sum_{y\in S(u)} (e^{-i\varphi(y+)} - e^{-i\varphi(y-)})\delta_y \\ &= -e^{-i\varphi}\bar{u}\left((\dot{u})^a + (\dot{u})^c\right) + \sum_{y\in S(u)} (e^{-i\varphi(y+)} - e^{-i\varphi(y-)})\delta_y.\end{aligned}$$

Remark that the space $BV(\Omega, \mathbb{C}) \cap L^\infty$ is an algebra. Differentiating the product $u\, e^{-i\varphi}$, we obtain

$$\begin{aligned}(u\, \dot{e^{-i\varphi}}) &= e^{-i\varphi}\left((\dot{u})^a + (\dot{u})^c\right) - u\, e^{-i\varphi}\bar{u}\left((\dot{u})^a + (\dot{u})^c\right) \\ &\quad + \sum_{y\in S(u)} (u(y+)\, e^{-i\varphi(y+)} - u(y-)\, e^{-i\varphi(y-)})\delta_y \\ &= 0.\end{aligned}$$

Thus, up to a constant, φ is a lifting of u, i.e. $u = e^{i\varphi}$ a.e. in Ω. If we compute the total variation of φ, we conclude

$$\begin{aligned}\int_\Omega |\dot{\varphi}| &= \int_\Omega (|(\dot{u})^a| + |(\dot{u})^c|) + \sum_{y\in S(u)} |d_y(u)| \\ &\le \int_\Omega (|(\dot{u})^a| + |(\dot{u})^c|) + \frac{\pi}{2}\sum_{y\in S(u)} |u(y+) - u(y-)| \\ &\le \frac{\pi}{2}\int_\Omega |\dot{u}|.\end{aligned}$$

It remains to prove that $\pi/2$ is the best constant in (1.4). For simplicity of the writing, let $\Omega = (-1, 1)$. Define $u \in BV(\Omega, S^1)$ as

$$u(x) = \begin{cases} 1 & \text{if } x \in (-1, 0), \\ -1 & \text{if } x \in (0, 1). \end{cases}$$

Then $|\dot{u}|(\Omega) = 2$. Let $\varphi \in BV(\Omega, \mathbb{R})$ be a lifting of u. We prove that

$$\int_\Omega |\dot{\varphi}| \ge \frac{\pi}{2}\int_\Omega |\dot{u}|.$$

By the chain rule, it follows that

$$\begin{aligned}&(\dot{\varphi})^a + (\dot{\varphi})^c = u \wedge ((\dot{u})^a + (\dot{u})^c) = 0 \\ &(\dot{\varphi})^j = (\varphi(0+) - \varphi(0-))\delta_0 + \sum_{y\in B}(\varphi(y+) - \varphi(y-))\delta_y\end{aligned} \tag{1.14}$$

where $B \subset \Omega$ is a finite set such that $0 \notin B$ and $\varphi(y+) - \varphi(y-) = -2\pi\alpha_y$, $\alpha_y \in \mathbb{Z}$, for every $y \in B$. Obviously, $|\varphi(0+) - \varphi(0-)| \geq |d_0(u)| = \pi$. According to (1.14), we have

$$\int_\Omega |\dot{\varphi}| = |\varphi(0+) - \varphi(0-)| + \sum_{y \in B} |\varphi(y+) - \varphi(y-)| \geq \pi = \frac{\pi}{2} \int_\Omega |\dot{u}|.$$

□

1.5 Two examples in the disc

We give two examples of BV functions u defined in the unit disc in $\mathbb{R}^2$ and taking values in S^1 for which 2 is the minimal constant in (1.1) for every possible lifting φ in BV. The first example is a $W^{1,1}$ function (and therefore, it belongs also in SBV) and the second one is a purely Cantor type function (by that, we mean that $u \in BV$ and $D^a u \equiv D^j u \equiv 0$). Though, both functions have the same topological defect at the origin that will generate a jump part of the lifting with larger variation than the BV seminorm of u.

The following result was already suggested in [22]:

Lemma 1.6 *Let Ω be the unit disc in $\mathbb{R}^2$. Define $u : \Omega \setminus \{0\} \to S^1$,*

$$u(x) = \frac{x}{|x|} \quad \text{for every } x \in \Omega \setminus \{0\}.$$

Let $\varphi \in BV(\Omega, \mathbb{R})$ be a lifting of u. Then $|D\varphi|(\Omega) \geq 4\pi = 2|u|_{BV}$.

First Proof: We have that $u \in \mathrm{W}^{1,p}(\Omega)$ for all $p \in [1,2)$ and $\int_\Omega |\nabla u| = 2\pi$. By the chain rule for BV functions applied to $u = e^{i\varphi}$ we obtain

$$Du = iu(D^a\varphi + D^c\varphi) + (e^{i\varphi^+} - e^{i\varphi^-})\nu_\varphi \, \mathcal{H}^1 \llcorner S(\varphi).$$

Since $D^c u = 0$, we get that $D^c\varphi = 0$ and then, $D^a\varphi = -i\bar{u}\, Du$ which leads to $|D^a\varphi|(\Omega) = 2\pi$. Because u has no jump, we have $e^{i\varphi^+} - e^{i\varphi^-} = 0$ and the size of the jump of φ is at least 2π. Now, it is sufficient to show that $|D^j\varphi|(\Omega) \geq 2\pi$. Let $g \in C_0^\infty(\Omega, \mathbb{R})$. The radial derivative of g is defined as

$$g_\tau(re^{i\theta}) = \lim_{t \to 0} \frac{g(re^{i(\theta+t)}) - g(re^{i\theta})}{t}.$$

Also for BV functions (see [11]), there exists an unique measure called the radial derivative of φ and denoted by $D_\tau\varphi$ such that

$$\int_\Omega \varphi g_\tau \, dx = -\int_\Omega g \, d\big(D_\tau\varphi\big), \, \forall g \in C_0^\infty(\Omega, \mathbb{R}).$$

Since $D^c\varphi = 0$, we write

$$D_\tau\varphi = D_\tau^a\varphi + D_\tau^j\varphi.$$

For every $r \in (0,1)$, if $S_r = \{x \in \mathbb{R}^2 : |x| = r\}$ is the circle of radius r, then we denote by $\varphi_r : S_r \to \mathbb{R}$ the restriction of φ to S_r. Using the Characterization Theorem of BV functions by sections and Theorem 3.108 in [11], it results that for a.e. $r \in (0,1)$, $\varphi_r \in BV(S_r, \mathbb{R})$,

$D^j_\tau \varphi = \left(\mathcal{L}^1 \llcorner (0,1)\right) \otimes D^j \varphi_r$ (as product of measures) and the discontinuity set of φ_r is $S(\varphi) \cap S_r$. Remark that $\deg(u, S_r) = 1$ for every $r \in (0,1)$. Hence, for a.e. $r \in (0,1)$, φ_r will have a jump on S_r and the size of the jump is not less than 2π. Finally,

$$|D^j \varphi|(\Omega) \geq |D^j_\tau \varphi|(\Omega) = \int_0^1 |D^j \varphi_r|(S_r)\, dr \geq 2\pi.$$

□

Second proof: Take $\varphi_n \in W^{1,1} \cap C^\infty(\Omega, \mathbb{R})$ such that $\varphi_n \to \varphi$ a.e. on Ω and $\int_\Omega |\nabla \varphi_n|\, dx \to |D\varphi|(\Omega)$ as $n \to \infty$. Set $u_n = e^{i\varphi_n} \in W^{1,1} \cap C^\infty(\Omega, S^1)$. As above, for every $r \in (0,1)$ we denote by S_r the circle of radius r. Up to a subsequence, for a.e. $r \in (0,1)$ we have $u_n \to u$ a.e. in S_r and $\sup_n |Du_n|(S_r) < \infty$. For those r, since $\deg(u, S_r) = 1$, it follows by Lemma 18 in [22],

$$\liminf_{n\to\infty} \int_{S_r} |\nabla u_n \cdot \tau|\, d\mathcal{H}^1 \geq \int_{S_r} |\nabla u \cdot \tau|\, d\mathcal{H}^1 + 2\pi = \int_{S_r} |\nabla u|\, d\mathcal{H}^1 + 2\pi$$

(here τ is the tangent vector in each point of S_r). Therefore, by Fatou's lemma,

$$|D\varphi|(\Omega) = \liminf_{n\to\infty} \int_\Omega |\nabla u_n| \geq \int_0^1 \liminf_{n\to\infty} \int_{S_r} |\nabla u_n|\, d\mathcal{H}^1\, dr \geq \int_\Omega |\nabla u| + 2\pi.$$

□

The second example in the unit disc is the following:

Lemma 1.7 *Let Ω be the unit disc in $\mathbb{R}^2$ and let $f : [0,1] \to [0,1]$ be the standard Cantor function. Define in polar coordinates the function $u : \Omega \setminus \{0\} \to S^1$,*

$$u(r,\theta) = e^{2\pi f(\frac{\theta}{2\pi})i} \quad \textit{for every } \theta \in [0, 2\pi) \ \textit{ and } r \in (0,1).$$

Let $\varphi \in BV(\Omega, \mathbb{R})$ be a lifting of u. Then $|D\varphi|(\Omega) \geq 4\pi = 2|u|_{BV}$.

Proof. It's easy to see that $u \in BV(\Omega, S^1)$ and $|u|_{BV} = |D^c u|(\Omega) = 2\pi$. By the chain rule we get that $|D^c \varphi|(\Omega) = |D^c u|(\Omega)$ and $|D^a \varphi|(\Omega) = |D^a u|(\Omega) = 0$. Repeating the same argument as before for the restrictions to the circles S_r, we deduce that $|D^j \varphi|(\Omega) \geq 2\pi$ and the conclusion follows. □

1.6 The constant 2 is optimal for $N > 1$

In this section we complete the proof of Theorem 1.4:

Proof of *(ii)* in Theorem 1.4. First, we make the construction of a dipole in the disc $\mathcal{R} = B(1,2) \subset \mathbb{R}^2$ (see also [22]): it is a function $u_\varepsilon \in W^{1,1}(\mathcal{R}, S^1)$, $\varepsilon > 0$ is small, u_ε has two topological singularities (called *poles*) in $P = (0,0) \in \mathbb{R}^2$ and $N = (2,0) \in \mathbb{R}^2$, i.e., $\det \nabla u_\varepsilon = \pi(\delta_P - \delta_N)$. Moreover,

$$\int_{\mathcal{R}} |\nabla u_\varepsilon|\, dx \leq 4\pi + 2\varepsilon \tag{1.15}$$

$$\text{and} \quad \int_{\mathcal{R}} |\nabla \varphi_\varepsilon|\, dx \geq 8\pi, \quad \text{for every lifting } \varphi_\varepsilon \in BV(\mathcal{R}, \mathbb{R}). \tag{1.16}$$

Then we adapt this construction for every domain $\Omega \in \mathbb{R}^N$, $N > 1$.

We distinguish two cases:

a) The case of the disc $\mathcal{R} = B(1,2) \subset \mathbb{R}^2$. Let $\varepsilon > 0$ be a small parameter. Set

$$\mathcal{R}_\varepsilon = \{(x_1, x_2) \in \mathcal{R} \,:\, 0 < x_1 < 2,\, |x_2| < \varepsilon \min(x_1, 2 - x_2)\}.$$

Then $\mathcal{H}^2(\mathcal{R}_\varepsilon) = 2\varepsilon$. We define the function $\varphi_{0,\varepsilon} : \mathcal{R}_\varepsilon \to \mathbb{R}$ by

$$\varphi_{0,\varepsilon}(x_1, x_2) = \pi \left(1 + \frac{x_2}{\varepsilon \min(x_1, 2 - x_2)}\right).$$

Now consider the function $u_\varepsilon : \mathcal{R} \to S^1$ given by

$$u_\varepsilon(x_1, x_2) = \begin{cases} e^{i\varphi_{0,\varepsilon}(x_1,x_2)} & \text{in} \quad \mathcal{R}_\varepsilon, \\ 1 & \text{in} \quad \mathcal{R} \setminus \mathcal{R}_\varepsilon. \end{cases}$$

We have that

$$\begin{aligned} \int_{\mathcal{R}} |\nabla u_\varepsilon| \, dx &= \int_{\mathcal{R}_\varepsilon} |\nabla \varphi_{0,\varepsilon}| \, dx \\ &= 4\pi \int_0^1 \sqrt{1 + \varepsilon^2 t^2} \, dt \in (4\pi, 4\pi + 2\varepsilon), \end{aligned}$$

and hence, (1.15) holds. An easy computation shows that the jacobian of u_ε (in the sense of distributions) is the dipole (P, N):

$$\det(\nabla u_\varepsilon) := \frac{1}{2}\text{curl}\left(u_\varepsilon \wedge \frac{\partial u_\varepsilon}{\partial x_1}, u_\varepsilon \wedge \frac{\partial u_\varepsilon}{\partial x_2}\right) = \pi(\delta_P - \delta_N).$$

Therefore, we expect that every lifting $\varphi_\varepsilon \in BV(\mathcal{R}, \mathbb{R})$ of u_ε has a jump of size 2π along a connection between the poles P and N. By the chain rule, the total variation of the absolutely continuous part of $D\varphi_\varepsilon$ is

$$\int_{\mathcal{R}} |D^a \varphi_\varepsilon| = \int_{\mathcal{R}} |\nabla u_\varepsilon| \, dx \geq 4\pi.$$

In order to have (1.16), it is sufficient to show that

$$\int_{B(0,1)} |D^j \varphi_\varepsilon| \geq 2\pi \quad \text{and} \quad \int_{B(2,1)} |D^j \varphi_\varepsilon| \geq 2\pi.$$

By symmetry, we only prove the first inequality. As in the proofs of Lemma 1.6, the argument is based on the restriction $\varphi_{r,\varepsilon}$ of φ_ε on the circle S_r, $r \in (0,1)$. We know that $\deg(u_\varepsilon, S_r) = 1$ for every $r \in (0,1)$. Hence, for a.e. $r \in (0,1)$, $\varphi_{r,\varepsilon}$ will have a jump on S_r and the size of the jump is larger than 2π. Since the discontinuity set of $\varphi_{r,\varepsilon}$ is $S(\varphi_\varepsilon) \cap S_r$, it follows that

$$|D^j \varphi_\varepsilon|(B(0,1)) \geq \int_0^1 |D^j \varphi_{r,\varepsilon}|(S_r) \, dr \geq 2\pi.$$

Finally, (1.15) and (1.16) yield the conclusion by letting $\varepsilon \to 0$.

b) The general case of a domain $\Omega \subset \mathbb{R}^N$*.* Let $\varepsilon > 0$ and set $\mathcal{D} = \mathcal{R} \times (0,1)^{N-2} \subset \mathbb{R}^N$. By translating and shrinking homotopically the cylinder $\mathcal{D}$, we may suppose that $\mathcal{D} \subset\subset \Omega$. Let u_ε be the function in $\mathcal{R}$ constructed above. We write $x = (x_1, x_2, \ldots, x_N) = (x_1, x_2, x') \in \mathbb{R}^N$. We consider the function $w_\varepsilon \in BV(\Omega, S^1)$ given by

$$w_\varepsilon(x) = \begin{cases} u_\varepsilon(x_1, x_2) & \text{in} \quad \mathcal{D}, \\ 1 & \text{in} \quad \Omega \setminus \mathcal{D}. \end{cases}$$

We have that $Dw_\varepsilon = D^a w_\varepsilon + D^j w_\varepsilon$ where $S(w_\varepsilon) \subset \mathcal{R}_\varepsilon \times \partial\big([0,1]^{N-2}\big)$. Moreover,

$$\int_\Omega |D^a w_\varepsilon| = \int_{(0,1)^{N-2}} dx' \int_{\mathcal{R}\times\{x'\}} |\nabla u_\varepsilon| \, dx_1 \, dx_2 \le 4\pi + 2\varepsilon,$$
$$|D^j w_\varepsilon|(\Omega) \le 2\mathcal{H}^{N-1}\left(\mathcal{R}_\varepsilon \times \partial\big([0,1]^{N-2}\big)\right) \le 8\varepsilon(N-2).$$

Let $\psi_\varepsilon \in BV(\Omega, \mathbb{R})$ be a lifting of w_ε. As above, the chain rule leads to the total variation of the absolutely continuous part of the lifting:

$$\int_\Omega |D^a \psi_\varepsilon| = \int_\Omega |D^a w_\varepsilon| \le 4\pi + 2\varepsilon.$$

We want to show that $|D^j \psi_\varepsilon|(\Omega) \ge 4\pi$. For that, we notice that the restriction of ψ_ε to $\mathcal{R} \times \{x'\}$ is a BV lifting of u_ε for almost every $x' \in (0,1)^{N-2}$. Therefore, by (1.16), we deduce that

$$\int_\Omega |D^j \psi_\varepsilon| \ge \int_{(0,1)^{N-2}} dx' \int_{\mathcal{R}\times\{x'\}} |D^j \psi_\varepsilon| \ge 4\pi.$$

We conclude that

$$\int_\Omega |D\psi_\varepsilon| \ge 4\pi + \int_\Omega |D^a w_\varepsilon| \ge \big(2 + o(\varepsilon)\big) \int_\Omega |Dw_\varepsilon|,$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. □

Remark 1.5 It would be interesting to know if for every domain $\Omega \subset \mathbb{R}^N$, $N > 1$, there exists a non-constant function $u \in BV(\Omega, S^1)$ such that

$$\int_\Omega |D\varphi| \ge 2 \int_\Omega |Du|, \text{ for any lifting } \varphi \in BV(\Omega, \mathbb{R}) \text{ of } u.$$

Chapter 2

Optimal lifting for $BV(S^1, S^1)$

Abstract

For each $g \in BV(S^1, S^1)$, we solve the following variational problem

$$E(g) = \inf\left\{ \int_{S^1} |\dot{\varphi}| \,:\, \varphi \in BV(S^1, \mathbb{R}), e^{i\varphi} = g \text{ a.e. on } S^1 \right\}$$

and we will deduce that $E(g) \le 2|g|_{BV}$.
This chapter is published in Calc. Var. Partial Differential Equations **23** (2005), 83–96 (cf. [52]).

2.1 Introduction

Let $g \in BV(S^1, S^1)$, i.e. $g \in BV(S^1, \mathbb{R}^2)$ and $|g(y)| = 1$ for a.e. $y \in S^1$. The aim of this chapter is to compute the total variation of an optimal lifting BV of g

$$E(g) = \inf\left\{ \int_{S^1} |\dot{\varphi}| \,:\, \varphi \in BV(S^1, \mathbb{R}), e^{i\varphi} = g \text{ a.e. on } S^1 \right\} \tag{2.1}$$

(here "$\dot{}$" stands for the tangential derivative operator). It is easy to see that the above infimum is achieved and it is equal to the relaxed energy

$$E_{\mathrm{rel}}(g) = \inf\left\{ \liminf_{n\to\infty} \int_{S^1} |\dot{g}_n| \mathrm{d}\mathcal{H}^1 \,:\, g_n \in C^\infty(S^1, S^1), \deg g_n = 0,\ g_n \to g \text{ a.e. on } S^1 \right\}$$

(see Remark 2.1).

In what follows, we will always identify g with its precise representative, which is a Borel function such that

$$g(y) = \frac{g(y+) + g(y-)}{2}, \ \forall y \in S^1.$$

In order to state the main results, we need to introduce some notations. We decompose the finite Radon measure $\dot{g}$ as

$$\dot{g} = (\dot{g})^a + (\dot{g})^c + (\dot{g})^j,$$
$$\text{with } (\dot{g})^j = \sum_{y \in S(g)} (g(y+) - g(y-))\delta_y.$$

Here, $(\dot{g})^a$, $(\dot{g})^c$ and $(\dot{g})^j$ are the absolutely continuous part, the Cantor part and the jump part of $\dot{g}$ and the set (at most countable)

$$S(g) = \{y \in S^1 : \dot{g}(\{y\}) \neq 0\}$$

is the set of jump points of g. For any $y \in S(g)$, let $d_y(g) \in (-\pi, \pi] \setminus \{0\}$ be such that

$$e^{i\, d_y(g)} = \frac{g(y+)}{g(y-)};$$

thus, $|d_y(g)| = d_{S^1}(g(y+), g(y-))$, where d_{S^1} is the geodesic distance on S^1. We denote

$$\begin{aligned} P(g) &= \sum_{y \in S(g)} d_y(g) \\ L(g) &= \int_{S^1} g \wedge \Big((\dot{g})^a + (\dot{g})^c\Big) \\ m(g) &= \frac{P(g) + L(g)}{2\pi}, \end{aligned}$$

where $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \wedge \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = u_1\mu_2 - u_2\mu_1$. Remark that the measure $g \wedge ((\dot{g})^a + (\dot{g})^c)$ is well-defined since the measure $(\dot{g})^a + (\dot{g})^c$ vanishes on most countable sets.

A preliminary result is the following:

Lemma 2.1 $m(g) \in \mathbb{Z}$, $\forall g \in BV(S^1, S^1)$.

The reason to introduce $m(g)$ is the following: if $y \in S(g)$ and φ is a lifting BV of g, then $|\varphi(y+) - \varphi(y-)| \geq |d_y(g)|$. It turns out that $m(g)$ is related to the number of times where the above inequality is strict.

Set

$$\tilde{E}(g) = \int_{S^1} \Big(|(\dot{g})^a| + |(\dot{g})^c|\Big).$$

This quantity represents the total variation of the diffuse part of the derivative of g and plays also the role of the total variation of the diffuse part of the derivative of a lifting of g.

If $S(g) = \varnothing$, set $E^j(g) = |L(g)|$; otherwise (i.e. $S(g) \neq \varnothing$), set

$$E^j(g) = \min_{\substack{\alpha_y \in \mathbb{Z},\, y \in S(g) \\ \#\{y \in S(g) :\, \alpha_y \neq 0\} < \infty \\ \sum_{y \in S(g)} \alpha_y = m(g)}} \sum_{y \in S(g)} |d_y(g) - 2\pi\alpha_y|. \tag{2.2}$$

As we will see, $E^j(g)$ is the total variation of the jump part of an optimal lifting of g. The analytic formula for $E^j(g)$ (when g has jumps) is given by:

Lemma 2.2 *If $S(g) \neq \varnothing$, then*

$$E^j(g) = \begin{cases} \operatorname{sgn}(m(g))L(g) + 2 \displaystyle\min_{\substack{B \subset S(g) \\ \#B = \min(|m(g)|, \#S(g))}} \sum_{\substack{y \in S(g) \setminus B \\ \operatorname{sgn}(d_y(g)) = \operatorname{sgn}(m(g))}} |d_y(g)| & \text{if } m(g) \neq 0 \\ \displaystyle\sum_{y \in S(g)} |d_y(g)| & \text{if } m(g) = 0 \end{cases}.$$

The above formula can be interpreted as follows: if $\operatorname{sgn}(m(g)) = 1$, then the minimal value in (2.2) is achieved by taking $\alpha_y \geq 1$ for the y's with the largest positive jump $d_y(g)$.

Our first main result is

Theorem 2.3 *For every $g \in BV(S^1, S^1)$, we have*

$$E(g) = \tilde{E}(g) + E^j(g).$$

In the case where g is a continuous function of bounded variation, the expression of $E(g)$ was already proved by Bourgain-Brezis-Mironescu [22]. In the general case, our result can presumably be proved using the theory of Cartesian Currents of Giaquinta-Modica-Soucek[47].

The next result yields an estimate of $E(g)$ in terms of the BV-seminorm of g. It is a straightforward variant of Theorem 1.1 for $BV(\Omega, S^1)$ functions (where $\Omega \subset \mathbb{R}^N$ is a bounded open set):

Theorem 2.4 *For every $g \in BV(S^1, S^1)$, we have*

$$E(g) \leq 2 \int_{S^1} |\dot{g}|. \tag{2.3}$$

The constant 2 in the above inequality is optimal (see the examples in Section 2.5). We present two different proofs for Theorem 2.4. The first one relies on the explicit formula obtained in Theorem 2.3, combined with the following trigonometrical inequality:

Lemma 2.5 *Let γ be the unique solution on $(0, \frac{\pi}{2})$ of the equation*

$$3 \sin \frac{\gamma + \pi}{3} = 2 \frac{\gamma + \pi}{3} \qquad (\gamma = 1.345752051076...).$$

For p integer, let $x_k \in [0, \frac{\pi}{2}]$, $\forall k \geq 1$ such that $\sum_{k \geq 1} x_k \leq p\pi + \gamma$. Then

$$\sum_{k \geq 1} \sin x_k \geq \sum_{k \geq 1} x_k - \max_{\substack{B \subset \mathbb{N} \\ \#B = p}} \sum_{k \in B} x_k.$$

The second proof of Theorem 2.4 is a straightforward adaptation of the proof given in Chapter 1; the idea is to use a special class of liftings of g. We discuss in Section 2.4 some properties of this class. The striking fact is that, although the lifting obtained using the technique in Theorem 1.1 is not optimal in general (i.e. this lifting is not a minimizer in (2.1)), inequality (2.3) is easier to prove using this lifting rather than using an optimal one.

2.2 Optimal lifting of $g \in BV(S^1, S^1)$

In this section we prove Lemma 2.1, Lemma 2.2 and Theorem 2.3; we also construct an optimal lifting of g. Finally, we present an estimate of $E(g)$ in terms of a more natural BV-seminorm $|g|_{BVS^1}$, defined below.

First, following [30], let us make some remarks about $E(g)$ and $E_{\text{rel}}(g)$:

Remark 2.1 i) $E(g)<\infty$ and $E_{\rm rel}(g)<\infty$ (the existence of a lifting BV for g is shown in the proof of Lemma 2.1);

ii) The infimum (2.1) is achieved; indeed, let $\varphi_n \in BV(S^1,\mathbb{R}), e^{i\varphi_n}=g$ a.e. on S^1 be such that

$$\lim_{n\to\infty}\int_{S^1}|\dot\varphi_n| = E(g)<\infty.$$

By Poincaré's inequality, there exists an universal constant $C>0$ such that

$$\int_{S^1}|\varphi_n - \fint_{S^1}\varphi_n| d\mathcal{H}^1 \le C\int_{S^1}|\dot\varphi_n|,\ \forall n\in\mathbb{N}$$

(where $\fint_{S^1}$ stands for the average). Therefore, by subtracting a suitable 2π integer multiple, we may assume that $(\varphi_n)_{n\in\mathbb{N}}$ is bounded in $BV(S^1,\mathbb{R})$. Up to a further subsequence, we may assume that $\varphi_n\to\varphi$ a.e. and L^1 for some $\varphi\in BV(S^1,\mathbb{R})$. It follows that φ is a lifting of g on S^1 and

$$E(g)=\lim_{n\to\infty}\int_{S^1}|\dot\varphi_n|\ge\int_{S^1}|\dot\varphi|\ge E(g);$$

iii) $E(g)=E_{\rm rel}(g)$. For $''\le''$, take $g_n\in C^\infty(S^1,S^1),\forall n\in\mathbb{N}$ such that $\deg g_n=0$, $g_n\to g$ a.e. on S^1 and $\sup_{n\in\mathbb{N}}\int_{S^1}|\dot g_n| d\mathcal{H}^1<\infty$. Then there exists $\varphi_n\in C^\infty(S^1,\mathbb{R})$ such that $e^{i\varphi_n}=g_n$. Since $\int_{S^1}|\dot\varphi_n| d\mathcal{H}^1=\int_{S^1}|\dot g_n| d\mathcal{H}^1$, using the same argument as above, we may assume that $\varphi_n\to\varphi$ a.e. and L^1 for some $\varphi\in BV(S^1,\mathbb{R})$. Therefore, $e^{i\varphi}=g$ a.e. on S^1 and

$$E(g)\le\int_{S^1}|\dot\varphi|\le\liminf_{n\to\infty}\int_{S^1}|\dot\varphi_n| d\mathcal{H}^1=\liminf_{n\to\infty}\int_{S^1}|\dot g_n| d\mathcal{H}^1.$$

For $''\ge''$, consider a BV lifting φ of g and take an approximating sequence $\varphi_n\in C^\infty(S^1,\mathbb{R})$ such that $\varphi_n\to\varphi$ a.e. and $|\dot\varphi|(S^1)=\lim_{n\to\infty}\int_{S^1}|\dot\varphi_n| d\mathcal{H}^1$. With $g_n=e^{i\varphi_n}\in C^\infty(S^1,S^1)$, we have $\deg g_n=0$, $g_n\to g$ a.e. on S^1 and

$$E_{\rm rel}(g)\le\lim_{n\to\infty}\int_{S^1}|\dot g_n| d\mathcal{H}^1=\lim_{n\to\infty}\int_{S^1}|\dot\varphi_n| d\mathcal{H}^1=\int_{S^1}|\dot\varphi|.$$

We next prove Lemmas 2.1 and 2.2:

Proof of Lemma 2.1. If $(\dot g)^j=0$, i.e. $S(g)=\varnothing$, then g is continuous on S^1. We claim that $m(g)=\deg g\in\mathbb{Z}$. This is clear when g is smooth; the general case is obtained by approximating g with a sequence $(g_n)_n\subset C^\infty(S^1,S^1)$ such that $g_n\to g$ uniformly and $\dot g_n\rightharpoonup\dot g$ weakly* as $n\to\infty$. Otherwise, let y_1 be a jump point of g on S^1. Consider $S^1\setminus\{y_1\}$ as an interval and on that interval take $\varphi^a,\varphi^c,\varphi^j$ the BV functions (unique up to constants) having as derivatives in $S^1\setminus\{y_1\}$ the finite Radon measures $g\wedge(\dot g)^a$, $g\wedge(\dot g)^c$ and $\sum_{y\in S(g)\setminus\{y_1\}} d_y(g)\delta_y$.

Let $\varphi=\varphi^a+\varphi^c+\varphi^j$. As in (1.13), by the chain rule, we have

$$\dot{(ge^{-i\varphi})}=0 \text{ on } S^1\setminus\{y_1\}$$

so that, up to a constant, φ is a lifting of g, i.e. $g = e^{i\varphi}$ a.e. on S^1. Clearly, $\varphi \in BV(S^1, \mathbb{R})$, $\varphi(y_1+) - \varphi(y_1-) = d_{y_1}(g) + 2\pi\alpha$, $\alpha \in \mathbb{Z}$ and

$$\dot{\varphi} = \dot{\varphi}\Big|_{S^1\setminus\{y_1\}} + (\varphi(y_1+) - \varphi(y_1-))\delta_{y_1} = g \wedge (\dot{g})^a + g \wedge (\dot{g})^c + \sum_{y\in S(g)} d_y(g)\delta_y + 2\pi\alpha\delta_{y_1}.$$

Since $\displaystyle\int_{S^1} \dot{\varphi} = 0$ we conclude that $P(g) + L(g) = -2\pi\alpha \in 2\pi\mathbb{Z}$. $\square$

Proof of Lemma 2.2. Suppose that $m(g) \geq 0$ (the case $m(g) < 0$ is analogous). We start by noting that

$$\inf_{\substack{\alpha_y\in\mathbb{Z},\, y\in S(g)\\ \#\{y\in S(g):\alpha_y\neq 0\}<\infty\\ \sum_{y\in S(g)}\alpha_y=m(g)}} \sum_{y\in S(g)} |d_y(g) - 2\pi\alpha_y| = \inf_{\substack{\alpha_y\in\mathbb{Z},\, y\in S(g)\\ \#\{y\in S(g):\alpha_y\neq 0\}<\infty\\ \sum_{y\in S(g)}\alpha_y=m(g)\\ |\alpha_y-\alpha_z|\leq 1,\forall y,z\in S(g)}} \sum_{y\in S(g)} |d_y(g) - 2\pi\alpha_y|. \tag{2.4}$$

Indeed, it suffices to observe that, if $d_1, d_2 \in (-\pi, \pi], \alpha_1, \alpha_2 \in \mathbb{Z}$ such that $\alpha_1 - \alpha_2 \geq 2$, then

$$|d_1 - 2\pi\alpha_1| + |d_2 - 2\pi\alpha_2| \geq |d_1 - 2\pi(\alpha_1 - 1)| + |d_2 - 2\pi(\alpha_2 + 1)|. \tag{2.5}$$

We distinguish in our analysis the following cases:
Case 1: $m(g) \geq \#S(g) > 0$. Then, by (2.4), we have that $\alpha_y \geq 1$, $\forall y \in S(g)$. It follows that $|d_y(g) - 2\pi\alpha_y| = 2\pi\alpha_y - d_y(g)$, $\forall y \in S(g)$. Therefore,

$$E^j(g) = 2\pi m(g) - P(g) = L(g) \geq 0.$$

The minimum is achieved in (2.2); consider, for example, the choice

$$(\alpha_y)_{y\in S(g)} = \Big(1, ..., 1, m(g) - \#S(g) + 1\Big).$$

Case 2: $0 < m(g) < \#S(g)$. By (2.4), we must have $\alpha_y \in \{0, 1\}$, $\forall y \in S(g)$. Therefore, the RHS of (2.4) is equal to

$$\inf_{\substack{\alpha_y\in\{0,1\}\\ \#\{y\in S(g):\alpha_y\neq 0\}<\infty\\ \sum_{y\in S(g)}\alpha_y=m(g)}} \sum_{y\in S(g)} |d_y(g) - 2\pi\alpha_y| = L(g) + 2 \inf_{\substack{B\subset S(g)\\ \#B=m(g)}} \sum_{\substack{y\in S(g)\setminus B\\ d_y(g)>0}} d_y(g);$$

this follows by noting that the y's for which $\alpha_y = 1$ have to be the ones with the largest positive jump $d_y(g)$. The infimum is achieved in (2.4). Indeed, set

$$\tilde{S}(g) = \{y \in S(g) : d_y(g) > 0\}.$$

If $\#\tilde{S}(g) \geq m(g)$, then choose $B = \{y_1, ..., y_{m(g)}\} \subset \tilde{S}(g)$ such that $d_{y_1}(g), ..., d_{y_{m(g)}}(g)$ are the biggest $m(g)$ elements of the set $\{d_y(g) \,:\, y \in \tilde{S}(g)\}$. If $\#\tilde{S}(g) < m(g)$, then choose $B \subset S(g)$ such that $\#B = m(g)$ and $\tilde{S}(g) \subset B$. Then an optimal choice is

$$\alpha_y = \begin{cases} 1 & \text{if } y \in B \\ 0 & \text{if } y \in S(g) \setminus B \end{cases}.$$

Case 3: $m(g) = 0$. Here the RHS of (2.4) is equal to $\sum_{y \in S(g)} |d_y(g)|$ and the infimum (2.4) is achieved for $\alpha_y = 0, \forall y \in S(g)$. $\square$

Proof of Theorem 2.3.

"$\geq$": Let $\varphi \in BV(S^1, \mathbb{R})$ be a lifting of g on S^1, i.e. $g = e^{i\varphi}$ a.e. on S^1. Then, by the chain rule,

$$(\dot\varphi)^a + (\dot\varphi)^c = g \wedge ((\dot g)^a + (\dot g)^c)$$

$$\text{and} \quad (\dot\varphi)^j = \sum_{y \in S(g)} (\varphi(y+) - \varphi(y-))\delta_y + \sum_{b \in B} (\varphi(b+) - \varphi(b-))\delta_b.$$

Here,

(i) $B \subset S^1$ is some finite set such that $S(g) \cap B = \varnothing$,

(ii) $\varphi(y+) - \varphi(y-) = d_y(g) - 2\pi\alpha_y$ with $\alpha_y \in \mathbb{Z}, \forall y \in S(g)$

(iii) $\varphi(b+) - \varphi(b-) = -2\pi\alpha_b, \alpha_b \in \mathbb{Z}, \forall b \in B$.

Clearly

$$\#\{y \in S(g) : \alpha_y \neq 0\} < \infty.$$

Since $\int_{S^1} \dot\varphi = 0$, we get $\sum_{y \in S(g) \cup B} \alpha_y = \frac{L(g) + P(g)}{2\pi} = m(g)$. We have

$$|\dot\varphi|(S^1) = \int_{S^1} \Big(|(\dot\varphi)^a| + |(\dot\varphi)^c| \Big) + \sum_{y \in S(g)} |d_y(g) - 2\pi\alpha_y| + 2\pi \sum_{b \in B} |\alpha_b|.$$

If $S(g) = \varnothing$, then

$$|\dot\varphi|(S^1) \geq \tilde E(g) + 2\pi |\sum_{b \in B} \alpha_b| = \tilde E(g) + |L(g)|,$$

which is the desired inequality. Otherwise, take $y_1 \in S(g)$ and observe that

$$|\dot\varphi|(S^1) \geq \tilde E(g) + \sum_{y \in S(g) \setminus \{y_1\}} |d_y(g) - 2\pi\alpha_y| + |d_{y_1}(g) - 2\pi\tilde\alpha_{y_1}|$$

where $\tilde\alpha_{y_1} = \alpha_{y_1} + \sum_{b \in B} \alpha_b$. Therefore, we conclude that

$$E(g) \geq \tilde E(g) + E^j(g).$$

"$\leq$" (**The construction of an optimal lifting**): If $S(g) = \varnothing$, then g is continuous on the simply connected set $S^1 \setminus \{1\}$ and so there is a unique (up to $2\pi\mathbb{Z}$ constants) lifting $\varphi \in BV(S^1 \setminus \{1\}, \mathbb{R}) \cap C^0$ of g on $S^1 \setminus \{1\}$. Moreover, $\varphi(1-) - \varphi(1+) = L(g)$ and we conclude that

$$|\dot\varphi|(S^1) = \tilde E(g) + |L(g)|.$$

Otherwise, take $y_1 \in S(g)$. By Lemma 2.2, we may take integers $\alpha_y \in \mathbb{Z}, \forall y \in S(g)$ (all zero except a finite number) such that $\sum_{y \in S(g)} \alpha_y = m(g)$ and (2.2) holds, i.e.

$$\sum_{y \in S(g)} |d_y(g) - 2\pi\alpha_y| = E^j(g).$$

As in the proof of Lemma 2.1, construct a lifting $\varphi \in BV(S^1, \mathbb{R})$ of g satisfying on $S^1 \setminus \{y_1\}$

$$(\dot{\varphi})^a = g \wedge (\dot{g})^a$$

$$(\dot{\varphi})^c = g \wedge (\dot{g})^c$$

$$\text{and} \quad (\dot{\varphi})^j \Big|_{S^1 \setminus \{y_1\}} = \sum_{y \in S(g) \setminus \{y_1\}} (d_y(g) - 2\pi\alpha_y)\delta_y \ \text{ on } \ S^1 \setminus \{y_1\}.$$

Since $\int_{S^1} \dot{\varphi} = 0$, we find that $\varphi(y_1+) - \varphi(y_1-) = d_{y_1}(g) - 2\pi\alpha_{y_1}$ which implies that $|\dot{\varphi}|(S^1) = \tilde{E}(g) + E^j(g)$. □

Note that the optimal lifting is not unique modulo 2π; indeed, if

$$g(e^{it}) = \begin{cases} 1 & \text{if } t \in (0, \pi) \\ -1 & \text{if } t \in (\pi, 2\pi) \end{cases}$$

then

$$\varphi_1(e^{it}) = \begin{cases} 0 & \text{if } t \in (0, \pi) \\ -\pi & \text{if } t \in (\pi, 2\pi) \end{cases} \quad \text{and} \quad \varphi_2(e^{it}) = \begin{cases} 0 & \text{if } t \in (0, \pi) \\ \pi & \text{if } t \in (\pi, 2\pi) \end{cases}$$

are optimal liftings and $\varphi_1 - \varphi_2 \neq const$ on S^1.

Remark 2.2 As we have proved, $E(g)$ depends on $(d_y(g))_{y \in S(g)}$ where $d_y(g)$ is the unique argument of the complex number $\frac{g(y+)}{g(y-)}$ in $(-\pi, \pi]$. Consider now, for each $y \in S(g)$, an arbitrary argument $d'_y(g)$ of $\frac{g(y+)}{g(y-)}$ such that $\sum_{y \in S(g)} |d'_y(g)| < \infty$. It is easy to see that

$$m'(g) = \frac{L(g) + \sum_{y \in S(g)} d'_y(g)}{2\pi} \in \mathbb{Z}.$$

Observe that if $S(g) \neq \varnothing$, then

$$E^j(g) = \min_{\substack{\alpha_y \in \mathbb{Z},\, y \in S(g) \\ \#\{y \in S(g) : \alpha_y \neq 0\} < \infty \\ \sum_{y \in S(g)} \alpha_y = m'(g)}} \sum_{y \in S(g)} |d'_y(g) - 2\pi\alpha_y|.$$

The analytic formula for $E^j(g)$ in Lemma 2.2 still holds for the $(d'_y(g))_{y \in S(g)}$ and $m'(g)$ provided $d'_y \in [-2\pi, 2\pi], \forall y \in S(g)$ and $|d'_y(g) - d'_z(g)| \leq 2\pi$, $\forall y, z \in S(g)$. This is a consequence of the fact that (2.5) holds if $d_1, d_2 \in [-2\pi, 2\pi]$ and $|d_1 - d_2| \leq 2\pi$.

As an immediate consequence of Lemma 2.2 and Theorem 2.3, we have:

Corollary 2.6 *For every $g \in BV(S^1, S^1)$,*

$$E(g) \leq 2|g|_{BV\,S^1}$$

where $|g|_{BV\,S^1} = \displaystyle\int_{S^1} \Big(|(\dot g)^a| + |(\dot g)^c|\Big) + \sum_{y\in S(g)} d_{S^1}(g(y+), g(y-)).$

Remark that $|\cdot|_{BV\,S^1}$ is a seminorm equivalent to the standard BV-seminorm $|\cdot|_{BV}$; in fact, we have

$$|g|_{BV} \leq |g|_{BV\,S^1} \leq \frac{\pi}{2}|g|_{BV}, \forall g \in BV(S^1, S^1).$$

Therefore, Theorem 2.4 is an improvement of the above corollary.

2.3 First proof of Theorem 2.4

We start by stating some trigonometrical inequalities:

Lemma 2.7 *Let $n \geq p \geq 1$ be two integers and let $x_k \in [0, \frac{\pi}{2}]$, $k = 1, ..., n$, be such that $\displaystyle\sum_{k=1}^n x_k \leq p\pi + \gamma$. Then*

$$\sum_{k=1}^n \sin x_k \geq \sum_{k=1}^n x_k - \max_{\substack{B\subset\{1,...,n\}\\ \#B=p}} \sum_{k\in B} x_k.$$

Proof. If $n = p$ then the conclusion is straightforward. Suppose now that $n > p$. By symmetry, we can assume that $B = \{x_{n-p+1}, ..., x_n\}$ contains the biggest p terms among $\{x_1, ..., x_n\}$. Set $z = \displaystyle\min_{n-p+1\leq k\leq n} x_k$. It is sufficient to prove that

$$\sum_{k=1}^{n-p} (\sin x_k - x_k) + p\sin z \geq 0.$$

Define the smooth symmetric function

$$f : [0, \frac{\pi}{2}]^{n-p} \to \mathbb{R},\ f(x_1, ..., x_{n-p}) = \sum_{k=1}^{n-p} (\sin x_k - x_k) + p\sin z.$$

Then f is a concave function. We want to find the minimum of f over the convex compact polyhedron

$$D = \{(x_1, ..., x_{n-p}) \in [0, z]^{n-p} : \sum_{k=1}^{n-p} x_k \leq p(\pi - z) + \gamma\}.$$

Since f is concave, its minimum on D is achieved in one of the extremal points (i.e. corners) of D. By a permutation of the coordinates, a corner $(x_1, ..., x_{n-p})$ of D has the following form: either

$$x_i \in \{0, z\}, \forall k = 1, ..., n - p$$

or

$$x_k \in \{0, z\}, \forall k = 1, ..., n-p-1 \text{ and } x_{n-p} = \gamma + p(\pi - z) - \sum_{k=1}^{n-p-1} x_k.$$

In order to prove that $f \geq 0$ on these points of D, we have to check that: if $k, p \geq 1$ are two integer numbers, $I_1 = [0, \frac{p\pi+\gamma}{k+p}] \cap [0, \frac{\pi}{2}]$ and $I_2 = [\frac{p\pi+\gamma}{k+p+1}, \frac{p\pi+\gamma}{k+p}] \cap [0, \frac{\pi}{2}]$, then

$$(k+p)\sin z - kz \geq 0, \ \forall z \in I_1 \tag{2.6}$$

and

$$(k+p)\sin z - kz + \sin\big(\gamma + p\pi - (k+p)z\big) - p\pi - \gamma + (k+p)z \geq 0, \ \forall z \in I_2. \tag{2.7}$$

Indeed, remark that the two LHS of each inequality from above represent concave functions in z and therefore, it is sufficient to show that they are positive on the extremities of the given intervals I_1 and I_2.

For (2.6), let us denote

$$h(z) = (k+p)\sin z - kz, \ \forall z \in I_1.$$

Case 1: $I_1 = [0, \frac{p\pi+\gamma}{k+p}]$, *i.e.* $\frac{p\pi+\gamma}{k+p} \leq \frac{\pi}{2}$. Then $k \geq p+1$. We have that $h(0) = 0$ and it remains to check that

$$h(\frac{p\pi+\gamma}{k+p}) \geq 0.$$

If $p = 1$ and $k = 2$ then $h(\frac{\pi+\gamma}{3}) = 0$. If $p = 1$ and $k \geq 3$, then the inequality

$$\sin z \geq z - \frac{z^3}{6} \tag{2.8}$$

yields, for $z = \frac{\pi+\gamma}{k+1}$,

$$h(\frac{\pi+\gamma}{k+1}) \geq \frac{\pi+\gamma}{k+1}\left(1 - \frac{(\pi+\gamma)^2}{6(k+1)}\right) \geq 0.$$

Otherwise, $p \geq 2$ and applying (2.8) for $z = \frac{p\pi+\gamma}{k+p}$, we obtain

$$h(\frac{p\pi+\gamma}{k+p}) \geq \frac{p\pi+\gamma}{k+p}\left(p - \frac{(p\pi+\gamma)^2}{6(k+p)}\right) \geq \frac{p\pi+\gamma}{k+p}\left(p - \frac{(p\pi+\gamma)^2}{6(2p+1)}\right) \geq 0.$$

Case 2: $I_1 = [0, \frac{\pi}{2}]$, *i.e.* $k \leq p$. Then $h(\frac{\pi}{2}) = k + p - k\frac{\pi}{2} \geq (2 - \frac{\pi}{2})k \geq 0$.

The proof of (2.7) follows the same lines. □

Remark 2.3 γ is optimal for the above inequality (consider $n = 3, p = 1, x_1 = x_2 = x_3 = \frac{\pi+\gamma}{3}$).

Proof of Lemma 2.5. We can assume that $B = \{x_1, ..., x_p\}$ contains the biggest p terms among $\{x_k : k \geq 1\}$. Let $\varepsilon > 0$. There exists $n > p$ such that $\sum_{k>n} x_k < \varepsilon$. By Lemma 2.7, we know that

$$\sum_{k=1}^{n} \sin x_k \geq \sum_{k=p+1}^{n} x_k.$$

Therefore,

$$\sum_{k\geq 1} \sin x_k \geq \sum_{k=1}^{n} \sin x_k \geq \sum_{k>p} x_k - \varepsilon.$$

Letting now $\varepsilon \to 0$, the conclusion follows. □

We now present:

Proof of Theorem 2.4. It suffices to prove that

$$E^j(g) \leq \int_{S^1} \Big(|(\dot{g})^a| + |(\dot{g})^c|\Big) + 2 \sum_{y\in S(g)} |g(y+) - g(y-)|. \tag{2.9}$$

If $S(g) = \varnothing$, the conclusion follows using the inequality $|L(g)| \leq \int_{S^1} \Big(|(\dot{g})^a|+|(\dot{g})^c|\Big)$. If $m(g) = 0$, (2.9) is a consequence of the fact that $|d_y(g)| \leq \frac{\pi}{2}|g(y+) - g(y-)|, \forall y \in S(g)$. Suppose now that $S(g) \neq \varnothing$ and $m(g) \neq 0$; assume that $m(g) > 0$ (the case $m(g) < 0$ is similar). As in the proof of Lemma 2.2, consider

$$\tilde{S}(g) = \{y \in S(g) : d_y(g) > 0\}.$$

If $\#\tilde{S}(g) \leq m(g)$ then, by Lemma 2.2, $E^j(g) = |L(g)|$ and so (2.9) holds.

Otherwise, we have $\#\tilde{S}(g) > m(g) \geq 1$. Rewrite $P(g) + L(g) = 2\pi m(g)$ as

$$\sum_{y\in\tilde{S}(g)} d_y(g) - \sum_{y\in S(g)\setminus\tilde{S}(g)} |d_y(g)| + L(g) = 2\pi m(g). \tag{2.10}$$

Let $B \subset \tilde{S}(g)$ consist of the largest $m(g)$ elements of the set $\{d_y(g) : y \in \tilde{S}(g)\}$. For each $y \in \tilde{S}(g)$, set $x_y = \frac{d_y(g)}{2} \in [0, \frac{\pi}{2}]$. Then $|g(y+) - g(y-)| = 2\sin x_y$. We distinguish two cases:

Case 1:

$$\sum_{y\in S(g)\setminus\tilde{S}(g)} |d_y(g)| - L(g) \leq 2\gamma.$$

By (2.10), $\sum_{y\in\tilde{S}(g)} d_y(g) \leq 2\pi m(g) + 2\gamma$. By Lemma 2.5, we have

$$\sum_{y\in\tilde{S}(g)\setminus B} d_y(g) \leq \sum_{y\in\tilde{S}(g)} |g(y+) - g(y-)|.$$

Using Lemma 2.2, we find that

$$E^j(g) = L(g) + 2 \sum_{y\in\tilde{S}(g)\setminus B} d_y(g) \leq |L(g)| + 2 \sum_{y\in S(g)} |g(y+) - g(y-)|.$$

Case 2:

$$\sum_{y\in S(g)\setminus\tilde{S}(g)} |d_y(g)| - L(g) > 2\gamma, \text{ i.e. } \sum_{y\in\tilde{S}(g)} d_y(g) > 2\pi m(g) + 2\gamma. \tag{2.11}$$

The following two situations can occur:

i) There exists $S_1 \subset \tilde{S}(g)$ such that $B \subset S_1$ and

$$2\pi m(g) \leq \sum_{y \in S_1} d_y(g) \leq 2\pi m(g) + 2\gamma. \tag{2.12}$$

By (2.12), using Lemma 2.5, we infer that

$$\sum_{y \in S_1 \setminus B} d_y(g) \leq \sum_{y \in S_1} |g(y+) - g(y-)|. \tag{2.13}$$

With $S_2 = \tilde{S}(g) \setminus S_1$, it follows from (2.10) and (2.12) that

$$\sum_{y \in S_2} d_y(g) - \sum_{y \in S(g) \setminus \tilde{S}(g)} |d_y(g)| + L(g) \leq 0.$$

By adding $\sum_{y \in S_2} d_y(g)$, we obtain

$$2 \sum_{y \in S_2} d_y(g) + L(g) \leq \sum_{y \in S_2 \cup (S(g) \setminus \tilde{S}(g))} |d_y(g)| \leq \frac{\pi}{2} \sum_{y \in S_2 \cup (S(g) \setminus \tilde{S}(g))} |g(y+) - g(y-)|. \tag{2.14}$$

Combining (2.13) and (2.14), we deduce

$$E^j(g) = L(g) + 2 \sum_{y \in \tilde{S}(g) \setminus B} d_y(g) \leq 2 \sum_{y \in S(g)} |g(y+) - g(y-)|.$$

ii) There exist $S_1 \subset \tilde{S}(g)$ and $\tilde{y} \in \tilde{S}(g) \setminus S_1$ such that $B \subset S_1$ and

$$2\pi m(g) + 2\gamma - d_{\tilde{y}}(g) < \sum_{y \in S_1} d_y(g) < 2\pi m(g).$$

Set $S_2 = \tilde{S}(g) \setminus (S_1 \cup \{\tilde{y}\})$. By (2.10), we have

$$\sum_{y \in S_2} d_y(g) - \sum_{y \in S(g) \setminus \tilde{S}(g)} |d_y(g)| + L(g) \leq -2\gamma. \tag{2.15}$$

By adding $\sum_{y \in S_2} d_y(g)$ to (2.15), we find that

$$\begin{aligned} 2 \sum_{y \in S_2} |d_y(g)| + L(g) &\leq -2\gamma + \sum_{y \in S_2 \cup (S(g) \setminus \tilde{S}(g))} |d_y(g)| \\ &= -2\gamma + \frac{4}{\pi} \sum_{y \in S_2 \cup (S(g) \setminus \tilde{S}(g))} |d_y(g)| - (\frac{4}{\pi} - 1) \sum_{y \in S_2 \cup (S(g) \setminus \tilde{S}(g))} |d_y(g)|. \end{aligned}$$

From (2.11), we get that

$$2\gamma + L(g) \leq \sum_{y \in S(g) \setminus \tilde{S}(g)} |d_y(g)| \leq \sum_{y \in S_2 \cup (S(g) \setminus \tilde{S}(g))} |d_y(g)|.$$

Therefore,

$$
\begin{aligned}
2\sum_{y\in S_2} d_y(g) + L(g) &\le -2\gamma + \frac{4}{\pi}\sum_{y\in S_2\cup(S(g)\setminus\tilde S(g))} |d_y(g)| - (\frac{4}{\pi}-1)\Big(2\gamma + L(g)\Big)\\
&\le \frac{4}{\pi}\sum_{y\in S_2\cup(S(g)\setminus\tilde S(g))} |d_y(g)| + |L(g)| - \frac{8\gamma}{\pi} \qquad (2.16)\\
&\le 2\sum_{y\in S_2\cup(S(g)\setminus\tilde S(g))} |g(y+)-g(y-)| + |L(g)| - \frac{8\gamma}{\pi};
\end{aligned}
$$

here, we have used the fact that $|d_y(g)| \le \dfrac{\pi}{2}|g(y+)-g(y-)|$, $\forall y \in S(g)$. On the other side, Lemma 2.5 yields

$$\sum_{y\in S_1\setminus B} d_y(g) \le \sum_{y\in S_1} |g(y+)-g(y-)|. \tag{2.17}$$

Remark also that

$$d_{\tilde y}(g) \le |g(\tilde y+)-g(\tilde y-)| + (\pi-2); \tag{2.18}$$

this amounts to the inequality $x \le 2\sin\frac{x}{2} + \pi - 2$, $\forall x\in[0,\pi]$. By combining (2.16), (2.17) and (2.18), we obtain

$$
\begin{aligned}
E^j(g) &= L(g) + 2\sum_{y\in S_2} d_y(g) + 2\sum_{y\in S_1\setminus B} d_y(g) + 2d_{\tilde y}(g)\\
&\le |L(g)| + 2\sum_{y\in S(g)} |g(y+)-g(y-)| + 2(\pi-2) - \frac{8\gamma}{\pi}.
\end{aligned}
$$

Since $2(\pi-2) - \frac{8\gamma}{\pi} < 0$, the conclusion follows. □

2.4 Another proof of Theorem 2.4

In Chapter 1, we proved the estimate (2.3) for $BV(\Omega, S^1)$ functions, where $\Omega\subset\mathbb{R}^N$ is a bounded open set. The idea was to consider the function $f : S^1 \to \mathbb{R}$ defined by

$$f(e^{i\theta}) = \theta \quad \text{for} \quad \forall -\pi \le \theta < \pi,$$

and to show that for an appropriate $\alpha\in\mathbb{R}$, the lifting

$$\varphi = f(e^{i\alpha}g) - \alpha$$

satisfies $|\varphi|_{BV} \le 2|g|_{BV}$. For that, one can repeat the same arguments and prove that

$$\int_0^{2\pi} |f(e^{i\alpha}g)-\alpha|_{BV}\, d\alpha \le 4\pi|g|_{BV}; \tag{2.19}$$

the conclusion is now straightforward.

Remark 2.4 i) Set $\mathscr{C}(g) = \{f(e^{i\alpha}g) - \alpha \,:\, \alpha \in \mathbb{R}\}$. Then $\mathscr{C}(g)$ need not be contained in BV. Here is an example. Consider the step function $g \in BV(S^1, S^1)$ defined by

$$g(e^{2\pi it}) = e^{ix_k},\ t \in (\frac{1}{2^k}, \frac{1}{2^{k-1}}),\ k = 1, 2, ...$$

where $x_k = (-1)^k 2^{-[\frac{k+1}{2}]}\pi$. It is easy to see that

$$|f(e^{i\pi}g) - \pi|_{BV} = |x_1 + 2\pi| + \sum_{k\geq 1} |x_{k+1} - x_k + (-1)^k 2\pi| = \infty.$$

ii) It follows from (2.19) that, for a.e. $\alpha \in [0, 2\pi]$, $f(e^{i\alpha}g) - \alpha \in BV(S^1, \mathbb{R})$; clearly, the same holds for a.e. $\alpha \in \mathbb{R}$.

iii) There exist some functions $g \in BV(S^1, S^1)$ such that no lifting in $\mathscr{C}(g)$ is optimal. For example, consider the step function $g : S^1 \to S^1$ be defined as:

$$g(e^{\pi it}) = \begin{cases} 1 & \text{if } t \in (0, \frac{1}{7}) \\ e^{i\frac{\pi}{2}} & \text{if } t \in (\frac{1}{7}, \frac{2}{7}) \\ e^{i\frac{3\pi}{4}} & \text{if } t \in (\frac{2}{7}, \frac{3}{7}) \\ e^{i\frac{(k+2)\pi}{4}} & \text{if } t \in (\frac{k}{7}, \frac{k+1}{7}),\ k = 3, ..., 13 \end{cases}.$$

So g has 2 jumps of length $\frac{\pi}{2}$ (with respect to d_{S^1}) and 12 jumps of length $\frac{\pi}{4}$. Then $m(g) = 2$ and

$$E(g) = E^j(g) = 12 \cdot \frac{\pi}{4} + 2 \cdot (2\pi - \frac{\pi}{2}).$$

Remark now that for every $\alpha \in \mathbb{R}$, the cut $\{z \in \mathbb{C} \,:\, arg(z) = \pi - \alpha \pmod{2\pi}\}$ of the function $z \to f(e^{i\alpha}z) - \alpha$ will affect two jumps of g and at least one of them has the size $\frac{\pi}{4}$ (with respect to the geodesic distance d_{S^1} on S^1). Therefore,

$$|f(e^{i\alpha}g) - \alpha|_{BV} \geq \frac{\pi}{2} + 11 \cdot \frac{\pi}{4} + (2\pi - \frac{\pi}{4}) + (2\pi - \frac{\pi}{2}) > E(g).$$

2.5 Some examples

We present some examples showing that the constant 2 in (2.3) is optimal (see also Brezis-Mironescu-Ponce[30]).

1. Let $g = Id : S^1 \to S^1$. Then $g \in BV(S^1, S^1) \cap C^0$. Remark that $(\dot{g})^c = (\dot{g})^j = 0$. Thus, $\deg g = 1$, $\tilde{E}(g) = E^j(g) = |g|_{BV} = 2\pi$ and so $E(g) = 2|g|_{BV}$.

2. Let $f : [0, 1] \to [0, 1]$ be the standard Cantor function. Define $g : S^1 \to S^1$ as

$$g(e^{2\pi i\, t}) = e^{2\pi i\, f(t)}, \forall t \in [0, 1].$$

Clearly, $g \in BV(S^1, S^1) \cap C^0$, $(\dot{g})^a = (\dot{g})^j = 0$ and $\deg g = 1$. As above, $\tilde{E}(g) = E^j(g) = |g|_{BV} = 2\pi$ and $E(g) = 2|g|_{BV}$.

3. For each $n \geq 2$, take $g_n(e^{2\pi i\, t}) = e^{2\pi i\, k/n}$ for $\frac{k}{n} \leq t < \frac{k+1}{n}$, $k = 0, 1, ..., n-1$. Then $g_n \in BV(S^1, S^1)$ and $(\dot{g}_n)^a = (\dot{g}_n)^c = 0$. We have that $\tilde{E}(g_n) = 0$, $m(g_n) = 1$, $E^j(g_n) = 4\pi(1 - \frac{1}{n})$ and $|g_n|_{BV} = 2n \sin \frac{\pi}{n}$. We deduce that

$$\lim_{n\to\infty} \frac{E(g_n)}{|g_n|_{BV}} = 2.$$

Chapter 3

The space $BV(S^2, S^1)$: minimal connection and optimal lifting

Abstract

We show that topological singularities of maps in $BV(S^2, S^1)$ can be detected by a special distribution. As an application, we construct an optimal lifting and we compute its total variation.

This chapter is published in Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), 283–302 (cf. [53]).

3.1 Introduction

Let $u \in BV(S^2, S^1)$, i.e. $u = (u_1, u_2) \in L^1(S^2, \mathbb{R}^2)$, $|u(x)| = 1$ for a.e. $x \in S^2$ and the derivative of u (in the sense of the distributions) is a finite $2 \times 2-$matrix Radon measure

$$\int_{S^2} |Du| = \sup \left\{ \int_{S^2} \sum_{k=1}^{2} u_k \operatorname{div} \zeta_k \, \mathrm{d}\mathcal{H}^2 \;:\; \zeta_k \in C^1(S^2, \mathbb{R}^2), \sum_{k=1}^{2} |\zeta_k(x)|^2 \leq 1, \forall x \in S^2 \right\} < \infty ,$$

where the norm in $\mathbb{R}^2$ is the Euclidean norm. Observe that the total variation of Du is independent of the choice of the orthonormal frame (x, y) on S^2; a frame (x, y) is always taken such that (x, y, e) is direct, where e is the outward normal to the sphere S^2.

We begin with the notion of minimal connection between point singularities of u. The concept of a minimal connection associated to a function from $\mathbb{R}^3$ into S^2 was originally introduced by Brezis, Coron and Lieb [27]. Following the ideas in [27] and [38], Brezis, Mironescu and Ponce [30] studied the topological singularities of functions $g \in W^{1,1}(S^2, S^1)$. They show that the distributional Jacobian of g describes the location and the topological charge of the singular set of g. More precisely, let $T(g) \in \mathcal{D}'(S^2, \mathbb{R})$ be the distribution on S^2 defined as

$$T(g) = 2 \det(\nabla g) = -(g \wedge g_x)_y + (g \wedge g_y)_x;$$

then there exist two sequences of points $(p_k), (n_k)$ in S^2 such that

$$\sum_k |p_k - n_k| < \infty \quad \text{and} \quad T(g) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}).$$

Our aim is to extend these notions for functions $u \in BV(S^2, S^1)$. In this case, the difficulty of the analysis of the singular set arises from the existence of more than one type of singularity: besides the point singularities carrying a degree, the jump singularities of u should be taken into account.

We start by introducing some notation. Write the finite Radon 2×2-matrix measure Du as

$$Du = D^a u + D^c u + D^j u,$$

where $D^a u, D^c u$ and $D^j u$ are the absolutely continuous part, the Cantor part and the jump part of Du (see e.g. [11]). We recall that $D^j u$ can be written as

$$D^j u = (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^1 \llcorner S(u),$$

where $S(u)$ denotes the set of jump points of u; $S(u)$ is a countably $\mathcal{H}^1$-rectifiable set on S^2 oriented by the Borel map $\nu_u : S(u) \to S^1$. The Borel functions $u^+, u^- : S(u) \to S^1$ are the traces of u on the jump set $S(u)$ with respect to the orientation ν_u. Throughout the chapter we identify u by its precise representative that is defined $\mathcal{H}^1$-a.e. on S^2.

We now introduce the distribution $T(u) \in \mathcal{D}'(S^2, \mathbb{R})$ as

$$\langle T(u), \zeta \rangle = \int_{S^2} \nabla^\perp \zeta \cdot (u \wedge (D^a u + D^c u)) + \int_{S(u)} \rho(u^+, u^-) \, \nu_u \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1, \;\; \forall \zeta \in C^1(S^2, \mathbb{R}). \quad (3.1)$$

Here, $\nabla^\perp \zeta = (\zeta_y, -\zeta_x)$,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \wedge \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = (u \wedge a, u \wedge b) = (u_1 a_2 - u_2 a_1, u_1 b_2 - u_2 b_1)$$

where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. The function $\rho(\cdot,\cdot) : S^1 \times S^1 \to [-\pi, \pi]$ is the signed geodesic distance on S^1 defined as

$$\rho(\omega_1, \omega_2) = \begin{cases} \mathrm{Arg}\left(\frac{\omega_1}{\omega_2}\right) & \text{if } \frac{\omega_1}{\omega_2} \neq -1 \\ \mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2) & \text{if } \frac{\omega_1}{\omega_2} = -1 \end{cases}, \; \forall \omega_1, \omega_2 \in S^1$$

where $\mathrm{Arg}\,(\omega) \in (-\pi, \pi]$ stands for the argument of the unit complex number $\omega \in S^1$. $T(u)$ represents the distributional determinant of the absolutely continuous part and the Cantor part of Du which is adjusted on $S(u)$ by the tangential derivative of $\rho(u^+, u^-)$. The second term in the RHS of (3.1) is motivated by the study of $BV(S^1, S^1)$ functions (see Chapter 2): we defined there a similar quantity that represents a pseudo-degree for $BV(S^1, S^1)$ functions.

Remark 3.1 i) The integrand in (3.1) is computed pointwise in any orthonormal frame (x, y) and the corresponding quantity is frame-invariant.

ii) The 2-vector measure

$$\mu = (\mu_1, \mu_2) = u \wedge (D^a u + D^c u) = (u \wedge (D^a u_x + D^c u_x), u \wedge (D^a u_y + D^c u_y))$$

is well-defined since $D^a u + D^c u$ vanishes on sets which are σ-finite with respect to $\mathcal{H}^1$.

iii) Notice that the function ρ is antisymmetric, i.e.

$$\rho(\omega_1, \omega_2) = -\rho(\omega_2, \omega_1),\ \forall \omega_1, \omega_2 \in S^1$$

and therefore, $T(u)$ does not depend of the choice of the orientation ν_u on the jump set $S(u)$. By Lemma 3.10 (see below), we obtain

$$|\langle T(u), \zeta \rangle| \leq |u|_{BV\, S^1},\ \forall \zeta \in C^1(S^2, \mathbb{R}) \text{ with } |\nabla \zeta| \leq 1$$

where $|u|_{BV\, S^1} = \displaystyle\int_{S^2} \Big(|D^a u| + |D^c u| \Big) + \int_{S(u)} d_{S^1}(u^+, u^-)\, \mathrm{d}\mathcal{H}^1$ and d_{S^1} stands for the geodesic distance on S^1. Therefore, $T(u)$ is indeed a distribution (of order 1) on S^2.

For a compact Riemannian manifold X with the induced distance d, let $\mathcal{Z}(X)$ be the set of distributions that can be written as a countable sum of dipoles:

$$\mathcal{Z}(X) = \left\{ \Lambda \in \left[C^1(X) \right]^* \,:\, \exists (p_k), (n_k) \subset X,\ \sum_k d(p_k, n_k) < \infty \text{ and } \Lambda = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}) \right\}.$$

Remark 3.2 i) In general, $\Lambda \in \mathcal{Z}(X)$ is not a measure. In fact, it can be shown that Λ is a measure if and only if Λ is a finite sum of dipoles (see Smets [85] and also Ponce [72]).

ii) $\Lambda \in \mathcal{Z}(X)$ has always infinitely many representations as a sum of dipoles and these representations need not be equivalent modulo a permutation of points. For example, a dipole $\delta_p - \delta_n$ may be represented as $\delta_p - \delta_{n_1} + \displaystyle\sum_{k \geq 1} (\delta_{n_k} - \delta_{n_{k+1}})$ for any sequence $(n_k)_k$ rapidly converging to n.

For each $\Lambda \in \mathcal{Z}(X)$, the length of a minimal connection between the singularities is defined as

$$\|\Lambda\| = \sup_{\substack{\zeta \in C^1(X) \\ |\nabla \zeta| \leq 1}} \langle \Lambda, \zeta \rangle.$$

For example, when $\Lambda = 2\pi \displaystyle\sum_{k=1}^m (\delta_{p_k} - \delta_{n_k})$ is a finite sum of dipoles, Brezis, Coron and Lieb [27] showed that

$$\|\Lambda\| = 2\pi \min_{\sigma \in S_m} \sum_{k=1}^m d(p_k, n_{\sigma(k)})$$

where S_m denotes the group of permutation of $\{1, 2, \ldots, m\}$. In general, for an arbitrary $\Lambda \in \mathcal{Z}(X)$, Bourgain, Brezis and Mironescu [22] proved the following characterization of the length of a minimal connection:

$$\|\Lambda\| = \inf_{(p_k), (n_k)} \left\{ 2\pi \sum_k d(p_k, n_k) \,:\, \Lambda = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}) \text{ and } \sum_k d(p_k, n_k) < \infty \right\}. \tag{3.2}$$

From (3.2), one can deduce that $\mathcal{Z}(X)$ is a complete metric space with respect to the distance induced by $\| \cdot \|$ (see e.g. [72]).

Our first theorem states that $T(u)$ is a countable sum of dipoles. It is the extension to the BV case of the result in [30] mentioned in the beginning.

Theorem 3.1 *For every $u \in BV(S^2, S^1)$, we have $T(u) \in \mathcal{Z}(S^2)$, i.e. there exist $(p_k), (n_k)$ in S^2 such that*

$$\sum_k |p_k - n_k| < \infty \quad \textit{and} \quad T(u) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}).$$

The proof relies on the fact that the derivative (in the sense of distributions) of the characteristic function of a bounded measurable set in $\mathbb{R}$ can be written as a sum of differences between Dirac masses:

Lemma 3.2 *Let $I \subset \mathbb{R}$ be a compact interval and $f : I \to 2\pi\mathbb{Z}$ be an integrable function. Define*

$$\langle \frac{\mathrm{d}f}{\mathrm{d}t}, \zeta \rangle := - \int_I f(t)\, \zeta'(t)\, \mathrm{d}t\,,\ \forall \zeta \in C^1(I).$$

Then

$$\frac{\mathrm{d}f}{\mathrm{d}t} \in \mathcal{Z}(I) \quad \textit{and} \quad \| \frac{\mathrm{d}f}{\mathrm{d}t} \| = \int_I |f|\, \mathrm{d}t.$$

The same property is valid for the distributional tangential derivative of an integrable function taking values in $2\pi\mathbb{Z}$ and defined on a C^1 1-graph (see Remark 3.3). Since every countably $\mathcal{H}^1$-rectifiable set $S \subset S^2$ can be covered $\mathcal{H}^1$-a.e. by a sequence of C^1 1-graphs, it makes sense to define for every $\Lambda \in \mathcal{Z}(S^2)$ the set

$$\mathcal{J}(\Lambda) = \left\{ (f, S, \nu) \ : \begin{array}{l} S \text{ is a countably } \mathcal{H}^1\text{- rectifiable set in } S^2,\ \nu \text{ is an orientation on } S, \\ f \in L^1(S, 2\pi\mathbb{Z}) \text{ is such that } \displaystyle\int_S f\, \nu \cdot \nabla^{\perp} \zeta \, \mathrm{d}\mathcal{H}^1 = \langle \Lambda, \zeta \rangle,\ \forall \zeta \in C^1(S^2) \end{array} \right\}.$$

We have the following reformulation of (3.2):

Lemma 3.3 *For every $\Lambda \in \mathcal{Z}(S^2)$, we have*

$$\|\Lambda\| = \min_{(f,S,\nu) \in \mathcal{J}(\Lambda)} \int_S |f|\, \mathrm{d}\mathcal{H}^1.$$

It is known that the infimum in (3.2) is not achieved in general (see [72]); the advantage of the above formula is that the minimum is always attained. It means that the length of Λ represents the minimal mass that an $\mathcal{H}^1$-integrable function with values into $2\pi\mathbb{Z}$ could carry between the dipoles of Λ.

In the sequel we are concerned with the lifting of $u \in BV(S^2, S^1)$. We call BV lifting of u every function $\varphi \in BV(S^2, \mathbb{R})$ such that

$$u = e^{i\varphi} \ \text{ a.e. on } S^2.$$

The existence of a BV lifting for functions $u \in BV(S^2, S^1)$ was initially shown by Giaquinta, Modica and Souček [47]. Adapting the argument in Chapter 1, one can prove the existence of a lifting $\varphi \in BV \cap L^\infty(S^2, \mathbb{R})$ such that

$$\int_{S^2} |D\varphi| \leq 2 \int_{S^2} |Du|; \tag{3.3}$$

moreover, the constant 2 in (3.3) is the best constant (see Example 3.1 and Proposition 3.13 below).

We give the following characterization for a lifting of u:

Lemma 3.4 *Let $u \in BV(S^2, S^1)$. For every lifting $\varphi \in BV(S^2,\mathbb{R})$ of u, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that*

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \, \mathcal{H}^1 \llcorner S(u) - f\nu \, \mathcal{H}^1 \llcorner S. \tag{3.4}$$

Conversely, for every triple $(f, S, \nu) \in \mathcal{J}(T(u))$ there exists a lifting $\varphi \in BV(S^2,\mathbb{R})$ of u such that (3.4) *holds.*

In this framework, it is natural to investigate the quantity

$$E(u) = \inf\left\{ \int_{S^2} |D\varphi| \,:\, \varphi \in BV(S^2,\mathbb{R}),\, e^{i\varphi} = u \text{ a.e. on } S^2 \right\}. \tag{3.5}$$

The infimum from above is achieved and it is equal to the relaxed energy

$$E_{\mathrm{rel}}(u) = \inf\left\{ \liminf_{k\to\infty} \int_{S^2} |\nabla u_k| \,\mathrm{d}\mathcal{H}^2 \,:\, u_k \in C^\infty(S^2, S^1),\, u_k \to u \text{ a.e. on } S^2 \right\} \tag{3.6}$$

(see Remark 3.4). A lifting $\varphi \in BV(S^2,\mathbb{R})$ of u is called *optimal* if

$$E(u) = \int_{S^2} |D\varphi|.$$

An optimal lifting need not be unique (see Proposition 3.13). Remark also that for $u \in BV(S^2, S^1)$, there could be no optimal BV lifting of u that belongs to L^∞ (see Example 3.3).

Our aim is to compute the total variation $E(u)$ of an optimal lifting and to construct an optimal lifting. Theorem 3.5 establishes the formula for $E(u)$ using the distribution $T(u)$.

Theorem 3.5 *For every $u \in BV(S^2, S^1)$, we have*

$$E(u) = \int_{S^2} (|D^a u| + |D^c u|) + \min_{(f,S,\nu)\in\mathcal{J}(T(u))} \int_{S\cup S(u)} \left| f\nu\,\chi_S - \rho(u^+, u^-)\nu_u\,\chi_{S(u)} \right| \mathrm{d}\mathcal{H}^1. \tag{3.7}$$

We refer the reader to [47] for related results in terms of cartesian currents.

As a consequence of Theorem 3.5, we recover the result of Brezis, Mironescu and Ponce [30] about the total variation of an optimal BV lifting for functions $g \in W^{1,1}(S^2, S^1)$: the gap

$$E(g) - \int_{S^2} |\nabla g| \,\mathrm{d}\mathcal{H}^2$$

is equal to the length of a minimal connection connecting the topological singularities of g.

Corollary 3.6 *For every $g \in W^{1,1}(S^2, S^1)$, we have*

$$E(g) = \int_{S^2} |\nabla g| \,\mathrm{d}\mathcal{H}^2 + \|T(g)\|.$$

From (3.7), we deduce an estimate for $E(u)$ (which is a weaker form of inequality (3.3)):

Corollary 3.7 *For every $u \in BV(S^2, S^1)$, we have*

$$E(u) \le 2|u|_{BV\,S^1}.$$

In the spirit of [30], we have the following interpretation of $\|T(u)\|$ as a distance:

Theorem 3.8 *For every $u \in BV(S^2,S^1)$, we have*

$$\|T(u)\| = \min_{\psi\in BV(S^2,\mathbb{R})} \int_{S^2} \Big| u\wedge (D^a u + D^c u) + \rho(u^+,u^-)\nu_u\,\mathcal{H}^1 \llcorner S(u) - D\psi\Big|. \tag{3.8}$$

Moreover, there is at least one minimizer $\psi \in BV(S^2,\mathbb{R})$ of (3.8) that is a lifting of u.

Remark that in general, $\|T(u)\|$ is not the distance of the measure

$$u\wedge (D^a u + D^c u) + \rho(u^+,u^-)\nu_u\,\mathcal{H}^1 \llcorner S(u)$$

to the class of gradient maps. In Example 3.4, we construct a function $u \in BV(S^2,S^1)$ such that

$$\|T(u)\| < \inf_{\psi\in C^\infty(S^2,\mathbb{R})} \int_{S^2} \Big| u\wedge (D^a u + D^c u) + \rho(u^+,u^-)\nu_u\,\mathcal{H}^1 \llcorner S(u) - D\psi\Big|.$$

In Section 3.2, we present the proofs of Lemmas 3.2, 3.3 and 3.4, Theorems 3.1, 3.5 and 3.8 and Corollaries 3.6 and 3.7. Some examples and interesting properties of $T(u)$ are given in Section 3.3. Among other things, we show that $T : BV(S^2,S^1) \to \mathcal{Z}(S^2)$ is discontinuous and we analyze some algebraic properties of $T(u)$. We also discuss the meaning of the point singularities of $T(u)$ and about their location on S^2.

All the results included here can be easily adapted for functions in $BV(\Omega,S^1)$ where Ω is a more general simply connected Riemannian manifold of dimension 2.

3.2 Remarks and proofs of the main results

We start by proving Lemma 3.2:

Proof of Lemma 3.2. Firstly, let us suppose that $f = 2\pi\chi_A$ where $A \subset I$ is an open set. Write $A = \bigcup_{j\in\mathbb{N}}(a_j,b_j)$ as a countable reunion of disjoint intervals. It is clear that

$$\langle \frac{\mathrm{d}\chi_A}{\mathrm{d}t},\zeta\rangle = \sum_{j\in\mathbb{N}}(\zeta(a_j)-\zeta(b_j)),\ \forall\zeta\in C^1(I)$$

and $\sum_{j\in\mathbb{N}}(b_j-a_j) = \mathcal{H}^1(A)$. Thus $2\pi\frac{\mathrm{d}\chi_A}{\mathrm{d}t} \in \mathcal{Z}(I)$ and

$$\|\frac{\mathrm{d}f}{\mathrm{d}t}\| = 2\pi \sup_{\substack{\zeta\in C^1(I)\\ |\zeta'|\le 1}} \int_I \chi_A\,\zeta'\,\mathrm{d}t = 2\pi \sup_{\substack{\psi\in C(I)\\ |\psi|\le 1}} \int_I \chi_A\,\psi\,\mathrm{d}t = 2\pi\mathcal{H}^1(A).$$

Moreover, let $A \subset I$ be a Lebesgue measurable set and $f = 2\pi\chi_A$. Using the regularity of the Lebesgue measure, there exists a decreasing sequence of open sets $A \subset A_{k+1} \subset A_k \subset I$, $k \in \mathbb{N}$ such that $\lim_{k\to\infty}\mathcal{H}^1(A_k) = \mathcal{H}^1(A)$. Observe that $\frac{\mathrm{d}\chi_{A_k}}{\mathrm{d}t} \to \frac{\mathrm{d}\chi_A}{\mathrm{d}t}$ in $\left[C^1(I)\right]^*$. Since $\mathcal{Z}(I)$ is a complete metric space, we conclude that $2\pi\frac{\mathrm{d}\chi_A}{\mathrm{d}t} \in \mathcal{Z}(I)$ and $\|2\pi\frac{\mathrm{d}\chi_A}{\mathrm{d}t}\| = 2\pi\mathcal{H}^1(A)$. In the general case of an integrable function $f : I \to 2\pi\mathbb{Z}$, write

$$f = 2\pi \sum_{k\in\mathbb{Z}} k\,\chi_{E_k} \ \text{ in } \ L^1, \tag{3.9}$$

where $E_k = \{x \in I \,:\, f(x) = 2\pi k\}$. Notice that $2\pi\frac{\mathrm{d}\,(k\,\chi_{E_k})}{\mathrm{d}t} \in \mathcal{Z}(I)$ and the series $\sum_{k\in\mathbb{Z}} 2\pi\frac{\mathrm{d}\,(k\,\chi_{E_k})}{\mathrm{d}t}$ converges absolutely; indeed, we have

$$\sum_{k\in\mathbb{Z}} \|2\pi\frac{\mathrm{d}\,(k\,\chi_{E_k})}{\mathrm{d}t}\| = 2\pi\sum_{k\in\mathbb{Z}} |k|\mathcal{H}^1(E_k) = \int_I |f|\,\mathrm{d}t < \infty.$$

By (3.9), we conclude that $\frac{\mathrm{d}f}{\mathrm{d}t} \in \mathcal{Z}(I)$ and

$$\|\frac{\mathrm{d}f}{\mathrm{d}t}\| = \sup_{\substack{\zeta\in C^1(I)\\ |\zeta'|\le 1}} \int_I f\,\zeta'\,\mathrm{d}t = \sup_{\substack{\psi\in C(I)\\ |\psi|\le 1}} \int_I f\,\psi\,\mathrm{d}t = \int_I |f|\,\mathrm{d}t.$$

□

Remark 3.3 The conclusion of Lemma 3.2 is also true for $\mathcal{H}^1$-integrable functions with values in $2\pi\mathbb{Z}$ that are defined on C^1 1-graphs. For simplicity, we restrict to C^1 1-graphs in S^2, i.e. for an orthonormal frame (x, y) on S^2, we consider the set

$$\Gamma = \{(x, y) \,:\, \phi(x) = y\}$$

where ϕ is a C^1 function. Suppose $c : [0, 1] \to \Gamma$ is a parameterization of Γ and set $\tau(c(t)) = \frac{c'(t)}{|c'(t)|}$ the tangent unit vector to the curve Γ at $c(t)$, $\forall t \in (0, 1)$. Let $f : \Gamma \to 2\pi\mathbb{Z}$ be an $\mathcal{H}^1$-integrable function on Γ. Define

$$\langle\frac{\partial f}{\partial\tau}, \zeta\rangle := -\int_0^1 f \circ c(t)\,(\zeta \circ c)'(t)\,\mathrm{d}t\,,\ \forall\zeta \in C^1(\Gamma).$$

By Lemma 3.2, we have

$$\frac{\partial f}{\partial\tau} \in \mathcal{Z}(\Gamma) \quad \text{and} \quad \|\frac{\partial f}{\partial\tau}\| = \int_0^1 |f|(c(t))\,|c'(t)|\,\mathrm{d}t.$$

Before proving Lemma 3.4, we give the following result:

Lemma 3.9 *For every $u \in BV(S^2, S^1)$, we have*

$$u \wedge (D^a u + D^c u) = \frac{1}{i}\bar{u}(D^a u + D^c u)$$
$$\textit{and} \quad |u \wedge (D^a u + D^c u)| = |D^a u| + |D^c u|.$$

Proof. Write $u = (u_1, u_2) = u_1 + i\, u_2$. We can consider the 2×2 matrix of real measures Du as a 2-vector of complex measures, i.e. $Du = Du_1 + i\, Du_2$. Since $u_1^2 + u_2^2 = 1$, it results $D(u_1^2 + u_2^2) = 0$. By the chain rule (see e.g. [11]), we obtain

$$u_1(D^a u_1 + D^c u_1) + u_2(D^a u_2 + D^c u_2) = 0,$$

i.e. the real part of the $\mathbb{C}^2$-measure $\bar{u}(D^a u + D^c u)$ vanishes. Therefore,

$$u \wedge (D^a u + D^c u) = \frac{1}{i}\bar{u}(D^a u + D^c u).$$

Hence, using the fact that the absolutely continuous part and the Cantor part of Du are mutually singular, we conclude that

$$|u \wedge (D^a u + D^c u)| = |u|(|D^a u| + |D^c u|) = |D^a u| + |D^c u|.$$

□

Proof of Lemma 3.4. Let $\varphi \in BV(S^2, \mathbb{R})$ be a lifting of u. Write

$$D\varphi = D^a\varphi + D^c\varphi + (\varphi^+ - \varphi^-)\nu_\varphi \mathcal{H}^1 \llcorner S(\varphi).$$

By the chain rule and Lemma 3.9, we obtain

$$D^a\varphi + D^c\varphi = \frac{1}{i}\bar{u}(D^a u + D^c u) = u \wedge (D^a u + D^c u).$$

Since $u = e^{i\varphi}$ a.e. on S^2, we have that $S(u) \subset S(\varphi)$ and by changing the orientation ν_φ, we may assume

$$\begin{cases} \nu_\varphi = \nu_u \\ e^{i\varphi+} = u^+ \quad \mathcal{H}^1\text{-a.e. on } S(u). \\ e^{i\varphi-} = u^- \end{cases}$$

Therefore,

$$\varphi^+ - \varphi^- \equiv \rho(u^+, u^-) \pmod{2\pi} \ \ \mathcal{H}^1\text{-a.e. on } S(u)$$
$$\text{and} \quad \varphi^+ - \varphi^- \equiv 0 \pmod{2\pi} \ \ \mathcal{H}^1\text{-a.e. on } S(\varphi) \setminus S(u).$$

Hence, there exists $f_\varphi : S(\varphi) \to 2\pi\mathbb{Z}$ a measurable function such that

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u\, \mathcal{H}^1 \llcorner S(u) - f_\varphi \nu_\varphi\, \mathcal{H}^1 \llcorner S(\varphi). \tag{3.10}$$

Observe that f_φ is an $\mathcal{H}^1$-integrable function since

$$|\rho(u^+, u^-)| = d_{S^1}(u^+, u^-) \le \frac{\pi}{2}|u^+ - u^-|.$$

Since $D\varphi$ is a measure, we have

$$\operatorname{curl} D\varphi = 0 \text{ in } \mathcal{D}',$$

i.e. for every $\zeta \in C^1(S^2, \mathbb{R})$,

$$\int_{S^2} \nabla^\perp \zeta\, D\varphi = 0.$$

By (3.10), it yields

$$\langle T(u),\zeta\rangle=\int_{S(\varphi)} f_\varphi\,\nabla^\perp\zeta\cdot\nu_\varphi\,\mathrm{d}\mathcal{H}^1,\ \forall\zeta\in C^1(S^2)$$

and therefore, $(f_\varphi,S(\varphi),\nu_\varphi)\in\mathcal{J}(T(u))$.

Conversely, take $(f,S,\nu)\in\mathcal{J}(T(u))$. Without loss of generality, we may consider $S=\{f\neq 0\}$. Consider the finite Radon $\mathbb{R}^2$-valued measure

$$\mu=u\wedge(D^a u+D^c u)+\rho(u^+,u^-)\nu_u\,\mathcal{H}^1\llcorner S(u)-f\,\nu\,\mathcal{H}^1\llcorner S.$$

We check that $\operatorname{curl}\mu=0$ in $\mathcal{D}'(S^2)$. Indeed, for every $\zeta\in C^1(S^2,\mathbb{R})$,

$$-\langle\operatorname{curl}\mu,\zeta\rangle=\int_{S^2}\nabla^\perp\zeta\,\mathrm{d}\mu=\langle T(u),\zeta\rangle-\int_S f\,\nabla^\perp\zeta\cdot\nu\,\mathrm{d}\mathcal{H}^1=0.$$

By the BV version of Poincare's lemma, there exists $\varphi\in BV(S^2,\mathbb{R})$ such that $D\varphi=\mu$ in $\mathcal{D}'(S^2,\mathbb{R}^2)$. Here, $S\cup S(u)$ is the jump set of φ. On the set $S\cup S(u)$, we choose an orientation ν_φ such that $\nu_\varphi=\nu_u$ on $S(u)$. We have

$$\begin{cases} D^a\varphi+D^c\varphi=u\wedge(D^a u+D^c u)=\frac{1}{i}\bar{u}(D^a u+D^c u)\\ \varphi^+-\varphi^-\equiv\rho(u^+,u^-)\pmod{2\pi}\ \ \mathcal{H}^1\text{- a.e. on } S(u)\\ \varphi^+-\varphi^-\equiv 0\pmod{2\pi}\ \ \mathcal{H}^1\text{- a.e. on } S\setminus S(u)\end{cases}.$$

We now show that

$$D(u\,e^{-i\varphi})=0.$$

By the chain rule, we get

$$\begin{aligned} D(e^{-i\varphi})&=-ie^{-i\varphi}(D^a\varphi+D^c\varphi)+(e^{-i\varphi^+}-e^{-i\varphi^-})\otimes\nu_u\,\mathcal{H}^1\llcorner S(u)\\ &=-e^{-i\varphi}\bar{u}(D^a u+D^c u)+(e^{-i\varphi^+}-e^{-i\varphi^-})\otimes\nu_u\,\mathcal{H}^1\llcorner S(u).\end{aligned}$$

Remark that the space $BV(S^2,\mathbb{C})\cap L^\infty$ is an algebra. Differentiating the product $u\,e^{-i\varphi}$, we obtain

$$D(u\,e^{-i\varphi})=e^{-i\varphi}(D^a u+D^c u)-u\,e^{-i\varphi}\bar{u}(D^a u+D^c u)+(u^+\,e^{-i\varphi^+}-u^-\,e^{-i\varphi^-})\otimes\nu_u\,\mathcal{H}^1\llcorner S(u)=0.$$

Thus, up to an additive constant, φ is a BV lifting of u and (3.4) is fulfilled. □

Proof of Theorem 3.1. Let $\varphi\in BV(S^2,\mathbb{R})$ be a lifting of u. By Lemma 3.4, there exists $(f,S,\nu)\in\mathcal{J}(T(u))$ such that (3.4) holds. Denote by $\tau:S\to S^1$ the tangent vector in $\mathcal{H}^1$-a.e. point of S such that (ν,τ,e) is direct. By (3.4),

$$\begin{aligned}\langle T(u),\zeta\rangle&=\int_S f\,\nabla^\perp\zeta\cdot\nu\,\mathrm{d}\mathcal{H}^1\\ &=\int_S f\,\frac{\partial\zeta}{\partial\tau}\,\mathrm{d}\mathcal{H}^1\\ &=\sum_{k\in\mathbb{N}}\int_{I_k}\chi_S f\,\frac{\partial\zeta}{\partial\tau}\,\mathrm{d}\mathcal{H}^1,\ \forall\zeta\in C^1(S^2)\end{aligned}$$

where $\{I_k\}_{k\in\mathbb{N}}$ is a family of disjoint compact C^1 1-graphs that covers $\mathcal{H}^1$-almost all of the countably rectifiable set S, i.e.

$$\mathcal{H}^1\left(S \setminus \bigcup_{k\in\mathbb{N}} I_k\right) = 0.$$

According to Lemma 3.2 and Remark 3.3, we conclude $T(u) \in \mathcal{Z}(S^2)$ and $\|T(u)\| \leq \int_S |f|\, d\mathcal{H}^1$.
□

Before proving Theorem 3.5, let us make some remarks about $E(u)$ and $E_{\text{rel}}(u)$ for $u \in BV(S^2, S^1)$ (see also [30]):

Remark 3.4 i) $E(u) < \infty$ and $E_{\text{rel}}(u) < \infty$ (the existence of a BV lifting of u was shown in Chapter 1 and [47]);

ii) The infimum in (3.5) is achieved; indeed, let $\varphi_k \in BV(S^2, \mathbb{R})$, $e^{i\varphi_k} = u$ a.e. on S^2, be such that

$$\lim_{k\to\infty} \int_{S^2} |D\varphi_k| = E(u) < \infty.$$

By Poincaré's inequality, there exists a universal constant $C > 0$ such that

$$\int_{S^2} \left|\varphi_k - \fint_{S^2} \varphi_k\right| \mathrm{d}\mathcal{H}^2 \leq C \int_{S^2} |D\varphi_k|, \ \forall k \in \mathbb{N}$$

(where $\fint_{S^2}$ stands for the average). Therefore, by subtracting a suitable integer multiple of 2π, we may assume that $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $BV(S^2, \mathbb{R})$. After passing to a subsequence if necessary, we may assume that $\varphi_k \to \varphi$ a.e. and L^1 for some $\varphi \in BV(S^2, \mathbb{R})$. It follows that φ is a lifting of u on S^2 and

$$E(u) = \lim_{k\to\infty} \int_{S^2} |D\varphi_k| \geq \int_{S^2} |D\varphi| \geq E(u);$$

iii) The infimum in (3.6) is also achieved; take $u_k^m \in C^\infty(S^2, S^1)$ such that for each $k \in \mathbb{N}$,

$$u_k^m \to u \text{ a.e. on } S^2 \text{ and } \int_{S^2} |\nabla u_k^m|\, \mathrm{d}\mathcal{H}^2 \searrow a_k \in \mathbb{R} \text{ as } m \to \infty$$

and $\lim_{k\to\infty} a_k = E_{\text{rel}}(u)$. Subtracting a subsequence, we may assume that for each $k \in \mathbb{N}$,

$$\int_{S^2} |u_k^m - u|\, \mathrm{d}\mathcal{H}^2 < \frac{1}{k} \text{ and } \int_{S^2} |\nabla u_k^m|\, \mathrm{d}\mathcal{H}^2 - a_k < \frac{1}{k}, \ \forall m \geq 1.$$

Therefore, $u_k^k \to u$ in L^1 and

$$\lim_{k\to\infty} \int_{S^2} |\nabla u_k^k|\, \mathrm{d}\mathcal{H}^2 = E_{\text{rel}}(u).$$

iv) $E(u) = E_{\text{rel}}(u)$. For "$\leq$", take $u_k \in C^\infty(S^2, S^1), \forall k \in \mathbb{N}$ such that $u_k \to u$ a.e. on S^2 and $\sup_{k\in\mathbb{N}} \int_{S^2} |\nabla u_k|\, \mathrm{d}\mathcal{H}^2 < \infty$. Since S^2 is simply connected, there exists $\varphi_k \in C^\infty(S^2, \mathbb{R})$ such that

$e^{i\varphi_k} = u_k$. Moreover, $\int_{S^2} |\nabla \varphi_k| \, d\mathcal{H}^2 = \int_{S^2} |\nabla u_k| \, d\mathcal{H}^2$. Using the same argument as in ii), we may assume that $\varphi_k \to \varphi$ a.e. and L^1 for some $\varphi \in BV(S^2, \mathbb{R})$. Therefore, $e^{i\varphi} = u$ a.e. on S^2 and

$$E(u) \leq \int_{S^2} |D\varphi| \leq \liminf_{k\to\infty} \int_{S^2} |\nabla \varphi_k| \, d\mathcal{H}^2 = \liminf_{k\to\infty} \int_{S^2} |\nabla u_k| \, d\mathcal{H}^2.$$

For "$\geq$", consider a BV lifting φ of u and take an approximating sequence $\varphi_k \in C^\infty(S^2, \mathbb{R})$ such that $\varphi_k \to \varphi$ a.e. and $|D\varphi|(S^2) = \lim_{k\to\infty} \int_{S^2} |\nabla \varphi_k| \, d\mathcal{H}^2$. With $u_k = e^{i\varphi_k} \in C^\infty(S^2, S^1)$, we have $u_k \to u$ a.e. on S^2 and

$$E_{\text{rel}}(u) \leq \lim_{k\to\infty} \int_{S^2} |\nabla u_k| \, d\mathcal{H}^2 = \lim_{k\to\infty} \int_{S^2} |\nabla \varphi_k| \, d\mathcal{H}^2 = \int_{S^2} |D\varphi|.$$

□

Proof of Theorem 3.5. For "$\leq$", take $(f, S, \nu) \in \mathcal{J}(T(u))$. By Lemma 3.4, there exists a lifting $\varphi \in BV(S^2, \mathbb{R})$ of u such that (3.4) holds. It follows that

$$E(u) \leq \int_{S^2} |D\varphi| = \int_{S^2} (|D^a u| + |D^c u|) + \int_{S \cup S(u)} \left| f\nu\chi_S - \rho(u^+, u^-)\nu_u \chi_{S(u)} \right| d\mathcal{H}^1.$$

Let us prove now "$\geq$". By Remark 3.4, there is an optimal BV lifting φ of u, i.e. $E(u) = \int_{S^2} |D\varphi|$. By Lemma 3.4, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that (3.4) holds. It results that

$$E(u) = \int_{S^2} |D\varphi| = \int_{S^2} (|D^a u| + |D^c u|) + \int_{S \cup S(u)} \left| f\nu\chi_S - \rho(u^+, u^-)\nu_u \chi_{S(u)} \right| d\mathcal{H}^1.$$

From here, we also deduce that the minimum inside the RHS of (3.7) is achieved. □

Remark 3.5 (Construction of an optimal lifting) Take $(f, S, \nu) \in \mathcal{J}(T(u))$ that achieves the minimum

$$\min_{(f,S,\nu)\in\mathcal{J}(T(u))} \int_{S \cup S(u)} \left| f\nu \, \chi_S - \rho(u^+, u^-)\nu_u \, \chi_{S(u)} \right| d\mathcal{H}^1. \tag{3.11}$$

By Lemma 3.4, there exists a lifting $\varphi \in BV(S^2, \mathbb{R})$ of u such that (3.4) holds. Then

$$\int_{S^2} |D\varphi| = \int_{S^2} (|D^a u| + |D^c u|) + \int_{S \cup S(u)} \left| f\nu\chi_S - \rho(u^+, u^-)\nu_u \chi_{S(u)} \right| d\mathcal{H}^1 = E(u)$$

and therefore, φ is an optimal lifting of u. □

Proof of Lemma 3.3. For "$\leq$", it is easy to see that if $(f, S, \nu) \in \mathcal{J}(\Lambda)$ then for every $\zeta \in C^1(S^2)$ with $|\nabla\zeta| \leq 1$,

$$\langle \Lambda, \zeta \rangle = \int_S f \, \nu \cdot \nabla^\perp \zeta \, d\mathcal{H}^1 \leq \int_S |f| \, d\mathcal{H}^1.$$

For "$\geq$", we use characterization (3.2) of the distribution $\Lambda \in \mathcal{Z}(S^2)$. We denote by d_{S^2} the geodesic distance on S^2. Let $\Lambda = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k})$ where $(p_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ belong to S^2 such that $\sum_k d_{S^2}(p_k, n_k) < \infty$. For every $k \in \mathbb{N}$, consider $\widehat{n_k p_k}$ a geodesic arc on S^2 oriented from n_k to p_k. Take ν_k the normal vector to $\widehat{n_k p_k}$ in the frame (x, y). Set $S = \bigcup_k \widehat{n_k p_k}$. Since $\sum_k d_{S^2}(p_k, n_k) < \infty$, there exist an orientation $\nu : S \to S^1$ on S and an $\mathcal{H}^1$-integrable function $f : S \to 2\pi\mathbb{Z}$ such that

$$f\nu\chi_S = \sum_k 2\pi\nu_k\chi_{\widehat{n_k p_k}} \quad \text{in } L^1(S, \mathbb{R}^2). \tag{3.12}$$

Then

$$\int_S f\nu \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1 = 2\pi \sum_k \int_{\widehat{n_k p_k}} \nu_k \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1 = 2\pi \sum_k (\zeta(p_k) - \zeta(n_k)) = \langle \Lambda, \zeta \rangle, \, \forall \zeta \in C^1(S^2).$$

It follows that $(f, S, \nu) \in \mathcal{J}(\Lambda)$ and by (3.12),

$$\int_S |f| \, \mathrm{d}\mathcal{H}^1 \leq \sum_k 2\pi d_{S^2}(n_k, p_k).$$

Minimizing after all suitable pairs $(p_k, n_k)_{k\in\mathbb{N}}$, it follows

$$\|\Lambda\| = \inf_{(f,S,\nu)\in\mathcal{J}(\Lambda)} \int_S |f| \, \mathrm{d}\mathcal{H}^1. \tag{3.13}$$

We now show that the infimum in (3.13) is indeed achieved. By a dipole construction (see [22], Lemma 16), there exists $u \in W^{1,1}(S^2, S^1)$ such that $\Lambda = T(u)$. We choose $(f_k, S_k, \nu_k) \in \mathcal{J}(T(u))$ such that

$$\|T(u)\| = \lim_k \int_{S_k} |f_k| \, \mathrm{d}\mathcal{H}^1.$$

By Lemma 3.4, we construct a lifting $\varphi_k \in BV(S^2, \mathbb{R})$ of u such that

$$D\varphi_k = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \, \mathcal{H}^1 \llcorner S(u) - f_k \, \nu_k \, \mathcal{H}^1 \llcorner S_k.$$

Remark that

$$\int_{S^2} |D\varphi_k| \leq \int_{S^2} (|D^a u| + |D^c u|) + \int_{S(u)} |\rho(u^+, u^-)| \, \mathrm{d}\mathcal{H}^1 + \int_{S_k} |f_k| \, \mathrm{d}\mathcal{H}^1.$$

Subtracting a suitable number in $2\pi\mathbb{Z}$, we may assume that $(\varphi_k)_k$ is a bounded sequence in $BV(S^2, \mathbb{R})$. Up to a subsequence, we find $\varphi \in BV(S^2, \mathbb{R})$ such that

$$\varphi_k \to \varphi \text{ a.e. in } S^2 \text{ and } D\varphi_k \overset{*}{\rightharpoonup} D\varphi \text{ in the measure sense.}$$

Therefore, φ is a BV lifting of u and by Lemma 3.4, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \, \mathcal{H}^1 \llcorner S(u) - f \, \nu \, \mathcal{H}^1 \llcorner S.$$

We conclude

$$\begin{aligned}\int_S |f|\,\mathrm{d}\mathcal{H}^1 &= \int_{S^2} \Big| u\wedge(D^a u + D^c u) + \rho(u^+,u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\varphi \Big| \\ &\le \liminf_k \int_{S^2} \Big| u\wedge(D^a u + D^c u) + \rho(u^+,u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\varphi_k \Big| \\ &= \lim_k \int_{S_k} |f_k|\,\mathrm{d}\mathcal{H}^1 \\ &= \|T(u)\|. \end{aligned}$$

□

Proof of Theorem 3.8. Let $\psi \in BV(S^2,\mathbb{R})$ and $\zeta \in C^1(S^2)$ be such that $|\nabla\zeta| \le 1$. Then

$$\int_{S^2} \Big| u\wedge(D^a u + D^c u) + \rho(u^+,u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\psi \Big| \ge \langle T(u),\zeta\rangle - \int_{S^2} D\psi\cdot\nabla^{\perp}\zeta = \langle T(u),\zeta\rangle.$$

By taking the supremum over ζ, we obtain

$$\int_{S^2} \Big| u\wedge(D^a u + D^c u) + \rho(u^+,u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\psi \Big| \ge \|T(u)\|.$$

We now show that there is a lifting $\varphi \in BV(S^2,\mathbb{R})$ of u such that the minimum in (3.8) is achieved. By Lemma 3.3, choose $(f,S,\nu) \in \mathcal{J}(T(u))$ such that

$$\|T(u)\| = \int_S |f|\,\mathrm{d}\mathcal{H}^1.$$

Using Lemma 3.4, we construct a lifting $\varphi \in BV(S^2,\mathbb{R})$ such that (3.4) holds. Thus,

$$\|T(u)\| = \int_S |f|\,\mathrm{d}\mathcal{H}^1 = \int_{S^2} \Big| u\wedge(D^a u + D^c u) + \rho(u^+,u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\varphi \Big|.$$

□

Proof of Corollary 3.6. The result is a straightforward consequence of Theorem 3.5 and Lemma 3.3. □

In order to prove Corollary 3.7, we need the following estimation of $\|T(u)\|$ in terms of the seminorm $|u|_{BVS^1}$:

Lemma 3.10 *We have* $\|T(u)\| \le |u|_{BVS^1}$, $\forall u \in BV(S^2,S^1)$.

Proof. By Lemma 3.9, it results that for every $\zeta \in C^1(S^2)$ with $|\nabla\zeta| \le 1$,

$$\begin{aligned}|\langle T(u),\zeta\rangle| &\le \int_{S^2} |u\wedge(D^a u + D^c u)| + \int_{S(u)} |\rho(u^+,u^-)|\,\mathrm{d}\mathcal{H}^1 \\ &= \int_{S^2} (|D^a u| + |D^c u|) + \int_{S(u)} d_{S^1}(u^+,u^-)\,\mathrm{d}\mathcal{H}^1;\end{aligned}$$

therefore

$$\|T(u)\| \le |u|_{BVS^1}.$$

□

Proof of Corollary 3.7. By Theorem 3.5, Lemmas 3.3 and 3.10, we conclude that

$$\begin{aligned} E(u) &\leq \int_{S^2} (|D^a u| + |D^c u|) + \int_{S(u)} |\rho(u^+, u^-)| \, \mathrm{d}\mathcal{H}^1 + \min_{(f,S,\nu)\in\mathcal{J}(T(u))} \int_S |f| \, \mathrm{d}\mathcal{H}^1 \\ &= |u|_{BVS^1} + \|T(u)\| \\ &\leq 2|u|_{BVS^1}. \end{aligned}$$

□

Let $|u|_{BV} = \int_{S^2} |Du| = \int_{S^2} (|D^a u| + |D^c u|) + \int_{S(u)} |u^+ - u^-| \, \mathrm{d}\mathcal{H}^1$; we deduce that

$$|u|_{BV} \leq |u|_{BV\,S^1} \leq \frac{\pi}{2} |u|_{BV}, \forall u \in BV(S^2, S^1).$$

Therefore, Corollary 3.7 is a weaker estimate of $E(u)$ than inequality (3.3) obtained in Chapter 1.

3.3 Some other properties of the distribution T

We start by observing that $T : BV(S^2, S^1) \to \mathcal{D}'(S^2, \mathbb{R})$ is not continuous, i.e. there exists a sequence of functions $u_k \in BV(S^2, S^1)$ such that $u_k \to u$ strongly in $BV(S^2, S^1)$ and $T(u_k) \nrightarrow T(u)$ in $\mathcal{D}'(S^2, \mathbb{R})$. The reason for that is the discontinuity of the function ρ that enters in the definition of T.

Proposition 3.11 *The map $T : BV(S^2, S^1) \to \mathcal{D}'(S^2, \mathbb{R})$ is discontinuous.*

Proof. Write

$$S^2 = \{(\cos\theta \sin\alpha, \sin\theta \sin\alpha, \cos\alpha) \,:\, \alpha \in [0, \pi], \, \theta \in (0, 2\pi]\}.$$

In the spherical coordinates $(\alpha, \theta) \in [0, \pi] \times [0, 2\pi]$, consider the BV functions φ and u defined as

$$\varphi(\alpha, \theta) = \begin{cases} -2\theta & \text{if } \theta \in (0, \frac{\pi}{2}), \, \alpha \in (0, \frac{\pi}{2}) \\ -\pi & \text{if } \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \, \alpha \in (0, \frac{\pi}{2}) \\ 2(\theta - 2\pi) & \text{if } \theta \in (\frac{3\pi}{2}, 2\pi), \, \alpha \in (0, \frac{\pi}{2}) \\ 0 & \text{if } \theta \in (0, 2\pi), \, \alpha \in (\frac{\pi}{2}, \pi) \end{cases} \quad \text{and} \quad u = e^{i\varphi}. \tag{3.14}$$

We have that the jump set of u and φ is concentrated on the equator $\{\alpha = \frac{\pi}{2}\}$ of the sphere S^2, i.e.

$$S(\varphi) = S(u) = \{\alpha = \frac{\pi}{2}\}.$$

On the equator we choose the orientation given by the normal vector $\vec{\alpha}$ oriented from the north to the south; so $(\vec{\alpha}, \vec{\theta}, \vec{e})$ is direct. We show that

$$T(u) = 2\pi(\delta_p - \delta_n) \tag{3.15}$$

where $n = (\frac{\pi}{2}, \frac{3\pi}{2})$ and $p = (\frac{\pi}{2}, \frac{\pi}{2})$ in the frame (α, θ). Indeed, we remark that

$$\varphi^+ - \varphi^- = \rho(u^+, u^-) + 2\pi \chi_{\overset{\frown}{np}} \quad \text{on } S(u);$$

by Lemma 3.4, we obtain

$$D\varphi = u \wedge \nabla u\,\mathcal{H}^2 + \rho(u^+,u^-)\vec{\alpha}\,\mathcal{H}^1 \llcorner S(u) + 2\pi\vec{\alpha}\,\mathcal{H}^1 \llcorner \widehat{np}$$

and it yields

$$\langle T(u),\zeta\rangle = -2\pi\int_{\widehat{np}} \vec{\alpha}\cdot\nabla^{\perp}\zeta\,\mathrm{d}\mathcal{H}^1 = -2\pi\int_p^n \frac{\partial\zeta}{\partial\theta}\,\mathrm{d}\mathcal{H}^1 = 2\pi(\zeta(p)-\zeta(n)),\ \forall\zeta\in C^1(S^2,\mathbb{R}).$$

Construct the approximation sequence $\varphi_\varepsilon \in BV(S^2,\mathbb{R})$, $\varepsilon\in(0,1)$ defined (in the spherical coordinates) as

$$\varphi_\varepsilon(\alpha,\theta) = \begin{cases} -2\theta & \text{if } \theta\in(0,\frac{\pi-\varepsilon}{2}),\ \alpha\in(0,\frac{\pi}{2}) \\ -\pi+\varepsilon & \text{if } \theta\in(\frac{\pi-\varepsilon}{2},\frac{3\pi+\varepsilon}{2}),\ \alpha\in(0,\frac{\pi}{2}) \\ 2(\theta-2\pi) & \text{if } \theta\in(\frac{3\pi+\varepsilon}{2},2\pi),\ \alpha\in(0,\frac{\pi}{2}) \\ 0 & \text{if } \theta\in(0,2\pi),\ \alpha\in(\frac{\pi}{2},\pi) \end{cases}.$$

and set $u_\varepsilon = e^{i\,\varphi_\varepsilon}$. An easy computation shows that $\varphi_\varepsilon\to\varphi$ strongly in BV; therefore, $u_\varepsilon\to u$ strongly in BV as $\varepsilon\to 0$. As before, we have

$$S(\varphi_\varepsilon) = S(u_\varepsilon) = \{\alpha=\frac{\pi}{2}\} \text{ and } \varphi_\varepsilon^+ - \varphi_\varepsilon^- = \rho(u_\varepsilon^+,u_\varepsilon^-) \ \text{ on } \{\alpha=\frac{\pi}{2}\}.$$

It follows that $T(u_\varepsilon)=0$ and we conclude

$$T(u_\varepsilon) \nrightarrow T(u) \text{ in } \mathcal{D}'(S^2,\mathbb{R}).$$

□

As Brezis, Mironescu and Ponce proved in [30], if we restrict ourselves to $W^{1,1}(S^2,S^1)$, then the map $T\big|_{W^{1,1}(S^2,S^1)} : W^{1,1}(S^2,S^1)\to\mathcal{Z}(S^2)$ is continuous, i.e. if $g,g_k\in W^{1,1}(S^2,S^1)$ such that $g_k\to g$ in $W^{1,1}$ then $\|T(g_k)-T(g)\|\to 0$ as $k\to\infty$. It is natural to ask if one could change the antisymmetric function ρ in order that the corresponding map T become continuous. The answer is negative:

Proposition 3.12 *There is no antisymmetric function $\gamma : S^1\times S^1\to\mathbb{R}$ such that the map $T_\gamma : BV(S^2,S^1)\to\mathcal{Z}(S^2)$ given for every $u\in BV(S^2,S^1)$ as*

$$\langle T_\gamma(u),\zeta\rangle = \int_{S^2}\nabla^{\perp}\zeta\cdot(u\wedge(D^a u + D^c u)) + \int_{S(u)}\gamma(u^+,u^-)\,\nu_u\cdot\nabla^{\perp}\zeta\,\mathrm{d}\mathcal{H}^1,\ \forall\zeta\in C^1(S^2,\mathbb{R})$$

is well-defined and continuous.

Proof. By contradiction, suppose that there exists such a function γ. First we show that

$$\gamma(\omega_1,\omega_2) \equiv \mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2) \pmod{2\pi},\ \forall\omega_1,\omega_2\in S^1. \tag{3.16}$$

Indeed, fix $\omega_1,\omega_2\in S^1$. Take $f:[0,2\pi]\to\mathbb{R}$ the linear function satisfying $f(0)=\mathrm{Arg}\,(\omega_1)$ and $f(2\pi)=\mathrm{Arg}\,(\omega_2)$; define $u\in BV(S^2,S^1)$ as

$$u(\alpha,\theta) = e^{if(\theta)},\ \forall\alpha\in(0,\pi),\theta\in(0,2\pi).$$

Consider the lifting $\varphi \in BV(S^2, \mathbb{R})$ of u given by

$$\varphi(\alpha, \theta) = f(\theta), \ \forall \alpha \in (0, \pi), \theta \in (0, 2\pi).$$

If $\omega_1 \neq \omega_2$, the jump set of u and φ is concentrated on the meridian $\{\theta = 0\}$ orientated counterclockwise by the unit vector $\vec{\theta}$. We have that

$$D\varphi = u \wedge \nabla u \mathcal{H}^2 + (\mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2))\vec{\theta}\,\mathcal{H}^1 \llcorner \{\theta = 0\}.$$

Since $\operatorname{curl} D\varphi = 0$ in $\mathcal{D}'$, it yields

$$\begin{aligned}
\int_{S^2} u \wedge \nabla u \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^2 &= -\int_{\{\theta=0\}} (\mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2))\vec{\theta} \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1 \\
&= (\mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2)) \int_p^n \frac{\partial \zeta}{\partial \alpha} \, \mathrm{d}\mathcal{H}^1 \\
&= (\mathrm{Arg}\,(\omega_2) - \mathrm{Arg}\,(\omega_1))(\zeta(p) - \zeta(n)), \ \forall \zeta \in C^1(S^2)
\end{aligned}$$

where $p = (0, 0)$ and $n = (\pi, 0)$ (in the spherical coordinates) are the north and the south pole of S^2. We obtain that

$$\begin{aligned}
\langle T_\gamma(u), \zeta \rangle &= \int_{S^2} \nabla^\perp \zeta \cdot (u \wedge \nabla u) \, \mathrm{d}\mathcal{H}^2 + \gamma(\omega_1, \omega_2) \int_{\{\theta=0\}} \vec{\theta} \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1 \\
&= (\mathrm{Arg}\,(\omega_2) - \mathrm{Arg}\,(\omega_1) + \gamma(\omega_1, \omega_2))(\zeta(p) - \zeta(n)), \ \forall \zeta \in C^1(S^2, \mathbb{R}).
\end{aligned}$$

From the definition we know that $T_\gamma(u) \in \mathcal{Z}(S^2)$ and therefore, (3.16) holds. If $\omega_1 = \omega_2$, by the antisymmetry of γ, we have $\gamma(\omega_1, \omega_2) = 0$ and so, (3.16) is obvious.

Second we prove that the continuity of T_γ implies that γ is continuous on $S^1 \times S^1$. Indeed, let $(\omega_1^\varepsilon)_\varepsilon$ and $(\omega_2^\varepsilon)_\varepsilon$ be two sequences in S^1 such that $\omega_1^\varepsilon \to \omega_1$ and $\omega_2^\varepsilon \to \omega_2$. We want that

$$\gamma(\omega_1^\varepsilon, \omega_2^\varepsilon) \to \gamma(\omega_1, \omega_2). \tag{3.17}$$

Take $\beta \in [0, 2\pi)$ such that $e^{i\beta}$ is different from ω_1 and ω_2. For each $\omega \in S^1$ denote by $\arg_\beta(\omega) \in (\beta - 2\pi, \beta]$ the argument of ω, i.e.

$$e^{i \arg_\beta(\omega)} = \omega. \tag{3.18}$$

As above, define $f_\varepsilon : [0, 2\pi] \to \mathbb{R}$ as the linear function satisfying $f_\varepsilon(0) = \arg_\beta(\omega_1^\varepsilon)$ and $f_\varepsilon(2\pi) = \arg_\beta(\omega_2^\varepsilon)$ and consider $u_\varepsilon \in BV(S^2, S^1)$ such that

$$u_\varepsilon(\alpha, \theta) = e^{if_\varepsilon(\theta)}, \ \forall \alpha \in (0, \pi), \theta \in (0, 2\pi).$$

It's easy to check that $u_\varepsilon \to u$ strongly in BV, where $u(\alpha, \theta) = e^{if(\theta)}$ and f is the linear function satisfying $f(0) = \arg_\beta(\omega_1)$ and $f(2\pi) = \arg_\beta(\omega_2)$. As before, we obtain

$$\begin{aligned}
T_\gamma(u_\varepsilon) &= (\arg_\beta(\omega_2^\varepsilon) - \arg_\beta(\omega_1^\varepsilon) + \gamma(\omega_1^\varepsilon, \omega_2^\varepsilon))(\delta_p - \delta_n) \\
\text{and} \quad T_\gamma(u) &= (\arg_\beta(\omega_2) - \arg_\beta(\omega_1) + \gamma(\omega_1, \omega_2))(\delta_p - \delta_n).
\end{aligned}$$

Since T_γ and $\arg_\beta$ are continuous on $BV(S^2, S^1)$, respectively on $S^1 \setminus \{e^{i\beta}\}$, we deduce that (3.17) holds.

Observe now that the function

$$(\omega_1, \omega_2) \mapsto \gamma(\omega_1, \omega_2) - \mathrm{Arg}\,(\omega_1) + \mathrm{Arg}\,(\omega_2)$$

is continuous on the connected set $S^1 \setminus \{-1\} \times S^1 \setminus \{-1\}$ and takes values in $2\pi\mathbb{Z}$. Therefore, there exists $k \in \mathbb{Z}$ such that

$$\gamma(\omega_1, \omega_2) = \mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2) - 2\pi k \text{ in } S^1 \setminus \{-1\} \times S^1 \setminus \{-1\}.$$

In fact, $k = 0$ if one takes $\omega_1 = \omega_2$. But $\mathrm{Arg}\,(\cdot)$ is not a continuous map on S^1 which is a contradiction with the continuity of γ on $S^1 \times S^1$. □

The algebraic properties of T restricted to $W^{1,1}(S^2, S^1)$ (see [30], Lemma 1) do not hold in general for $BV(S^2, S^1)$ functions.

Remark 3.6 a) There exists $u \in BV(S^2, S^1)$ such that $T(\bar{u}) \neq -T(u)$. Indeed, take the function u defined in (3.14). A similar computation gives us that $T(\bar{u}) = 0 \neq -T(u)$.
b) The relation $T(gh) = T(g) + T(h)$, $\forall g, h \in W^{1,1}(S^2, S^1)$ need not hold for $BV(S^2, S^1)$ functions. As before, consider the function u in (3.14). Then $T(-u) = 0$. Since $T(-1) = 0$, we conclude $T(-u) \neq T(u) + T(-1)$. □

In the following we discuss the nature of the singularities of the distribution $T(u)$. As it was mentioned in the beginning, we deal with two types of singularity:
i) topological singularities carrying a degree which are created by the absolutely continuous part and the Cantor part of the distributional determinant of u;
ii) point singularities coming from the jump part of the derivative Du.

We give some examples in order to point out these two different kind of singularity. In Example 3.1, $T(u)$ is a dipole made up by two vortices of degree 1 and -1; these two vortices are generated by the absolutely continuous part of $\det(\nabla u)$ in a), respectively by the Cantor part of the distributional Jacobian of u in b).

Example 3.1 a) Let us analyze the function $g \in W^{1,1}(S^2, S^1)$,

$$g(\alpha, \theta) = e^{i\theta}, \forall \alpha \in (0, \pi), \theta \in [0, 2\pi).$$

Denote p and n the north and respectively the south pole of the unit sphere. We consider the lifting $\varphi \in BV(S^2, \mathbb{R})$ of u given by $\varphi(\alpha, \theta) = \theta$ for every $\alpha \in (0, \pi), \theta \in (0, 2\pi)$. Then the jump set of φ is concentrated on the meridian $\{\theta = 0\}$ oriented counterclockwise by the unit vector $\vec{\theta}$. We have

$$D\varphi = g \wedge \nabla g\, \mathcal{H}^2 - 2\pi\vec{\theta}\, \mathcal{H}^1 \llcorner \widehat{np}.$$

Therefore, $T(g) = 2\pi(\delta_p - \delta_n)$. The two poles are the vortices of the function g.

b) The same situation may occur for some purely Cantor functions. Let us consider the standard Cantor function $f : [0,1] \to [0,1]$; f is a continuous, nondecreasing function with $f(0) = 0$, $f(1) = 1$ and $f'(x) = 0$ a.e. $x \in (0,1)$. Take $v \in BV(S^2, S^1)$ defined as

$$v(\alpha, \theta) = e^{2\pi i f(\theta/2\pi)}, \forall \alpha \in (0, \pi), \theta \in [0, 2\pi).$$

The lifting $\varphi \in BV(S^2, \mathbb{R})$ given by $\varphi(\alpha, \theta) = 2\pi f(\theta/2\pi)$ for every $\alpha \in (0, \pi), \theta \in (0, 2\pi)$ has the jump set concentrated on the meridian $\{\theta = 0\}$ and

$$D\varphi = v \wedge D^c v - 2\pi\vec{\theta}\mathcal{H}^1 \llcorner \widehat{np}.$$

As before, we obtain that $T(v) = 2\pi(\delta_p - \delta_n)$ where p and n are the poles of S^2.

Remark also that for the two functions constructed in Example 3.1, the constant 2 in inequality (3.3) is optimal and we have a specific structure for an optimal lifting:

Proposition 3.13 *Let $u \in BV(S^2, S^1)$ be one of the two functions defined in Example 3.1. Then for every lifting $\varphi \in BV(S^2, \mathbb{R})$ of u we have*

$$\int_{S^2} |D\varphi| \geq 2 \int_{S^2} |Du|.$$

Moreover, the set of all optimal liftings of u is given by

$$\{\arg_\beta(u) + 2\pi k \,:\, \beta \in [0, 2\pi),\ k \in \mathbb{Z}\}$$

where $\arg_\beta(\omega) \in (\beta - 2\pi, \beta]$ stands for the argument of $\omega \in S^1$ (as in (3.18)).

Proof. First remark that

$$\int_{S^2} |Du| = 2\pi^2 \quad \text{and} \quad \|T(u)\| = 2\pi d_{S^2}(n, p) = 2\pi^2$$

where n and p are the two poles of S^2.

Let $\varphi \in BV(S^2, \mathbb{R})$ be a lifting of u. By Theorem 3.5 and Lemma 3.3, we obtain

$$\int_{S^2} |D\varphi| \geq E(u) = \int_{S^2} |Du| + \|T(u)\| = 4\pi^2 = 2 \int_{S^2} |Du|.$$

Take now $\varphi \in BV(S^2, \mathbb{R})$ an optimal lifting of u. By Lemma 3.4, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ that achieves the minimum in (3.11) and satisfies

$$D\varphi = u \wedge Du - f\nu\mathcal{H}^1 \llcorner S.$$

That means

$$D^j\varphi = -f\nu\mathcal{H}^1 \llcorner S \quad \text{and} \quad \int_S |f| = 2\pi d_{S^2}(n, p). \tag{3.19}$$

We may assume here that $S = \{f \neq 0\}$. For every $\alpha \in (0, \pi)$ we denote L_α the latitude on S^2 corresponding to α and $\varphi_\alpha : L_\alpha \to \mathbb{R}$ the restriction of φ to L_α. Using the Characterization Theorem of BV functions by sections and Theorem 3.108 in [11], it results that for a.e. $\alpha \in (0, \pi)$, $\varphi_\alpha \in BV(L_\alpha; \mathbb{R})$ and the discontinuity set of φ_α is $S \cap L_\alpha$. Remark that $\deg(u; L_\alpha) = 1$ for every $\alpha \in (0, \pi)$. Thus, for a.e. $\alpha \in (0, \pi)$, φ_α will have at least one jump on L_α and the length of a jump is not less than 2π. It yields $\mathcal{H}^1(S) \geq \pi$ and $|f| \geq 2\pi\ \mathcal{H}^1 -$ a.e. on S. By (3.19), we deduce that

$$|f| = 2\pi\ \mathcal{H}^1 - \text{a.e. on } S \quad \text{and} \quad \mathcal{H}^1(S) = \pi.$$

We know that
$$\int_S \frac{f}{2\pi}\nu \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1 = \zeta(p) - \zeta(n), \; \forall \zeta \in C^1(S^2).$$
By [44](Section 4.2.25), it results that S covers $\mathcal{H}^1$-almost all of a Lipschitz univalent path c between the two poles. Since $\mathcal{H}^1(S) = d_{S^2}(n,p)$ we deduce that S is a geodesic arc on S^2 between n and p and $\frac{f}{2\pi}\nu$ is the normal unit vector to the curve c. Take $\beta \in [0, 2\pi)$ such that $S = \{\theta = \beta\}$ in the spherical coordinates. We have that $\varphi - \arg_\beta(u) : S^2 \setminus S \to 2\pi\mathbb{Z}$ is continuous on the connected set $S^2 \setminus S$. Therefore, there exists $k \in \mathbb{Z}$ such that
$$\varphi = \arg_\beta(u) + 2\pi k$$
and the conclusion follows. □

The appearance of non-topological singularities in the writing of $T(u)$ for $u \in BV(S^2, S^1)$ was already seen in the example (3.14); there the distribution $T(u)$ is a dipole even if the function u does not have any vortex. One should notice that the dipole (3.15) is created on the jump set of u by the discontinuity of the chosen argument Arg. In Remark 3.7, we will see that a dipole could disappear if we change the choice of the argument.

Remark 3.7 Let $\beta \in [0, 2\pi)$. Define the antisymmetric function $\gamma_\beta(\cdot,\cdot) : S^1 \times S^1 \to [-\pi, \pi]$ as
$$\gamma_\beta(\omega_1, \omega_2) = \begin{cases} \mathrm{Arg}\left(\frac{\omega_1}{\omega_2}\right) & \text{if } \frac{\omega_1}{\omega_2} \neq -1 \\ \arg_\beta(\omega_1) - \arg_\beta(\omega_2) & \text{if } \frac{\omega_1}{\omega_2} = -1 \end{cases}, \; \forall \omega_1, \omega_2 \in S^1.$$
Consider now the distribution $T_{\gamma_\beta}(u) \in \mathcal{D}'(S^2, \mathbb{R})$ given as in Proposition 3.12:
$$\langle T_{\gamma_\beta}(u), \zeta \rangle = \int_{S^2} \nabla^\perp \zeta \cdot (u \wedge (D^a u + D^c u)) + \int_{S(u)} \gamma_\beta(u^+, u^-)\nu_u \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1, \; \forall \zeta \in C^1(S^2, \mathbb{R}).$$
Observe that T_{γ_β} inherits the properties of T given in Theorems 3.1, 3.5 and 3.8. However, the structure of the singularities of $T_{\gamma_\beta}(u)$ may be different from $T(u)$. Indeed, consider $u \in BV(S^2, S^1)$ the function constructed in (3.14). We saw that $T(u) = 2\pi(\delta_p - \delta_n)$ where $n = (\frac{\pi}{2}, \frac{3\pi}{2})$ and $p = (\frac{\pi}{2}, \frac{\pi}{2})$ (in the spherical coordinates). The same computation gives us $T_{\gamma_{\pi/2}}(u) = 0$. The difference between $T(u)$ and $T_{\gamma_{\pi/2}}(u)$ arises from the choice of the argument.

An interesting phenomenon is observed in Example 3.2 where the two types of singularity are mixed: some topological vortices may be located on the jump set of u.

Example 3.2 a) An example that points out the mixture of the two type of singularity is given by functions with pseudo-vortices: define $u \in BV(S^2, S^1)$ as
$$u(\alpha, \theta) = e^{3i\theta/2}, \; \forall \alpha \in (0, \pi), \theta \in (0, 2\pi).$$
The jump set of u is the meridian $\{\theta = 0\}$. We have
$$T(u) = 2\pi(\delta_p - \delta_n) \text{ and } T_{\gamma_{\pi/2}}(u) = 4\pi(\delta_p - \delta_n).$$

The two poles p and n arise on the jump set of u and behave like some pseudo-vortices, i.e. after a complete turn, the function u rotates 3/2 times around the poles (with different signs: '+' around p and '−' around n). According to the choice of the argument in the definition of γ_β, the distribution $T_{\gamma_\beta}(u)$ will count once or twice the dipole.

b) A piecewise constant function $u \in BV(S^2, S^1)$ may create a dipole for $T(u)$. Indeed, let us define $\varphi \in BV(S^2, \mathbb{R})$ as

$$\varphi(\alpha, \theta) = \begin{cases} 0 & \text{if } \theta \in (0, 2\pi/3),\ \alpha \in (0, \pi) \\ 2\pi/3 & \text{if } \theta \in (2\pi/3, 4\pi/3),\ \alpha \in (0, \pi) \\ 4\pi/3 & \text{if } \theta \in (4\pi/3, 2\pi),\ \alpha \in (0, \pi) \end{cases}$$

and set $u = e^{i\varphi}$. The jump set of u and φ is the union of three meridians

$$S(u) = S(\varphi) = \{\theta = 0\} \cup \{\theta = 2\pi/3\} \cup \{\theta = 4\pi/3\}.$$

We have

$$\varphi^+ - \varphi^- = \rho(u^+, u^-) - 2\pi\chi_{\{\theta=0\}}.$$

We obtain $T(u) = 2\pi(\delta_p - \delta_n)$ where p and n are the two poles of the unit sphere. For every $\beta \in [0, 2\pi)$, T_{γ_β} has the same behavior, i.e. $T_{\gamma_\beta}(u) = 2\pi(\delta_p - \delta_n)$.

c) Let $u \in BV(S^2, S^1)$ be the function defined above in b) and take g the function constructed in Example 3.1 a). Set $w = gu \in BV(S^2, S^1)$. We have $S(w) = \{\theta = 0\} \cup \{\theta = 2\pi/3\} \cup \{\theta = 4\pi/3\}$. We show that $T(w) = 4\pi(\delta_p - \delta_n)$. Indeed, construct the lifting $\psi \in BV(S^2, \mathbb{R})$ of w as

$$\psi(\alpha, \theta) = \begin{cases} \theta & \text{if } \theta \in (0, 2\pi/3),\ \alpha \in (0, \pi) \\ \theta + 2\pi/3 & \text{if } \theta \in (2\pi/3, 4\pi/3),\ \alpha \in (0, \pi) \\ \theta - 2\pi/3 & \text{if } \theta \in (4\pi/3, 2\pi),\ \alpha \in (0, \pi) \end{cases}.$$

Observe that

$$\psi^+ - \psi^- = \rho(w^+, w^-) - 2\pi\chi_{\{\theta=0\}} - 2\pi\chi_{\{\theta=4\pi/3\}} \text{ on } S(w)$$

and conclude that $T(w) = 4\pi(\delta_p - \delta_n)$. So, the north pole p and the south pole n which are the vortices of g remain singularities for the function w; they appear now on the jump part of w. The same behavior happens to T_{γ_β} for every $\beta \in [0, 2\pi)$, i.e. $T_{\gamma_\beta}(w) = 4\pi(\delta_p - \delta_n)$.

As we mentioned before, for every $u \in BV(S^2, S^1)$ there exists a bounded lifting $\varphi \in BV \cap L^\infty(S^2, \mathbb{R})$ (see Chapter 1). The striking fact is that we can construct functions $u \in BV(S^2, S^1)$ such that no optimal lifting belongs to L^∞. We give such an example in the following:

Example 3.3 On the interval $(0, 2\pi)$ we consider

$$p_1 = 1,\ n_k = p_k + \frac{1}{4^k} \text{ and } p_{k+1} = n_k + \frac{1}{2^k},\ \forall k \geq 1.$$

Suppose that this configuration of points lies on the equator $\{\frac{\pi}{2}\} \times [0, 2\pi]$ (in the spherical coordinates) of S^2 and we consider that each dipole (p_k, n_k) appears k times. Since $\sum_{k\geq 1} k d_{S^2}(p_k, n_k) <$

∞, set

$$\Lambda = 2\pi \sum_{k\geq 1} k(\delta_{p_k} - \delta_{n_k}) \in \mathcal{Z}(S^2).$$

By [22] (Lemma 16),

$$T\left(W^{1,1}(S^2, S^1)\right) = \mathcal{Z}(S^2).$$

Thus, take $g \in W^{1,1}(S^2, S^1)$ such that $T(g) = \Lambda$. Using (3.2), it follows that

$$\|T(g)\| = 2\pi \sum_{k\geq 1} k d_{S^2}(p_k, n_k).$$

Let $\varphi \in BV(S^2, \mathbb{R})$ be an optimal lifting of g. Then there is a triple $(f, S, \nu) \in \mathcal{J}(T(g))$ such that

$$D\varphi = g \wedge \nabla g \mathcal{H}^2 - f\,\nu\,\mathcal{H}^1 \llcorner S \quad \text{and} \quad \int_S |f|\,\mathrm{d}\mathcal{H}^1 = \|T(g)\|. \tag{3.20}$$

We may assume that $S = \{f \neq 0\}$.

We know that $\displaystyle\int_S f\nu \cdot \nabla^\perp \zeta \,\mathrm{d}\mathcal{H}^1 = 2\pi \sum_{k\geq 1} k(\zeta(p_k) - \zeta(n_k))$, $\forall \zeta \in C^1(S^2)$. For each $k \geq 1$, we denote $V_k = (0, \pi) \times (p_k - \dfrac{1}{8^k}, n_k + \dfrac{1}{8^k})$. Then

$$\int_S f\nu \cdot \nabla^\perp \zeta \,\mathrm{d}\mathcal{H}^1 = 2\pi k(\zeta(p_k) - \zeta(n_k)),\ \forall \zeta \in C^1(S^2) \text{ with } \operatorname{supp}\zeta \subset V_k.$$

By (3.20), it follows that

$$\int_{S\cap V_k} |f|\,\mathrm{d}\mathcal{H}^1 = 2\pi k\, d_{S^2}(p_k, n_k).$$

Using the same argument as in the proof of Proposition 3.13, we deduce that for each $k \in \mathbb{N}$,

$$S(\varphi) \cap V_k = S \cap V_k = \widehat{n_k p_k} \quad \text{and} \quad |\varphi^+ - \varphi^-| = |f| = 2k\pi\ \mathcal{H}^1\text{-a.e. on } \widehat{n_k p_k}$$

where $\widehat{n_k p_k}$ is the geodesic arc connecting n_k and p_k. It yields that $\varphi \notin L^\infty$. So, every optimal BV lifting of g does not belong to L^∞.

In the next example, we show that Theorem 3.8 fails if we minimize the energy in (3.8) just over the class of gradient maps:

Example 3.4 Let $u \in BV(S^2, S^1)$ be defined as

$$u(\alpha, \theta) = e^{i\theta/3},\ \forall \alpha \in (0, \pi), \theta \in (0, 2\pi).$$

The jump set of u is the meridian $\{\theta = 0\}$ orientated counterclockwise and $\rho(u^+, u^-) = -2\pi/3$ on $S(u)$. We have that $T(u) = 0$. On the other hand, for every $\psi \in C^\infty(S^2, \mathbb{R})$, we have

$$\begin{aligned}
&\int_{S^2} |u \wedge \nabla u\,\mathcal{H}^2 + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - \nabla\psi\,\mathcal{H}^2| \\
&= \int_{S^2} |u \wedge \nabla u - \nabla\psi|\,\mathrm{d}\mathcal{H}^2 + \int_{S(u)} |\rho(u^+, u^-)|\,\mathrm{d}\mathcal{H}^1 \\
&\geq \int_{S(u)} 2\pi/3\,\mathrm{d}\mathcal{H}^1 = 2\pi^2/3 > \|T(u)\|.
\end{aligned}$$

Chapter 4

On the relation between minimizers of a Γ-limit energy and optimal lifting in BV

Abstract

We study the minimizers of an energy functional which is obtained as the Γ-limit of a family of functionals depending on a small parameter $\varepsilon > 0$, associated with a function $u \in BV(\Omega, S^1)$ and a positive parameter p. We find necessary and sufficient conditions on p and the dimension under which these minimizers coincide with the optimal liftings of u, for *every* $u \in BV(\Omega, S^1)$.

This chapter is written in collaboration with A. Poliakovsky and it is published in Commun. Contemp. Math. **9** (2007), pp. 447–472 (cf. [58]).

4.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $u \in BV(\Omega, S^1)$, i.e., $u = (u_1, u_2) \in L^1(\Omega, \mathbb{R}^2)$, $|u(x)| = 1$ for almost every $x \in \Omega$ and the derivative of u (in the distributional sense) is a finite $2\times N-$matrix Radon measure. The BV-seminorm of u is given by

$$\int_\Omega |Du| = \sup\left\{\int_\Omega \sum_{k=1}^{2} u_k \operatorname{div}\zeta_k \, dx \;:\; \zeta_k \in C_c^1(\Omega, \mathbb{R}^2),\, \sum_{k=1}^{2} |\zeta_k(x)|^2 \leq 1, \forall x \in \Omega\right\} < \infty\,,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$. A BV lifting of u is a function $\varphi \in BV(\Omega, \mathbb{R})$ such that

$$u = e^{i\varphi} \quad \text{a.e. in } \Omega.$$

The existence of a BV lifting for any $u \in BV(\Omega, S^1)$ was first proved by Giaquinta, Modica and Soucek [47]. In general, we may have that

$$\min\left\{\int_\Omega |D\varphi| \;:\; \varphi \in BV(\Omega, \mathbb{R}),\, e^{i\varphi} = u \text{ a.e. in } \Omega\right\} > \int_\Omega |Du|.$$

The optimal control of a BV lifting was given in Chapter 1: there exists a lifting $\varphi \in BV \cap L^\infty(\Omega, \mathbb{R})$ such that

$$\int_\Omega |D\varphi| \leq 2 \int_\Omega |Du|. \tag{4.1}$$

The constant 2 in the inequality (4.1) is optimal for $N \geq 2$ (for example, consider

$$u(x) = \frac{x}{|x|} \tag{4.2}$$

in the unit disc in $\mathbb{R}^2$).

It is natural to investigate the quantity

$$E(u) = \min\left\{ \int_\Omega |D\varphi| \,:\, \varphi \in BV(\Omega, \mathbb{R}),\, e^{i\varphi} = u \text{ a.e. in } \Omega \right\}. \tag{4.3}$$

The case $u \in W^{1,1}$ was previously studied in [30] while the more general case $u \in BV$ was studied in [47, 52, 53]. We shall say that a lifting $\varphi \in BV(\Omega, \mathbb{R})$ of u is *optimal* if $E(u) = \int_\Omega |D\varphi|$, i.e., if φ is a minimizer in (4.3). An optimal lifting of u always exists but in general it is not unique (i.e., there might exist two optimal BV liftings φ_1 and φ_2 such that $\varphi_1 - \varphi_2$ is not identically constant). For example, for the function u given in (4.2), every optimal lifting is an argument function whose jump set is a radius of the unit disc, see [53]. The structure of an optimal lifting of u is described in [47, 52, 53] using the notion of minimal connection between singularity sets of dimension $N-2$ of u.

A natural way to approximate liftings of u is to consider, for a fixed parameter $0 < p < +\infty$, the family of energy functionals $\{F_\varepsilon^{(u,p)}\}_{\varepsilon>0}$ defined by

$$F_\varepsilon^{(u,p)}(\varphi) = \varepsilon \int_\Omega |\nabla\varphi|^2 + \frac{1}{\varepsilon}\int_\Omega |u - e^{i\varphi}|^p, \quad \forall \varphi \in H^1(\Omega, \mathbb{R}). \tag{4.4}$$

Due to the penalizing term in (4.4), sequences of minimizers φ_ε of $F_\varepsilon^{(u,p)}$ are expected to converge to a lifting φ_0 of u as $\varepsilon \to 0$. More precisely, Poliakovsky [70] proved that for $p > 1$ and for bounded domains Ω with Lipschitz boundary, any sequence of minimizers $\varphi_\varepsilon \in H^1(\Omega, \mathbb{R})$ of $F_\varepsilon^{(u,p)}$, satisfying $|\int_\Omega \varphi_\varepsilon| \leq C$, converges strongly in L^1 (up to a subsequence) to a lifting $\varphi_0 \in BV(\Omega, \mathbb{R})$ of u as $\varepsilon \to 0$ and φ_0 is a minimizer of the $\Gamma-$limit energy $F_0^{(u,p)} : L^1(\Omega, \mathbb{R}) \to \mathbb{R}$ given by

$$F_0^{(u,p)}(\varphi) = \begin{cases} 2\displaystyle\int_{S(\varphi)} f^{(p)}(|\varphi^+ - \varphi^-|)\, d\mathcal{H}^{N-1} & \text{if } \varphi \text{ is a } BV \text{ lifting of } u, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.5}$$

Here, $S(\varphi)$ is the jump set of $\varphi \in BV(\Omega, \mathbb{R})$ and φ^-, φ^+ are the traces of φ on each of the sides of the jump set and $f^{(p)} : [0, +\infty) \to \mathbb{R}$ is the function defined by

$$f^{(p)}(\theta) = \inf_{t \in \mathbb{R}} \int_t^{\theta+t} |e^{is} - 1|^{p/2}\, ds, \; \forall \theta \geq 0.$$

Notice that $F_0^{(u,p)}(\varphi) < +\infty$ for a BV lifting φ of u since $f^{(p)}$ is an increasing Lipschitz function (see Lemma 4.3). Due to the fact that the energies $\{F_\varepsilon^{(u,p)}\}_{\varepsilon>0}$ and $F_0^{(u,p)}$ are invariant with respect to translations by $2\pi k$, $k \in \mathbb{Z}$, uniqueness of minimizers has a meaning up to additive constants in $2\pi\mathbb{Z}$.

The goal of this chapter is to study the question whether the minimizers of $F_0^{(u,p)}$ are necessarily optimal liftings of u, for any p. Surprisingly, this turns out to be the case (in general) only in dimension one, while in dimension $N \geq 2$ this holds only for $p = 4$. Our main result is the following:

Theorem 4.1 *Let Ω be a bounded domain in $\mathbb{R}^N$.*

(i) If $N = 1$ then for every $u \in BV(\Omega, S^1)$ and $p \in (0, +\infty)$, φ is a minimizer of $F_0^{(u,p)}$ if and only if φ is an optimal lifting of u ;

(ii) If $N \geq 2$ then only for $p = 4$ it is true that for every $u \in BV(\Omega, S^1)$, any minimizer of $F_0^{(u,p)}$ is an optimal lifting of u.

We recall that for a function u in the smaller class $W^{1,1}(\Omega, S^1)$, a lifting of u is optimal if and only if it is a minimizer of $F_0^{(u,p)}$, for every $p \in (0, +\infty)$ (see Proposition 4.10).

The chapter is organized as follows. In Section 4.2 we recall some basic notions of BV spaces that will be needed throughout this chapter. Section 4.3 is devoted to the one dimensional case. In Section 4.4 we treat the case $p = 4$, which was already studied in [70]. In Section 4.5 we construct counterexamples needed for the proof of assertion (ii) of Theorem 4.1 in the case $0 < p < 4$. For any domain Ω we construct a piecewise constant function $u \in BV(\Omega, S^1)$ depending on p such that $F_0^{(u,p)}$ has a unique minimizer ξ_0 (up to $2\pi\mathbb{Z}$ constants), u has a unique optimal lifting ζ_0 (up to $2\pi\mathbb{Z}$ constants) and $\xi_0 - \zeta_0$ is not a constant function. In Section 4.6, we deal with the general case $p \neq 4$. For any bounded domain G, we construct a family of functions $\{U_t\}_{t\in(-1/4,1/4)}$ that contains elements U_t with a unique optimal lifting whose energy $F_0^{(U_t,p)}$ is strictly larger than the minimal energy $\min F_0^{(U_t,p)}$. (In addition, for those functions U_t, we will prove that $F_0^{(U_t,p)}$ has a unique minimizer up to a $2\pi\mathbb{Z}$ translation.)

For the sake of simplicity of notations we shall often suppress the dependence on u and p when referring to the energies $\{F_\varepsilon^{(u,p)}\}_{\varepsilon>0}$, $F_0^{(u,p)}$ and $f^{(p)}$.

4.2 Preliminaries about the space BV

In this section we present some known results on BV functions that can be found in the book [11] by Ambrosio, Fusco and Pallara (see also Giusti [48] and Evans and Gariepy [42]). Let $v \in BV(\Omega, \mathbb{R}^m)$. A point $x \in \Omega$ is a point of *approximate continuity* of v if there exists $\tilde{v}(x) \in \mathbb{R}^m$ such that $\tilde{v}(x) = \operatorname*{ap-lim}_{y\to x} v(y)$, that is:

$$\lim_{r\to 0} \frac{\mathcal{H}^N\big(B_r(x) \cap \{y \in \Omega \,:\, |v(y) - \tilde{v}(x)| > \varepsilon\}\big)}{\mathcal{H}^N(B_r(x))} = 0, \quad \forall \varepsilon > 0.$$

The complement of the set of points of *approximate continuity* is denoted by $S(v)$. It is known (see [11]) that the set $S(v)$ is a countably $\mathcal{H}^{N-1}$-rectifiable Borel set, i.e., $S(v)$ is σ-finite with respect to the Hausdorff measure $\mathcal{H}^{N-1}$ and there exist countably many $N-1$ dimensional C^1-hypersurfaces $\{S_k\}_{k=1}^{\infty}$ such that $\mathcal{H}^{N-1}\Big(S(v) \setminus \bigcup_{k=1}^{\infty} S_k\Big) = 0$. Moreover, for $\mathcal{H}^{N-1}$-a.e. $x \in S(v)$ there exist $v^+(x), v^-(x) \in \mathbb{R}^m$ and a unit vector $\nu_v(x)$ such that

$$\operatorname*{ap-lim}_{y\to x,\ \langle y-x,\nu_v(x)\rangle>0} v(y) = v^+(x) \quad \text{and} \quad \operatorname*{ap-lim}_{y\to x,\ \langle y-x,\nu_v(x)\rangle<0} v(y) = v^-(x). \tag{4.6}$$

In the sequel we shall refer to $S(v)$ as the *jump set* of v, although (4.6) is valid only for $\mathcal{H}^{N-1}$-a.e. $x \in S(v)$. The vector field ν_v is called the orientation of the jump set $S(v)$. Dv is a $m \times N$ matrix valued Radon measure which can be decomposed as $Dv = D^a v + D^j v + D^c v$, where $D^a v$ is the absolutely continuous part of Dv with respect to the Lebesgue measure, while $D^j v$ and $D^c v$ are defined by

$$D^j v = Dv \llcorner S(v) \quad \text{and} \quad D^c v = (Dv - D^a v) \llcorner (\Omega \setminus S(v)).$$

We shall call $D^j v$ and $D^c v$ the jump part and the Cantor part, respectively, of Dv. We have:

1. $D^a v = \nabla v\, \mathcal{H}^N$ where $\nabla v \in L^1(\Omega, \mathbb{R}^{m\times N})$ is the approximate differential of v;
2. $(D^c v)(B) = 0$ for any Borel set $B \subset \Omega$ which is σ-finite with respect to $\mathcal{H}^{N-1}$;
3. $D^j v = (v^+ - v^-) \otimes \nu_v\, \mathcal{H}^{N-1} \llcorner S(v)$.

Throughout this chapter we identify the function v with its precise representative $v^* : \Omega \mapsto \mathbb{R}^m$ given by

$$v^*(x) = \lim_{r\to 0} \fint_{B_r(x)} v(y)\, dy\,,$$

if this limit exists, and $v^*(x) = 0$ otherwise. Note that v^* specifies the values of v except on a $\mathcal{H}^{N-1}$-negligible set.

We also recall Vol'pert's chain rule. Let Ω be a bounded domain and assume that $v \in BV(\Omega, \mathbb{R}^m)$ and $g \in [C^1(\mathbb{R}^m)]^q$ is a Lipschitz function. Then $w = g \circ v$ belongs to $BV(\Omega, \mathbb{R}^q)$ and

$$D^a w = \nabla g(v)\, \nabla v\, \mathcal{H}^N,\ D^c w = \nabla g(v)\, D^c v,\ D^j w = \big[g(v^+) - g(v^-)\big] \otimes \nu_v\, \mathcal{H}^{N-1} \llcorner S(v)\,. \tag{4.7}$$

4.3 The one-dimensional case

In this section we shall show that the optimal liftings of u coincide with the minimizers of $F_0^{(u,p)}$ in the one-dimensional case, for every parameter $p > 0$ and any function $u \in BV(\Omega, S^1)$. The proof uses the same method as in [52].

Proof of *(i)* in Theorem 4.1. Let Ω be an interval in $\mathbb{R}$ and let $\varphi \in BV(\Omega, \mathbb{R})$ be a lifting of u. By the chain rule (4.7), it follows that

$$(\dot\varphi)^a + (\dot\varphi)^c = u \wedge ((\dot u)^a + (\dot u)^c) \ \text{ and } \ (\dot\varphi)^j = \sum_{a\in S(u)} (\varphi(a+) - \varphi(a-))\delta_a + \sum_{b\in B} (\varphi(b+) - \varphi(b-))\delta_b \tag{4.8}$$

where $B \subset \Omega$ is a finite set such that $S(u) \cap B = \emptyset$ and $\varphi(b+) - \varphi(b-) = -2\pi\alpha_b$, $\alpha_b \in \mathbb{Z}$, for every $b \in B$. For any $a \in S(u)$, we denote $d_a(u) = \operatorname{Arg} \dfrac{u(a+)}{u(a-)}$ where $\operatorname{Arg} \omega \in (-\pi, \pi]$ is the argument of the unit complex number ω. Since $f^{(p)}$ is increasing and $|\varphi(a+) - \varphi(a-)| \geq |d_a(u)|$ in $S(u)$, it follows that

$$f^{(p)}(|\varphi(a+) - \varphi(a-)|) \geq f^{(p)}(|d_a(u)|) \text{ if } a \in S(u) \text{ and } f^{(p)}(|\varphi(b+) - \varphi(b-)|) \geq 0 \text{ if } b \in B \quad (4.9)$$

with equality if and only if

$$|\varphi(a+) - \varphi(a-)| = |d_a(u)| \text{ for } a \in S(u) \quad \text{and} \quad \alpha_b = 0 \text{ for } b \in B. \quad (4.10)$$

According to (4.8), we have

$$\int_\Omega \Big(|(\dot\varphi)^a| + |(\dot\varphi)^c| \Big) = \int_\Omega \Big(|(\dot u)^a| + |(\dot u)^c| \Big).$$

By [52], it follows that

$$E(u) = \int_\Omega \Big(|(\dot u)^a| + |(\dot u)^c| \Big) + \sum_{a \in S(u)} |d_a(u)|,$$

i.e., φ is an optimal lifting if $\displaystyle\int_\Omega |(\dot\varphi)^j| = \sum_{a \in S(u)} |d_a(u)|$. Therefore, by (4.9) and (4.10), we obtain that

$$\min F_0^{(u,p)} = 2 \sum_{a \in S(u)} f^{(p)}(|d_a(u)|).$$

Finally, we conclude that φ is a minimizer of $F_0^{(u,p)}$ if and only if φ is an optimal lifting of u. □

4.4 The case $p = 4$

In this section we shall recall the proof from [70] of the result that states that for $p = 4$ minimizers of the Γ-limit energy $F_0^{(u,p)}$ coincide with those of the energy $E(u)$ in (4.3) for every $u \in BV(\Omega, S^1)$. We also derive an asymptotic upper bound for the minimal energy of $F_\varepsilon^{(u,4)}$ in terms of the mass of the measure $|Du|$.

Proof of *(ii)* of Theorem 4.1 for $p = 4$. Let $\varphi \in BV(\Omega, \mathbb{R})$ be a lifting of u. Then $|u^+ - u^-| = 2\Big|\sin \dfrac{\varphi^+ - \varphi^-}{2}\Big|$ $\mathcal{H}^{N-1}$-a.e. in $S(u)$. A simple computation yields

$$f^{(4)}(\theta) = 2\theta - 4\Big|\sin\frac{\theta}{2}\Big|, \ \forall \theta \geq 0.$$

This implies that

$$F_0^{(u,4)}(\varphi) = 4 \int_{S(\varphi)} |\varphi^+ - \varphi^-| \, d\mathcal{H}^{N-1} - 4 \int_{S(u)} |u^+ - u^-| \, d\mathcal{H}^{N-1}.$$

On the other hand, the chain rule (4.7) yields that

$$D^a\varphi = u \wedge D^a u \quad \text{and} \quad D^c\varphi = u \wedge D^c u \tag{4.11}$$

and therefore, the total variation of the diffuse part of $D\varphi$ is completely determined by Du, i.e.,

$$\int_\Omega (|D^a\varphi| + |D^c\varphi|) = \int_\Omega (|D^a u| + |D^c u|). \tag{4.12}$$

Hence, φ is a minimizer of $F_0^{(u,4)}$ if and only if φ is an optimal lifting of u. □

As a consequence, we deduce an estimate for the energy $F_\varepsilon^{(u,4)}$ which relies on some results from [37] and [70].

Corollary 4.2 *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $u \in BV(\Omega, S^1)$. Then*

$$\min F_\varepsilon^{(u,4)} \leq 4 \int_\Omega |Du| + o(1)$$

where $o(1)$ is a quantity that tends to 0 as $\varepsilon \to 0$.

Proof. By contradiction, assume that there exist a constant $\delta > 0$ and a sequence $\{\varepsilon_k\}_{k\geq 1}$ tending to 0 as $k \to \infty$, such that

$$F_{\varepsilon_k}^{(u,4)}(\varphi_{\varepsilon_k}) \geq 4 \int_\Omega |Du| + \delta\,, \tag{4.13}$$

where $\varphi_{\varepsilon_k} \in H^1(\Omega, \mathbb{R})$ is a minimizer of $F_{\varepsilon_k}^{(u,4)}$. Since the value of $F_{\varepsilon_k}^{(u,4)}(\varphi_{\varepsilon_k})$ does not change by adding a constant multiple of 2π to φ_{ε_k}, we may assume that $0 \leq \int_\Omega \varphi_{\varepsilon_k}\, dx \leq 2\pi\mathcal{H}^N(\Omega)$. According to [70] it follows that, up to a subsequence,

$$\varphi_{\varepsilon_k} \to \varphi_0 \ \text{ in } \ L^1 \quad \text{and} \quad \lim_{k\to\infty} F_{\varepsilon_k}^{(u,4)}(\varphi_{\varepsilon_k}) = F_0^{(u,4)}(\varphi_0)\,,$$

where φ_0 is a BV lifting of u that minimizes the $\Gamma-$limit energy $F_0^{(u,4)}$. Using (4.13), it follows that

$$F_0^{(u,4)}(\varphi_0) \geq 4 \int_\Omega |Du| + \delta. \tag{4.14}$$

On the other hand, by assertion *(ii)* of Theorem 4.1 in the case $p = 4$, we know that φ_0 is an optimal lifting and

$$F_0^{(u,4)}(\varphi_0) = 4 \int_{S(\varphi_0)} |\varphi_0^+ - \varphi_0^-|\, d\mathcal{H}^{N-1} - 4 \int_{S(u)} |u^+ - u^-|\, d\mathcal{H}^{N-1}.$$

By (4.1) we deduce that $\int_\Omega |D\varphi_0| \leq 2 \int_\Omega |Du|$ and therefore, it implies by (4.12),

$$F_0^{(u,4)}(\varphi_0) \leq 4 \int_\Omega |Du|$$

which contradicts (4.14). □

It would be interesting to have a direct proof of Corollary 4.2 which does not use the results in [37] and [70]. That will lead to a new proof of the inequality (4.1).

4.5 The case $p \in (0,4)$

In this section we prove the case $p < 4$ of assertion *(ii)* of Theorem 4.1. We shall first construct, for each $0 < p < 4$, a piecewise constant function $u \in BV(\mathcal{R}, S^1)$ in a rectangle $\mathcal{R} \subset \mathbb{R}^2$ such that no minimizer of $F_0^{(u,p)}$ is an optimal lifting of u. Then, we shall adapt this example to the case of an arbitrary bounded domain Ω.

We start by two preliminary results about the function $f^{(p)}$:

Lemma 4.3 *Let $0 < p < \infty$. The function $f^{(p)}$ is an increasing Lipschitz continuous function. Moreover,*

$$f^{(p)}(\theta) = \begin{cases} \displaystyle\int_{-\theta/2}^{\theta/2} |e^{is} - 1|^{p/2}\, ds & \text{if } \theta \in [2\pi k, 2\pi(k+1)],\ k \text{ even}, \\ \displaystyle\int_{-\theta/2+\pi}^{\theta/2+\pi} |e^{is} - 1|^{p/2}\, ds & \text{if } \theta \in [2\pi k, 2\pi(k+1)],\ k \text{ odd}. \end{cases} \tag{4.15}$$

Proof. In the sequel we shall write for short f instead of $f^{(p)}$. The function

$$s \in \mathbb{R} \mapsto |e^{is} - 1|^{p/2} = 2^{p/2} \Big| \sin \frac{s}{2} \Big|^{p/2}$$

is 2π-periodic, increasing on $(0, \pi)$ and symmetric with respect to π. Hence, if $\theta \in [0, 2\pi]$, then $f(\theta) = \displaystyle\int_{-\theta/2}^{\theta/2} |e^{is} - 1|^{p/2}\, ds$. In general, if $\theta = 2\pi k + \tilde{\theta}$ with $\tilde{\theta} \in [0, 2\pi]$ and $k \in \mathbb{N}$, we have $f(\theta) = f(2\pi k) + f(\tilde{\theta})$ and (4.15) is now straightforward. In particular, we deduce that

$$f(2\pi k) = k f(2\pi)\,, \quad \forall k \in \mathbb{N}. \tag{4.16}$$

From here, we conclude that almost everywhere in $(0, +\infty)$, f is differentiable and $0 < f' \leq 2^{p/2}$. □

Lemma 4.4 *Let $0 < p < 4$. Then the function $\theta \in (0, \pi) \mapsto \dfrac{f^{(p)}(2\pi - \theta) - f^{(p)}(\theta)}{\pi - \theta}$ is increasing.*

Proof. It is sufficient to prove that the function $g : (0, \pi) \to \mathbb{R}$ defined by

$$g(\theta) = f(2\pi - \theta) - f(\theta) - (\pi - \theta)\Big(f'(2\pi - \theta) + f'(\theta) \Big)$$

is positive, where we denoted $f = f^{(p)}$ as above. Indeed, by Lemma 4.3 we have for every $\theta \in (0, \pi)$,

$$g'(\theta) = (\pi - \theta)\big(f''(2\pi - \theta) - f''(\theta)\big) = p\, 2^{p/2-4}\, (\pi - \theta)\, \sin \frac{\theta}{2} \bigg(\cos^{p/2-2} \frac{\theta}{4} - \sin^{p/2-2} \frac{\theta}{4} \bigg).$$

Since $p < 4$ it follows that $g'(\theta) < 0$, $\forall \theta \in (0, \pi)$; hence g is decreasing. Since $\lim_{\theta \to \pi} g(\theta) = 0$, we deduce that g must be positive on $(0, \pi)$. □

Construction of a counter-example u when Ω is a rectangle. Let $p \in (0,4)$. We first construct our function u in a certain rectangle $\mathcal{R}$. Let $\theta_1 = \frac{4\pi}{5}$ and $\theta_2 = \frac{3\pi}{4}$. Thanks to Lemma 4.4 we can choose $L_3 > L_1 > 0$ such that

$$\frac{5}{4} = \frac{\pi - \theta_2}{\pi - \theta_1} > \frac{L_3}{L_1} > \frac{f^{(p)}(2\pi - \theta_2) - f^{(p)}(\theta_2)}{f^{(p)}(2\pi - \theta_1) - f^{(p)}(\theta_1)} > 1. \tag{4.17}$$

Set also $L_2 = L_3$ and $L_4 = L_3$. We consider the rectangle

$$\mathcal{R} = \Big\{(x,y) \in \mathbb{R}^2 \,:\, -L_2 < x < L_4,\, -L_3 < y < L_1\Big\}.$$

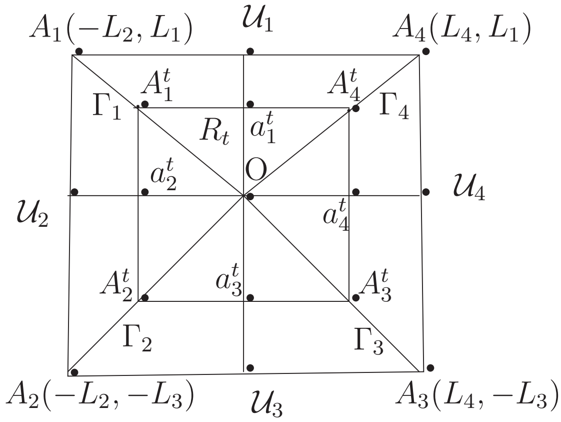


Figure 4.1: The rectangle construction for $p \in (0,4)$

Notice that the rectangle $\mathcal{R}$ depends on p by the choice of the edges; moreover, the choice (4.17) is no longer possible for $p \geq 4$. In the rectangle $\mathcal{R}$, we denote the vertices $A_1 = (-L_2, L_1)$, $A_2 = (-L_2, -L_3)$, $A_3 = (L_4, -L_3)$ and $A_4 = (L_4, L_1)$ and also the interior full triangles $\mathcal{U}_k = \triangle\, A_k O A_{k-1}$ and the segments $\Gamma_k = (OA_k)$ for $1 \leq k \leq 4$ where $O = (0,0)$ is the origin and we use the convention that $A_0 = A_4$, see Figure 1.

Let $\varphi_0 \in BV(\mathcal{R}, \mathbb{R})$ be the piecewise constant function defined by

$$\varphi_0(x,y) = \begin{cases} \frac{\pi}{2} & \text{if} \quad 0 < x < L_4, \quad 0 < y < L_1, \\ \frac{5\pi}{4} & \text{if} \quad -L_2 < x < 0, \quad 0 < y < L_1, \\ \frac{3\pi}{2} & \text{if} \quad -L_2 < x < 0, \quad -L_3 < y < 0, \\ \frac{3\pi}{10} & \text{if} \quad 0 < x < L_4, \quad -L_3 < y < 0 \end{cases}$$

and set $u = e^{i\varphi_0} \in BV(\mathcal{R}, S^1)$.

In Lemmas 4.5 and 4.6 below we shall prove that φ_0 is the unique optimal lifting of u (up to a $2\pi\mathbb{Z}$ constant) and φ_0 is not a minimizer of $F_0^{(u,p)}$. Actually, we prove that the lifting

$\psi_0 \in BV(\mathcal{R}, \mathbb{R})$ of u defined as

$$\psi_0(x,y) = \begin{cases} \frac{\pi}{2} & \text{if} \quad 0 < x < L_4, \quad 0 < y < L_1, \\ -\frac{3\pi}{4} & \text{if} \quad -L_2 < x < 0, \quad 0 < y < L_1, \\ -\frac{\pi}{2} & \text{if} \quad -L_2 < x < 0, \quad -L_3 < y < 0, \\ \frac{3\pi}{10} & \text{if} \quad 0 < x < L_4, \quad -L_3 < y < 0 \end{cases}$$

is the unique minimizer of $F_0^{(u,p)}$ (up to $2\pi\mathbb{Z}$ constants).

Lemma 4.5 *The function φ_0 is the unique optimal lifting of u (up to a $2\pi\mathbb{Z}$ constant).*

Proof. Let $\varphi \in BV(\mathcal{R}, \mathbb{R})$ be a lifting of u. Then

$$\int_{\mathcal{R}} |D\varphi| = \sum_{k=1}^{4} \left(\int_{\mathcal{U}_k} |D\varphi| + \int_{\Gamma_k} |\varphi^+_{\Gamma_k} - \varphi^-_{\Gamma_k}| \, d\mathcal{H}^1 \right)$$

where $\varphi^+_{\Gamma_k}$ and $\varphi^-_{\Gamma_k}$ are the traces of φ on Γ_k. Let us consider the one-dimensional sections

$$\mathcal{R}_t = \Big\{ (tx, ty) \, : \, (x,y) \in \partial\mathcal{R} \Big\}, \, \forall t \in (0,1)$$

where we denote the vertices of the rectangle $\mathcal{R}_t$ by $\{A^t_k\}_{1 \le k \le 4}$. By the characterization of BV functions by sections (see Theorem 3.103 in [11]), the restriction $\varphi_t = \varphi\big|_{\mathcal{R}_t}$ belongs to $BV(\mathcal{R}_t, \mathbb{R})$ for almost any $t \in (0,1)$. We define the following rescaled variation of φ_t on $\mathcal{R}_t$ as

$$V(\varphi_t, \mathcal{R}_t) = \sum_{k=1}^{4} \left(L_k \int_{\mathcal{R}_t \cap \mathcal{U}_k} \Big| \frac{\partial \varphi_t}{\partial \tau} \Big| + \sqrt{L_k^2 + L_{k+1}^2} \, \big| \varphi^+_{\Gamma_k}(A^t_k) - \varphi^-_{\Gamma_k}(A^t_k) \big| \right) \quad \text{for a.e. } t \in (0,1)$$

so that

$$\int_0^1 V(\varphi_t, \mathcal{R}_t) \, dt \le \int_{\mathcal{R}} |D\varphi|$$

(here τ is the tangent vector of straight lines). An easy computation yields

$$\int_{\mathcal{R}} |D\varphi_0| = L_1 \frac{3\pi}{4} + L_2 \frac{\pi}{4} + L_3 \frac{6\pi}{5} + L_4 \frac{\pi}{5}.$$

In order to prove that φ_0 is an optimal lifting, it is sufficient to prove that

$$V(\varphi_t, \mathcal{R}_t) \ge L_1 \frac{3\pi}{4} + L_2 \frac{\pi}{4} + L_3 \frac{6\pi}{5} + L_4 \frac{\pi}{5} \quad \text{for a.e. } t \in (0,1). \tag{4.18}$$

We shall use a method from [52]. Denoting the restriction of u to $\mathcal{R}_t$ by $u_t = u\big|_{\mathcal{R}_t}$, we have for almost every $t \in (0,1)$: $u_t = e^{i\varphi_t}$ $\mathcal{H}^1 -$ a.e. in $\mathcal{R}_t$ and $S(u_t) = \{a^t_k \, : \, 1 \le k \le 4\}$ where $a^t_k = \mathcal{R}_t \cap \mathcal{U}_k \cap \{x = 0\}$ for $k \in \{1,3\}$ and $a^t_k = \mathcal{R}_t \cap \mathcal{U}_k \cap \{y = 0\}$ for $k \in \{2,4\}$. The chain rule (4.7) leads to

$$\left(\frac{\partial \varphi_t}{\partial \tau} \right)^a = u_t \wedge \left(\frac{\partial u_t}{\partial \tau} \right)^a = 0 \quad \text{and} \quad \left(\frac{\partial \varphi_t}{\partial \tau} \right)^c = u_t \wedge \left(\frac{\partial u_t}{\partial \tau} \right)^c = 0;$$

hence,

$$\frac{\partial \varphi_t}{\partial \tau} = \left(\frac{\partial \varphi_t}{\partial \tau}\right)^j = \sum_{a\in S(u_t)} (\varphi_t(a+) - \varphi_t(a-))\delta_a + \sum_{b\in\mathcal{B}} (\varphi_t(b+) - \varphi_t(b-))\delta_b.$$

Here, the Lipschitz curve $\mathcal{R}_t$ is considered oriented counterclockwise and the traces of φ_t are taken with respect to this orientation. We have that

1. $\mathcal{B} \subset \mathcal{R}_t$ is a finite set such that $S(u_t) \cap \mathcal{B} = \emptyset$ and $\varphi_t(b+) - \varphi_t(b-) = -2\pi\alpha_b$ where $\alpha_b \in \mathbb{Z}, \forall b \in \mathcal{B}$;

2. $\varphi_t(a+) - \varphi_t(a-) = \text{Arg}\,\frac{u_t(a+)}{u_t(a-)} - 2\pi\alpha_a$ with $\alpha_a \in \mathbb{Z}, \forall a \in S(u_t)$.

Therefore, setting $L_5 = L_1$, it follows that

$$V(\varphi_t, \mathcal{R}_t) = \sum_{k=1}^{4}\Bigg(\sum_{a\in(S(u_t)\cup\mathcal{B})\cap\mathcal{U}_k} L_k\big|\varphi_t(a+) - \varphi_t(a-)\big| + \sqrt{L_k^2 + L_{k+1}^2}\,\big|\varphi^+_{\Gamma_k}(A_k^t) - \varphi^-_{\Gamma_k}(A_k^t)\big|\Bigg). \tag{4.19}$$

Since $\displaystyle\int_{\mathcal{R}_t} \frac{\partial \varphi_t}{\partial \tau} = 0$, we get

$$\sum_{a\in S(u_t)\cup\mathcal{B}} \alpha_a = \frac{1}{2\pi}\sum_{a\in S(u_t)} \text{Arg}\,\frac{u_t(a+)}{u_t(a-)} = 1. \tag{4.20}$$

Obviously,

$$|\varphi_t(a_k^t+) - \varphi_t(a_k^t-)| \geq \big|\,\text{Arg}\,\frac{u_t(a_k^t+)}{u_t(a_k^t-)}\big|,\ \forall 1 \leq k \leq 4.$$

By (4.19), the inequality (4.18) will follow from the surplus of the variation induced by the condition (4.20), i.e.,

$$V(\varphi_t, \mathcal{R}_t) \geq L_3\frac{2\pi}{5} + \sum_{k=1}^{4} L_k\big|\,\text{Arg}\,\frac{u_t(a_k^t+)}{u_t(a_k^t-)}\big|. \tag{4.21}$$

Indeed, suppose that there is $b \in \mathcal{B}$ such that $\alpha_b \neq 0$. If $b \in \mathcal{U}_k$ for some $1 \leq k \leq 4$ then by (4.17),

$$L_k|\varphi_t(b+) - \varphi_t(b-)| \geq 2\pi L_k > L_3\frac{2\pi}{5}.$$

If $b = A_k^t$ for some $1 \leq k \leq 4$, then

$$\sqrt{L_k^2 + L_{k+1}^2}\,|\varphi^+_{\Gamma_k}(A_k^t) - \varphi^-_{\Gamma_k}(A_k^t)| \geq 2\pi\sqrt{L_k^2 + L_{k+1}^2} > L_3\frac{2\pi}{5}$$

(here we used the fact that the traces of φ_t on Γ_k coincide with $\varphi^{\pm}_{\Gamma_k}(A_k^t)$ for a.e. $t \in (0,1)$). Otherwise, according to (4.20), there exists $\alpha_a \neq 0$ for some $a = a_k^t$ and by (4.17), we easily check that

$$L_k|\varphi_t(a_k^t+) - \varphi_t(a_k^t-)| \geq L_3\frac{2\pi}{5} + L_k\big|\,\text{Arg}\,\frac{u_t(a_k^t+)}{u_t(a_k^t-)}\big|$$

with equality if and only if $k = 3$. Therefore, (4.21) holds, i.e., φ_0 is an optimal lifting of u.

It remains to prove the uniqueness of the optimal lifting φ_0 (up to a $2\pi\mathbb{Z}$ constant). Let φ be an optimal lifting. From above, we deduce that the restriction φ_t on $\mathcal{R}_t$ satisfies for almost $t \in (0,1)$ that

$$S(\varphi_t) = S(u_t) \quad \text{and} \quad \alpha_{a_k^t} = \begin{cases} 0 & \text{if } k \in \{1,2,4\}, \\ 1 & \text{if } k = 3. \end{cases} \tag{4.22}$$

It follows that

$$\int_{\mathcal{R}} |D\varphi| \geq \int_{S(\varphi)} |\varphi^+ - \varphi^-| \, d\mathcal{H}^1 \geq \int_{S(u)} |\varphi^+ - \varphi^-| \, d\mathcal{H}^1$$
$$\geq \int_0^1 \sum_{k=1}^4 L_k |\varphi_t(a_k^t+) - \varphi_t(a_k^t-)| \, dt = \int_{\mathcal{R}} |D\varphi_0|.$$

Since φ is an optimal lifting, we deduce that $S(\varphi) = S(u)$. By (4.11), we have $D^a\varphi = D^c\varphi = 0$. It follows that φ is constant on each connected component of $\mathcal{R} \setminus S(u)$. By (4.22), we conclude that $\varphi - \varphi_0$ is a constant function, for some constant in $2\pi\mathbb{Z}$. □

Lemma 4.6 *The function ψ_0 is the unique minimizer of $F_0^{(u,p)}$ (up to $2\pi\mathbb{Z}$ constants).*

Proof. We use the same argument and notations as in the proof of Lemma 4.5. Let $\varphi \in BV(\mathcal{R}, \mathbb{R})$ be a lifting of u. By (4.11), we have $D^a\varphi = D^c\varphi = 0$ and $D\varphi = D^j\varphi = (\varphi^+ - \varphi^-)\nu_\varphi \mathcal{H}^1 \llcorner S(\varphi)$. We define for almost every $t \in (0,1)$ the following variation of φ_t on $\mathcal{R}_t$:

$$G(\varphi_t, \mathcal{R}_t) = \sum_{k=1}^4 \Bigg(\sum_{a \in (S(u_t) \cup \mathcal{B}) \cap \mathcal{U}_k} L_k f^{(p)}\big(|\varphi_t(a+) - \varphi_t(a-)|\big)$$
$$+ \sqrt{L_k^2 + L_{k+1}^2}\, f^{(p)}\big(|\varphi^+_{\Gamma_k}(A_k^t) - \varphi^-_{\Gamma_k}(A_k^t)|\big) \Bigg)$$

so that

$$2 \int_0^1 G(\varphi_t, \mathcal{R}_t) \, dt \leq F_0^{(u,p)}(\varphi).$$

In order to prove that ψ_0 is a minimizer of $F_0^{(u,p)}$, it is sufficient to verify that

$$G(\varphi_t, \mathcal{R}_t) \geq L_1 f^{(p)}(\frac{5\pi}{4}) + L_2 f^{(p)}(\frac{\pi}{4}) + L_3 f^{(p)}(\frac{4\pi}{5}) + L_4 f^{(p)}(\frac{\pi}{5}) = \frac{F_0^{(u,p)}(\psi_0)}{2} \quad \text{for a.e. } t \in (0,1). \tag{4.23}$$

Indeed, suppose that there is $b \in \mathcal{B}$ such that $\alpha_b \neq 0$. If $b \in \mathcal{U}_k$ for some $1 \leq k \leq 4$ then by (4.17) and Lemma 4.3,

$$L_k f^{(p)}(|\varphi_t(b+) - \varphi_t(b-)|) + L_1 f^{(p)}(|\varphi_t(a_1^t+) - \varphi_t(a_1^t-)|) > L_1 f^{(p)}(\frac{5\pi}{4})$$

and then, we use that

$$f^{(p)}(|\varphi_t(a_k^t+) - \varphi_t(a_k^t-)|) \geq f^{(p)}\Bigg(\Big|\operatorname{Arg} \frac{u_t(a_k^t+)}{u_t(a_k^t-)}\Big|\Bigg), \ 2 \leq k \leq 4.$$

If $b = A_k^t$ for some $1 \le k \le 4$, then

$$\sqrt{L_k^2 + L_{k+1}^2}\, f^{(p)}(|\varphi^+_{\Gamma_k}(A_k^t) - \varphi^-_{\Gamma_k}(A_k^t)|) + L_1 f^{(p)}(|\varphi_t(a_1^t+) - \varphi_t(a_1^t-)|) > L_1 f^{(p)}(\frac{5\pi}{4}).$$

Otherwise, according to (4.20), there exists $\alpha_a \neq 0$ for some $a = a_k^t$. By Lemma 4.3, we notice that the map $\theta \in (0,\pi) \mapsto f^{(p)}(2\pi - \theta) - f^{(p)}(\theta)$ is decreasing. Then, by (4.17), we easily check that

$$L_k f^{(p)}(|\varphi_t(a_k^t+) - \varphi_t(a_k^t-)|) + L_1 f^{(p)}\left(\left|\operatorname{Arg}\frac{u_t(a_1^t+)}{u_t(a_1^t-)}\right|\right) \ge L_k f^{(p)}\left(\left|\operatorname{Arg}\frac{u_t(a_k^t+)}{u_t(a_k^t-)}\right|\right) + L_1 f^{(p)}(\frac{5\pi}{4})$$

with equality if and only if $k = 1$. Therefore, (4.23) holds and we also deduce that if φ is a minimizer of $F_0^{(u,p)}$, then for almost every $t \in (0,1)$,

$$S(\varphi_t) = S(u_t) \quad \text{and} \quad \alpha_{a_k^t} = \begin{cases} 0 & \text{if } 2 \le k \le 4, \\ 1 & \text{if } k = 1. \end{cases} \tag{4.24}$$

The uniqueness of the minimizer ψ_0 (up to $2\pi\mathbb{Z}$ constants) follows by (4.24) as in the proof of Lemma 4.5. □

Proof of *(ii)* in Theorem 4.1 for $p \in (0,4)$**.** Let Ω be an arbitrary bounded domain in $\mathbb{R}^N$, for $N \ge 2$. Denote by $\mathcal{D} = (2\mathcal{R}) \times (-2,2)^{N-2} \subset \mathbb{R}^N$. By translating and shrinking homotopically the rectangular parallelepiped $\mathcal{D}$, we may suppose that $\mathcal{D} \subset\subset \Omega$. Let u, φ_0 and ψ_0 be the functions in $\mathcal{R}$ constructed above and denote $\mathcal{D}_1 = \mathcal{R} \times (-1,1)^{N-2}$. We write $x = (x_1, x_2, \dots, x_N) = (x_1, x_2, x') \in \mathbb{R}^N$. We define in Ω,

$$w(x) = \begin{cases} u(x_1,x_2) & \text{in } \mathcal{D}_1, \\ 1 & \text{in } \big(\mathcal{D}\setminus\mathcal{D}_1\big) \cap \{x_1 > 0\}, \\ -1 & \text{otherwise.} \end{cases}$$

Consider the liftings

$$\zeta_0(x) = \begin{cases} \varphi_0(x_1,x_2) & \text{in } \mathcal{D}_1, \\ 0 & \text{in } \big(\mathcal{D}\setminus\mathcal{D}_1\big) \cap \{x_1 > 0\}, \\ \pi & \text{otherwise} \end{cases}$$

and

$$\xi_0(x) = \begin{cases} \psi_0(x_1,x_2) & \text{in } \mathcal{D}_1, \\ 0 & \text{in } \big(\mathcal{D}\setminus\mathcal{D}_1\big) \cap \{x_1 > 0\}, \\ -\pi & \text{otherwise.} \end{cases}$$

We prove that ζ_0 is the unique optimal lifting of w and ξ_0 is the unique minimizer of $F_0^{(w,p)}$, but $\zeta_0 - \xi_0$ is not constant since

$$\zeta_0 = \begin{cases} \xi_0 & \text{in } \mathcal{D} \cap \{x_1 > 0\}, \\ \xi_0 + 2\pi & \text{otherwise.} \end{cases}$$

Step 1. The function ζ_0 is the unique optimal lifting of w (up to a $2\pi\mathbb{Z}$ constant).

Indeed, let $\zeta \in BV(\Omega, \mathbb{R})$ be a lifting of w. Obviously, $|\zeta^+ - \zeta^-| \geq d_{S^1}(w^+, w^-) = |\zeta_0^+ - \zeta_0^-|$ $\mathcal{H}^{N-1}$-a.e. in $S(w) \cap \big(\Omega \setminus \mathcal{D}_1\big)$. The restriction of ζ to $\mathcal{R} \times \{x'\}$ is a BV lifting of u for almost every $x' \in (-1,1)^{N-2}$. Therefore, by Lemma 4.5, we obtain

$$\begin{aligned}\int_\Omega |D\zeta| &= \int_{\Omega\setminus\mathcal{D}_1} |D\zeta| + \int_{\mathcal{D}_1} |D\zeta| \\ &\geq \int_{S(w)\cap(\Omega\setminus\mathcal{D}_1)} |\zeta^+ - \zeta^-|\, d\mathcal{H}^{N-1} + \int_{(-1,1)^{N-2}} dx' \int_{\mathcal{R}\times\{x'\}} \left|\left(\frac{\partial\zeta}{\partial x_1}, \frac{\partial\zeta}{\partial x_2}\right)\right| \\ &\geq \int_{S(w)\cap(\Omega\setminus\mathcal{D}_1)} d_{S^1}(w^+, w^-)\, d\mathcal{H}^{N-1} + 2^{N-2}\int_{\mathcal{R}} |D\varphi_0| = \int_\Omega |D\zeta_0|,\end{aligned}$$

i.e., ζ_0 is an optimal lifting of w. Let now ζ be an optimal lifting. From the above it follows that

$$\int_{\Omega\setminus\mathcal{D}_1} |D\zeta| = \int_{S(w)\cap(\Omega\setminus\mathcal{D}_1)} d_{S^1}(w^+, w^-)\, d\mathcal{H}^{N-1}$$

and for almost every $x' \in (-1,1)^{N-2}$, the restriction of ζ to $\mathcal{R} \times \{x'\}$ is an optimal lifting of u, i.e.,

$$\int_{\mathcal{R}\times\{x'\}} |D\zeta| = \int_{\mathcal{R}} |D\varphi_0|.$$

As in the proof of Lemma 4.5, it follows that $\zeta - \zeta_0 \equiv 2\pi m$ in $\mathcal{D}_1$ where $m \in \mathbb{Z}$. Since the size of the jump of ζ must satisfy $0 < d_{S^1}(w^+, w^-) < \pi$ on $\partial\mathcal{D}$, we deduce that

$$\zeta - \zeta_0 \equiv 2\pi m \quad \text{in} \quad \Omega.$$

Hence, ζ_0 is the unique optimal lifting of w (up to $2\pi\mathbb{Z}$ constants).

Step 2. The function ξ_0 is the unique minimizer of $F_0^{(w,p)}$ (up to $2\pi\mathbb{Z}$ constants).

As in *Step 1*, using Lemma 4.6, we have that for every BV lifting ζ of w,

$$\begin{aligned}\frac{F_0^{(w,p)}(\zeta)}{2} &= \int_{S(\zeta)\cap(\Omega\setminus\mathcal{D}_1)} f^{(p)}(|\zeta^+ - \zeta^-|)\, d\mathcal{H}^{N-1} + \int_{S(\zeta)\cap\mathcal{D}_1} f^{(p)}(|\zeta^+ - \zeta^-|)\, d\mathcal{H}^{N-1} \\ &\geq \int_{S(w)\cap(\Omega\setminus\mathcal{D}_1)} f^{(p)}(|\zeta^+ - \zeta^-|)\, d\mathcal{H}^{N-1} \\ &\quad + \int_{(-1,1)^{N-2}} dx' \int_{S(\zeta)\cap(\mathcal{R}\times\{x'\})} f^{(p)}(|\zeta^+ - \zeta^-|)\, d\mathcal{H}^1 \\ &\geq \int_{S(w)\cap(\Omega\setminus\mathcal{D}_1)} f^{(p)}\big(d_{S^1}(w^+, w^-)\big)\, d\mathcal{H}^{N-1} + 2^{N-3}F_0^{(u,p)}(\psi_0) = \frac{F_0^{(w,p)}(\xi_0)}{2}\end{aligned}$$

i.e., ξ_0 is a minimizer of $F_0^{(w,p)}$. The uniqueness of the minimizer follows by the same argument as above. ☐

4.6 Proof of *(ii)* in Theorem 4.1 for $p \neq 4$

In this section we shall complete the proof of our main result in the general case $p \in (0,4) \cup (4,+\infty)$. The strategy will be to construct a family of functions $\mathcal{U} = \{U_t\}_{t\in(-\frac{1}{4},\frac{1}{4})}$ in $BV(\Omega, S^1)$

with the following property: for every $p \neq 4$, there exists a function U_t in the family $\mathcal{U}$ such that U_t has a unique optimal lifting (up to translations in $2\pi\mathbb{Z}$) and the energy $F_0^{(U_t,p)}$ of the optimal lifting is larger than the minimal energy $\min F_0^{(U_t,p)}$. First of all, we make that construction in the special case of the two-dimensional disc

$$\Omega := \{z \in \mathbb{C} : \, |z| < 2\}.$$

Construction of the family $\mathcal{U} = \{U_t\}_{t \in (-\frac{1}{4},\frac{1}{4})}$ in the disc $\Omega = B(0,2) \subset \mathbb{R}^2$. For any $z \in \Omega \setminus \{0\}$, we denote the argument $\bar{\theta}(z) \in [0, 2\pi)$, i.e., $\frac{z}{|z|} = e^{i\bar{\theta}(z)}$. Let $t \in (-\frac{1}{4}, \frac{1}{4})$. We define the set

$$A_t := \{\, z \in \Omega : \; z = re^{i\theta}, \; r \in (1,2), \; 0 < \theta < (\frac{3}{4} + t) \ln r \,\}$$

and we consider the function $\hat{\theta}_t : \Omega \to \mathbb{R}$ given by

$$\hat{\theta}_t(z) := \bar{\theta}(z) + 2\pi\chi_{A_t}(z), \quad \forall z \in \Omega, \tag{4.25}$$

where χ_{A_t} is the characteristic function associated to the set A_t. Now let $U_t \in BV(\Omega, S^1)$ be defined by

$$U_t(z) := e^{i\frac{9}{10}\hat{\theta}_t(z)}, \quad \forall z \in \Omega. \tag{4.26}$$

Set the liftings $\varphi_{1,t}, \varphi_{2,t} \in BV(\Omega, \mathbb{R})$ of U_t:

$$\varphi_{1,t} := \frac{9}{10}\hat{\theta}_t = \frac{9}{10}\bar{\theta} + \frac{9\pi}{5}\chi_{A_t} \quad \text{and} \quad \varphi_{2,t} := \frac{9}{10}\hat{\theta}_t - 2\pi\chi_{A_t} = \frac{9}{10}\bar{\theta} - \frac{\pi}{5}\chi_{A_t}. \tag{4.27}$$

We will show that:

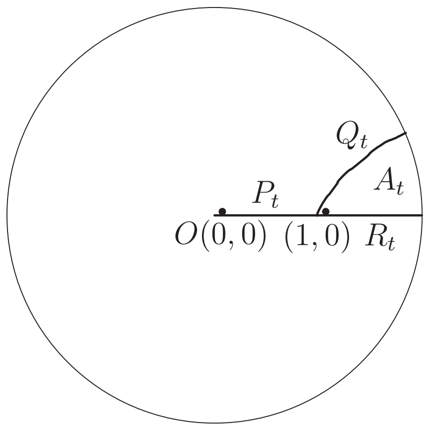


Figure 4.2: The construction for the general case $p \neq 4$

Lemma 4.7

(i) For any $t \in (-\frac{1}{4}, 0)$, $\varphi_{1,t}$ is the unique optimal lifting of U_t (up to $2\pi\mathbb{Z}$ additive constants);

(ii) For any $t \in (0, \frac{1}{4})$, $\varphi_{2,t}$ is the unique optimal lifting of U_t (up to $2\pi\mathbb{Z}$ additive constants).

The conclusion of Theorem 4.1 (in the case of the disc) will then follow from the next result:

Lemma 4.8

(i) For every $0 < p < 4$ there exists a positive number $\rho_p \in (0, \frac{1}{4})$ such that for any $t \in (-\rho_p, 0)$ we have that $F_0^{(U_t,p)}(\varphi_{1,t}) > F_0^{(U_t,p)}(\varphi_{2,t})$, i.e., the optimal lifting $\varphi_{1,t}$ of U_t is not a minimizer of $F_0^{(U_t,p)}$. Moreover, $\varphi_{2,t}$ is the unique minimizer of $F_0^{(U_t,p)}$ (up to a $2\pi\mathbb{Z}$ translation), for every $t \in (-\rho_p, \rho_p)$.

(ii) For any $p > 4$ there exists $\rho_p \in (0, \frac{1}{4})$ such that $F_0^{(U_t,p)}(\varphi_{2,t}) > F_0^{(U_t,p)}(\varphi_{1,t})$, for each $t \in (0, \rho_p)$, i.e., the optimal lifting $\varphi_{2,t}$ of U_t is not a minimizer of $F_0^{(U_t,p)}$. Moreover, $\varphi_{1,t}$ is the unique minimizer of $F_0^{(U_t,p)}$ (up to a $2\pi\mathbb{Z}$ translation), for every $t \in (-\rho_p, \rho_p)$.

Before proving the above Lemmas, we shall introduce some notations (see Figure 2). Set

$$P_t := \{z \in \mathbb{C} : z = r, r \in (0,1)\} \quad \text{and} \quad Q_t := \{z \in \mathbb{C} : z = re^{i(3/4+t)\ln r}, r \in (1,2)\}. \tag{4.28}$$

Then the jump set of U_t is given by

$$S(U_t) = P_t \cup Q_t \cup \{(0,0),(1,0)\}; \tag{4.29}$$

moreover, we have that

$$\mathcal{H}^1(P_t) = 1 \quad \text{and} \quad \mathcal{H}^1(Q_t) = \sqrt{1 + (3/4+t)^2}. \tag{4.30}$$

We choose the orientation of the jump set $S(U_t)$ to be given by the unit normal vector $\nu_{U_t} \in S^1$ defined by

$$\nu_{U_t}(z) = \begin{cases} (0,1) & z \in P_t, \\ \dfrac{1}{|\gamma_t'(|z|)|}\big(-\gamma_{t,2}'(|z|), \gamma_{t,1}'(|z|)\big) & z \in Q_t, \end{cases}$$

where $\gamma_t(r) = \gamma_{t,1}(r) + i\gamma_{t,2}(r) := re^{i(3/4+t)\ln r}$. Then for any $z \in S(U_t)$ we consider the traces

$$U_t^+(z) = e^{i\frac{9}{10}\bar{\theta}(z)} \quad \text{and} \quad U_t^-(z) = e^{i\frac{9}{10}(\bar{\theta}(z)+2\pi)} = e^{i\left(\frac{9}{10}\bar{\theta}(z) - \frac{\pi}{5}\right)}.$$

We start by giving a useful characterization of a general lifting $\varphi \in BV(\Omega, \mathbb{R})$ of U_t. We can choose the orientation of $S(\varphi)$ to coincide with the orientation of $S(U_t)$ on $S(\varphi) \cap S(U_t)$. Then, we have

$$\varphi^+(z) - \varphi^-(z) = \frac{\pi}{5} + 2\pi n(z), \forall z \in S(U_t) \quad \text{and} \quad \varphi^+(z) - \varphi^-(z) = 2\pi n(z), \forall z \in S(\varphi) \setminus S(U_t),$$

where $n : S(\varphi) \to \mathbb{Z}$ is an integrable function. We define the sets

$$L_\varphi := \{z \in S(\varphi) : n(z) \neq 0\} \quad \text{and} \quad L_\varphi^r := \{r \in (0,2) : \exists \theta \in \mathbb{R}, re^{i\theta} \in L_\varphi\}. \tag{4.31}$$

We next prove the following property:

Lemma 4.9 *For any lifting $\varphi \in BV(\Omega, \mathbb{R})$ of U_t, we have $\mathcal{H}^1(L_\varphi^r) = 2$.*

Proof. By contradiction, assume that $\mathcal{H}^1(L^r_\varphi) < 2$. Then, there exists a compact set $K \subset (0,2)$ such that $\mathcal{H}^1(K) > 0$ and $L^r_\varphi \cap K = \emptyset$. Consider a sequence of open sets $V_k \subset\subset (0,2)$ such that $K \subset V_k \subset\subset (0,2)$ and $\bigcap_{k=1}^\infty V_k = K$. Now take a sequence of functions $\sigma_k \in C^1_c\big((0,2),\mathbb{R}\big)$ that satisfy $0 \le \sigma_k \le 1$, $\sigma_k(r) = 1$ for any $r \in K$ and $\sigma_k(r) = 0$ for any $r \in (0,2) \setminus V_k$. Define the functions $\delta_k \in C^2_c(\Omega,\mathbb{R})$ by

$$\delta_k(z) := \int_{|z|}^2 \sigma_k(t)dt.$$

For $z = (x,y)$, we denote $\nabla^\perp \delta_k := (-\partial_y \delta_k, \partial_x \delta_k)$. Then we have

$$\int_\Omega \nabla^\perp \delta_k(z)\, d[D\varphi](z) = 0. \tag{4.32}$$

Since $U_t = e^{i\varphi}$, we obtain from the chain rule (4.7),

$$D\varphi = D^a\varphi + D^j\varphi = \frac{9}{10} D^a\bar\theta + \frac{\pi}{5}\nu_{U_t}\, \mathcal{H}^1 \llcorner S(U_t) + 2\pi n(\cdot)\nu_\varphi\, \mathcal{H}^1 \llcorner L_\varphi.$$

Therefore, by (4.32) we infer

$$-2\pi\delta_k(0) + 2\pi \int_{L_\varphi} n(z) \nabla^\perp \delta_k(z) \cdot \nu_\varphi(z)\, d\mathcal{H}^1(z) = 0. \tag{4.33}$$

Define the sets $W_k := \{z \in \Omega :\, |z| \in V_k \setminus K\}$, $\forall k \ge 1$. Then by the construction of δ_k, we deduce from (4.33),

$$\delta_k(0) = \int_{L_\varphi \cap W_k} n(z) \nabla^\perp \delta_k(z) \cdot \nu_\varphi(z)\, d\mathcal{H}^1(z).$$

Since $|\nabla^\perp \delta_k| \le 1$, it follows that

$$|\delta_k(0)| \le \int_{L_\varphi \cap W_k} |n(z)|\, d\mathcal{H}^1(z) \le \frac{1}{\pi} \int_{L_\varphi \cap W_k} |\varphi^+(z) - \varphi^-(z)|\, d\mathcal{H}^1(z) \le \frac{1}{\pi} \int_{W_k} |D\varphi|.$$

Using $\cap_{k=1}^\infty W_k = \emptyset$, we get that

$$\lim_{k\to\infty} \delta_k(0) = 0. \tag{4.34}$$

On the other hand, according to the definition of δ_k, we have

$$\delta_k(0) = \int_0^2 \sigma_k(t)dt \ge \int_K 1\, dt = \mathcal{H}^1(K) > 0,$$

which leads to a contradiction to (4.34). This completes the proof of Lemma 4.9. □

We now present the proofs of Lemmas 4.7 and 4.8:

Proof of Lemma 4.7. The jump set of $\varphi_{1,t}$ and $\varphi_{2,t}$ are

$$S(\varphi_{1,t}) = S(U_t) = P_t \cup Q_t \cup \{(0,0),(1,0)\} \quad \text{and} \quad S(\varphi_{2,t}) = P_t \cup Q_t \cup R_t \cup \{(0,0),(1,0)\}, \tag{4.35}$$

where $R_t := \{z \in \mathbb{C} :\, z = r,\, r \in (1,2)\}$. Moreover, the size of the jump is

$$|\varphi^+_{1,t}(z) - \varphi^-_{1,t}(z)| = \frac{9\pi}{5}, \quad \forall z \in P_t \cup Q_t$$

and

$$|\varphi^+_{2,t}(z) - \varphi^-_{2,t}(z)| = \begin{cases} \frac{9\pi}{5} & \text{if} \quad z \in P_t, \\ \frac{\pi}{5} & \text{if} \quad z \in Q_t, \\ 2\pi & \text{if} \quad z \in R_t. \end{cases}$$

Therefore, by (4.30), it follows that

$$\begin{aligned} \int_\Omega |D^j \varphi_{1,t}| &= \frac{9\pi}{5} + \frac{9\pi}{5}\sqrt{1+(3/4+t)^2}; \\ \int_\Omega |D^j \varphi_{2,t}| &= \frac{9\pi}{5} + \frac{\pi}{5}\sqrt{1+(3/4+t)^2} + 2\pi. \end{aligned} \tag{4.36}$$

Hence, we have

$$\begin{aligned} &\int_\Omega |D^j \varphi_{1,t}| < \int_\Omega |D^j \varphi_{2,t}|, \quad \forall t \in (-1/4, 0), \\ &\int_\Omega |D^j \varphi_{1,t}| > \int_\Omega |D^j \varphi_{2,t}|, \quad \forall t \in (0, 1/4), \\ &\int_\Omega |D^j \varphi_{1,0}| = \int_\Omega |D^j \varphi_{2,0}|. \end{aligned} \tag{4.37}$$

Let now $\varphi \in BV(\Omega, \mathbb{R})$ be an arbitrary lifting of U_t. From (4.11) it follows that $\int_\Omega |D^a\varphi| = \int_\Omega |D^a U_t|$ and $\int_\Omega |D^c\varphi| = \int_\Omega |D^c U_t| = 0$. We choose an orientation of $S(\varphi)$ that coincides with the orientation of $S(U_t)$ on $S(\varphi) \cap S(U_t)$. Put

$$\begin{cases} x_\varphi := \mathcal{H}^1(L_\varphi \cap P_t), \quad y_\varphi := \mathcal{H}^1(L_\varphi \cap Q_t), \\ w_\varphi := \mathcal{H}^1(S(\varphi) \setminus S(U_t)) = \mathcal{H}^1\big(L_\varphi \setminus (P_t \cup Q_t)\big), \\ z_\varphi := w_\varphi + x_\varphi + \frac{y_\varphi}{\sqrt{1+(3/4+t)^2}}, \end{cases} \tag{4.38}$$

where P_t and Q_t are defined in (4.28) and L_φ is given in (4.31). Consider the following decomposition of L^r_φ (defined in (4.31)):

$$L^r_\varphi = A^r_\varphi \cup B^r_\varphi \cup D^r_\varphi \quad \text{a.e. in } (0,2),$$

where

$$\begin{cases} A^r_\varphi := \{r \in (0,1) : \exists\, \theta \in \mathbb{R},\ re^{i\theta} \in L_\varphi \cap P_t\}, \\ B^r_\varphi := \{r \in (1,2) : \exists\, \theta \in \mathbb{R},\ re^{i\theta} \in L_\varphi \cap Q_t\}, \\ D^r_\varphi := \{r \in (0,2) : \exists\, \theta \in \mathbb{R},\ re^{i\theta} \in L_\varphi \setminus (P_t \cup Q_t)\}. \end{cases} \tag{4.39}$$

Note that $A^r_\varphi \cap B^r_\varphi = \emptyset$, but A^r_φ (resp. B^r_φ) and D^r_φ are not necessarily disjoint. We have

$$\mathcal{H}^1(A^r_\varphi) = x_\varphi \quad \text{and} \quad \mathcal{H}^1(B^r_\varphi) = \frac{y_\varphi}{\sqrt{1+(3/4+t)^2}},$$

where the last equality follows by the construction of Q_t. It is clear then that

$$w_\varphi \geq \mathcal{H}^1\big(D^r_\varphi\big) \geq \mathcal{H}^1\big(L^r_\varphi \setminus (A^r_\varphi \cup B^r_\varphi)\big) = \mathcal{H}^1\big(L^r_\varphi\big) - x_\varphi - \frac{y_\varphi}{\sqrt{1+(3/4+t)^2}}.$$

By Lemma 4.9 we have $\mathcal{H}^1(L^r_\varphi) = 2$. Therefore,

$$w_\varphi \geq 2 - x_\varphi - \frac{y_\varphi}{\sqrt{1+(3/4+t)^2}}, \quad \text{i.e.,} \quad z_\varphi \geq 2. \tag{4.40}$$

By (4.30), we deduce that

$$(x_\varphi, y_\varphi, z_\varphi) \in M_t := \{(x,y,z) \in \mathbb{R}^3 : \ 0 \leq x \leq 1, \ 0 \leq y \leq \sqrt{1+(3/4+t)^2}, \ z \geq 2\}. \tag{4.41}$$

We define the function $\Phi_t : M_t \to \mathbb{R}$ by

$$\Phi_t(x,y,z) := 2\pi z - \frac{2\pi}{5}x + \frac{2\pi\big(4\sqrt{1+(3/4+t)^2} - 5\big)}{5\sqrt{1+(3/4+t)^2}}y + \frac{\pi}{5}\Big(1 + \sqrt{1+(3/4+t)^2}\Big).$$

It is easy to check that for $t > 0$ the unique minimum point of Φ_t on the set M_t is achieved at the point $(1,0,2)$. Similarly, if $t < 0$ then Φ_t attains its unique minimum on the set M_t at $(x,y,z) = \big(1, \sqrt{1+(3/4+t)^2}, 2\big)$.

On the other hand, from (4.29) we infer

$$\begin{aligned}\int_\Omega |D^j\varphi| &\geq \int_{S(\varphi)\setminus S(U_t)} |\varphi^+ - \varphi^-| + \int_{(L_\varphi\cap P_t)\cup(L_\varphi\cap Q_t)} |\varphi^+ - \varphi^-| + \int_{(P_t\cup Q_t)\setminus L_\varphi} |\varphi^+ - \varphi^-| \\ &\geq 2\pi w_\varphi + \Big(2\pi - \frac{\pi}{5}\Big)(x_\varphi + y_\varphi) + \frac{\pi}{5}\Big(1 + \sqrt{1+(3/4+t)^2} - x_\varphi - y_\varphi\Big) \\ &= \Phi_t(x_\varphi, y_\varphi, z_\varphi).\end{aligned} \tag{4.42}$$

Therefore,

$$\begin{aligned}&\int_\Omega |D^j\varphi| \geq \Phi_t(x_\varphi,y_\varphi,z_\varphi) \geq \Phi_t\big(1, \sqrt{1+(3/4+t)^2}, 2\big) = \int_\Omega |D^j\varphi_{1,t}|, \quad \text{if } t \in (-1/4, 0), \\ &\int_\Omega |D^j\varphi| \geq \Phi_t(x_\varphi,y_\varphi,z_\varphi) \geq \Phi_t(1,0,2) = \int_\Omega |D^j\varphi_{2,t}|, \quad \text{if } t \in (0, 1/4).\end{aligned} \tag{4.43}$$

We conclude that for $t \in (-1/4, 0)$, $\varphi_{1,t}$ is an optimal lifting of U_t while for $t \in (0, 1/4)$, $\varphi_{2,t}$ is an optimal lifting of U_t.

It remains to prove the uniqueness of the optimal lifting of U_t. Let φ be an arbitrary optimal lifting of U_t. Then all inequalities in (4.42) and (4.43) become equalities.

(i) In the case of $t \in (-1/4, 0)$, we deduce that $x_\varphi = 1$, $y_\varphi = \sqrt{1+(3/4+t)^2}$, $w_\varphi = 0$ (hence, $S(\varphi) = S(U_t)$). Moreover, by (4.42),

$$|\varphi^+ - \varphi^-| = \frac{9\pi}{5} \quad \mathcal{H}^1\text{-a.e. in } S(\varphi).$$

Since every lifting has the same diffuse part (see (4.11)), it follows that

$$D(\varphi - \varphi_{1,t}) = 0 \quad \text{in } \Omega.$$

Since Ω is connected, we conclude that $\varphi - \varphi_{1,t}$ is constant in Ω.

(ii) In the case $t \in (0, 1/4)$ we obtain $x_\varphi = 1$, $y_\varphi = 0$, $w_\varphi = 1$. Moreover, by (4.42),

$$|\varphi^+ - \varphi^-| = \begin{cases} \frac{9\pi}{5} & \mathcal{H}^1\text{-a.e. in } S(\varphi) \cap P_t, \\ \frac{\pi}{5} & \mathcal{H}^1\text{-a.e. in } S(\varphi) \cap Q_t, \\ 2\pi & \mathcal{H}^1\text{-a.e. in } S(\varphi) \setminus (P_t \cup Q_t). \end{cases}$$

Then, according to (4.11), it follows that

$$D(\varphi - \varphi_{2,t}) = 2\pi\Big(\nu_{\varphi_{2,t}}\mathcal{H}^1 \llcorner R_t - \nu_\varphi \mathcal{H}^1 \llcorner \big(S(\varphi) \setminus S(U_t)\big)\Big).$$

We deduce that for every function $\delta \in C^1_c(\Omega)$,

$$\int_{S(\varphi)\setminus S(U_t)} \frac{\partial \delta}{\partial \tau_\varphi}\, d\mathcal{H}^1 = \int_{S(\varphi)\setminus S(U_t)} \nabla^\perp \delta \cdot \nu_\varphi \, d\mathcal{H}^1 = \delta(1,0),$$

where τ_φ stands for the tangent vector to the $\mathcal{H}^1$-rectifiable set $S(\varphi) \setminus S(U_t)$. Using the same technique as in [53], since $\mathcal{H}^1\big(S(\varphi)\setminus S(U_t)\big) = \operatorname{dist}((0,1), \partial\Omega) = 1$, we conclude that $S(\varphi) \setminus S(U_t)$ coincides with R_t (which is the geodesic line between the point $(0,1)$ and $\partial\Omega$). Thus, $D(\varphi - \varphi_{2,t}) = 0$ in Ω, i.e., $\varphi - \varphi_{2,t}$ is constant in Ω. This completes the proof of Lemma 4.7. $\square$

Proof of Lemma 4.8. Let $p > 0$. By Lemma 4.3 we compute

$$\begin{aligned}
F_0^{(U_t,p)}(\varphi_{1,t}) &= \big(1 + \sqrt{1 + (3/4+t)^2}\big) \int_{-9\pi/10}^{9\pi/10} 2|e^{is} - 1|^{p/2} ds \\
&= 2^{p/2+3}\big(1 + \sqrt{1 + (3/4+t)^2}\big) \int_0^{9\pi/20} \sin^{p/2} s\, ds \\
&= 2^{p/2+3} \int_0^{9\pi/20} \sin^{p/2} s\, ds + 2^{p/2+3}\sqrt{1 + (3/4+t)^2} \int_{\pi/20}^{\pi/2} \cos^{p/2} s\, ds.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
F_0^{(U_t,p)}(\varphi_{2,t}) &= \int_0^{9\pi/10} 4|e^{is} - 1|^{p/2} ds + \sqrt{1 + (3/4+t)^2} \int_0^{\pi/10} 4|e^{is} - 1|^{p/2} ds \\
&\quad + \int_0^{\pi} 4|e^{is} - 1|^{p/2} ds \\
&= 2^{p/2+3}\Big(\int_0^{9\pi/20} \sin^{p/2} s\, ds + \sqrt{1 + (3/4+t)^2} \int_0^{\pi/20} \sin^{p/2} s\, ds + \int_0^{\pi/2} \cos^{p/2} s\, ds\Big).
\end{aligned}$$

Therefore, we infer that

$$\begin{aligned}
&2^{-p/2-3}\big(F_0^{(U_t,p)}(\varphi_{1,t}) - F_0^{(U_t,p)}(\varphi_{2,t})\big) = \\
&= \big(\sqrt{1 + (3/4+t)^2} - 1\big) \int_0^{\pi/2} \cos^{p/2} s\, ds - \sqrt{1 + (3/4+t)^2} \int_0^{\pi/20} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds \\
&= \big(\sqrt{1 + (3/4+t)^2} - 1\big) \int_0^{\pi/4} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds - \sqrt{1 + (3/4+t)^2} \int_0^{\pi/20} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds \\
&= \frac{1}{5} \int_0^{\pi/4} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds \cdot \Big(5\big(\sqrt{1 + (3/4+t)^2} - 1\big) - c_p \sqrt{1 + (3/4+t)^2}\Big), \qquad (4.44)
\end{aligned}$$

where we denoted

$$c_p := \frac{5\int_0^{\pi/20}\big(\cos^{p/2} s + \sin^{p/2} s\big)ds}{\int_0^{\pi/4}\big(\cos^{p/2} s + \sin^{p/2} s\big)ds} \in (0,5).$$

Since the function

$$s \in (0, \frac{\pi}{4}) \mapsto \big(\cos^{p/2} s + \sin^{p/2} s\big)$$

is increasing for $0 < p < 4$ and decreasing for $p > 4$, it turns out that

$$c_p < 1, \quad \forall p \in (0,4) \qquad \text{and} \qquad c_p > 1, \quad \forall p \in (4,\infty).$$

Therefore, by (4.44), for any $p \in (0,4)$ there exists $0 < \rho_p < 1/4$ such that

$$F_0^{(U_t,p)}(\varphi_{1,t}) > F_0^{(U_t,p)}(\varphi_{2,t}) \qquad \forall t \in (-\rho_p, \rho_p). \tag{4.45}$$

Similarly, for any $p \in (4,\infty)$, there exists $0 < \rho_p < 1/4$ such that

$$F_0^{(U_t,p)}(\varphi_{1,t}) < F_0^{(U_t,p)}(\varphi_{2,t}) \qquad \forall t \in (-\rho_p, \rho_p). \tag{4.46}$$

Now we prove that for any $t \in (-\rho_p, \rho_p)$, $\varphi_{2,t}$ (resp. $\varphi_{1,t}$) is the unique minimizer of $F_0^{(U_t,p)}$ if $p \in (0,4)$ (resp. $p > 4$). Let $\varphi \in BV(\Omega,\mathbb{R})$ be an arbitrary lifting of U_t. We choose an orientation on $S(\varphi)$ that coincides with the orientation of $S(U_t)$ on $S(\varphi) \cap S(U_t)$. In the following we use the same notations as in the proof of Lemma 4.7 (see (4.38), (4.39) and (4.41)). We define the function $\Psi_t : M_t \to \mathbb{R}$ by

$$\begin{aligned}
\Psi_t(x,y,z) :&= f^{(p)}(2\pi)z - \Big(f^{(p)}(2\pi) + f^{(p)}(\frac{\pi}{5}) - f^{(p)}(\frac{9\pi}{5})\Big)x \\
&\quad + \Big(f^{(p)}(\frac{9\pi}{5}) - \frac{f^{(p)}(2\pi)}{\sqrt{1+(3/4+t)^2}} - f^{(p)}(\frac{\pi}{5})\Big)y + f^{(p)}(\frac{\pi}{5})\Big(1 + \sqrt{1+(3/4+t)^2}\Big) \\
&= f^{(p)}(2\pi)z - \Big(f^{(p)}(2\pi) + f^{(p)}(\frac{\pi}{5}) - f^{(p)}(\frac{9\pi}{5})\Big)x \\
&\quad + \frac{y}{\sqrt{1+(3/4+t)^2}}\Big(F_0^{(U_t,p)}(\varphi_{1,t}) - F_0^{(U_t,p)}(\varphi_{2,t})\Big) + f^{(p)}(\frac{\pi}{5})\Big(1 + \sqrt{1+(3/4+t)^2}\Big).
\end{aligned}$$

By (4.45) and (4.46), it can be easily checked that: if $p \in (0,4)$ and $t \in (-\rho_p, \rho_p)$ then the unique minimal point of Ψ_t in the set M_t is achieved in $(1,0,2)$, while if $p > 4$ and $t \in (-\rho_p, \rho_p)$ then Ψ_t has also a unique minimal point in M_t for $(x,y,z) = \big(1, \sqrt{1+(3/4+t)^2}, 2\big)$. Using the same argument as in the proof of Lemma 4.7, it follows that

$$\begin{aligned}
\frac{F_0^{(U_t,p)}(\varphi)}{2} &\geq \int_{S(\varphi)\setminus S(U_t)} f^{(p)}(|\varphi^+ - \varphi^-|)\, d\mathcal{H}^1 + \int_{(L_\varphi \cap P_t)\cup(L_\varphi \cap Q_t)} f^{(p)}(|\varphi^+ - \varphi^-|)\, d\mathcal{H}^1 \\
&\quad + \int_{(P_t \cup Q_t)\setminus L_\varphi} f^{(p)}(|\varphi^+ - \varphi^-|)\, d\mathcal{H}^1 \\
&\geq f^{(p)}(2\pi)w_\varphi + f^{(p)}\Big(2\pi - \frac{\pi}{5}\Big)(x_\varphi + y_\varphi) + f^{(p)}\Big(\frac{\pi}{5}\Big)\Big(1 + \sqrt{1+(3/4+t)^2} - x_\varphi - y_\varphi\Big) \\
&= \Psi_t(x_\varphi, y_\varphi, z_\varphi).
\end{aligned} \tag{4.47}$$

Therefore, for every $t \in (-\rho_p, \rho_p)$,

$$\begin{cases} F_0^{(U_t,p)}(\varphi) \geq 2\Psi_t(x_\varphi, y_\varphi, z_\varphi) \geq 2\Psi_t\big(1, \sqrt{1+(3/4+t)^2}, 2\big) = F_0^{(U_t,p)}(\varphi_{1,t}) & \text{if } p > 4, \\ F_0^{(U_t,p)}(\varphi) \geq 2\Psi_t(x_\varphi, y_\varphi, z_\varphi) \geq 2\Psi_t(1,0,2) = F_0^{(U_t,p)}(\varphi_{2,t}) & \text{if } p \in (0,4). \end{cases} \tag{4.48}$$

It follows that for any $t \in (-\rho_p, \rho_p)$, $\varphi_{1,t}$ is a minimizer of $F_0^{(U_t,p)}$ if $p > 4$, and $\varphi_{2,t}$ is a minimizer of $F_0^{(U_t,p)}$ if $p \in (0,4)$. It remains to prove the uniqueness of the minimizer of $F_0^{(U_t,p)}$ for any $t \in (-\rho_p, \rho_p)$. Let φ be a lifting of U_t that minimizes the energy $F_0^{(U_t,p)}$. Then all inequalities in (4.47) and (4.48) become equalities. Next we distinguish two cases:

(i) In the case of $p > 4$ we deduce that $x_\varphi = 1$, $y_\varphi = \sqrt{1+(3/4+t)^2}$, $w_\varphi = 0$ (hence, $S(\varphi) = S(U_t)$). Moreover, by Lemma 4.3 and (4.47),

$$|\varphi^+ - \varphi^-| = \frac{9\pi}{5} \quad \mathcal{H}^1\text{-a.e. in } S(\varphi).$$

Since every lifting has the same diffuse part (see (4.11)), it follows that

$$D(\varphi - \varphi_{1,t}) = 0 \quad \text{in } \Omega.$$

Since Ω is connected, we conclude that $\varphi - \varphi_{1,t}$ is constant in Ω.

(ii) In the case $p \in (0,4)$ we obtain that $x_\varphi = 1$, $y_\varphi = 0$, $w_\varphi = 1$. Moreover, by (4.47)

$$|\varphi^+ - \varphi^-| = \begin{cases} \frac{9\pi}{5} & \mathcal{H}^1\text{-a.e. in } S(\varphi) \cap P_t, \\ \frac{\pi}{5} & \mathcal{H}^1\text{-a.e. in } S(\varphi) \cap Q_t, \\ 2\pi & \mathcal{H}^1\text{-a.e. in } S(\varphi) \setminus (P_t \cup Q_t). \end{cases}$$

Then, by the same argument as in the end of the proof of Lemma 4.7, we conclude that $\varphi - \varphi_{2,t}$ is constant in Ω. □

In the following, we shall adapt our construction of the family $\mathcal{U}$ to the general case of an arbitrary domain G:

Proof of *(ii)* in Theorem 4.1. Assume that G is an arbitrary bounded domain in $\mathbb{R}^N$ for $N \geq 2$. We construct a family of functions $\tilde{\mathcal{U}} = \{\tilde{U}_t\}_{t \in (-1/4,1/4)}$ in $BV(G, S^1)$ that will have the same behavior as the family $\mathcal{U} = \{U_t\}_{t \in (-1/4,1/4)}$, defined in (4.26) over the set $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 4\}$. Let us introduce the sets

$$\Omega_1 := \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 16\},$$

$$G_1 := \Omega \times (-1/2, 1/2)^{N-2} \subset \mathbb{R}^N \quad \text{and} \quad G_2 := \Omega_1 \times (-1,1)^{N-2} \subset \mathbb{R}^N.$$

For $t \in (-1/4, 1/4)$, set also

$$H_t := \{\, (x_1,x_2) \in \Omega_1 : \ (x_1,x_2) = re^{i\theta}, \ r \in (1,4), \ 0 < \theta < (3/4+t)\ln r \,\},$$

and define $\tilde{H}_t := H_t \times (-1,1)^{N-2} \subset \mathbb{R}^N$. As before, by translating and shrinking homotopically the set G_2, we may suppose that $G_2 \subset G$. We write $x = (x_1, x_2, \dots, x_N) = (x_1, x_2, x') \in \mathbb{R}^N$. Next we define the function $\tilde{U}_t \in BV(G, S^1)$ by

$$\tilde{U}_t(x) := \begin{cases} U_t(x_1, x_2) & x \in G_1, \\ 1 & x \in \tilde{H}_t \setminus G_1, \\ -1 & \text{otherwise.} \end{cases} \tag{4.49}$$

Recall the liftings $\varphi_{1,t}, \varphi_{2,t} \in BV(\Omega, \mathbb{R})$ of U_t defined in (4.27). Then, consider the liftings $\Phi_{1,t}, \Phi_{2,t} \in BV(G, \mathbb{R})$ of $\tilde{U}_t$ given by

$$\Phi_{1,t}(x) \ := \begin{cases} \varphi_{1,t}(x_1, x_2) & x \in G_1, \\ 2\pi & x \in \tilde{H}_t \setminus G_1, \\ \pi & \text{otherwise} \end{cases} \quad \text{and} \quad \Phi_{2,t}(x) \ := \begin{cases} \varphi_{2,t}(x_1, x_2) & x \in G_1, \\ 0 & x \in \tilde{H}_t \setminus G_1, \\ \pi & \text{otherwise.} \end{cases} \tag{4.50}$$

The jump part of these liftings enjoys the following property: for every $j = 1, 2$, and every $t \in (-1/4, 1/4)$ we have

$$S(\Phi_{j,t}) \backslash G_1 = S(\tilde{U}_t) \backslash G_1 \ \text{ and } \ \left|\Phi^+_{j,t}(x) - \Phi^-_{j,t}(x)\right| = d_{S^1}\big(\tilde{U}^+_t(x), \tilde{U}^-_t(x)\big) \ \mathcal{H}^{N-1}\text{-a.e. in } S(\Phi_{j,t}) \backslash G_1. \tag{4.51}$$

In the sequel we will prove that the analog results to those of Lemmas 4.7 and 4.8 hold for the functions $\Phi_{j,t}$, $j = 1, 2$.

Step 1. For $j = 1, 2$, $\Phi_{j,t}$ is the unique optimal lifting of $\tilde{U}_t$ (up to $2\pi\mathbb{Z}$ constants) if t is between 0 and $(-1)^j/4$.

Indeed, let $\Phi : G \to \mathbb{R}$ be an arbitrary lifting of $\tilde{U}_t$ on G. First notice that by (4.12), we have that

$$\int_{G \backslash G_1} |D^a \Phi| + \int_{G \backslash G_1} |D^c \Phi| = \int_{G \backslash G_1} |D^a \tilde{U}_t| + \int_{G \backslash G_1} |D^c \tilde{U}_t| = 0.$$

Using Lemma 4.7 it follows that

$$\begin{aligned} \int_G |D\Phi| &= \int_{G \backslash G_1} |D\Phi| + \int_{G_1} |D\Phi| \\ &= \int_{S(\Phi) \backslash G_1} |\Phi^+ - \Phi^-| \, d\mathcal{H}^{N-1} + \int_{G_1} |D\Phi| \\ &\geq \int_{S(\tilde{U}_t) \backslash G_1} d_{S^1}(\tilde{U}^+_t, \tilde{U}^-_t) d\mathcal{H}^{N-1} + \int\limits_{(-1/2,1/2)^{N-2}} dx' \int\limits_{\Omega \times \{x'\}} \Big| \Big(\frac{\partial \Phi}{\partial x_1}, \frac{\partial \Phi}{\partial x_2}\Big) \Big| \\ &\geq \int_{S(\tilde{U}_t) \backslash G_1} d_{S^1}(\tilde{U}^+_t, \tilde{U}^-_t) d\mathcal{H}^{N-1} + \int\limits_{\Omega} |D\varphi_{j,t}| = \int_G |D\Phi_{j,t}|, \end{aligned} \tag{4.52}$$

i.e., $\Phi_{j,t}$ is an optimal lifting of $\tilde{U}_t$ if t is between 0 and $(-1)^j/4$. It remains to show the uniqueness of the optimal lifting. For that, let Φ be an arbitrary optimal lifting of $\tilde{U}_t$. Then we must have equalities in (4.52) and therefore we obtain:

$$S(\Phi)\backslash G_1 = S(\tilde{U}_t)\backslash G_1 \quad \text{and} \quad \left|\Phi^+(x) - \Phi^-(x)\right| = d_{S^1}\big(\tilde{U}_t^+(x), \tilde{U}_t^-(x)\big) \quad \mathcal{H}^{N-1}\text{-a.e. in } S(\Phi_{j,t})\backslash G_1\,, \tag{4.53}$$

and for almost every $x' \in (-1/2, 1/2)^{N-2}$, the restriction of Φ to $\Omega \times \{x'\}$ is an optimal lifting of U_t. Therefore, the jump set of Φ satisfies:

$$S(\Phi) \cap G_1 = S(\varphi_{j,t}) \times (-1/2, 1/2)^{N-2} = S(\Phi_{j,t}) \cap G_1.$$

By (4.11), it follows that $D(\Phi - \Phi_{j,t}) = 0$ in $G_1 \setminus S(\Phi_{j,t})$, i.e., $\Phi - \Phi_{j,t}$ is constant on all j connected components of $G_1 \setminus S(\Phi_{j,t})$, $j = 1, 2$. The optimality of Φ does not allow any jumps for $\Phi - \Phi_{j,t}$ on $S(\Phi_{j,t}) \cap G_1$. Hence, by (4.53), we conclude that $\Phi - \Phi_{j,t}$ is constant in G.

Step 2. For every $p \in (4, \infty)$ (resp. $p \in (0, 4)$), there exists $\rho_p \in (0, \frac{1}{4})$ such that for any $0 < t < \rho_p$ (resp. $-\rho_p < t < 0$), we have

$$F_0^{(\tilde{U}_t,p)}(\Phi_{2,t}) > F_0^{(\tilde{U}_t,p)}(\Phi_{1,t}) \quad (\text{resp. } F_0^{(\tilde{U}_t,p)}(\Phi_{1,t}) > F_0^{(\tilde{U}_t,p)}(\Phi_{2,t})\ ),$$

i.e., the optimal lifting of $\tilde{U}_t$ is not a minimizer of $F_0^{(\tilde{U}_t,p)}$ for the above ranges of p and t.

Indeed, let us prove the claim for $p > 4$ (the other case follows using the same argument). Take $\rho_p \in (0, 1/4)$ as given by Lemma 4.8. Then, by Step 1 and Lemma 4.8, we deduce that for $t \in (0, \rho_p)$,

$$\begin{aligned}
F_0^{(\tilde{U}_t,p)}(\Phi_{2,t}) &= \int_{S(\Phi_{2,t})\backslash G_1} f^{(p)}(|\Phi_{2,t}^+ - \Phi_{2,t}^-|)\, d\mathcal{H}^{N-1} + \int_{G_1 \cap S(\Phi_{2,t})} f^{(p)}(|\Phi_{2,t}^+ - \Phi_{2,t}^-|) d\mathcal{H}^{N-1} \\
&= \int_{S(\tilde{U}_t)\backslash G_1} f^{(p)}\big(d_{S^1}(\tilde{U}_t^+, \tilde{U}_t^-)\big) d\mathcal{H}^{N-1} + \int_{\Omega \cap S(\varphi_{2,t})} f^{(p)}(|\varphi_{2,t}^+ - \varphi_{2,t}^-|) d\mathcal{H}^1 \\
&> \int_{S(\tilde{U}_t)\backslash G_1} f^{(p)}\big(d_{S^1}(\tilde{U}_t^+, \tilde{U}_t^-)\big) d\mathcal{H}^{N-1} + \int_{\Omega \cap S(\varphi_{1,t})} f^{(p)}(|\varphi_{1,t}^+ - \varphi_{1,t}^-|)\, d\mathcal{H}^1 \\
&= F_0^{(\tilde{U}_t,p)}(\Phi_{1,t}).
\end{aligned}$$

As before, one can also obtain that for any $t \in (-\rho_p, \rho_p)$, $\Phi_{2,t}$ (resp. $\Phi_{1,t}$) is the unique minimizer of $F_0^{(\tilde{U}_t,p)}$ if $p \in (0, 4)$ (resp. $p > 4$). □

We remark that for $W^{1,1}$ functions, the minimizers of the energy $E(\cdot)$ defined in (4.3) coincide with those of $F_0^{(\cdot,p)}$, for any $p > 0$:

Proposition 4.10 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $0 < p < \infty$ and $u \in W^{1,1}(\Omega, S^1)$. Then a lifting φ of u is a minimizer of $F_0^{(u,p)}$ if and only if φ is an optimal lifting of u.*

Proof. Let $\varphi \in BV(\Omega, \mathbb{R})$ be a lifting of u. Then

$$\varphi^+ - \varphi^- \equiv 0 \pmod{2\pi} \quad \text{in } S(\varphi).$$

We denote $n(x) = \dfrac{|\varphi^+(x) - \varphi^-(x)|}{2\pi} \in \mathbb{N}$ for $\mathcal{H}^{N-1}$-a.e. $x \in S(\varphi)$. By (4.12), we have

$$\int_\Omega |D\varphi| = \int_\Omega |\nabla u|\, dx + 2\pi \int_{S(\varphi)} n(x)\, d\mathcal{H}^{N-1}(x).$$

According to (4.16), we deduce that

$$F_0^{(u,p)}(\varphi) = 2f^{(p)}(2\pi) \int_{S(\varphi)} n(x)\, d\mathcal{H}^{N-1}(x).$$

Therefore, φ is a minimizer of $F_0^{(u,p)}$ if and only if it minimizes the energy $E(u)$. □

Chapter 5

On an open problem about how to recognize constant functions

Abstract

We find necessary and sufficient conditions for the function ω in order that any measurable function $f : \Omega \to \mathbb{R}$ which satisfies

$$\int_\Omega \int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} < +\infty, \tag{5.1}$$

is constant a.e. in Ω. We also study what regularity on f should be assumed so that for any function ω which is continuous, $\omega(0) = 0$ and $\omega(t) > 0$ for every $t > 0$, if (5.1) holds, then f is a constant.

The first part of this chapter is published in Houston J. Math. **31** (2005), 285–304 (cf. [51]) and the second part is a work in progress in collaboration with R.-A. Todor.

5.1 Introduction

In this chapter, we investigate an open question posed by Brezis in [25]. The starting point is the following result (see [21], [25]):

Theorem 5.1 *(Bourgain, Brezis, Mironescu) Let Ω be a domain (i.e. a connected open set) in $\mathbb{R}^N$. If $f : \Omega \to \mathbb{R}$ is a measurable function which satisfies*

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|} \frac{dx\,dy}{|x-y|^N} < +\infty,$$

then f is a constant a.e. in Ω. More generally, if $p \geq 1$ and

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \frac{dx\,dy}{|x-y|^N} < +\infty,$$

then the same conclusion holds.

The motivation comes from the theory of Ginzburg-Landau equation where the problem of existence and uniqueness of lifting in Sobolev spaces is essential. More precisely, if $\Omega \subset \mathbb{R}^N$ is an open set and $u \in W^{s,p}(\Omega, S^1)$, is there a lifting $\varphi \in W^{s,p}(\Omega, \mathbb{R})$ of u (i.e. $u = e^{i\varphi}$ a.e. in Ω)? Is this lifting unique in $W^{s,p}$ (up to $2\pi\mathbb{Z}$ constants)? Here, $0 < s < \infty$ et $1 < p < \infty$. The answer to the question of existence of lifting was given by Bourgain, Brezis and Mironescu (see [20]). The uniqueness of lifting holds if $sp \geq 1$ and is a direct consequence of Theorem 5.1 (see Corollary 5.10). Another motivation comes from the degree theory for classes of discontinuous maps: if the degree $\deg h_t(\cdot)$ remains constant along a homotopy $h_t(\cdot)$ as t varies in some connected parameter space, then it is possible to define a degree. For the case of Sobolev maps, we refer to the work of Brezis and Coron [26] and Brezis, Li, Mironescu and Nirenberg [28].

We denote

$$\mathcal{W} = \{\omega \in C(\mathbb{R}_+, \mathbb{R}_+) \,|\, \omega(0) = 0,\, \omega(t) > 0,\, \forall t > 0\}\,.$$

The following problem now arises:

Problem 1 *Find a necessary and sufficient condition for $\omega \in \mathcal{W}$ so that any measurable function $f : \Omega \to \mathbb{R}$ which satisfies*

$$\int_\Omega \int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x - y|}\right) \frac{dx\,dy}{|x - y|^N} < +\infty, \tag{5.2}$$

is constant (a.e. in Ω).

Observe that the restriction $\omega \in \mathcal{W}$ is natural. Indeed, the continuity of ω is needed to make the left hand side of (5.2) well-defined. Also, $\omega(0) = 0$ (since for any constant function f, (5.2) should hold) and $\omega(t) > 0, \forall t > 0$ (if $\omega(t) = 0$ for some $t > 0$, take $N = 1$ and $f(x) = tx$). Henceforth it is assumed that $\omega \in \mathcal{W}$.

Three theorems are established concerning Problem 1. Theorem 5.2 gives a necessary condition and Theorems 5.3 and 5.4 provide sufficient conditions. The question whether the necessary condition in Theorem 5.2 is also sufficient remains open.

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $\omega \in \mathcal{W}$ be such that any measurable function $f : \Omega \to \mathbb{R}$ that satisfies (5.2) is constant a.e. in Ω. Then $\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt = +\infty$.*

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^N$ be a domain, $f : \Omega \to \mathbb{R}$ be a measurable function and $\omega \in \mathcal{W}$ such that $\liminf_{t\to+\infty} \frac{\omega(t)}{t} > 0$. If (5.2) holds, then f is constant a.e. in Ω.*

Theorem 5.4 *Let $\Omega \subset \mathbb{R}^N$ be a domain, $f : \Omega \to \mathbb{R}$ be a measurable function and $\omega \in \mathcal{W}$. Define $\phi : (0, +\infty) \mapsto (0, +\infty)$, $\phi(t) = t^{-1}\omega(t)$ for all $t > 0$. Assume that ω is a non-decreasing function such that*

$$\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt = +\infty \text{ and } \sup_{0<s\leq t} \frac{\phi(t)}{\phi(s)} < +\infty.$$

If (5.2) holds, then f is constant a.e. in Ω.

Open question 1 *Is the condition* $\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt = +\infty$ *sufficient for Problem 1 (of course, under the assumption* $\omega \in \mathcal{W}$*)?*

In the second part of the chapter, we investigate the following problem:

Problem 2 *What regularity on* f *should be assumed so that for any* $\omega \in \mathcal{W}$, (5.2) *imply* f *is a constant?*

The motivation is clear: if we don't want any restriction on $\omega \in \mathcal{W}$, we need to impose an additional condition on f in order that (5.2) yields f to be a constant. We establish the following results for Problem 2. Theorem 5.5 says that the condition $f \in W^{1,1}_{loc}(\Omega)$ guarantees that Problem 2 has a positive answer. The other two theorems deal with the question raised by Brezis in [25]: *Is the continuity (or even the* $\mathcal{C}^{0,\alpha}_{loc}$ *regularity) of* f *sufficient for Problem 2?* The answer is negative in general. In the end, we state another open question (related to the previous one).

Theorem 5.5 *Let* Ω *be a domain in* $\mathbb{R}^N$ *and* $f \in W^{1,1}_{loc}(\Omega)$. *For any* $\omega \in \mathcal{W}$, *if* (5.2) *holds, then* f *is constant a.e in* Ω.

Theorem 5.6 *Let* Ω *be the unit cube in* $\mathbb{R}^N$ *i.e.* $\Omega = (0,1)^N$. *For every* $0 < \alpha < 1$, *there is a nonconstant* α*-Hölder continuous function* $f : [0,1]^N \mapsto \mathbb{R}$ *of bounded variation which satisfies* (5.2), *for every bounded function* $\omega \in \mathcal{W}$.

Theorem 5.7 *Let* $\Omega = (0,1)^N$. *For every* $0 < \alpha < 1$, *there is a nonconstant* α*-Hölder continuous function* $f : [0,1]^N \mapsto \mathbb{R}$ *of bounded variation which satisfies*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^\theta}{|x-y|^\theta} \frac{dx\, dy}{|x-y|^N} < +\infty, \quad \forall \theta \in (0,1).$$

Open question 2 *Let* $\omega \in \mathcal{W}$ *be such that* $\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt = +\infty$. *Suppose* f *is continuous (or even* $C^{0,\alpha}_{loc}$ *for some* $0 < \alpha < 1$*) and satisfies* (5.2). *Is* f *constant?*

The outline of the chapter is the following: In Section 5.2, we prove the necessary condition for Problem 1 stated in Theorem 5.2. In Section 5.3, we show the sufficient conditions for Problem 1 announced in Theorems 5.3 and 5.4. In Section 5.3, we prove the $W^{1,1}$ case for Problem 2. In Section 5.5 we present some remarkable properties concerning a generalized Cantor set and Cantor function, results that we use in the proof of Theorems 5.6 and 5.7 in Section 5.6. In Sections 5.7-5.9, we present some further results about Problem 1 that will appear in [59]: first we prove a dimension reduction theorem, then we show that the necessary condition in Theorem 5.2 prevents the function f to be a non-trivial indicator function in Ω and also from being a Cantor function.

5.2 Necessary condition for Problem 1

In this section we prove Theorem 5.2 i.e., the condition

$$\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt = +\infty$$

is necessary for Problem 1. Firstly, we present a preliminary result. It states that the above condition is needed in order to prevent f from being a step function.

Lemma 5.8 *Let $\Omega = (-1,1) \times (0,1)^{N-1}$ and $\omega \in \mathcal{W}$. Let f be the characteristic function of the unit cube i.e. $f = \chi_{(0,1)^N}$. Then (5.2) holds if and only if $\int_1^\infty \frac{\omega(t)}{t^2}\, dt < +\infty$.*

Proof: We denote $x = (x_1, x_2, \ldots, x_N) = (x_1, x') \in \mathbb{R}^N$ and

$$I = \int_\Omega \int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|^N}.$$

After a change of variable $t = x_1 - y_1$ we get $I = 2(I_1 + I_2)$ where

$$I_1 = \int_{(0,1)^{N-1}} \int_{(0,1)^{N-1}} dx'\, dy' \int_0^1 \omega\left(\frac{1}{\sqrt{|x'-y'|^2 + t^2}}\right) \frac{t}{(|x'-y'|^2+t^2)^{\frac{N}{2}}}\, dt$$

$$I_2 = \int_{(0,1)^{N-1}} \int_{(0,1)^{N-1}} dx'\, dy' \int_1^2 \omega\left(\frac{1}{\sqrt{|x'-y'|^2 + t^2}}\right) \frac{2-t}{(|x'-y'|^2+t^2)^{\frac{N}{2}}}\, dt.$$

We remark that $|I_2| \leq ||\omega||_{L^\infty[0,1]}$ and

$$I_1 = 2^{N-1} \underbrace{\int_0^1 \ldots \int_0^1}_{\text{N times}} \omega\left(\frac{1}{|x|}\right) \frac{x_1 \prod_{i=2}^{N} (1 - x_i)}{|x|^N}\, dx.$$

If $N = 1$, then $I_1 = \int_0^1 \omega\left(\frac{1}{x}\right) dx = \int_1^\infty \frac{\omega(z)}{z^2}\, dz$. If $N \geq 2$, after the change of variable $z = \frac{1}{\sqrt{x_1^2 + |x'|^2}}$ for each x', we get $I_1 = 2^{N-1}(I_3 + I_4)$ where

$$I_3 = \int_{\frac{1}{\sqrt{N}}}^1 \omega(z) z^{N-3} \int_{(0,1)^{N-1}} \prod_{i=2}^{N} (1-x_i) \cdot \chi_{\left(\frac{1}{\sqrt{|x'|^2+1}}, \frac{1}{|x'|}\right)}(z)\, dx'\, dz$$

$$I_4 = \int_1^\infty \omega(z) z^{N-3} \int_{\substack{|x'| \leq \frac{1}{z} \\ x' \in [0,1]^{N-1}}} \prod_{i=2}^{N} (1-x_i)\, dx'\, dz.$$

Note that $|I_3| \leq ||\omega||_{L^\infty[0,1]}$. Therefore it is sufficient to show that $I_4 < +\infty$ if and only if $\int_1^\infty \frac{\omega(t)}{t^2} dt < +\infty$. For $0 < t < 1$, define

$$T_N(t) = \int_{\substack{x \in [0,1]^N \\ |x| \leq t}} \prod_{i=1}^{N} (1-x_i)\, dx.$$

Then

$$\int_{[0,\frac{t}{\sqrt{N}}]^N} \prod_{i=1}^{N}(1-x_i)\,dx \le T_N(t) \le \int_{[0,t]^N} \prod_{i=1}^{N}(1-x_i)\,dx;$$

so there is a constant $c_N = (\frac{1}{2\sqrt{N}})^N$ such that

$$c_N t^N \le T_N(t) \le t^N \text{ for all } t \in (0,1).$$

This yields $I_4 \approx \int_1^\infty \frac{\omega(z)}{z^2}\,dz$. □

Proof of Theorem 5.2: Assume the contrary i.e. $\int_1^{+\infty} \frac{\omega(t)}{t^2}\,dt < +\infty$. Since Ω is bounded, $\Omega \subset (-r,r)^N$ for some $r > 0$. For the simplicity, we suppose that $0 \in \Omega$. Take now the characteristic function $f = \chi_{(0,r)\times(-r,r)^{N-1}}$. By Lemma 5.8,

$$\int_{(-r,r)^N} \int_{(-r,r)^N} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} < +\infty.$$

Therefore (5.2) holds which contradicts the hypothesis that f is not constant on Ω. □

5.3 Sufficient conditions for Problem 1

In this section, the proofs of Theorem 5.3 and Theorem 5.4 are presented. We call *mollifiers* in $\mathbb{R}^N$, any family $(\rho_\varepsilon)_{\varepsilon>0}$ of functions in $L^1_{loc}(0,\infty)$ satisfying the following properties

$$\begin{cases} \rho_\varepsilon \ge 0 \text{ a.e. in } (0,+\infty), \\ \displaystyle\int_0^\infty \rho_\varepsilon(t)\, t^{N-1}\,dt = 1 \quad \forall \varepsilon > 0, \\ \displaystyle\lim_{\varepsilon\to 0} \int_\delta^\infty \rho_\varepsilon(t)\, t^{N-1}\,dt = 0 \quad \forall \delta > 0. \end{cases}$$

Recall the following result of Brezis (see e.g. [71] Proposition 1 and Lemma 4):

Theorem 5.9 *(Brezis) Let $\Omega \subset \mathbb{R}^N$ be a domain, (ρ_ε) be mollifiers in $\mathbb{R}^N$, $f \in L^1_{loc}(\Omega)$ and $\omega \in \mathcal{W}$ be a convex function. If*

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \rho_\varepsilon(|x-y|)\,dx\,dy = 0$$

then f is constant a.e. in Ω.

First proof of Theorem 5.3: Since $\omega \in \mathcal{W}$ we can construct a convex function $\tilde{\omega} \in \mathcal{W}$ such that $\tilde{\omega}(t) \le \omega(t), \forall t \in [0,1]$ and $\tilde{\omega}(t) = at + b, \forall t \ge 1$ for some $a, b > 0$. The hypothesis $\liminf_{t\to\infty} \frac{\omega(t)}{t} > 0$ implies the existence of a constant $c > 0$ such that $\omega(t) \ge c\,\tilde{\omega}(t), \forall t \ge 0$. Therefore

$$\int_\Omega \int_\Omega \tilde{\omega}\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} < +\infty.$$

Consider the mollifiers in $\mathbb{R}^N$

$$\rho_\varepsilon(t) = \begin{cases} \frac{\varepsilon}{t^{N-\varepsilon}} & \text{if } 0 < t < 1 \\ 0 & \text{if } t \geq 1 \end{cases}. \tag{5.3}$$

By the dominated convergence theorem,

$$\lim_{\varepsilon \to 0} \int_\Omega \int_\Omega \tilde{\omega}\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \rho_\varepsilon(|x-y|)\, dx\, dy = 0.$$

If $f \in L^1_{loc}(\Omega)$, we conclude by Theorem 5.9. In the general case of a measurable function f, we consider

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq n \\ n & \text{if } f(x) \geq n \\ -n & \text{if } f(x) \leq -n \end{cases}.$$

So $f_n \in L^1_{loc}(\Omega)$, $f_n \to f$ a.e. in Ω and

$$|f_n(x) - f_n(y)| \leq |f(x) - f(y)| \quad \forall x, y \in \Omega.$$

Since $\tilde{\omega}$ is increasing, we get for all $n \geq 1$,

$$\lim_{\varepsilon \to 0} \int_\Omega \int_\Omega \tilde{\omega}\left(\frac{|f_n(x) - f_n(y)|}{|x-y|}\right) \rho_\varepsilon(|x-y|)\, dx\, dy = 0.$$

This yields $f_n \equiv c_n$ et $c_n \to f$ a.e. in Ω. Thus f is constant a.e. □

We now present a second method[2] of proving Theorem 5.3 without making use of Theorem 5.9:

Second proof of Theorem 5.3: By the same argument as above, we may assume that ω is convex and $f \in L^\infty(\Omega)$. Let $x_0 \in \Omega$ and $r > 0$ be such that $B(x_0, 3r) \subset \Omega$. We denote $B_1 = B(x_0, r)$ and $B_2 = B(x_0, 2r)$. Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of mollifiers with $\operatorname{supp} \rho_n \subset B(0, \frac{1}{n})$. Set

$$f_n = \rho_n * f \quad \text{on} \quad B_2.$$

Then $f_n \in C^1(B_2)$ and $f_n \to f$ a.e. on B_2. Using that w is an increasing convex function, it follows by Jensen's inequality

$$\begin{aligned}
\int_{B_2} \int_{B_2} \omega\left(\frac{|f_n(x) - f_n(y)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|^N} &\leq \int_{B_2} \int_{B_2} \omega\left(\int_{B(0,\frac{1}{n})} \rho_n(z) \frac{|f(x-z) - f(y-z)|}{|x-y|}\, dz\right) \frac{dx\, dy}{|x-y|^N} \\
&\leq \int_{B_2} \int_{B_2} \int_{B(0,\frac{1}{n})} \rho_n(z)\, \omega\left(\frac{|f(x-z) - f(y-z)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|^N}\, dz \\
&\leq \int_{B(0,\frac{1}{n})} \rho_n(z) \int_{B_2 - z} \int_{B_2 - z} \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|^N}\, dz \\
&\leq \int_\Omega \int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|^N}.
\end{aligned}$$

[2]This part does not appear in the published version of the paper [51]

Write

$$\int_{B_2}\int_{B_2} \omega\Big(\frac{|f_n(x)-f_n(y)|}{|x-y|}\Big)\frac{dx\,dy}{|x-y|^N} \geq \int_{B_1} dx \int_{S^{N-1}} d\sigma \int_0^r \omega\Big(\frac{|f_n(x+t\sigma)-f_n(x)|}{t}\Big)\frac{dt}{t}$$

and deduce that for a.e. $x \in B_1$ and for a.e. $\sigma \in S^{N-1}$,

$$\int_0^r \omega\left(\frac{|f_n(x+t\sigma)-f_n(x)|}{t}\right)\frac{dt}{t} < +\infty.$$

Since $\int_0^r \frac{dt}{t} = \infty$, we get

$$\liminf_{t\to 0} \omega\left(\frac{|f_n(x+t\sigma)-f_n(x)|}{t}\right) = 0, \text{ i.e. } \omega(|\nabla f_n(x)\cdot\sigma|) = 0 \text{ for a.e. } x \in B_1 \text{ and } \sigma \in S^{N-1}$$

We conclude that f_n is a constant on B_1. Hence, f is constant. □

As consequence, we obtain the uniqueness of lifting in $W^{s,p}$ for $sp \geq 1$:

Corollary 5.10 *Let $\Omega \subset \mathbb{R}^N$ be a connected open set, $s > 0$ and $p > 1$ such that $sp \geq 1$. Consider $u \in W^{s,p}(\Omega, S^1)$. If $\varphi_1, \varphi_2 \in W^{s,p}(\Omega, \mathbb{R})$ are two liftings of u, then $\varphi_1 - \varphi_2$ is a constant function.*

Proof. Let $f := \varphi_1 - \varphi_2 \in W^{s,p}(\Omega, \mathbb{Z})$. Let B be an arbitrary ball in Ω. Since $W^{s,p}(B)$ is embedded in $W^{\frac{1}{p},p}(B)$, it is enough to prove the statement for $s = \frac{1}{p}$. Recall the Gagliardo seminorm in $W^{1/p,p}$ (see [2])

$$|f|^p_{W^{\frac{1}{p},p}(B)} = \int_B\int_B \frac{|f(x)-f(y)|^p}{|x-y|^{N+1}}\,dx\,dy.$$

Since f takes values in $\mathbb{Z}$, we have that $|f(x)-f(y)| \leq |f(x)-f(y)|^p$. Then

$$\int_B\int_B \frac{|f(x)-f(y)|}{|x-y|}\frac{dx\,dy}{|x-y|^N} < \infty.$$

The conclusion follows from Theorem 5.3 for $\omega(t) = t$. □

Proof of Theorem 5.4: Since ω is non-decreasing, using the same argument as in the first proof of Theorem 5.3, it is sufficient to show that the conclusion holds for $f \in L^\infty_{loc}(\Omega)$. Firstly, assume that the function ϕ is non-increasing on $(0, +\infty)$. Take an arbitrary ball $\bar{B} \subset \Omega$. For simplicity, we suppose that $|f| \leq \frac{1}{2}$ a.e. in B. By these assumptions we get

$$\int_B\int_B \frac{|f(x)-f(y)|}{|x-y|}\phi\left(\frac{1}{|x-y|}\right)\frac{dx\,dy}{|x-y|^N} < +\infty.$$

For each $\varepsilon > 0$, set

$$0 < c_\varepsilon := \int_0^1 \phi\left(\frac{1}{t}\right)\frac{\varepsilon}{t^{1-\varepsilon}}dt \leq \phi(1).$$

Consider the functions

$$\rho_\varepsilon(t) = \begin{cases} \frac{1}{c_\varepsilon}\phi\left(\frac{1}{t}\right)\frac{\varepsilon}{t^{N-\varepsilon}} & \text{if } 0 < t < 1 \\ 0 & \text{if } t \geq 1 \end{cases} \qquad \forall \varepsilon > 0.$$

Using the hypothesis that $\int_0^1 \phi\left(\frac{1}{t}\right)\frac{dt}{t} = +\infty$, we see that (ρ_ε) are mollifiers in $\mathbb{R}^N$. We also notice that $\lim_{\varepsilon\to 0}\frac{\varepsilon}{c_\varepsilon} = 0$. By dominated convergence theorem we obtain

$$\lim_{\varepsilon\to 0}\int_B\int_B \frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(|x-y|)dxdy = 0.$$

Hence Theorem 5.9 implies f is constant a.e. in B and since Ω is connected, we conclude that f is constant a.e. in Ω. We now consider the general case when $c := \sup_{0<s\le t}\frac{\phi(t)}{\phi(s)} < +\infty$. Set $\phi(0) = \frac{\phi(1)}{c}$ and define

$$\tilde{\phi} : [0,+\infty) \mapsto (0,+\infty),\ \tilde{\phi}(t) = \min_{s\in[0,t]}\phi(s) \quad \forall t \ge 0.$$

So $\tilde{\phi}$ is continuous and non-increasing on $[0,+\infty)$ and $\tilde{\phi}(t) \le \phi(t), \forall t > 0$. From here,

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\tilde{\phi}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{dx\,dy}{|x-y|^N} < +\infty.$$

We also have that $\phi(t) \le c^2\,\tilde{\phi}(t), \forall t \ge 1$ and thus $\int_0^1 \tilde{\phi}\left(\frac{1}{t}\right)\frac{dt}{t} = +\infty$. By the previous case, f is constant a.e. in Ω. $\square$

5.4 The case of $W^{1,1}_{loc}$ functions

In this section, we show that for $f \in W^{1,1}_{loc}(\Omega)$ (in particular for Lipschitz functions), the answer to Problem 2 is positive. We will present two different approaches for solving this case.

Proof of Theorem 5.5: Let $x_0 \in \Omega$. Take $r > 0$ such that $\tilde{B} = B(x_0, 2r) \subset \Omega$ and denote $B = B(x_0, r)$. Then $f \in W^{1,1}(B)$ i.e. $f \in L^1(B)$ and $\nabla f \in \left(L^1(B)\right)^N$. So it makes sense to speak of $f(x)$ and $\nabla f(x)$ for a.e. $x \in B$. Let $\sigma \in S^{N-1}$. By Fubini's theorem we find that for a.e. $x \in B$ there is a small $t_x > 0$ such that $I_x = \{x + t\sigma \,|\, t \in (-t_x, t_x)\} \subset B$ and $f \in W^{1,1}(I_x)$ i.e., f is absolutely continuous on I_x. Therefore for every $\sigma \in S^{N-1}$,

$$\lim_{t\to 0}\frac{f(x+t\sigma)-f(x)}{t} = \nabla f(x)\cdot\sigma \quad \text{for a.e.} \quad x \in B. \tag{5.4}$$

Write

$$\int_{\tilde{B}}\int_{\tilde{B}} \omega\Big(\frac{|f(x)-f(y)|}{|x-y|}\Big)\frac{dx\,dy}{|x-y|^N} \ge \int_B dx \int_{S^{N-1}} d\sigma \int_0^r \omega\Big(\frac{|f(x+t\sigma)-f(x)|}{t}\Big)\frac{dt}{t}$$

and by (5.2) deduce that for a.e. $x \in B$ and for a.e. $\sigma \in S^{N-1}$,

$$\int_0^r \omega\left(\frac{|f(x+t\sigma)-f(x)|}{t}\right)\frac{dt}{t} < +\infty.$$

Using $\int_0^r \frac{dt}{t} = \infty$, we get

$$\liminf_{t\to 0}\omega\left(\frac{|f(x+t\sigma)-f(x)|}{t}\right) = 0.$$

ω being continuous, by (5.4) one can find N linear independent directions $(\sigma_i)_{1\le i\le N}$ such that $\omega(|\nabla f(x)\cdot\sigma_i|)=0$ for a.e. $x\in B$ and for every $i\in\{1,...,N\}$. This implies $\nabla f=0$ a.e. in B. By the Poincaré-Wirtinger inequality, we have that

$$\left\| f-\frac{1}{|B|}\int_B f\right\|_{L^1(B)}\le C\,\|\nabla f\|_{L^1(B)}=0$$

i.e. f is constant a.e. in B. Since x_0 was arbitrarily chosen and Ω is connected, we conclude that f is constant a.e. in Ω. □

Remark: One could prove this result using another method, as follows. Define $\tilde\omega:[0,+\infty)\mapsto[0,1]$, $\tilde\omega(t)=\min(\omega(t),1)$ for every $t\ge 0$. Take an arbitrary ball $\bar B\subset\Omega$. Then

$$\int_B\int_B \tilde\omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{dx\,dy}{|x-y|^N}<+\infty.$$

Consider the mollifiers (5.3) in $\mathbb{R}^N$. By the dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to 0}\int_B\int_B \tilde\omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(|x-y|)\,dx\,dy=0.$$

On the other hand, one can show that for a bounded continuous function $\tilde\omega$ on $[0,+\infty)$ and $f\in W^{1,1}(B)$,

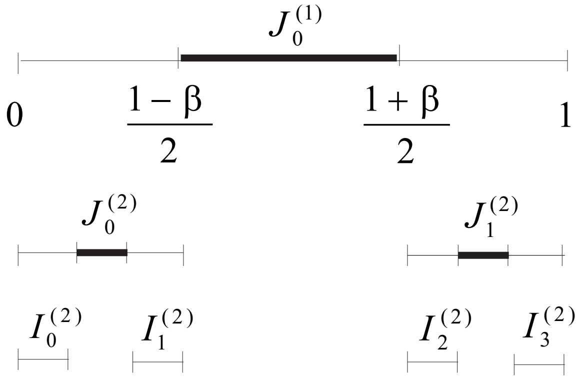
$$\lim_{\varepsilon\to 0}\int_B\int_B \tilde\omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(|x-y|)\,dx\,dy=\int_B\int_{S^{N-1}}\tilde\omega(|\nabla f(x)\cdot\sigma|)\,dx\,d\sigma$$

(see e.g. [71] Lemma 5). As before, this yields $\nabla f=0$ a.e. in B for every ball $\bar B\subset\Omega$; since $f\in W^{1,1}_{loc}(\Omega)$ and Ω is connected, f is constant a.e. in Ω. □

5.5 Some generalized Cantor sets and Cantor functions

Let $0<\beta<1$. We recall the definition of some general Cantor sets, called here **β-Cantor sets**, all homeomorphic to the standard one and which can be obtained by deleting a sequence of pairwise disjoint open intervals from the interior of the segment $I_0^{(0)}=[0,1]$, as follows (see [50]). Firstly, remove the centered open interval from $I_0^{(0)}$ which has length $\beta=\beta\cdot\left|I_0^{(0)}\right|$ i.e., delete the interval $J_0^{(1)}=\left(\frac{1-\beta}{2},\frac{1+\beta}{2}\right)$ and leave two segments $I_0^{(1)}=\left[0,\frac{1-\beta}{2}\right]$ and $I_1^{(1)}=\left[\frac{1+\beta}{2},1\right]$. The second step consists in deleting the open subinterval of length $\beta\cdot\left|I_0^{(1)}\right|=\beta\cdot\left|I_1^{(1)}\right|=\beta\frac{1-\beta}{2}$ from the center of each of the segments $I_0^{(1)}$ and $I_1^{(1)}$, namely $J_0^{(2)}=\left(\frac{(1-\beta)^2}{4},\frac{1-\beta^2}{4}\right)$ and $J_1^{(2)}=1-J_0^{(2)}$; thus, there remains 2^2 segments, denoted $I_0^{(2)},I_1^{(2)},I_2^{(2)}$ and $I_3^{(2)}$. We iterate this procedure; at the $(n+1)$ step, remove the centered open subinterval $J_k^{(n+1)}$ of length $\beta\cdot\left|I_k^{(n)}\right|$ from each segment $I_k^{(n)}=[a_k^{(n)},b_k^{(n)}]$ and leave the two segments

$$I_{2k}^{(n+1)}=[a_{2k}^{(n+1)},b_{2k}^{(n+1)}]\text{ and }I_{2k+1}^{(n+1)}=[a_{2k+1}^{(n+1)},b_{2k+1}^{(n+1)}]\text{ for }k=0,1,\ldots,2^n-1.$$



The limit set is the β**-Cantor set**, denoted by C_β. It is a compact set, containing an uncountable infinity of points; it has Lebesgue measure zero and it is nowhere dense (i.e. it has no interior). We will give the specific form of C_β. In order to do that, let us consider σ_n and δ_n the length of the removed interval $J_k^{(n)}$ and respectively, of the remaining segment $I_k^{(n)}$ at the n step. A simple computation yields

$$\delta_n = \left(\frac{1-\beta}{2}\right)^n, \ \sigma_n = \beta\delta_{n-1} \quad \forall n \geq 1 \text{ (here } \delta_0 = 1).$$

Set $\varepsilon_n = \delta_n + \sigma_n$. Then one can deduce (see [50]) that

$$C_\beta = \left\{ \sum_{k=1}^{\infty} \alpha_k \varepsilon_k \,|\, \alpha_k \in \{0,1\}, k = 0, 1, \dots \right\}.$$

In fact, the binary decomposition

$$j = \alpha_n + 2\alpha_{n-1} + \cdots + 2^{n-1}\alpha_1 = (\alpha_1 \dots \alpha_n)_2$$

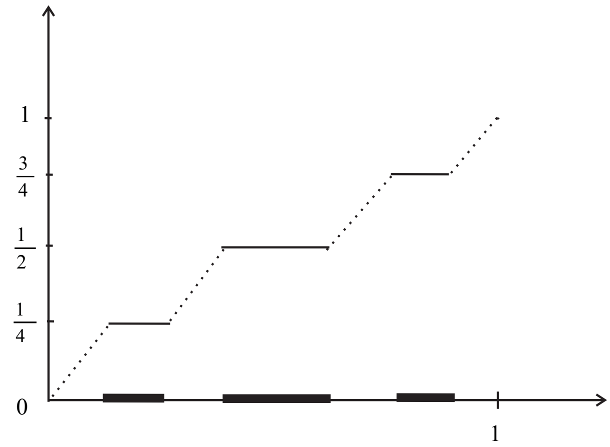
gives $a_j^{(n)} = \sum\limits_{k=1}^{n} \alpha_k \varepsilon_k$ and $b_j^{(n)} = a_j^{(n)} + \sum\limits_{k \geq n+1} \varepsilon_k$.

We define now the β**-Cantor function**, denoted here by f_β (see [36]). Set $f_\beta(0) = 0$ and $f_\beta(1) = 1$. So f_β is specified at the endpoints of $I_0^{(0)}$. Define $f_\beta(x) = \frac{1}{2}$ if $x \in clJ_0^{(1)}$. Thus $f_\beta(x)$ is the average of the values of f_β at the endpoints of $I_0^{(0)}$ when x belongs to the removed interval $J_0^{(1)}$ and f_β is specified at the endpoints of $I_0^{(1)}$ and $I_1^{(1)}$. At the $n+1$ step, define $f_\beta \equiv \frac{f_\beta(b_k^{(n)}) - f_\beta(a_k^{(n)})}{2}$ on the closure of each $J_k^{(n+1)}$, the removed interval from $I_k^{(n)} = [a_k^{(n)}, b_k^{(n)}]$. By that, f_β is defined in every endpoint of $I_{2k}^{(n+1)}$ and $I_{2k+1}^{(n+1)}$ for $k = 0, 1, \dots, 2^n - 1$; then we can iterate the process.

Suppose f_β is not yet defined at x. At each n step, x is in the interior of exactly one of the 2^n retained segments, say $[a_n, b_n]$ of length δ_n. Moreover, $b_n = a_n + \delta_n$, $f_\beta(b_n) = f_\beta(a_n) + 2^{-n}$, $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ and $f_\beta(a_n) \leq f_\beta(a_{n+1}) < f_\beta(b_{n+1}) \leq f_\beta(b_n)$; then $f_\beta(x)$ is defined by

$$\lim_{n\to\infty} f_\beta(a_n) = f_\beta(x) = \lim_{n\to\infty} f_\beta(b_n).$$

Furthermore, f_β is a continuous, nondecreasing map of $[0,1]$ onto $[0,1]$ (so f_β is a function of bounded variation on $[0,1]$) and $f'_\beta(x) = 0$ for a.e. $x \in [0,1]$. One can easily check that on



the β-Cantor set we have

$$f_\beta\left(\sum_{k=1}^{\infty}\alpha_k\varepsilon_k\right)=\sum_{k=1}^{\infty}\alpha_k 2^{-k}.$$

We now show that each β-Cantor function is Hölder continuous with Hölder exponent equal to the Hausdorff dimension of C_β i.e. $H_\beta=\frac{1}{1-\log_2(1-\beta)}$ (see also [46]).

Theorem 5.11 *The β-Cantor function is α-Hölder if and only if $0<\alpha\le H_\beta$.*

Proof: Since C_β is nowhere dense and f_β is continuous, it is sufficient to prove that for every $\alpha\le H_\beta$, there exists $l_\alpha>0$ such that

$$|f_\beta(x)-f_\beta(y)|\le l_\alpha|x-y|^\alpha \quad \forall x,y\in[0,1]\backslash C_\beta. \tag{5.5}$$

Take $x<y, x,y\in[0,1]\backslash C_\beta$ i.e. x and y are in the interior of two removed intervals in the construction of C_β, say (b,a) and $(\tilde{b},\tilde{a})$. Write $a=\sum\limits_{k=1}^{n}\alpha_k\varepsilon_k, \alpha_k\in\{0,1\}$, $\alpha_n=1$ and $\tilde{a}=\sum\limits_{j=1}^{m}\gamma_j\varepsilon_j, \gamma_j\in\{0,1\}, \gamma_m=1$. Then $b=a-\sigma_n$, $\tilde{b}=\tilde{a}-\sigma_m$. If the two removed intervals coincide, then $f_\beta(x)=f_\beta(y)$ and (5.5) is obvious. Otherwise, $a<\tilde{b}$. Take $s\ge 1$ such that $\alpha_j=\gamma_j$ for $j=1,\ldots,s-1$ and $\alpha_s\ne\gamma_s$ (we may consider $\alpha_j=0,\forall j>n$). Thus $\gamma_s=1$, $\alpha_s=0$ and $s\le m$.

If $s < n$, we get

$$\begin{aligned} f_\beta(y) - f_\beta(x) &= \sum_{j=1}^{m} \gamma_j 2^{-j} - \sum_{k=1}^{n} \alpha_k 2^{-k} \\ &= 2^{-n} + \sum_{j=s+1}^{m} \gamma_j 2^{-j} + \sum_{k=s+1}^{n} (1-\alpha_k) 2^{-k}, \\ y - x \geq \tilde{b} - a &= \sum_{j=1}^{m} \gamma_j \varepsilon_j - \sigma_m - \sum_{k=1}^{n} \alpha_k \varepsilon_k \\ &\geq \delta_n + \sum_{j=s+1}^{m} \gamma_j \delta_j + \sum_{k=s+1}^{n} (1-\alpha_k) \delta_k \end{aligned}$$

(here we used $\varepsilon_s = \sigma_s + \delta_s = \sigma_s + \varepsilon_{s+1} + \cdots + \varepsilon_n + \delta_n$). Otherwise, $s > n$ (since $s \neq n$) and we have

$$\begin{aligned} f_\beta(y) - f_\beta(x) &= \sum_{j=s}^{m} \gamma_j 2^{-j}, \\ y - x \geq \tilde{b} - a &= \sum_{j=s}^{m} \gamma_j \varepsilon_j - \sigma_m \geq \sum_{j=s}^{m} \gamma_j \delta_j. \end{aligned}$$

So in both cases, we can write

$$f_\beta(y) - f_\beta(x) = \sum_{j=1}^{M} h_j 2^{-j} \text{ and } y - x \geq \sum_{j=1}^{M} h_j \delta_j$$

where $M \geq 1, h_j \in \{0,1,2\}, j = 1, \ldots, M$. We distinguish three cases:
Case 1: $0 < \alpha < H_\beta$. Set $\varepsilon = H_\beta - \alpha > 0$. By Hölder's inequality, we get

$$\sum_{j=1}^{M} h_j 2^{-j} = \sum_{j=1}^{M} h_j^\alpha \delta_j^\alpha h_j^{1-\alpha} \delta_j^\varepsilon \leq \left(\sum_{j=1}^{M} h_j \delta_j \right)^\alpha \left(\sum_{j=1}^{M} h_j \delta_j^{\frac{\varepsilon}{1-\alpha}} \right)^{1-\alpha}.$$

Since $h_j \in \{0,1,2\}$, we deduce

$$\sum_{j=1}^{M} h_j \delta_j^{\frac{\varepsilon}{1-\alpha}} \leq 2 \sum_{j \geq 1} \left(\delta_1^{\frac{\varepsilon}{1-\alpha}} \right)^j =: l_\alpha{}^{\frac{1}{1-\alpha}} < +\infty.$$

So $|f(x) - f(y)| \leq l_\alpha |x-y|^\alpha$.
Case 2: $\alpha = H_\beta$ i.e. $\delta_j^\alpha = 2^{-j}, \forall j \geq 0$. Take the smallest $j_0 \geq 1$ such that $h_{j_0} \neq 0$. Then

$$\frac{\sum\limits_{j=j_0}^{M} h_j \delta_j^\alpha}{\left(\sum\limits_{j=j_0}^{M} h_j \delta_j \right)^\alpha} \leq \frac{2 \sum\limits_{j \geq j_0} \delta_j^\alpha}{\delta_{j_0}^\alpha} = 2 \sum_{j \geq 0} 2^{-j} = 4.$$

Thus, (5.5) is satisfied.
Case 3: $\alpha > H_\beta$. Take $x = \varepsilon_n$ and $y = \delta_{n-1} = \sum_{k\geq n} \varepsilon_k$. Then

$$\frac{f(y)-f(x)}{|y-x|^\alpha} = \frac{2^{-n}}{|\delta_{n-1}-\varepsilon_n|^\alpha} = \frac{2^{-n}}{\delta_n^\alpha} \to \infty \text{ if } n \to \infty.$$

So, in this case, f_β is not an α-Hölder continuous function. □

5.6 Some counter-examples

In this section, we present some counter-examples for Problem 2 in the case of regularity $C^{0,\alpha}$. We will assume that Ω is the unit cube in $\mathbb{R}^N$ i.e. $\Omega = (0,1)^N$.

Theorem 5.12 *For every $\alpha \in (0,1)$, there is a nonconstant α-Hölder function $f : [0,1]^N \mapsto \mathbb{R}$ of bounded variation which satisfies (5.2), for all $\omega \in \mathcal{W}$ with the property that $\omega(t) \leq \frac{1}{t}, \forall t > 0$.*

Proof: Let $\alpha \in (0,1)$. Consider the unique $\beta \in (0,1)$ such that $\alpha = H_\beta$.
Case 1: $N = 1$. Let f be the β-Cantor function. Take an arbitrary $\omega \in \mathcal{W}$ such that $\omega(t) \leq \frac{1}{t}, \forall t > 0$. Denote by $\mathcal{J}$ the (countable) set of all removed intervals in the construction of the β-Cantor set i.e.

$$\mathcal{J} = \left\{ J_k^{(n+1)} : n \geq 0, k = 0, 1, \dots, 2^n - 1 \right\}.$$

We have

$$\begin{aligned} I &= \int_0^1 \int_0^1 \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|} \\ &= \sum_{J\in\mathcal{J}} \sum_{\tilde{J}\in\mathcal{J}} \int_J \int_{\tilde{J}} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|} \\ &= 2 \sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \int_J \int_{\tilde{J}} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|} \end{aligned}$$

(we denote $J = (b,a) < \tilde{J} = (\tilde{b},\tilde{a})$ if $a < \tilde{b}$). We want to prove that $I < +\infty$. Take two removed intervals $J = (b,a)$ and $\tilde{J} = (\tilde{b},\tilde{a})$ such that $J < \tilde{J}$. Write $a = \sum_{k=1}^n \alpha_k \varepsilon_k$, $\alpha_k \in \{0,1\}$, $\alpha_n = 1$ and $\tilde{a} = \sum_{j=1}^m \gamma_j \varepsilon_j$, $\gamma_j \in \{0,1\}$, $\gamma_m = 1$; here $b = a - \sigma_n$, $\tilde{b} = \tilde{a} - \sigma_m$. Take $r = f|_{\tilde{J}} - f|_J = \sum_{j=1}^m \gamma_j 2^{-j} - \sum_{k=1}^n \alpha_k 2^{-k} > 0$. **We use these notations in the rest of the chapter.** Since $\omega(t) \leq \frac{1}{t}, \forall t > 0$ we get

$$\int_J \int_{\tilde{J}} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|} \leq \int_J \int_{\tilde{J}} \frac{dx\,dy}{r} = \frac{|J|\cdot|\tilde{J}|}{r} = \frac{\sigma_n \sigma_m}{r}.$$

The aim is to estimate

$$S = \sum_{\substack{J<\tilde{J} \\ J,\tilde{J}\in\mathcal{J}}} \frac{|J|\cdot|\tilde{J}|}{f|_{\tilde{J}} - f|_J}.$$

Firstly, consider the interval $J=(b,a)$ fix. Let $\tilde{J}=(\tilde{b},\tilde{a})$ be a variable removed interval (in the construction of C_β) such that $\tilde{J}>J$ (i.e. $a<\tilde{a}$). Each time, we consider the first s step (in the construction of C_β) when J and $\tilde{J}$ do not belong anymore to the same remaining interval; that means the biggest $1\le s\le n$ such that $\alpha_j=\gamma_j$ for $j=1,\dots,s-1$ (if $\alpha_1\neq\gamma_1$ then $s=1$). Notice that $s\le m$, $\gamma_s=1$ and $\alpha_s=\gamma_s \iff s=n$.

If $s<m$ i.e. $dist(J,\tilde{J})\ge\delta_m$ then

$$r = f|_{\tilde{J}} - f|_J = \sum_{j=1}^{m}\gamma_j 2^{-j} - \sum_{k=1}^{n}\alpha_k 2^{-k} \ge \sum_{j=s+1}^{m}\gamma_j 2^{-j}.$$

If we sum up over these $\tilde{J}$, we get:

$$\begin{aligned}
\sum_{\substack{\tilde{J}\in\mathcal{J},J<\tilde{J} \\ dist(J,\tilde{J})\ge\delta_m}} \frac{|\tilde{J}|}{f|_{\tilde{J}} - f|_J} &= \sum_{s=1}^{n}\sum_{m\ge s+1}\sum_{\substack{\gamma_j\in\{0,1\},\gamma_s=\gamma_m=1 \\ s+1\le j\le m-1}} \frac{\sigma_m}{r} \\
&\le \sum_{s=1}^{n}\sum_{m\ge s+1}\sigma_m \sum_{\substack{\gamma_j\in\{0,1\},\gamma_m=1 \\ s+1\le j\le m-1}} \frac{1}{\sum_{j=s+1}^{m}\gamma_j 2^{-j}} \\
&\le \sum_{s=1}^{n}\sum_{m\ge s+1}\sigma_m 2^m \sum_{j=1}^{2^{m-s}-1}\frac{1}{j} \\
&\le \sum_{s=1}^{n}\sum_{m\ge s+1}\sigma_m 2^m (m-s) \\
&\le nL
\end{aligned}$$

where $L=\sum_{m\ge1}\sigma_m 2^m m = \frac{\beta}{\delta_1}\sum_{m\ge1}(1-\beta)^m m < +\infty$.

Otherwise, $s=m$ i.e. $dist(J,\tilde{J})<\delta_m$. Thus $s<n$ and

$$r = f|_{\tilde{J}} - f|_J = 2^{-s} - \sum_{k=s+1}^{n}\alpha_k 2^{-k} = \sum_{k=s+1}^{n-1}(1-\alpha_k)2^{-k} + 2^{-n}.$$

We get

$$\sum_{\substack{\tilde{J}\in\mathcal{J},J<\tilde{J} \\ dist(J,\tilde{J})<\delta_m}} \frac{|\tilde{J}|}{f|_{\tilde{J}} - f|_J} = \sum_{s=1}^{n-1}\frac{\sigma_s}{\sum_{k=s+1}^{n-1}(1-\alpha_k)2^{-k} + 2^{-n}}.$$

Finally, if we let J be variable in $\mathcal{J}$, we deduce

$$\begin{aligned}
S &\le \sum_{n\ge 1} \sum_{\substack{\alpha_k\in\{0,1\}\\ 1\le k\le n-1}} \sigma_n\left(nL + \sum_{s=1}^{n-1} \frac{\sigma_s}{\sum\limits_{k=s+1}^{n-1}(1-\alpha_k)2^{-k}+2^{-n}}\right)\\
&= \sum_{n\ge 1} n\sigma_n 2^{n-1}L + \sum_{n\ge 1}\sigma_n 2^n \sum_{s=1}^{n-1}\sigma_s \sum_{\substack{\tilde\alpha_k\in\{0,1\}\\ 1\le k\le n-1}} \frac{1}{1+\sum\limits_{k=1}^{n-s-1}\tilde\alpha_k 2^k}\\
&\le L^2 + \sum_{n\ge 1}\sigma_n\cdot 2^n \sum_{s=1}^{n-1}\sigma_s 2^s(n-s)\\
&\le 2L^2.
\end{aligned}$$

Case 2: $N\ge 2$. We denote $x=(x_1,x')=(x_1,x_2,\ldots,x_N)\in[0,1]^N$. Take $f(x)=f_\beta(x_1), \forall x\in[0,1]^N$. So $f\in C^{0,\alpha}\cap BV(\Omega)$. Choose any $\omega\in\mathcal{W}$ with the property that $\omega(t)\le\frac{1}{t}$ for all $t>0$. Firstly, remark that

$$\begin{aligned}
I &= \int_{(0,1)^N}\int_{(0,1)^N} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{dx\,dy}{|x-y|^N}\\
&= 2^{N-1}\int_0^1\int_0^1\int_{(0,1)^{N-1}} \omega\left(\frac{|f_\beta(x_1)-f_\beta(y_1)|}{\sqrt{|x'|^2+(x_1-y_1)^2}}\right)\frac{\prod\limits_{i=2}^{N}(1-x_i)\,dx_1\,dy_1\,dx'}{(|x'|^2+(x_1-y_1)^2)^{\frac{N}{2}}}\\
&\le 2^N \sum_{\substack{J,\tilde J\in\mathcal{J}\\ J<\tilde J}} \int_J\int_{\tilde J}\int_{(0,1)^{N-1}} \omega\left(\frac{|f_\beta(x_1)-f_\beta(y_1)|}{\sqrt{|x'|^2+(x_1-y_1)^2}}\right)\frac{dx_1\,dy_1\,dx'}{(|x'|^2+(x_1-y_1)^2)^{\frac{N}{2}}}\\
&\le 2^N|S^{N-2}|\sum_{\substack{J,\tilde J\in\mathcal{J}\\ J<\tilde J}} \frac{1}{f_\beta|_{\tilde J}-f_\beta|_J}\int_J\int_{\tilde J} dx_1\,dy_1 \int_0^{N-1}\frac{t^{N-2}}{(t^2+(x_1-y_1)^2)^{\frac{N-1}{2}}}\,dt.
\end{aligned}$$

On the other hand, we have

$$\int_0^{N-1}\frac{t^{N-2}\,dt}{(t^2+(x_1-y_1)^2)^{\frac{N-1}{2}}} \le 2\int_0^{N-1}\frac{dt}{y_1-x_1+t} \le 2\left(\ln N + \ln\frac{1}{y_1-x_1}\right)$$

for every $0\le x_1<y_1\le 1$. Therefore there is a constant $c=c(N)>0$ such that

$$I\le c(N)\left(\sum_{\substack{J,\tilde J\in\mathcal{J}\\ J<\tilde J}}\frac{|J|\cdot|\tilde J|}{f_\beta|_{\tilde J}-f_\beta|_J} + \sum_{\substack{J,\tilde J\in\mathcal{J}\\ J<\tilde J}}\frac{|J|\cdot|\tilde J|}{f_\beta|_{\tilde J}-f_\beta|_J}\ln\frac{1}{dist(J,\tilde J)}\right).$$

We have already proved that the first sum converges; it remains to show that the second one is convergent, too. As before, fix $J=(b,a)$ and let $\tilde J=(\tilde b,\tilde a)$ be such that $J<\tilde J$; write

$a = \sum_{k=1}^{n} \alpha_k \varepsilon_k$, $b = a - \sigma_n$ and $\tilde{a} = \sum_{j=1}^{m} \gamma_j \varepsilon_j$, $\tilde{b} = \tilde{a} - \sigma_m$. Set $r = f_\beta|_{\tilde{J}} - f_\beta|_J$. We have that $dist(J, \tilde{J}) = \tilde{b} - a$. Using the same argument as in the case $N = 1$, we get

$$\begin{aligned}\sum_{\substack{\tilde{J}\in\mathcal{J}, J<\tilde{J} \\ dist(J,\tilde{J})\geq\delta_m}} \frac{|\tilde{J}|}{f_\beta|_{\tilde{J}} - f_\beta|_J} \ln \frac{1}{dist(J,\tilde{J})} &\leq \sum_{s=1}^{n} \sum_{m\geq s+1} \sum_{\substack{\gamma_j\in\{0,1\},\gamma_m=1 \\ s+1\leq j\leq m-1}} \frac{\sigma_m}{r} \ln \frac{1}{\delta_m} \\ &\leq \sum_{s=1}^{n} \sum_{m\geq s+1} m\sigma_m 2^m \sum_{j=1}^{2^{m-s}-1} \frac{1}{j} \ln \frac{1}{\delta_1} \\ &\leq n\tilde{L} \ln \frac{1}{\delta_1}\end{aligned}$$

where $\tilde{L} = \sum_{m\geq 1} \sigma_m 2^m m^2 < +\infty$. Since $dist(J, \tilde{J}) \geq min(\delta_n, \delta_m)$, it results

$$\sum_{\substack{\tilde{J}\in\mathcal{J}, J<\tilde{J} \\ dist(J,\tilde{J})<\delta_m}} \frac{|\tilde{J}|}{f_\beta|_{\tilde{J}} - f_\beta|_J} \ln \frac{1}{dist(J,\tilde{J})} \leq \sum_{s=1}^{n-1} \frac{\sigma_s}{\sum_{k=s+1}^{n-1} (1-\alpha_k)2^{-k} + 2^{-n}} \ln \frac{1}{\delta_n}.$$

Similarly, allowing J to be variable in $\mathcal{J}$ we conclude that:

$$\sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \frac{|J|\cdot|\tilde{J}|}{f_\beta|_{\tilde{J}} - f_\beta|_J} \ln \frac{1}{dist(J,\tilde{J})} \leq 2L\,\tilde{L} \ln \frac{1}{\delta_1}.$$

□

We now prove Theorem 5.7:

Proof of Theorem 5.7: Let $\alpha \in (0,1)$. Take $\beta \in (0,1)$ such that $\alpha = H_\beta$.

Case 1: $N = 1$. Let f be the β-Cantor function. Choose an arbitrary $\theta \in (0,1)$ and set $\omega(t) = t^\theta, \forall t \geq 0$. Like in the previous proof, we want to show that

$$\sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \int_J \int_{\tilde{J}} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|} < +\infty.$$

As before, consider the interval $J = (b,a)$ fix. Let $\tilde{J} = (\tilde{b}, \tilde{a})$ be a variable removed interval such that $a < \tilde{a}$. Each time, we consider the first s step (in the construction of C_β) when J and $\tilde{J}$ do not belong anymore to the same remaining interval. Let us denote $p = \frac{1}{\delta_1} > 2$ and we use the same notations $r = f|_{\tilde{J}} - f|_J$, $b = a - \sigma_n$, $\tilde{b} = \tilde{a} - \sigma_m$, $a = \sum_{k=1}^{n} \alpha_k \varepsilon_k$, $\alpha_k \in \{0,1\}$, $\alpha_n = 1$ and $\tilde{a} = \sum_{j=1}^{m} \gamma_j \varepsilon_j$, $\gamma_j \in \{0,1\}$, $\gamma_m = 1$.

If $dist(J,\tilde{J}) \geq \delta_m$ i.e. $s < m$, we distinguish two cases:

i) $dist(J,\tilde{J}) \geq \delta_n$ i.e. $s < n$. Here we have $\tilde{b} - a \geq \sigma_s$ and $r \leq 2^{-s+1}$. We write:

$$\begin{aligned}E(J,\tilde{J}) &= \int_J \int_{\tilde{J}} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|} \\ &= \int_0^1 \int_0^1 \frac{\omega(r)\sigma_n\sigma_m\,dt\,dz}{(\tilde{b}-a+t\sigma_n+z\sigma_m)^{1+\theta}} \leq \frac{r^\theta \sigma_n \sigma_m}{(\tilde{b}-a)^{1+\theta}}.\end{aligned}$$

If we sum up over these $\tilde{J}$, we get:

$$\sum_{\substack{\tilde{J}\in\mathcal{J},J<\tilde{J}\\ dist(J,\tilde{J})\geq max\{\delta_m,\delta_n\}}} E(J,\tilde{J}) \leq \sigma_n \sum_{s=1}^{n-1}\sum_{m\geq s+1}\sum_{\substack{\gamma_j\in\{0,1\}\\ s+1\leq j\leq m-1}} \frac{\sigma_m}{\sigma_s}\frac{1}{(2^{s-1}\sigma_s)^\theta}$$
$$\leq \sigma_n \sum_{s=1}^{n-1}\frac{1}{(2^{s-1}\sigma_s)^\theta}\sum_{m\geq s+1}\left(\frac{2}{p}\right)^{m-s}$$
$$\leq c\sigma_n \sum_{s=0}^{n-2}\left(\frac{p}{2}\right)^{s\theta} L_1$$
$$\leq c\sigma_n L_1 \left(\frac{p}{2}\right)^{\theta(n-1)}$$

where for $q>0$ we denote $L_q = \sum\limits_{m\geq 0}\left(\frac{2}{p}\right)^{mq} < +\infty$ and $c=c(\beta,\theta)$ is a constant that depends only on β and θ.
ii) $dist(J,\tilde{J})<\delta_n$ i.e. $s=n$. In this case,

$$E(J,\tilde{J}) \leq \int_0^1 \frac{r^\theta\sigma_n\sigma_m\, dt}{(\tilde{b}-a+t\sigma_n)^{1+\theta}}.$$

We have $\tilde{b}-a = \sum\limits_{j=n+1}^{m}\gamma_j\varepsilon_j - \sigma_m \geq \sum\limits_{j=n+1}^{m}\gamma_j\delta_j$ and $r=\sum\limits_{j=n+1}^{m}\gamma_j 2^{-j}$. From here, we obtain

$$\sum_{\substack{\tilde{J}\in\mathcal{J},\tilde{J}>J\\ \delta_m\leq dist(J,\tilde{J})<\delta_n}} E(J,\tilde{J}) \leq c\, L_\theta\, L_{1-\theta}\,\sigma_n\left(\frac{p}{2}\right)^{n\theta}$$

where $c=c(\beta,\theta)$ is a constant that depends only on β and θ. If we let J be variable in $\mathcal{J}$, we deduce

$$\sum_{\substack{J,\tilde{J}\in\mathcal{J},J<\tilde{J}\\ dist(J,\tilde{J})\geq\delta_m}} E(J,\tilde{J}) \leq c(\beta,\theta)\sum_{n\geq 1}\sum_{\substack{\alpha_k\in\{0,1\}\\ 1\leq k\leq n-1}}\sigma_n\left(\frac{p}{2}\right)^{n\theta}$$
$$\leq c(\beta,\theta)\sum_{n\geq 1}\left(\frac{2}{p}\right)^{n(1-\theta)}$$
$$< +\infty.$$

Otherwise, $dist(J,\tilde{J})<\delta_m$ i.e. $s=m$. Thus $m<n$,

$$\tilde{b}-a = \delta_m - \sum_{k=m+1}^{n}\alpha_k\varepsilon_k \geq \sum_{k=m+1}^{n-1}(1-\alpha_k)\delta_k + \delta_n$$

$$r = \sum_{k=m+1}^{n-1}(1-\alpha_k)2^{-k} + 2^{-n} \text{ and } E(J,\tilde{J}) \leq \int_0^1 \frac{r^\theta\sigma_n\sigma_m\, dz}{(\tilde{b}-a+z\sigma_m)^{1+\theta}}.$$

We get

$$\sum_{\substack{\tilde J\in\mathcal{J},J<\tilde J\\ dist(J,\tilde J)<\delta_m}} E(J,\tilde J)\le \sigma_n\sum_{m=1}^{n-1}\int_0^1 \frac{\sigma_m\Big(\sum\limits_{k=m+1}^{n-1}(1-\alpha_k)2^{-k}+2^{-n}\Big)^{\theta}dz}{\Big(\sum\limits_{k=m+1}^{n-1}(1-\alpha_k)\delta_k+\delta_n+z\sigma_m\Big)^{1+\theta}}.$$

Finally, if we let J be variable in $\mathcal{J}$, we find

$$\sum_{\substack{J,\tilde J\in\mathcal{J},J<\tilde J\\ dist(J,\tilde J)<\delta_m}} E(J,\tilde J)\le c(\beta,\theta)L_\theta M_{1-\theta}$$

where $M_{1-\theta}=\sum\limits_{n\ge1} n\left(\frac{2}{p}\right)^{n(1-\theta)}<+\infty$.

Case 2: $N\ge 2$. Let $f(x)=f_\beta(x_1),\forall x\in[0,1]^N$. As before, take $\theta\in(0,1)$ and set $\omega(t)=t^\theta,\forall t\ge0$. Write

$$\begin{aligned}
I&=\int\limits_{(0,1)^N}\int\limits_{(0,1)^N}\omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{dx\,dy}{|x-y|^N}\\
&\le 2^N\sum_{\substack{J,\tilde J\in\mathcal{J}\\ J<\tilde J}}\int_J\int_{\tilde J}\int\limits_{(0,1)^{N-1}}\omega\left(\frac{|f_\beta(x_1)-f_\beta(y_1)|}{\sqrt{|x'|^2+(x_1-y_1)^2}}\right)\frac{dx_1\,dy_1\,dx'}{(|x'|^2+(x_1-y_1)^2)^{\frac N2}}\\
&\le 2^N|S^{N-2}|\sum_{\substack{J,\tilde J\in\mathcal{J}\\ J<\tilde J}}\int_J\int_{\tilde J}\omega(r)\,dx_1\,dy_1\int_0^{N-1}\frac{t^{N-2}}{(t^2+(x_1-y_1)^2)^{\frac{N+\theta}{2}}}\,dt
\end{aligned}$$

(here we denote $r=f_\beta|_{\tilde J}-f_\beta|_J$). On the other hand, we have

$$\int_0^{N-1}\frac{t^{N-2}\,dt}{(t^2+(x_1-y_1)^2)^{\frac{N+\theta}{2}}}\le 4\int_0^{N-1}\frac{dt}{(y_1-x_1+t)^{2+\theta}}\le\frac{4}{(y_1-x_1)^{1+\theta}}$$

for every $0\le x_1<y_1\le1$. Therefore there is a constant $c=c(N)>0$ such that

$$I\le c(N)\sum_{\substack{J,\tilde J\in\mathcal{J}\\ J<\tilde J}}\int_J\int_{\tilde J}\omega\left(\frac{|f_\beta(x_1)-f_\beta(y_1)|}{|x_1-y_1|}\right)\frac{dx_1dy_1}{|x_1-y_1|}.$$

By Case 1, the conclusion follows. □

Theorem 5.6 is a consequence of the previous two 'counter-examples'; indeed, for some $0<\theta<1$ a bounded function ω satisfies $\omega(t)\le||\omega||_{L^\infty}\cdot\left(\frac1t+t^\theta\right)$ for every $t>0$.

5.7 Dimension reduction

The following result permits to reduce the proof of Problem 1 to one-dimensional domains[3]:

[3]This result is part of the forthcoming paper [59]

Theorem 5.13 *Let $\omega \in \mathcal{W}$. Suppose that Problem 1 holds true for any interval in $\mathbb{R}$, i.e., for any interval $\Omega \subset \mathbb{R}$, if $f : \Omega \to \mathbb{R}$ is a measurable function with (5.2), then f is a constant. Then Problem 1 holds true for any domain $\Omega \subset \mathbb{R}^N$, $N > 1$.*

Proof. Let $\Omega \subset \mathbb{R}^N$ and $x_0 \in \Omega$. Take $r > 0$ such that $\tilde{B} = B(x_0, (\sqrt{N}+1)r) \subset \Omega$. Denote the balls $B = B(x_0, r)$ and $\check{B} = B(x_0, \sqrt{N}r)$. Write

$$\int_{\tilde{B}} \int_{\tilde{B}} \omega\Big(\frac{|f(x)-f(y)|}{|x-y|}\Big) \frac{dx\,dy}{|x-y|^N} \geq \int_{S^{N-1}} d\sigma \int_{\check{B}} dx \int_0^r \omega\Big(\frac{|f(x+t\sigma)-f(x)|}{t}\Big)\frac{dt}{t}.$$

By (5.2), we deduce that for a.e. $\sigma \in S^{N-1}$,

$$\int_{\check{B}} dx \int_0^r \omega\left(\frac{|f(x+t\sigma)-f(x)|}{t}\right)\frac{dt}{t} < +\infty.$$

Therefore, one can find an orthonormal basis $(\sigma_i)_{1\leq i \leq N}$ for which the above inequality holds. For simplicity of the writing, we assume that $x_0 = 0$ and σ_i are the cartesian directions in $\mathbb{R}^N$, i.e., $\sigma_i = x_i$ for every $i = 1, \dots, N$. Set the $N-$cube $P = (-r, r)^N \subset \check{B}$. Then for each $1 \leq i \leq N$ and for a.e. $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in (-r, r)^{N-1}$, we have that

$$\int_{-r}^{r} \int_0^r \omega\left(\frac{|f(x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_N) - f(x_1,\dots,x_i,\dots,x_N)|}{t}\right)\frac{dt\,dx_i}{t} < +\infty.$$

Hence, by our assumption on one-dimensional domains, it follows that

$$x_i \in (-r, r) \to f(x_1, \dots, x_i, \dots, x_N)$$

is constant a.e. in $(-r, r)$. According to Lemma 2 in [28], we deduce that f is constant a.e. in the cube P. □

5.8 The case of an indicator function

In this section[4], we prove that the condition

$$\omega \in \mathcal{W}, \qquad \int_1^{+\infty} \frac{\omega(t)}{t^2}\,dt = +\infty \tag{5.6}$$

prevents a measurable $f : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{1}{|x-y|}\,dxdy < +\infty$$

from being the indicator function of a measurable subset $A \subseteq \Omega$ with $\lambda(A), \lambda(CA) > 0$ (λ denotes the Lebesgue measure on Ω and $CA := \Omega \setminus A$) . Note that if $\omega(t) = t$, $\forall t \in \mathbb{R}_+$ (satisfying (5.6)), the following result has been already proved, using a different approach, in [25]. Notice that in Theorem 5.2, it was proved that the condition (5.6) is necessary to prevent f from being a step function.

[4] This section is part of the forthcoming paper [59]

Proposition 5.14 *Let $\Omega \subset \mathbb{R}^N$ be a domain in $\mathbb{R}^N$. If ω satisfies* (5.6) *and $A \subseteq \Omega$ is measurable such that*

$$\int_A \int_{CA} \omega\left(\frac{1}{|x-y|}\right) \frac{1}{|x-y|}\, dxdy < +\infty, \tag{5.7}$$

then $\lambda(A) \in \{0,1\}$.

Proof. By the argument in Theorem 5.13, we may reduce the proof to the one-dimensional case: let us assume that $\Omega = (0,1) \subset \mathbb{R}$. Suppose that $\lambda(A), \lambda(CA) > 0$ and (5.7) holds. We show that

$$\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt < +\infty. \tag{5.8}$$

Let $x_0 \in (0,1)$ be an arbitrary point of density of CA. We recall that almost all points of a measurable set are points of density (see [11, 86]), so that such an x_0 exists due to $\lambda(CA) > 0$, and the corresponding density condition reads,

$$\lim_{t \searrow 0} \frac{\lambda(CA \cap [x_0 - t, x_0 + t])}{2t} = 1. \tag{5.9}$$

Let us introduce the notations

$$A^- := A \cap (0, x_0), \quad A^+ := A \cap (x_0, 1),$$

$$CA^- := CA \cap (0, x_0), \quad CA^+ := CA \cap (x_0, 1).$$

W.l.o.g. we assume that $\lambda(A^-) > 0$, so that from (5.7) we deduce

$$+\infty > \int_{A^-} \int_{CA^-} \omega\left(\frac{1}{|x-y|}\right) \frac{1}{|x-y|}\, dxdy \geq \int_0^{+\infty} \omega\left(\frac{1}{t}\right) \frac{1}{t} \psi_{A^-}(t)\, dt, \tag{5.10}$$

where the measurable function $\psi_{A^-} : \mathbb{R}_+ \to [0,1]$ is given by

$$\begin{aligned} \psi_{A^-}(t) &:= \lambda(\{x \in A^- : (x + t \in CA^-) \vee (x - t \in CA^-)\}) \\ &= \lambda(\{x \in A^- : x + t \in CA^-\} \cup \{x \in A^- : x - t \in CA^-\}) \quad \forall t \geq 0. \end{aligned} \tag{5.11}$$

Obviously, $\psi_{A^-}(0) = 0$.

The main idea is to investigate the behavior of ψ_{A^-} at $t = 0$. If we are able to show that ψ_{A^-} vanishes at $t = 0$ of order at most 1 (in the sense $t \lesssim \psi_{A^-}(t)$ on $[0, \varepsilon]$ for some $\varepsilon > 0$) we are done, since

$$\int_0^{\varepsilon} \omega\left(\frac{1}{t}\right) dt \overset{t=1/s}{=} \int_{1/\varepsilon}^{+\infty} \frac{\omega(s)}{s^2}\, ds.$$

We formulate therefore

Claim 1.

ψ_{A^-} vanishes at $t = 0$ of order at most 1, that is, there exist $\varepsilon_A, c_A > 0$ such that

$$\psi_{A^-}(t) \geq c_A t \quad \forall t \in [0, \varepsilon_A].$$

To see this, let us introduce also the measurable function $\phi_{A^-} : \mathbb{R} \to [0,1]$ given by

$$\phi_{A^-}(t) := \lambda(\{x \in A^- : x + t \in CA^-\}) \quad \forall t \in \mathbb{R}, \tag{5.12}$$

so that due to (5.11) we have,

$$\max\{\phi_{A^-}(t), \phi_{A^-}(-t)\} \leq \psi_{A^-}(t) \leq \phi_{A^-}(t) + \phi_{A^-}(-t) \quad \forall t \in \mathbb{R}_+. \tag{5.13}$$

Claim 1 follows from (5.13), if we show

Claim 2.

$\phi_{A^-}|_{\mathbb{R}_+}$ is continuous and vanishes at $t=0$ of order at most 1, that is, there exist $\varepsilon_A, c_A > 0$ such that

$$\phi_{A^-}(t) \geq c_A t \quad \forall t \in [0, \varepsilon_A].$$

Note first that

$$\begin{aligned} \phi_{A^-}(t) \quad &:= \quad \lambda(\{x \in A^- : x + t \in CA^-\}) = \lambda((t + A^-) \cap CA^-) = \int_{\mathbb{R}} 1_{A^-}(x) 1_{CA^-}(x+t)\, dx \\ &\overset{y=-x}{=} \quad \int_{\mathbb{R}} 1_{-A^-}(y) 1_{CA^-}(t-y)\, dy = (1_{-A^-} \star 1_{CA^-})(t), \end{aligned}$$

so that ϕ_{A^-} is positive and continuous (by the dominated convergence theorem), vanishing at $t = 0$.

Next note that

$$\begin{aligned} \phi_{A^-}(t) = \lambda((t + A^-) \cap CA^-) \quad &= \quad \lambda((t + A^-) \cap (0, x_0)) - \lambda((t + A^-) \cap A^-). \\ &= \quad \phi_{A^-,1}(t) + \phi_{A^-,2}(t) \end{aligned} \tag{5.14}$$

with

$$\phi_{A^-,1}(t) := \lambda((t + A^-) \cap (0, x_0)) - \lambda(A^-)$$
$$\phi_{A^-,2}(t) := \lambda(A^-) - \lambda((t + A^-) \cap A^-).$$

The continuity of $\phi_{A^-,1}$, $\phi_{A^-,2}$ on $\mathbb{R}$ follows again by the dominated convergence theorem, but more refined analysis of the behavior of $\phi_{A^-,1}$, $\phi_{A^-,2}$ at $t = 0$ is needed to prove *Claim 2.*

For $\phi_{A^-,1}$ we write for any $t \geq 0$, due to the invariance of the Lebesgue measure under translation,

$$\phi_{A^-,1}(t) = -\lambda((x_0 - t, x_0) \cap A),$$

so that

$$\lim_{t \searrow 0} \frac{\phi_{A^-,1}(t)}{t} = 0, \tag{5.15}$$

since x_0 is a point of density of CA (see (5.9)).

As for $\phi_{A^-,2}$, using the notation

$$A_t^- := (t + A^-) \cap A^- \subseteq A^-,$$

we write

$$\begin{aligned} \lambda(A_{t+s}^-) = \lambda((t + s + A^-) \cap A^-) \quad &\geq \quad \lambda((t + A_s^-) \cap A^-) \\ &= \quad \lambda((t + A^-) \cap A^-) - \lambda((t + (A^- \setminus A_s^-)) \cap A^-) \\ &\geq \quad \lambda(A_t^-) - \lambda(A^- \setminus A_s^-) \\ &= \quad \lambda(A_t^-) + \lambda(A_s^-) - \lambda(A^-), \end{aligned}$$

which ensures that the positive, continuous function $\phi_{A^-,2} : \mathbb{R} \to \mathbb{R}_+$ which satisfies

$$\phi_{A^-,2}(t) = \lambda(A^-) - \lambda(A_t^-) \in \mathbb{R}_+ \quad \forall t \in \mathbb{R} \tag{5.16}$$

is subadditive,

$$\phi_{A^-,2}(s+t) \leq \phi_{A^-,2}(s) + \phi_{A^-,2}(t) \quad \forall s, t \in \mathbb{R}. \tag{5.17}$$

It is easy to see that this property and the continuity of $\phi_{A^-,2}$ imply a linear growth estimate from above on $\phi_{A^-,2}$ in the neighbourhood of $t = +\infty$,

$$\phi_{A^-,2}(t) \leq c_A \cdot t \quad \forall t \geq 1,$$

but this is not really useful in our context.
We prove that in fact the subadditivity condition (5.17) ensures also a lower bound of this type on $\phi_{A^-,2}$, in the neighbourhood of $t = 0$. More precisely, we show that

Claim 3. Either $\phi_{A^-,2} = 0$ on $[0, \infty]$ *or there exist* $\varepsilon_A, c_A > 0$ *such that*

$$\phi_{A^-,2}(t) \geq c_A \cdot t \quad \forall t \in [0, \varepsilon_A]. \tag{5.18}$$

Indeed, if the latter does not hold, we have that there exists a sequence $(t_n)_{n \in \mathbb{N}_+} \searrow 0$ such that

$$\phi_{A^-,2}(t) \leq \frac{t_n}{n} \quad \forall n \in \mathbb{N}_+. \tag{5.19}$$

Taking $t \geq 0$ arbitrary, we have by continuity, subadditivity and (5.19),

$$\phi_{A^-,2}(t) = \lim_{n \to \infty} \phi_{A^-,2}\left(\left\lfloor \frac{t}{t_n} \right\rfloor t_n\right) \leq \left\lfloor \frac{t}{t_n} \right\rfloor \phi_{A^-,2}(t_n) \overset{(5.19)}{\leq} \frac{t}{n},$$

for any n such that $t_n \leq t$. Letting $n \to \infty$ we obtain the desired conclusion, $\phi_{A^-,2} = 0$ on $[0, \infty]$.

Note now that $\phi_{A^-,2} = 0$ on $[0, \infty]$ immediately implies via (5.16) that $\lambda(A^-) = 0$, contradiction. Combining (5.18) and (5.15) we thus obtain the existence of $\varepsilon_A, c_A > 0$ such that

$$\phi_{A^-}(t) \geq c_A \cdot t \quad \forall t \in [0, \varepsilon_A]. \tag{5.20}$$

This concludes the proof of *Claim 2* and, via (5.13), of *Claim 1* and Proposition 5.14 too. □

Remark 5.1 The method we use to prove Proposition 5.14 is based on the control of the behavior at $t = 0$ of

$$\mathbb{R} \ni t \to \lambda((t + A) \cap CA) \in \mathbb{R}_+,$$

and allows therefore a similar treatment of a more general condition than (5.7),

$$\int_A \int_{CA} \omega(|f(x) - f(y)|, |x - y|)\, dxdy < +\infty.$$

5.9 The case of a Cantor function

In this section[5] we show that condition (5.6) prevents f from being a Cantor function.

Proposition 5.15 *If ω satisfies (5.6) and $0 < \beta < 1$, then for the Cantor function $f_\beta : [0,1] \to [0,1]$ it holds:*

$$\int_{[0,1]} \int_{[0,1]} \omega\left(\frac{|f_\beta(x) - f_\beta(y)|}{|x-y|}\right) \frac{1}{|x-y|}\, dxdy = +\infty. \tag{5.21}$$

Proof. Let $J, \tilde{J}$ be two arbitrary, disjoint intervals removed in the construction of the Cantor set C_β. We assume w.l.o.g. $J < \tilde{J}$ (that is, $J = (b,a), \tilde{J} = (\tilde{b}, \tilde{a})$ with $b < a < \tilde{b} < \tilde{a}$) and we aim next at estimating from below the integral

$$I_{J\tilde{J}}(f_\beta) := \int_J \int_{\tilde{J}} \omega\left(\frac{|f_\beta(x) - f_\beta(y)|}{|x-y|}\right) \frac{1}{|x-y|}\, dxdy.$$

Using the same substitution as in the case of an indicator function we write

$$I_{J\tilde{J}}(f_\beta) = \int_0^\infty \omega\left(\frac{r_{J\tilde{J}}}{t}\right) \frac{1}{t} \psi_{J\tilde{J}}(t)\, dt, \tag{5.22}$$

where

$$r_{J\tilde{J}} = f|_{\tilde{J}} - f|_J, \quad \psi_{J\tilde{J}}(t) = \lambda(\{x \in J, x + t \in \tilde{J}\}).$$

We now remark that, in contrast to the case of an indicator function, $\psi_{J\tilde{J}}$ vanishes in a neighbourhood of $t = 0$. But, since $J, \tilde{J}$ are intervals, we can write $\psi_{J\tilde{J}}$ explicitly. To this end we introduce the notations,

$$d_{J\tilde{J}} := \tilde{b} - a, \quad m_{J\tilde{J}} := \min\{|J|, |\tilde{J}|\}, \quad M_{J\tilde{J}} := \max\{|J|, |\tilde{J}|\}, \quad r := r_{J\tilde{J}}.$$

Note that d, m, M, r depend on the intervals $J, \tilde{J}$. For notational ease however we do not indicate here and in the following this dependence using sub/superscripts.
With the substitution $t := r/z$, (5.22) becomes,

$$I_{J\tilde{J}}(f_\beta) = \int_0^\infty \frac{\omega(z)}{z^2} \cdot z\psi_{J\tilde{J}}(r/z)\, dz,$$

so that

$$\begin{aligned} \int_{[0,1]} \int_{[0,1]} \omega\left(\frac{|f_\beta(x) - f_\beta(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|} &= \sum_{J,\tilde{J}} I_{J\tilde{J}}(f_\beta) \\ &= \int_0^\infty \frac{\omega(z)}{z^2} \cdot \Psi(1/z)\, dz, \end{aligned} \tag{5.23}$$

where

$$\Psi(x) := \frac{\sum_{J,\tilde{J}} \tilde{\psi}_{J,\tilde{J}}(x)}{x}, \quad \tilde{\psi}_{J,\tilde{J}}(x) := \psi_{J,\tilde{J}}(rx) \quad \forall x \in (0,\infty).$$

Returning now to the explicit form of $\psi_{J\tilde{J}}$, we have that $\psi_{J\tilde{J}}$ vanishes on $[0,d]$, increases linearly with slope 1 on $[d, d+m]$, stays constant equal to m on $[d+m, d+M]$, decreases linearly

[5] This section is part of the forthcoming paper [59]

with slope -1 on $[d+M, d+M+m]$, and stays equal to 0 afterwards. More formally we can write for $\tilde{\psi}_{J\tilde{J}}$ (appearing in the definition of Ψ),

$$\tilde{\psi}_{J\tilde{J}}(x) = \begin{cases} 0 & 0 \leq x \leq d/r \\ rx - d & d/r \leq x \leq (d+m)/r \\ m & (d+m)/r \leq x \leq (d+M)/r \\ m - (rx - d - M) & (d+M)/r \leq x \leq (d+m+M)/r \\ 0 & (d+m+M)/r \leq x < +\infty \end{cases} . \tag{5.24}$$

The proof is concluded from (5.23) if we can show that

$$\liminf_{x \searrow 0} \Psi(x) > 0. \tag{5.25}$$

The argument we present in the following for the proof of (5.25) is based on the self-similarity of the Cantor set C_β and of the corresponding Cantor function f_β. More precisely, let us denote by $S(x)$ the numerator in the definition of $\Psi(x)$,

$$S(x) := \sum_{J,\tilde{J}} \tilde{\psi}_{J,\tilde{J}}(x) \quad \forall x \in [0, \infty).$$

Recall that $I_0^{(1)} := [0, (1-\beta)/2]$, $J_0^{(1)} := ((1-\beta)/2, 1+\beta)/2)$, $I_1^{(1)} := [(1+\beta)/2, 1]$ the three intervals into which $[0,1]$ is divided in the first step of the construction of C_β, and split the sum in the definition of S accordingly,

$$S = \sum_{\substack{J,\tilde{J} \\ J,\tilde{J} \subset I_0^{(1)}}} \tilde{\psi}_{J,\tilde{J}} + \sum_{\substack{J,\tilde{J} \\ (J = J_0^{(1)}) \vee (\tilde{J} = J_0^{(1)})}} \tilde{\psi}_{J,\tilde{J}} + \sum_{\substack{J,\tilde{J} \\ J \subset I_0^{(1)}, \tilde{J} \subset I_1^{(1)}}} \tilde{\psi}_{J,\tilde{J}} + \sum_{\substack{J,\tilde{J} \\ J \subset I_1^{(1)}, \tilde{J} \subset I_0^{(1)}}} \tilde{\psi}_{J,\tilde{J}} + \sum_{\substack{J,\tilde{J} \\ J,\tilde{J} \subset I_1^{(1)}}} \tilde{\psi}_{J,\tilde{J}}.$$

Denoting by $S_1, \ldots, S_5$ the sums above, we remark that $S_1, \ldots, S_5$ are all positive and $S_1 = S_5$, so that

$$S(x) \geq 2S_1(x) \quad \forall x \in [0, \infty). \tag{5.26}$$

But now a rescaling argument allows us to express S_1 in terms of S itself. More precisely, the stretching

$$\left[0, \frac{1-\beta}{2}\right] = I_0^{(1)} \ni t \overset{\phi_\beta}{\to} \frac{2}{1-\beta} t \in [0,1]$$

gives a bijection between the intervals $J \subset I_0^{(1)}$ and all intervals $J \subset [0,1]$ removed in the construction of the Cantor set. Additionally, the explicit form (5.24) of $\tilde{\psi}_{J,\tilde{J}}$ allows us to write for any pair $J, \tilde{J} \subset I_0^{(1)}$,

$$\tilde{\psi}_{J,\tilde{J}}(x) = \frac{1-\beta}{2} \tilde{\psi}_{\phi_\beta(J),\phi_\beta(\tilde{J})}((1-\beta)^{-1}x) \quad \forall x \in [0, \infty). \tag{5.27}$$

Indeed, let d, m, M, r and d', m', M', r' be the sets of parameters describing via (5.24) $\tilde{\psi}_{J,\tilde{J}}$ and $\tilde{\psi}_{\phi_\beta(J),\phi_\beta(\tilde{J})}$ respectively. By stretching we have $d' = \frac{2}{1-\beta} d$, $m' = \frac{2}{1-\beta} m$, $M' = \frac{2}{1-\beta} M$, whereas the definition of the Cantor function ensures $r' = 2r$. The scaling property (5.27) follows then

using these relations in (5.24).
Summing (5.27) over $J, \tilde{J} \subset I_0^{(1)}$ we obtain,

$$S_1(x) = \frac{1-\beta}{2} S((1-\beta)^{-1}x) \quad \forall x \in [0,\infty). \tag{5.28}$$

From (5.26) and (5.28) we obtain

$$S(x) \geq (1-\beta) S((1-\beta)^{-1}x) \quad \forall x \in [0,\infty),$$

which then ensures

$$\Psi(x) \geq \Psi((1-\beta)^{-1}x) \quad \forall x \in (0,\infty),$$

so that

$$\Psi(x) \geq \inf_{y \in [1,(1-\beta)^{-1}]} \Psi(y) \quad \forall x \in (0,1),$$

and the conclusion follows due to the fact that Ψ satisfies the condition:

$$\Psi(x) \geq c_K > 0, \quad \text{for every compact } K \subset (0,\infty). \tag{5.29}$$

Indeed, let $J := J_0^{(1)}$ be the first removed interval in the construction of the Cantor set and $\tilde{J} := J_n$ be the closest removed interval at the right side of J at the step n, $n > 1$ (see notations in Section 5.5). We have that

$$d_{J\tilde{J}} = \delta_n,\ r_{J\tilde{J}} = \frac{1}{2^n},\ m_{J\tilde{J}} = \sigma_n,\ M_{J\tilde{J}} = \sigma_1.$$

Now let $x \in K$. Then there exists $n_K > 0$ such that

$$\tilde{\psi}_{J,J_n}(x) = m_{J,J_n}, \text{ for every } n \geq n_K.$$

Therefore, (5.29) holds since

$$\sum_{n \geq n_K} \tilde{\psi}_{J,J_n}(x) \geq C \sum_{n \geq n_K} (1-\beta)^n \geq C_K > 0.$$

□

Part II

Vortices in a $2d$ rotating Bose-Einstein condensate

Chapter 6

The critical velocity for vortex existence in a two dimensional rotating Bose-Einstein condensate

Abstract

We investigate a model corresponding to the experiments for a two dimensional rotating Bose-Einstein condensate. It consists in minimizing a Gross-Pitaevskii functional defined in $\mathbb{R}^2$ under the unit mass constraint. We estimate the critical rotational speed Ω_1 for vortex existence in the bulk of the condensate and we give some fundamental energy estimates for velocities close to Ω_1.

This chapter is written in collaboration with V. Millot; the original text is published in J. Funct. Anal. **233** (2006), 260–306 (cf. [55]) and some of these results were annouced in C. R. Math. Acad. Sci. Paris **340** (2005), 571–576 (cf. [54]).

6.1 Introduction

The phenomenon of Bose-Einstein condensation has given rise to an intense research, both experimentally and theoretically, since its first realization in alkali gases in 1995. One of the most beautiful experiments was carried out by the ENS group and consisted in rotating the trap holding the atoms [65, 66] (see also [1]). Since a Bose-Einstein condensate (BEC) is a quantum gas, it can be described by a single complex-valued wave function (order parameter) and it rotates as a superfluid: above a critical velocity, it rotates through the existence of vortices, i.e., zeroes of the wave function around which there is a circulation of phase. In an experiment where a harmonic trap strongly confines the atoms in the direction of the rotation axis, the mathematical analysis becomes two-dimensional by the decoupling of the wave function (see [33, 34, 79]). We restrict our study to this two-dimensional model used in [33, 34]. After the nondimensionalization of the energy (see [4]), the wave function u_ε minimizes the Gross-Pitaevskii energy

$$\int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} V(x)|u|^2 + \frac{1}{4\varepsilon^2}|u|^4 - \Omega\, x^{\perp}\cdot(iu, \nabla u) \right\} dx \qquad (6.1)$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1, \tag{6.2}$$

where $\varepsilon > 0$ is small and represents a ratio of two characteristic lengths and $\Omega = \Omega(\varepsilon) \geq 0$ denotes the rotational velocity. We consider here the harmonic trapping case, that is $V(x) = |x|_\Lambda^2 := x_1^2 + \Lambda^2 x_2^2$ for a fixed parameter $0 < \Lambda \leq 1$. In [34], the equilibrium configurations are studied by looking for the minimizers in a reduced class of functions and some numerical simulations are presented.

Our aim is to estimate the critical velocity above which the wave function has vortices, and in Chapter 7 to analyze in more details the vortex patterns in the bulk of the condensate. According to numerical and theoretical predictions (see [4, 34]), we expect to find the critical speed in the regime $\Omega = \mathcal{O}(|\ln \varepsilon|)$ so that we restrict our study to this situation.

Due to the constraint (6.2), we may rewrite the energy in the equivalent form

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\big[(|u|^2 - a(x))^2 - (a^-(x))^2\big] - \Omega x^\perp \cdot (iu, \nabla u) \right\} dx \tag{6.3}$$

where $a(x) = a_0 - |x|_\Lambda^2$ and a_0 is determined by $\int_{\mathbb{R}^2} a^+(x) = 1$ so that $a_0 = \sqrt{2\Lambda/\pi}$. Here a^+ and a^- represent respectively the positive and the negative part of a. Then we consider the wave function u_ε as a solution of the variational problem

$$\text{Min}\big\{F_\varepsilon(u) \,:\, u \in \mathcal{H},\, \|u\|_{L^2(\mathbb{R}^2)} = 1\big\} \quad \text{where} \quad \mathcal{H} = \big\{u \in H^1(\mathbb{R}^2, \mathbb{C}) \,:\, \int_{\mathbb{R}^2} |x|^2|u|^2 < +\infty\big\}.$$

In the limit $\varepsilon \to 0$, the minimization of F_ε strongly forces $|u_\varepsilon|^2$ to be close to a^+ which means that the resulting density is asymptotically localized in the ellipsoidal region

$$\mathcal{D} := \big\{x \in \mathbb{R}^2 \,:\, a(x) > 0\big\} = \big\{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1^2 + \Lambda^2 x_2^2 < a_0\big\}.$$

We will also see that $|u_\varepsilon|$ decays exponentially fast outside $\mathcal{D}$. Actually, the domain $\mathcal{D}$ represents the region occupied by the condensate and consequently, vortices will be sought inside $\mathcal{D}$.

The main tools for studying vortices were developed by Béthuel, Brezis and Hélein [17] for "Ginzburg-Landau type" problems. We also refer to Sandier [75] and Sandier and Serfaty [76, 77, 78] for complementary techniques. In the case $a(x) \equiv 1$ and for a disc in $\mathbb{R}^2$, Serfaty proved the existence of local minimizers having vortices for different ranges of rotational velocity (see [83]). In [4], Aftalion and Du follow the strategy in [83] for the study of global minimizers of the Gross-Pitaevskii energy (6.3) where $\mathbb{R}^2$ is replaced by $\mathcal{D}$. In [3], Aftalion, Alama and Bronsard analyze the global minimizers of (6.3) for potentials of different nature leading to an annular region of confinement. We finally refer to [5, 6, 61] for mathematical studies on 3D models.

We emphasize that we tackle here the problem which corresponds exactly to the physical model. In particular, we minimize F_ε under the unit mass constraint and the admissible configurations are defined in the whole space $\mathbb{R}^2$. Several difficulties arise, especially in the proof of the existence results and the construction of test functions. We point out that we do not assume any implicit bound on the number of vortices. The singular and degenerate behavior of $\sqrt{a^+}$

near $\partial\mathcal{D}$ induces a cost of order $|\ln\varepsilon|$ in the energy and requires specific tools to detect vortices in the boundary region. Therefore we shall restrict our analysis to vortices lying down in the interior domain

$$\mathcal{D}_\varepsilon = \{x \in \mathcal{D} \,:\, a(x) > \nu_\varepsilon |\ln\varepsilon|^{-3/2}\} \tag{6.4}$$

where ν_ε is a chosen parameter in the interval $(1,2)$ (see Proposition 6.16).

We now start to describe our main results. We prove that

$$\Omega_1 := \frac{\Lambda^2+1}{a_0}|\ln\varepsilon| = \frac{\sqrt{\pi}(\Lambda^2+1)}{\sqrt{2\Lambda}}|\ln\varepsilon|$$

is the asymptotic estimate as $\varepsilon \to 0$ of the critical angular speed for nucleation of vortices in $\mathcal{D}$. The critical angular velocity Ω_1 coincides with the one found in [4, 34]. We observe that a very stretched condensate, i.e., $\Lambda \ll 1$, yields a very large value of Ω_1 and that the smallest Ω_1 is reached for $\Lambda = 1/\sqrt{3}$ (and surprisingly not for the symmetric case, i.e., $\Lambda = 1$). For subcritical velocities, we will see that u_ε behaves as the "vortex-free" profile $\tilde{\eta}_\varepsilon e^{i\Omega S}$ where $\tilde{\eta}_\varepsilon$ is the positive minimizer of

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right] \right\} dx$$

under the constraint (6.2) and the phase S is given by

$$S(x) = \frac{\Lambda^2-1}{\Lambda^2+1} x_1 x_2 \,. \tag{6.5}$$

For rotational speeds larger than Ω_1, we show the existence of vortices close to the origin. We also give some fundamental energy estimates in the regime $\Omega = \Omega_1 + \mathcal{O}(\ln|\ln\varepsilon|)$ which will allow to study the precise vortex structure of u_ε in Chapter 7.

Theorem 6.1 *Let u_ε be any minimizer of F_ε in $\mathcal{H}$ under the mass constraint* (6.2)*.*

(i) There exists a constant $\omega_1^\star < 0$ such that if $\Omega \le \Omega_1 + \omega_1 \ln|\ln\varepsilon|$ with $\omega_1 < \omega_1^\star$ then $|u_\varepsilon| \to \sqrt{a^+}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus \partial\mathcal{D})$ as $\varepsilon \to 0$. Moreover,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon\big(\tilde{\eta}_\varepsilon e^{i\Omega S}\big) + o(1) \tag{6.6}$$

and for any sequence $\varepsilon_n \to 0$, there exists a subsequence (still denoted by ε_n) and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $u_{\varepsilon_n} e^{-i\Omega S} \to \alpha\sqrt{a^+}$ in $H^1_{\mathrm{loc}}(\mathcal{D})$ as $n \to +\infty$.

(ii) If there exists some constant $\delta > 0$ such that $\Omega_1 + \delta\ln|\ln\varepsilon| \le \Omega \le \mathcal{O}(|\ln\varepsilon|)$, then u_ε has at least one vortex $x_\varepsilon \in \mathcal{D}$ such that $\operatorname{dist}(x^\varepsilon, \partial\mathcal{D}) \ge C > 0$ with C independent of ε. If in addition, $\Omega \le \Omega_1 + \mathcal{O}(\ln|\ln\varepsilon|)$, then x^ε remains close to the origin, i.e., $|x^\varepsilon| \le \mathcal{O}(|\ln\varepsilon|^{-1/6})$.

(iii) Set $v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$ and assume that $\Omega \le \Omega_1 + \omega_1\ln|\ln\varepsilon|$ for some $\omega_1 > 0$. Then there exist two positive constants $\mathcal{M}_1$ and $\mathcal{M}_2$ depending only on ω_1 such that

$$\int_{\mathcal{D}_\varepsilon} a(x)|\nabla v_\varepsilon|^2 + \frac{a^2(x)}{\varepsilon^2}(|v_\varepsilon|^2 - 1)^2 \le \mathcal{M}_1|\ln\varepsilon|,$$

$$\int_{\mathcal{D}_\varepsilon \setminus \{|x|_\Lambda < 2|\ln\varepsilon|^{-1/6}\}} a(x)|\nabla v_\varepsilon|^2 + \frac{a^2(x)}{\varepsilon^2}(|v_\varepsilon|^2 - 1)^2 \le \mathcal{M}_2 \ln|\ln\varepsilon|.$$

From the estimates in *(iii)* in Theorem 6.1, we are going to determine in Chapter 7 the number and the location of vortices in function of the angular speed Ω as $\varepsilon \to 0$. More precisely, we will compute the asymptotic expansion of the energy $F_\varepsilon(u_\varepsilon)$ in order to estimate the critical velocity Ω_d for having d vortices in the bulk and to exhibit the configuration of vortices by a certain renormalized energy. We also mention that the techniques used in Chapter 7 will permit to prove that the best constant in *(i)* in Theorem 6.1 is $\omega_1^\star = 0$. The proof will rely mostly on the study of "bad discs" in [17].

Sketch of the proof. We now describe briefly the content of this chapter.

Section 2 is devoted to the study of the density profile $\tilde{\eta}_\varepsilon$. We first introduce the real positive minimizer η_ε of E_ε, i.e.,

$$E_\varepsilon(\eta_\varepsilon) = \mathrm{Min}\left\{E_\varepsilon(\eta)\ :\ \eta \in \mathcal{H}\right\}. \tag{6.7}$$

We show the existence and uniqueness of η_ε (see Theorem 6.2) and we have that $E_\varepsilon(\eta_\varepsilon) \leq C|\ln\varepsilon|$ and $\eta_\varepsilon \to \sqrt{a^+}$ in $L^\infty(\mathbb{R}^2) \cap C^1_{\mathrm{loc}}(\mathcal{D})$ as $\varepsilon \to 0$ (see Proposition 6.3). Then we prove that there is a unique positive solution of the problem

$$\mathrm{Min}\ \left\{E_\varepsilon(\eta)\ :\ \eta \in \mathcal{H}\,,\ \|\eta\|_{L^2(\mathbb{R}^2)} = 1\right\} \tag{6.8}$$

called $\tilde{\eta}_\varepsilon$, which can be obtained from η_ε by a change of scale (see Theorem 6.7). This relationship yields an important estimate on the Lagrange multiplier k_ε associated to $\tilde{\eta}_\varepsilon\ :\ |k_\varepsilon| \leq \mathcal{O}(|\ln\varepsilon|)$, as well as the asymptotic properties of $\tilde{\eta}_\varepsilon$ from those of η_ε (see Proposition 6.8). In particular, we have $\tilde{\eta}_\varepsilon \to \sqrt{a^+}$ in $L^\infty(\mathbb{R}^2) \cap C^1_{\mathrm{loc}}(\mathcal{D})$ as $\varepsilon \to 0$.

In Section 3, we prove the existence of minimizers u_ε under the mass constraint (6.2) (see Proposition 6.10) and some general results about their behavior: $E_\varepsilon(u_\varepsilon) \leq C|\ln\varepsilon|^2$, u_ε decreases exponentially quickly to 0 outside $\mathcal{D}$, $|\nabla u_\varepsilon| \leq C_K\varepsilon^{-1}$ and $|u_\varepsilon| \lesssim \sqrt{a^+}$ in any compact $K \subset \mathcal{D}$ (see Proposition 6.11). Using a method introduced by Lassoued and Mironescu [63], we show that $F_\varepsilon(u_\varepsilon)$ splits into two independent pieces (see Lemma 6.12): the energy of the "vortex-free" profile $F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S})$ and the reduced energy of $v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$:

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \tag{6.9}$$

where

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon), \tag{6.10}$$

$$\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) = \int_{\mathbb{R}^2} \frac{\tilde{\eta}_\varepsilon^2}{2}|\nabla v_\varepsilon|^2 + \frac{\tilde{\eta}_\varepsilon^4}{4\varepsilon^2}(|v_\varepsilon|^2-1)^2\ , \quad \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon) = \frac{\Omega}{\Lambda^2+1}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 \nabla^\perp a \cdot (iv_\varepsilon, \nabla v_\varepsilon)\ , \tag{6.11}$$

$$\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\Omega^2|\nabla S|^2 - 2\Omega^2 x^\perp \cdot \nabla S + k_\varepsilon\right)\tilde{\eta}_\varepsilon^2(|v_\varepsilon|^2-1). \tag{6.12}$$

The motivation of S is explained in [4]: S satisfies $\mathrm{div}\left(a^+(\nabla S - x^\perp)\right) = 0$ in $\mathbb{R}^2$ and corresponds to the limit as $\varepsilon \to 0$ of the phase (globally defined in $\mathbb{R}^2$) divided by Ω, of any solution of $\mathrm{Min}\left\{F_\varepsilon(u)\ :\ u = \eta e^{i\varphi} \in \mathcal{H},\ \eta > 0\right\}$. The existence of the global limiting phase S is new in this type of variational problems related to the "Ginzburg-Landau" energy. We point out that the anisotropy carried by the phase S, leads to a negative term of order Ω^2 for $\Lambda \in (0,1)$ in the energy (see Remark 6.5):

$$F_\varepsilon\big(\tilde\eta_\varepsilon e^{i\Omega S}\big) = E_\varepsilon(\tilde\eta_\varepsilon) - \frac{\sqrt{2}(1-\Lambda^2)^2}{12\sqrt{\pi}(1+\Lambda^2)\Lambda^{3/2}}\Omega^2 + o(1).$$

We will prove that $|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)| = \mathcal{O}(\varepsilon|\ln\varepsilon|^3)$. Thus, we may focus on the reduced energy $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon)$. We study the vortex structure of u_ε via the map v_ε applying the Ginzburg-Landau techniques to the weighted energy $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon)$; the difficulty will arise in the region where $\tilde\eta_\varepsilon$ is small. We notice that v_ε inherits from u_ε and $\tilde\eta_\varepsilon$, the following properties (see Proposition 6.13): $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le C|\ln\varepsilon|^2$, $|\nabla v_\varepsilon| \le C_K\varepsilon^{-1}$ and $|v_\varepsilon| \lesssim 1$ in any compact $K \subset \mathcal{D}$. Using $\tilde\eta_\varepsilon e^{i\Omega S}$ as a test function and (6.9), we obtain in Proposition 6.14, a crucial upper bound of the reduced energy inside $\mathcal{D}_\varepsilon$:

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le o(1). \tag{6.13}$$

Motivated by the behavior $\tilde\eta_\varepsilon^2 \sim a^+$ (see (6.98) and (6.99)), we will use in the sequel the energies $\mathcal{F}_\varepsilon$, $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ in the interior of $\mathcal{D}$ (see Notations below).

In Section 4, we compute a first lower bound of $\mathcal{E}_\varepsilon(v_\varepsilon)$ using a method due to Sandier and Serfaty (see [76, 78]). We start with the construction of small disjoint balls $\{B(p_i,r_i)\}_{i\in I_\varepsilon}$ in the domain $\mathcal{D}_\varepsilon$ (given by (6.4)): outside these balls $|v_\varepsilon|$ is close to 1, so that v_ε carries a degree d_i on $\partial B(p_i,r_i)$ (see Proposition 6.16) and

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \ge \sum_{i\in I_\varepsilon}\mathcal{E}_\varepsilon(v_\varepsilon, B(p_i,r_i)) \gtrsim \pi\sum_{i\in I_\varepsilon} a(p_i)|d_i|\,|\ln\varepsilon|. \tag{6.14}$$

Then we prove an asymptotic expansion of the rotational energy outside the balls $\{B(p_i,r_i)\}_{i\in I_\varepsilon}$ (see Proposition 6.17),

$$\mathcal{R}_\varepsilon\big(v_\varepsilon, \mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i\big) \approx -\frac{\pi\Omega}{\Lambda^2+1}\sum_{i\in I_\varepsilon} a^2(p_i)\,d_i. \tag{6.15}$$

The presence of $a^2(p_i)$ is due to the harmonic type of the potential. In fact, for slightly more general potentials $a(x)$, we compute the solution ξ of the problem (see [4])

$$\operatorname{div}\Big(\frac{1}{a}\nabla\xi\Big) = -2 \text{ in } \mathcal{D} \quad\text{and}\quad \xi = 0 \text{ on } \partial\mathcal{D} \tag{6.16}$$

and the rotational energy will exhibit the terms $\xi(p_i)$ in (6.15). For our harmonic potential $a(x)$, an easy computation leads to $\xi = \frac{a^2}{2(\Lambda^2+1)}$. By (6.14) and (6.15), the first term in the lower expansion of the energy is

$$\pi\sum_{i\in I_\varepsilon} a(p_i)\bigg(|d_i||\ln\varepsilon| - d_i\Omega\frac{2\xi(p_i)}{a(p_i)}\bigg). \tag{6.17}$$

For having a vortex ball B_i with nonzero degree, Ω has to be larger than $\Omega_1 = \frac{1+\Lambda^2}{a_0}|\ln\varepsilon|$, p_i maximizes ξ/a and d_i is positive. Indeed, we obtain the subcritical case *(i)* in Theorem 6.1 matching (6.13) with (6.17). For velocities larger than Ω_1, we use an improvement of the upper estimate (6.13) using a test function having a single vortex at the origin. From here, we deduce *(ii)* in Theorem 6.1. We also prove that for $\Omega \leq \Omega_1 + \mathcal{O}(\ln|\ln\varepsilon|)$, the number of vortex balls with nonzero degree is uniformly bounded in ε and they appear close to the origin (see Proposition 6.19). We conclude by the two fundamental energy estimates stated in *(iii)* in Theorem 6.1.

Our analysis deals with vortices inside $\mathcal{D}$. However, we believe that for Ω small ($\Omega = \mathcal{O}(1)$), the solution should not have any vortices in $\mathbb{R}^2$. For Ω larger ($\Omega \sim \Omega_1$), vortices may exist in the region where u_ε is small. The study of the vortex structure in the region where $|u_\varepsilon|$ is small requires the development of other tools than energy estimates.

We recall that the choice of the harmonic potential is motivated by the physical experiments. For some other potentials a such that ξ/a has a unique maximum point at the origin, our method can be applied and the critical speed is given by

$$\Omega_1 = \frac{a(0)}{2\xi(0)}|\ln\varepsilon|.$$

If the set of maximum points of $\frac{\xi}{a}$ is not finite (it can be a curve, see Remark 6.6), the techniques are different and it will be the topic of a future work.

Notations. Throughout the chapter, we denote by C a positive constant independent of ε and we use the subscript to point out a possible dependence on the argument. For $x = (x_1, x_2) \in \mathbb{R}^2$, we write

$$x^\perp = (-x_2, x_1),\ |x|_\Lambda = \sqrt{x_1^2 + \Lambda^2 x_2^2} \quad \text{and} \quad B_R^\Lambda = \left\{x \in \mathbb{R}^2 \,:\, |x|_\Lambda < R\right\}$$

and for $\mathcal{A} \subset \mathbb{R}^2$,

$$\tilde{\mathcal{E}}_\varepsilon(v,\mathcal{A}) = \int_{\mathcal{A}} \frac{1}{2}\tilde{\eta}_\varepsilon^2|\nabla v|^2 + \frac{\tilde{\eta}_\varepsilon^4}{4\varepsilon^2}(1-|v|^2)^2\,, \quad \mathcal{E}_\varepsilon(v,\mathcal{A}) = \int_{\mathcal{A}} \frac{1}{2}\,a|\nabla v|^2 + \frac{a^2}{4\varepsilon^2}(1-|v|^2)^2,$$

$$\tilde{\mathcal{R}}_\varepsilon(v,\mathcal{A}) = \frac{\Omega}{1+\Lambda^2}\int_{\mathcal{A}} \tilde{\eta}_\varepsilon^2 \nabla^\perp a \cdot (iv, \nabla v)\,, \quad \mathcal{R}_\varepsilon(v,\mathcal{A}) = \frac{\Omega}{1+\Lambda^2}\int_{\mathcal{A}} a\nabla^\perp a \cdot (iv,\nabla v),$$

$$\tilde{\mathcal{F}}_\varepsilon(v,\mathcal{A}) = \tilde{\mathcal{E}}_\varepsilon(v,\mathcal{A}) + \tilde{\mathcal{R}}_\varepsilon(v,\mathcal{A})\,, \quad \mathcal{F}_\varepsilon(v,\mathcal{A}) = \mathcal{E}_\varepsilon(v,\mathcal{A}) + \mathcal{R}_\varepsilon(v,\mathcal{A}). \tag{6.18}$$

We do not write the dependence on $\mathcal{A}$ when $\mathcal{A} = \mathbb{R}^2$.

6.2 Analysis of the density profiles

In this section, we establish some preliminary results on η_ε and $\tilde{\eta}_\varepsilon$ defined respectively by (6.7) and (6.8). We will show that the shapes of η_ε and $\tilde{\eta}_\varepsilon$ are similar.

We notice that the space $\mathcal{H}$ in which we perform the minimization, is exactly the set of finiteness for E_ε. In the sequel, we endow $\mathcal{H}$ with the scalar product

$$\langle u, v\rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} \nabla u \cdot \nabla v + (1+|x|^2)(u\cdot v) \quad \text{for } u, v \in \mathcal{H};$$

obviously, $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a Hilbert space.

6.2.1 The free profile

We start by proving the existence and uniqueness for small ε of η_ε defined as the real positive solution of (6.7). Hence η_ε has to satisfy the associated Euler-Lagrange equation

$$\begin{cases}\varepsilon^2\Delta\eta_\varepsilon + (a(x) - \eta_\varepsilon^2)\eta_\varepsilon = 0 & \text{in } \mathbb{R}^2,\\ \eta_\varepsilon > 0 & \text{in } \mathbb{R}^2.\end{cases} \tag{6.19}$$

We denote by λ, the first eigenvalue of the elliptic operator $-\Delta + |x|_\Lambda^2$ in $\mathbb{R}^2$, i.e.,

$$\lambda = \mathrm{Inf}\left\{\int_{\mathbb{R}^2} |\nabla\phi|^2 + |x|_\Lambda^2|\phi|^2 \;:\; \phi \in \mathcal{H}\,,\; \|\phi\|_{L^2(\mathbb{R}^2)} = 1\right\}.$$

We have the following result:

Theorem 6.2 *If $0 < \varepsilon < \frac{a_0}{\lambda}$, there exists a unique classical solution η_ε of (6.19). Moreover, $\eta_\varepsilon \le \sqrt{a_0}$ and η_ε is the unique minimizer of E_ε in $\mathcal{H}$ up to a complex multiplier of modulus one. If $\varepsilon \ge \frac{a_0}{\lambda}$, then zero is the unique critical point of E_ε in $\mathcal{H}$.*

The method that we use for solving (6.19) involves several classical arguments generally used for a bounded domain. The main difficulty here is due to the fact that the equation is posed in the entire space $\mathbb{R}^2$ without any condition at infinity. We start with the construction of the *minimal solution*: we consider the solution $\eta_{R,\varepsilon}$ of the same equation posed in a ball of large radius R with homogeneous Dirichlet boundary condition and then we pass to the limit in R. We prove the uniqueness by estimating the ratio between the constructed solution and any other solution. A crucial point in the proof is an L^∞-bound of any weak solution.

Before proving Theorem 6.2, we present the asymptotic properties of η_ε as $\varepsilon \to 0$. We show that η_ε decays exponentially fast outside $\mathcal{D}$ and that η_ε^2 tends uniformly to a^+. The following estimates will be essential at several steps of our analysis.

Proposition 6.3 *For ε sufficiently small, we have*

6.3.a) $E_\varepsilon(\eta_\varepsilon) \le C|\ln\varepsilon|$,

6.3.b) $0 < \eta_\varepsilon(x) \le C\varepsilon^{1/3}\exp\left(\dfrac{a(x)}{4\varepsilon^{2/3}}\right)$ *in* $\mathbb{R}^2\setminus\mathcal{D}$,

6.3.c) $0 \le \sqrt{a(x)} - \eta_\varepsilon(x) \le C\varepsilon^{1/3}\sqrt{a(x)}$ *for* $x \in \mathcal{D}$ *with* $|x|_\Lambda < \sqrt{a_0} - \varepsilon^{1/3}$,

6.3.d) $\|\nabla\eta_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{-1}$,

6.3.e) $\|\eta_\varepsilon - \sqrt{a}\,\|_{C^1(K)} \le C_K\varepsilon^2$ *for any compact subset* $K \subset \mathcal{D}$.

Remark 6.1 *We observe that* 6.3.a) *in Proposition 6.3 implies*

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} |\eta_\varepsilon|^4 + 2a^-(x)|\eta_\varepsilon|^2 + \int_{\mathcal{D}}(a(x) - |\eta_\varepsilon|^2)^2 \le C\varepsilon^2|\ln\varepsilon|. \tag{6.20}$$

Proof of Theorem 6.2. Step 1: Existence for $0 < \varepsilon < \frac{a_0}{\lambda}$. For $R > 0$, we consider the equation

$$\begin{cases} \varepsilon^2 \Delta \eta_R + (a(x) - \eta_R^2)\eta_R = 0 & \text{in } B_R, \\ \eta_R > 0 & \text{in } B_R, \\ \eta_R = 0 & \text{on } \partial B_R. \end{cases} \tag{6.21}$$

By a result of Brezis and Oswald (see [32]), we have the existence and uniqueness of weak solutions of (6.21) if and only if the following first eigenvalue condition holds

$$\text{Inf}\left\{ \int_{B_R} |\nabla \phi|^2 - \frac{a(x)|\phi|^2}{\varepsilon^2} \ : \ \phi \in H_0^1(B_R)\ ,\ \|\phi\|_{L^2(B_R)} = 1 \right\} < 0, \text{ i.e.,}$$

$$\lambda_1(L_\varepsilon, B_R) = \text{Inf}\left\{ \int_{B_R} |\nabla \phi|^2 + \frac{|x|_\Lambda^2 |\phi|^2}{\varepsilon^2} \ : \ \phi \in H_0^1(B_R)\ ,\ \|\phi\|_{L^2(B_R)} = 1 \right\} < \frac{a_0}{\varepsilon^2} \tag{6.22}$$

where we denoted the elliptic operator $L_\varepsilon = -\Delta + \frac{|x|_\Lambda^2}{\varepsilon^2}$. We claim that for R sufficiently large, (6.22) is fulfilled. Indeed, let ψ be an eigenfunction of L_ε in $\mathbb{R}^2$ associated to the first eigenvalue $\lambda_1(L_\varepsilon, \mathbb{R}^2)$ with $\|\psi\|_{L^2(\mathbb{R}^2)} = 1$ (the existence of ψ is a direct consequence of the compact embedding $\mathcal{H} \hookrightarrow L^2(\mathbb{R}^2)$ proved in Lemma 6.4). For any integer $n \geq 1$, set $\psi_n(x) = c_n \zeta\left(\frac{|x|}{n}\right)\psi(x)$, where $\zeta : \mathbb{R} \to \mathbb{R}$ is the "cut-off" type function given by

$$\zeta(t) = \begin{cases} 1 & \text{if } t \leq 1, \\ 2 - t & \text{if } t \in (1, 2), \\ 0 & \text{if } t \geq 2 \end{cases} \tag{6.23}$$

and the constant c_n is chosen such that $\|\psi_n\|_{L^2(\mathbb{R}^2)} = 1$. We easily check that

$$\lambda_1(L_\varepsilon, B_{2n}) \leq \int_{B_{2n}} \left(|\nabla \psi_n|^2 + \frac{|x|_\Lambda^2}{\varepsilon^2}|\psi_n|^2 \right) \underset{n \to +\infty}{\longrightarrow} \int_{\mathbb{R}^2} \left(|\nabla \psi|^2 + \frac{|x|_\Lambda^2}{\varepsilon^2}|\psi|^2 \right) = \lambda_1(L_\varepsilon, \mathbb{R}^2)$$

and we deduce that the sequence $\{\lambda_1(L_\varepsilon, B_R)\}_{R>0}$ (which is decreasing in R) tends to $\lambda_1(L_\varepsilon, \mathbb{R}^2)$ as $R \to \infty$. Since

$$\lambda_1(L_\varepsilon, \mathbb{R}^2) = \frac{\lambda}{\varepsilon},$$

we conclude that there exists $R_\varepsilon > 0$ such that for every $R > R_\varepsilon$, condition (6.22) is fulfilled and equation (6.21) admits a unique weak solution $\eta_{R,\varepsilon}$.

By standard methods, it results that $\eta_{R,\varepsilon}$ is a smooth classical solution of (6.21). We notice that, for any $R_\varepsilon < R < \tilde{R}$, $\eta_{\tilde{R},\varepsilon}$ is a supersolution of (6.21) in B_R and thus $\eta_{R,\varepsilon} \leq \eta_{\tilde{R},\varepsilon}$ in B_R by the uniqueness of $\eta_{R,\varepsilon}$. By the maximum principle, we infer that $\eta_{R,\varepsilon} \leq \sqrt{a_0}$ in $\mathbb{R}^2$. For every $R > R_\varepsilon$, we extend $\eta_{R,\varepsilon}$ by 0 in $\mathbb{R}^2 \setminus B_R$. Since the function $R \to \eta_{R,\varepsilon}(x)$ is non-decreasing for any $x \in \mathbb{R}^2$, we may define for $x \in \mathbb{R}^2$, $\eta_\varepsilon(x) = \lim_{R \to +\infty} \eta_{R,\varepsilon}(x)$. It results that η_ε satisfies $0 < \eta_\varepsilon \leq \sqrt{a_0}$ and

$$\varepsilon^2 \Delta \eta_\varepsilon + (a(x) - \eta_\varepsilon^2)\eta_\varepsilon = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{6.24}$$

Since $\eta_\varepsilon \in L^\infty(\mathbb{R}^2)$, we derive by standard methods that η_ε is a smooth classical solution of (6.2).

Step 2. L^∞-bound for solutions of (6.19). The method we use in this step is due to Farina (see [43]) and relies on a result of Brezis (see [24]). We present the proof for convenience. Let η be any weak solution of (6.19) in $L^3_{\mathrm{loc}}(\mathbb{R}^2)$. We claim that

$$\eta \leq \sqrt{a_0} \quad \text{a.e. in } \mathbb{R}^2.$$

Indeed, if we consider $w = \varepsilon^{-1}(\eta - \sqrt{a_0})$, then $w \in L^3_{\mathrm{loc}}(\mathbb{R}^2)$ and since η satisfies (6.19), we infer that $\Delta w \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. By Kato's inequality, we have

$$\Delta(w^+) \geq \mathrm{sgn}^+(w)\Delta w \geq \frac{\mathrm{sgn}^+(w)}{\varepsilon^3}(\eta^2 - a_0)\eta = \frac{1}{\varepsilon^2}\, w^+(\varepsilon w + 2\sqrt{a_0})(\varepsilon w + \sqrt{a_0}) \geq (w^+)^3.$$

Therefore $w^+ \in L^3_{\mathrm{loc}}(\mathbb{R}^2)$ and w^+ satisfies

$$-\Delta(w^+) + (w^+)^3 \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

By Lemma 2 in [24], it leads to $w^+ \leq 0$ a.e. in $\mathbb{R}^2$ and thus $w^+ \equiv 0$.

Step 3. Uniqueness for $0 < \varepsilon < \frac{a_0}{\lambda}$. Let η_ε be the solution constructed at Step 1 and let η be any weak solution of (6.19) in $L^3_{\mathrm{loc}}(\mathbb{R}^2)$. By the previous step, $\eta \in L^\infty(\mathbb{R}^2)$ and using standard arguments, we derive that η is smooth and defines a classical solution of (6.19). We observe that η is a supersolution of (6.21) for every $R > R_\varepsilon$. Since $\eta_{R,\varepsilon}$ is extended by 0 outside B_R, $\eta_{R,\varepsilon} \leq \eta$ in $\mathbb{R}^2$. Passing to the limit in R, we get that $0 < \eta_\varepsilon \leq \eta$ in $\mathbb{R}^2$. Hence the function $\rho : \mathbb{R}^2 \to \mathbb{R}$ defined by $\rho = \eta_\varepsilon/\eta$ is smooth and takes values in $(0, 1]$. We easily check that ρ satisfies

$$\mathrm{div}(\eta^2 \nabla\rho) + \frac{\eta^4}{\varepsilon^2}(1 - \rho^2)\rho = 0 \quad \text{in } \mathbb{R}^2. \tag{6.25}$$

For every integer $n \geq 1$, we set $\zeta_n(x) = \zeta\left(n^{-1}|x|\right)$, where ζ is given by (6.23). Multiplying (6.25) by $(1 - \rho)\zeta_n^2$ and integrating by parts, we derive

$$\int_{\mathbb{R}^2} \left(\frac{\eta^4}{\varepsilon^2}\, \rho(1-\rho)^2(1+\rho)\zeta_n^2 + \eta^2\zeta_n^2|\nabla\rho|^2\right) = 2\int_{\mathbb{R}^2} \eta^2(1-\rho)\zeta_n(\nabla\rho \cdot \nabla\zeta_n). \tag{6.26}$$

Since ρ is bounded, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\int_{\mathbb{R}^2} \eta^2(1-\rho)\zeta_n(\nabla\rho\cdot\nabla\zeta_n) &= \int_{B_{2n}\setminus B_n} \eta^2(1-\rho)\zeta_n(\nabla\rho\cdot\nabla\zeta_n) \\
&\leq \left(\int_{B_{2n}} \eta^2(1-\rho)^2|\nabla\zeta_n|^2\right)^{1/2} \left(\int_{B_{2n}\setminus B_n} \eta^2\zeta_n^2|\nabla\rho|^2\right)^{1/2} \\
&\leq 2\sqrt{\pi}\,\|\eta\|_{L^\infty(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2\setminus B_n} \eta^2\zeta_n^2|\nabla\rho|^2\right)^{1/2}.
\end{aligned}$$

Using (6.26) and the L^∞-bound on η obtained in Step 2, we infer that

$$\int_{\mathbb{R}^2} \eta^2\zeta_n^2|\nabla\rho|^2 \leq 4\sqrt{\pi a_0}\left(\int_{\mathbb{R}^2\setminus B_n} \eta^2\zeta_n^2|\nabla\rho|^2\right)^{1/2}. \tag{6.27}$$

It follows

$$16\pi a_0 \geq \int_{\mathbb{R}^2} \eta^2 \zeta_n^2 |\nabla \rho|^2 \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} \eta^2 |\nabla \rho|^2$$

by monotone convergence. Since $\eta^2|\nabla\rho|^2 \in L^1(\mathbb{R}^2)$, the right hand side in (6.27) tends to 0 as $n \to +\infty$ and we finally deduce that $\int_{\mathbb{R}^2} \eta^2|\nabla\rho|^2 = 0$. Hence ρ is constant in $\mathbb{R}^2$ and by (6.26), we necessarily have $\rho = 1$, i.e., $\eta = \eta_\varepsilon$.

Step 4. End of the proof. The existence of a minimizer η of E_ε in $\mathcal{H}$ is standard. Since $E_\varepsilon(|\hat\eta|) \leq E_\varepsilon(\hat\eta)$ for any $\hat\eta \in \mathcal{H}$, we infer that $\hat\eta := |\eta|$ is also a minimizer and therefore $\hat\eta$ satisfies the equation

$$\begin{cases} \varepsilon^2 \Delta\hat\eta + (a(x) - \hat\eta^2)\hat\eta = 0 & \text{in } \mathbb{R}^2, \\ \hat\eta \geq 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{6.28}$$

By the maximum principle, it follows that either $\hat\eta > 0$ in $\mathbb{R}^2$ or $\hat\eta \equiv 0$.

If $0 < \varepsilon < \frac{a_0}{\lambda}$, we claim that $\hat\eta > 0$. Indeed, for $R > 0$ sufficiently large, we consider the unique solution $\eta_{R,\varepsilon}$ of (6.21). By [32], $\eta_{R,\varepsilon}$ is the unique non-negative minimizer of $E_\varepsilon(\cdot, B_R)$ in $H^1_0(B_R, \mathbb{R})$. Since $\eta_{R,\varepsilon}$ is extended by 0 outside B_R, we have

$$E_\varepsilon(\hat\eta) \leq E_\varepsilon(\eta_{R,\varepsilon}) = E_\varepsilon(\eta_{R,\varepsilon}, B_R) < E_\varepsilon(0, B_R) = E_\varepsilon(0)$$

which implies that $\hat\eta$ is not identically equal to 0. Then $\hat\eta$ solves (6.19) and by Step 3, we conclude that $|\eta| = \hat\eta = \eta_\varepsilon$. From the equality $E_\varepsilon(|\eta|) = E_\varepsilon(\eta)$, we easily deduce that there exists a real constant α such that $\eta = |\eta|e^{i\alpha} = \eta_\varepsilon e^{i\alpha}$.

If $\varepsilon \geq \frac{a_0}{\lambda}$, we prove that $\hat\eta \equiv 0$. Multiplying (6.28) by $\hat\eta$, it results

$$\int_{\mathbb{R}^2} |\nabla\hat\eta|^2 + \frac{|x|_\Lambda^2}{\varepsilon^2}\hat\eta^2 + \frac{1}{\varepsilon^2}\hat\eta^4 = \frac{a_0}{\varepsilon^2}\int_{\mathbb{R}^2} \hat\eta^2 \leq \frac{\lambda}{\varepsilon}\int_{\mathbb{R}^2} \hat\eta^2.$$

On the other hand,

$$\int_{\mathbb{R}^2} |\nabla\hat\eta|^2 + \frac{|x|_\Lambda^2}{\varepsilon^2}\hat\eta^2 \geq \lambda_1(L_\varepsilon, \mathbb{R}^2)\int_{\mathbb{R}^2} \hat\eta^2 = \frac{\lambda}{\varepsilon}\int_{\mathbb{R}^2} \hat\eta^2.$$

It follows that $\int_{\mathbb{R}^2} \hat\eta^4 = 0$, i.e., $\hat\eta \equiv 0$. Thus, in this range of ε, zero is the unique minimizer of E_ε.

Now it remains to show that zero is the unique critical point of E_ε when $\varepsilon \geq \frac{a_0}{\lambda}$. Indeed, let $\tilde\eta$ be any critical point of E_ε in $\mathcal{H}$, i.e., $\tilde\eta$ satisfies the equation (6.24). Then

$$\int_{\mathbb{R}^2} |\nabla\tilde\eta|^2 = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} a(x)\tilde\eta^2 - \tilde\eta^4. \tag{6.29}$$

Since zero is the global minimizer, we have that $E_\varepsilon(\tilde\eta) \geq E_\varepsilon(0)$, so that

$$\int_{\mathbb{R}^2} |\nabla\tilde\eta|^2 + \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^2} \tilde\eta^4 - 2a(x)\tilde\eta^2 \geq 0. \tag{6.30}$$

Combining (6.29) and (6.30), we derive that $\int_{\mathbb{R}^2} \tilde\eta^4 = 0$, i.e., $\tilde\eta \equiv 0$. □

We recall the following classical result:

Lemma 6.4 *The embedding $\mathcal{H} \hookrightarrow L^2(\mathbb{R}^2, \mathbb{C})$ is compact.*

Proof. Let $u_n \rightharpoonup 0$ weakly in $\mathcal{H}$ as $n \to \infty$. Extracting a subsequence if necessary, by the Sobolev embedding theorem, we may assume that $u_n \to 0$ strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$. Obviously, $\int_{\mathbb{R}^2} |x|^2|u_n|^2 \leq C$. For any $R > 0$, we have

$$R^2 \limsup_{n\to\infty} \int_{\mathbb{R}^2 \setminus B_R} |u_n|^2 \leq \limsup_{n\to\infty} \int_{\mathbb{R}^2} |x|^2|u_n|^2 \leq C.$$

Letting $R \to +\infty$ in this inequality, we conclude that $u_n \to 0$ strongly in $L^2(\mathbb{R}^2)$. □

Remark 6.2 *We emphasize that from the proof of Theorem 6.2, it follows that any smooth function η satisfying*

$$\begin{cases} -\varepsilon^2 \Delta \eta \geq (a(x) - |\eta|^2)\eta & \text{in } \mathbb{R}^2, \\ \eta > 0 & \text{in } \mathbb{R}^2, \end{cases}$$

verifies $\eta \geq \eta_\varepsilon$ in $\mathbb{R}^2$.

Proof of Proposition 6.3. Proof of 6.3.a). We construct an explicit test function $\varphi \in H^1(\mathbb{R}^2)$ such that $E_\varepsilon(\varphi) \leq C|\ln \varepsilon|$. Since η_ε minimizes E_ε, we deduce $E_\varepsilon(\eta_\varepsilon) \leq E_\varepsilon(\varphi) \leq C|\ln \varepsilon|$. The function φ is defined as in [61]: let

$$\gamma(s) = \begin{cases} \sqrt{s} & \text{if } s \geq \varepsilon^{2/3}, \\ \dfrac{s}{\varepsilon^{1/3}} & \text{otherwise} \end{cases}$$

and set $\varphi(x) = \gamma(a^+(x))$ for $x \in \mathbb{R}^2$. It results that

$$\int_{\mathbb{R}^2} |\nabla \varphi|^2 \leq C|\ln \varepsilon| \quad \text{and} \quad \int_{\mathbb{R}^2} (a^+ - \varphi^2)^2 \leq C\varepsilon^2 \tag{6.31}$$

for a positive constant C independent of ε.

Proof of 6.3.b). We construct a supersolution $\overline{\eta}$ of (6.19) of the form:

$$\overline{\eta}(x) = \begin{cases} \sqrt{a(x)} & \text{if } |x|_\Lambda \leq \sqrt{a_0 - \delta}\,, \\ \dfrac{-|x|_\Lambda \sqrt{a_0 - \delta} + a_0}{\sqrt{\delta}} & \text{if } \sqrt{a_0 - \delta} \leq |x|_\Lambda \leq r_\delta, \\ \beta \exp(-|x|_\Lambda^2 / 2\sigma) & \text{otherwise,} \end{cases} \tag{6.32}$$

where $\delta > 0$ will be determined later,

$$r_\delta = \frac{a_0}{2\sqrt{a_0 - \delta}} + \frac{\sqrt{a_0}}{2}$$

and β, σ are chosen such that $\overline{\eta} \in C^1(\mathbb{R}^2)$, i.e.,

$$\beta = \frac{a_0 - \sqrt{a_0(a_0 - \delta)}}{2\sqrt{\delta}} \exp(r_\delta^2 / 2\sigma) \quad \text{and} \quad \sigma = \frac{a_0 \delta}{4(a_0 - \delta)}.$$

A straightforward computation shows that for $\delta = 4a_0^{1/3}\varepsilon^{2/3}$, $\overline{\eta}$ is a supersolution of (6.19) and we also have

$$r_\delta - \sqrt{a_0} = \mathcal{O}(\varepsilon^{2/3}), \quad \sigma = \mathcal{O}(\varepsilon^{2/3}) \quad \text{and} \quad \beta = \mathcal{O}(\varepsilon^{1/3} e^{a_0/2\sigma}).$$

By Remark 6.2, it results that $\eta_\varepsilon \leq \overline{\eta}$ in $\mathbb{R}^2$ which leads to *6.3.b)*. Notice that we also obtain

$$\begin{cases} \eta_\varepsilon(x) \leq \sqrt{a(x)} & \text{for } |x|_\Lambda \leq \sqrt{a_0 - \delta}\,, \\ \eta_\varepsilon(x) \leq C\varepsilon^{1/3} & \text{for } \sqrt{a_0 - \delta} \leq |x|_\Lambda \leq \sqrt{a_0}\,. \end{cases} \tag{6.33}$$

Proof of 6.3.c). The estimate *6.3.c)* follows the ideas in Proposition 2.1 in [3]. Let $x_0 \in \mathcal{D}$ be such that

$$\operatorname{dist}(x_0, \partial\mathcal{D}) \geq \varepsilon^{1/3} \tag{6.34}$$

and set $\alpha = \min\left\{a(y),\, y \in B(x_0,\, \varepsilon^{2/3})\right\}$. We want to construct a subsolution in $B_\delta(x_0)$. For $\tilde{\varepsilon} = \varepsilon^{1/3}/\sqrt{\alpha}$, we denote by $\tilde{w}$ the unique solution of

$$\begin{cases} -\Delta\tilde{w} + \dfrac{1}{\tilde{\varepsilon}^2}\,(\tilde{w}^2 - 1)\tilde{w} = 0 & \text{in } B_1, \\ \tilde{w} > 0 & \text{in } B_1, \\ \tilde{w} = 0 & \text{on } \partial B_1. \end{cases} \tag{6.35}$$

From Proposition 2.1 in [14], we know that

$$0 \leq 1 - \tilde{w}(x) \leq C \exp\left(-\frac{1 - |x|^2}{2\tilde{\varepsilon}}\right).$$

Then we map $\tilde{w}$ to $B(x_0, \varepsilon^{2/3})$, namely

$$w(x) = \sqrt{\alpha}\,\tilde{w}\left(\frac{x - x_0}{\varepsilon^{2/3}}\right).$$

From (6.35) we derive

$$-\Delta w + \frac{1}{\varepsilon^2}(w^2 - a(x))w \leq -\Delta w + \frac{1}{\varepsilon^2}(w^2 - \alpha)w = 0 \quad \text{in} \quad B(x_0, \varepsilon^{2/3}).$$

Since $\eta_\varepsilon > 0$ on $\partial B(x_0, \varepsilon^{2/3})$, by the uniqueness of $\tilde{w}$, we deduce that

$$w \leq \eta_\varepsilon \quad \text{in} \quad B(x_0, \varepsilon^{2/3}).$$

The decay estimate on $\tilde{w}$ implies $0 \leq \sqrt{\alpha} - w(x_0) \leq C\sqrt{\alpha}\,\exp\left(-\frac{\sqrt{\alpha}}{2\varepsilon^{1/3}}\right) \ll C\sqrt{\alpha}\,\varepsilon^{1/3}$. By (6.34), we have

$$\sqrt{a(x_0)} - \sqrt{\alpha} \leq C\sqrt{a(x_0)}\varepsilon^{1/3}.$$

Then (6.33) yields

$$0 \leq \frac{\sqrt{a(x_0)} - \eta_\varepsilon(x_0)}{\sqrt{a(x_0)}} \leq \frac{\sqrt{a(x_0)} - w(x_0)}{\sqrt{a(x_0)}} = \frac{\sqrt{a(x_0)} - \sqrt{\alpha}}{\sqrt{a(x_0)}} + \frac{\sqrt{\alpha} - w(x_0)}{\sqrt{a(x_0)}} \leq C\varepsilon^{1/3},$$

for a constant C independent of x_0.

Proof of 6.3.d). Taking $x_0 \in \mathbb{R}^2$ arbitrarily, it suffices to show that $|\nabla \eta_\varepsilon| \le C\varepsilon^{-1}$ in $B(x_0,\varepsilon)$ with a constant C independent of x_0. We define the re-scaled function $\phi_\varepsilon : B_2(0) \to \mathbb{R}$ by $\phi_\varepsilon(y) = \eta_\varepsilon(x_0+\varepsilon y)$. From estimates *6.3.b)* and *6.3.c)*, we derive that $|\Delta\phi_\varepsilon| = |\left(a(x_0+\varepsilon y) - \phi_\varepsilon^2\right)\phi_\varepsilon| \le C$ in $B_2(0)$ for a constant C independent of x_0. By elliptic regularity, we deduce that for any $1 \le p < \infty$, $\|\phi_\varepsilon\|_{W^{2,p}(B_1(0))} \le C_p$ for a constant C_p independent of ε and x_0. Taking some $p > 2$, it implies that $\|\nabla\phi_\varepsilon\|_{L^\infty(B_1(0))} \le C$ for a constant C independent of ε and x_0 which yields the result.

Proof of 6.3.e). The idea of the proof is due to Shafrir [84]. First we prove that $|\nabla\eta_\varepsilon|$ remains bounded with respect to ε in any compact set $K \subset \mathcal{D}$. We choose some radii $0 < r < R < \sqrt{a_0}$ such that $K \subset B_r^\Lambda \subset B_R^\Lambda \subset \mathcal{D}$. We claim that

$$|\eta_\varepsilon - \sqrt{a}\,| \le C_R\,\varepsilon^2 \quad \text{in } B_r^\Lambda. \tag{6.36}$$

Indeed, we infer from (6.19) that

$$-\varepsilon^2\Delta(\sqrt{a} - \eta_\varepsilon) + \eta_\varepsilon(\eta_\varepsilon + \sqrt{a}\,)(\sqrt{a} - \eta_\varepsilon) = -\varepsilon^2\Delta(\sqrt{a}\,) = \mathcal{O}(\varepsilon^2) \quad \text{in } B_R^\Lambda.$$

By estimate *6.3.c)*, we have $|\sqrt{a} - \eta_\varepsilon| \le \frac{\sqrt{a}}{2}$ in B_R^Λ for ε small. Thus $\eta_\varepsilon(\eta_\varepsilon + \sqrt{a}) \ge A_R > 0$ in B_R^Λ for some positive constant A_R which only depends on R. Then (6.36) follows from Lemma 6.5 below (which is a slight modification of Lemma 2 in [16]).

Lemma 6.5 *Assume that $A > 0$ and $0 < r < R$. Let w_ε be a smooth function satisfying*

$$\begin{cases} -\varepsilon^2\Delta w_\varepsilon + Aw_\varepsilon \le B\varepsilon^2 & \text{in } B_R^\Lambda, \\ w_\varepsilon \le 1 & \text{on } \partial B_R^\Lambda, \end{cases}$$

for some constant $B \in \mathbb{R}$. Then $w_\varepsilon \le C\varepsilon^2$ in B_r^Λ with $C = C(R,r,A,B)$.

Proof of 6.3.e) completed. By (6.19) and (6.36), we deduce that η_ε is uniformly bounded in $W^{2,p}(B_r^\Lambda)$ for any $1 \le p < \infty$. In particular, it implies

$$\|\nabla\eta_\varepsilon\|_{L^\infty(K)} \le C_K\,. \tag{6.37}$$

We repeat the above argument with the functions $z_\varepsilon = \frac{\partial\eta_\varepsilon}{\partial x_j}$ and $z_0 = \frac{\partial\sqrt{a}}{\partial x_j}$, $j = 1,2$. Obviously, we can assume that (6.36) and (6.37) hold in B_R^Λ. Using (6.36), we easily check that

$$-\varepsilon^2\Delta(z_\varepsilon - z_0) + (3\eta_\varepsilon^2 - a)(z_\varepsilon - z_0) = \mathcal{O}(\varepsilon^2).$$

By (6.37), we can apply Lemma 6.5 which yields the announced result. □

We now state a result that we will require in Section 2.2. We follow here a technique introduced by Struwe (see [87]).

Lemma 6.6 *Let $I : (0,\infty) \mapsto \mathbb{R}_+$ defined by*

$$I(\varepsilon) = \mathrm{Min}\left\{E_\varepsilon(\eta) \,:\, \eta \in \mathcal{H}\right\}. \tag{6.38}$$

Then $I(\cdot)$ is locally Lipschitz continuous and non-increasing in $(0,\infty)$. Moreover,

$$|I'(\varepsilon)| \leq C\left(\frac{|\ln \varepsilon|}{\varepsilon} + 1\right) \quad \textit{for almost every } \varepsilon \in (0,\infty). \tag{6.39}$$

Proof. For every $\varepsilon \geq \frac{a_0}{\lambda}$, we know by Theorem 6.19 that $I(\varepsilon) = E_\varepsilon(0) = \frac{C}{\varepsilon^2}$ and $|I'(\varepsilon)| = \frac{C}{\varepsilon^3}$. Hence it remains to prove that the conclusion holds for $0 < \varepsilon < \frac{a_0}{\lambda} + 1$. By convention, we set $\eta_\varepsilon \equiv 0$ if $\varepsilon \geq \frac{a_0}{\lambda}$. Naturally, we have

$$I(\varepsilon) = E_\varepsilon(\eta_\varepsilon) \leq E_\varepsilon(0) = \frac{C}{\varepsilon^2} \text{ for every } \varepsilon > 0. \tag{6.40}$$

If ε is small, we infer from *6.3.b)* in Proposition 6.3 that we can find some radius $R > \frac{\sqrt{a_0}}{\Lambda}$ such that

$$\int_{\mathbb{R}^2\setminus B_R} |\eta_\varepsilon|^4 + 2a^-(x)|\eta_\varepsilon|^2 \leq C\varepsilon^3. \tag{6.41}$$

Using (6.40), we deduce that (6.41) holds for $0 < \varepsilon < \frac{a_0}{\lambda} + 1$. Let us now fix some $\varepsilon_0 \in (0, \frac{a_0}{\lambda} + 1)$ and $0 < h \ll 1$. We have

$$E_{\varepsilon_0+h}(\eta_{\varepsilon_0+h}) = I(\varepsilon_0 + h) \leq E_{\varepsilon_0+h}(\eta_{\varepsilon_0-h}) \leq E_{\varepsilon_0-h}(\eta_{\varepsilon_0-h}) = I(\varepsilon_0 - h) \leq E_{\varepsilon_0-h}(\eta_{\varepsilon_0+h}).$$

Hence, I is a non-increasing function and

$$E_{\varepsilon_0-h}(\eta_{\varepsilon_0-h}) - E_{\varepsilon_0+h}(\eta_{\varepsilon_0-h}) \leq I(\varepsilon_0 - h) - I(\varepsilon_0 + h) \leq E_{\varepsilon_0-h}(\eta_{\varepsilon_0+h}) - E_{\varepsilon_0+h}(\eta_{\varepsilon_0+h}).$$

By (6.41), it leads to

$$\frac{I(\varepsilon_0 + h) - I(\varepsilon_0 - h)}{2h} \geq \frac{-\varepsilon_0}{2(\varepsilon_0 + h)^2(\varepsilon_0 - h)^2}\left(\int_{B_R} (a(x) - |\eta_{\varepsilon_0+h}|^2)^2 - (a^-(x))^2\right) - C \tag{6.42}$$

and

$$\frac{I(\varepsilon_0 + h) - I(\varepsilon_0 - h)}{2h} \leq \frac{-\varepsilon_0}{2(\varepsilon_0 + h)^2(\varepsilon_0 - h)^2}\int_{B_R} \left[(a(x) - |\eta_{\varepsilon_0-h}|^2)^2 - (a^-(x))^2\right] \tag{6.43}$$

which proves with (6.40) that $I(\cdot)$ is locally Lipschitz continuous in $(0, \frac{a_0}{\lambda} + 1)$. Therefore $I(\cdot)$ is differentiable almost everywhere in $(0, \frac{a_0}{\lambda} + 1)$. We easily check using standard arguments that $\eta_{\varepsilon_0-h} \to \eta_{\varepsilon_0}$ and $\eta_{\varepsilon_0+h} \to \eta_{\varepsilon_0}$ in $L^4(B_R)$ as $h \to 0$. Assuming that ε_0 is a point of differentiability of $I(\cdot)$, we obtain letting $h \to 0$ in (6.42) and (6.43),

$$I'(\varepsilon_0) = \frac{-1}{2\varepsilon_0^3}\int_{B_R} \left[(a(x) - |\eta_{\varepsilon_0}|^2)^2 - (a^-(x))^2\right] + \mathcal{O}(1). \tag{6.44}$$

Then we deduce (6.39) combining (6.20) and (6.44). □

6.2.2 The profile under the mass constraint

In this section, we study the minimization problem (6.8). The motivation is to define the "vortex-free" profile

$$\tilde{\eta}_\varepsilon e^{i\Omega S} \tag{6.45}$$

and to construct admissible test functions for the model. Existence and uniqueness results for general potentials a are also presented in [64]. Our contribution consists in proving the identity (6.47) between η_ε and $\tilde{\eta}_\varepsilon$. By this formula, we obtain a precise information about the asymptotic behavior of the profile $\tilde{\eta}_\varepsilon$.

Theorem 6.7 *For every $\varepsilon > 0$, problem* (6.8) *admits a unique solution $\tilde{\eta}_\varepsilon$ up to a complex multiplier of modulus one. Moreover, there exists $k_\varepsilon \in \mathbb{R}$ such that*

$$-\Delta\tilde{\eta}_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |\tilde{\eta}_\varepsilon|^2)\tilde{\eta}_\varepsilon + k_\varepsilon\tilde{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2 \tag{6.46}$$

and $\tilde{\eta}_\varepsilon$ is characterized by

$$\tilde{\eta}_\varepsilon(x) = \frac{\sqrt{a_0 + k_\varepsilon\varepsilon^2}}{\sqrt{a_0}}\,\eta_{\tilde{\varepsilon}}\Big(\frac{\sqrt{a_0}\,x}{\sqrt{a_0 + k_\varepsilon\varepsilon^2}}\Big) \quad \text{with} \quad \tilde{\varepsilon} = \frac{a_0\varepsilon}{a_0 + k_\varepsilon\varepsilon^2} \in (0, \frac{a_0}{\lambda})\,. \tag{6.47}$$

In addition, for small $\varepsilon > 0$,

$$|k_\varepsilon| \le C|\ln\varepsilon| \tag{6.48}$$

and

$$\big|E_\varepsilon(\tilde{\eta}_\varepsilon) - E_\varepsilon(\eta_\varepsilon)\big| \le C\varepsilon^2|\ln\varepsilon|^2. \tag{6.49}$$

Identity (6.47) gives us automatically the asymptotic properties of $\tilde{\eta}_\varepsilon$ from those of η_ε by a change of scale and hence we obtain the analogue of Proposition 6.3 for $\tilde{\eta}_\varepsilon$:

Proposition 6.8 *For ε sufficiently small, we have*

6.8.a) $E_\varepsilon(\tilde{\eta}_\varepsilon) \le C|\ln\varepsilon|$,

6.8.b) $0 < \tilde{\eta}_\varepsilon(x) \le C\varepsilon^{1/3}\exp\left(\frac{a(x)}{4\varepsilon^{2/3}}\right)$ *for* $|x|_\Lambda \ge \sqrt{a_0} + \varepsilon$,

6.8.c) $\big|\sqrt{a(x)} - \tilde{\eta}_\varepsilon(x)\big| \le C\varepsilon^{1/3}\sqrt{a(x)}$ *for* $x \in \mathcal{D}$ *with* $|x|_\Lambda < \sqrt{a_0} - 2\varepsilon^{1/3}$,

6.8.d) $\|\nabla\tilde{\eta}_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{-1}$,

6.8.e) $\|\tilde{\eta}_\varepsilon - \sqrt{a}\,\|_{C^1(K)} \le C_K\varepsilon^2|\ln\varepsilon|$ *for any compact subset* $K \subset \mathcal{D}$.

Remark 6.3 *We observe that* 6.8.a) *in Proposition 6.8 implies for small $\varepsilon > 0$,*

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} |\tilde{\eta}_\varepsilon|^4 + 2a^-(x)|\tilde{\eta}_\varepsilon|^2 + \int_{\mathcal{D}} (a(x) - |\tilde{\eta}_\varepsilon|^2)^2 \le C\varepsilon^2|\ln\varepsilon| \tag{6.50}$$

Proof of Theorem 6.7. Step 1: Existence. Let $(\eta_n)_{n\in\mathbb{N}}$ be a minimizing sequence for (6.8). Extracting a subsequence if necessary, by Lemma 6.4, we may assume that $\eta_n \rightharpoonup \tilde{\eta}_\varepsilon$ weakly in $\mathcal{H}$ and strongly in $L^2(\mathbb{R}^2)$ as $n \to \infty$. Then we derive from (6.2) that $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. We easily check that E_ε is lower semi-continuous on $\mathcal{H}$ with respect to the weak $\mathcal{H}$-topology and therefore $E_\varepsilon(\tilde{\eta}_\varepsilon) \le \liminf_{n\to\infty} E_\varepsilon(\eta_n)$, i.e., $\tilde{\eta}_\varepsilon$ is a minimizer of (6.8). Since $E_\varepsilon(|\tilde{\eta}_\varepsilon|) = E_\varepsilon(\tilde{\eta}_\varepsilon)$, we infer that $\tilde{\eta}_\varepsilon = |\tilde{\eta}_\varepsilon| e^{i\alpha}$ for some constant α. Hence we may assume that $\tilde{\eta}_\varepsilon \ge 0$ in $\mathbb{R}^2$.

Step 2: Proof of (6.47). Let $\tilde{\eta}_\varepsilon$ be a solution of (6.8). As in *Step 1*, we may assume that $\tilde{\eta}_\varepsilon \ge 0$. Since $\tilde{\eta}_\varepsilon$ is a minimizer of E_ε under the constraint $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, there exists $k_\varepsilon \in \mathbb{R}$ such that $\tilde{\eta}_\varepsilon$ satisfies (6.46) and we necessarily have $\tilde{\eta}_\varepsilon > 0$ in $\mathbb{R}^2$ by the maximum principle. We rewrite equation (6.46) as

$$-\Delta\tilde{\eta}_\varepsilon = \frac{1}{\varepsilon^2}(a_\varepsilon(x) - |\tilde{\eta}_\varepsilon|^2)\tilde{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2, \tag{6.51}$$

with

$$a_\varepsilon(x) = a_0 + k_\varepsilon\varepsilon^2 - |x|^2_\Lambda. \tag{6.52}$$

Multiplying (6.51) by $\tilde{\eta}_\varepsilon$, integrating by parts and using that $\int_{\mathbb{R}^2} |\tilde{\eta}_\varepsilon|^2 = 1$, we obtain that

$$\frac{a_0 + k_\varepsilon\varepsilon^2}{\varepsilon^2} = \int_{\mathbb{R}^2} |\nabla\tilde{\eta}_\varepsilon|^2 + \frac{|x|^2_\Lambda}{\varepsilon^2}|\tilde{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}|\tilde{\eta}_\varepsilon|^4 > \lambda_1(L_\varepsilon, \mathbb{R}^2) = \frac{\lambda}{\varepsilon}$$

and therefore, $\tilde{\varepsilon} = \frac{a_0\varepsilon}{a_0 + k_\varepsilon\varepsilon^2} \in \big(0, \frac{a_0}{\lambda}\big)$. Setting

$$\vartheta_\varepsilon(x) = \frac{\sqrt{a_0}}{\sqrt{a_0 + k_\varepsilon\varepsilon^2}}\,\tilde{\eta}_\varepsilon\Big(\frac{\sqrt{a_0 + k_\varepsilon\varepsilon^2}\,x}{\sqrt{a_0}}\Big), \tag{6.53}$$

a straightforward computation shows that

$$\begin{cases} -\tilde{\varepsilon}^2\Delta\vartheta_\varepsilon = (a(x) - |\vartheta_\varepsilon|^2)\vartheta_\varepsilon & \text{in } \mathbb{R}^2, \\ \vartheta_\varepsilon > 0 & \text{in } \mathbb{R}^2. \end{cases}$$

By Theorem 6.2, it leads to

$$\vartheta_\varepsilon \equiv \eta_{\tilde{\varepsilon}}. \tag{6.54}$$

Combining this identity with (6.53) we obtain (6.47).

Step 3: Uniqueness. Let $\hat{\eta}_\varepsilon$ be another solution of (6.8). As for $\tilde{\eta}_\varepsilon$, we may assume that $\hat{\eta}_\varepsilon$ is a real positive function. Let $\hat{k}_\varepsilon$ be the Lagrange multiplier associated to $\hat{\eta}_\varepsilon$, i.e., $\hat{\eta}_\varepsilon$ satisfies

$$-\Delta\hat{\eta}_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |\hat{\eta}_\varepsilon|^2)\hat{\eta}_\varepsilon + \hat{k}_\varepsilon\hat{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2.$$

By *Step 2*, the solution $\hat{\eta}_\varepsilon$ is characterized by

$$\hat{\eta}_\varepsilon(x) = \frac{\sqrt{a_0 + \hat{k}_\varepsilon\varepsilon^2}}{\sqrt{a_0}}\,\eta_{\hat{\varepsilon}}\Big(\frac{\sqrt{a_0}\,x}{\sqrt{a_0 + \hat{k}_\varepsilon\varepsilon^2}}\Big) \quad \text{with} \quad \hat{\varepsilon} = \frac{a_0\varepsilon}{a_0 + \hat{k}_\varepsilon\varepsilon^2} \in (0, \frac{a_0}{\lambda}).$$

Hence it suffices to prove that $\hat{k}_\varepsilon = k_\varepsilon$. We proceed by contradiction. Assume for instance that $k_\varepsilon < \hat{k}_\varepsilon$. Then $\hat{\eta}_\varepsilon$ satisfies

$$-\Delta\hat{\eta}_\varepsilon \ge \frac{1}{\varepsilon^2}(a(x) - |\hat{\eta}_\varepsilon|^2)\hat{\eta}_\varepsilon + k_\varepsilon\hat{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{6.55}$$

We consider the function

$$\hat{\vartheta}_\varepsilon(x) = \frac{\sqrt{a_0}}{\sqrt{a_0 + k_\varepsilon \varepsilon^2}}\, \hat{\eta}_\varepsilon\Big(\frac{\sqrt{a_0 + k_\varepsilon \varepsilon^2}\, x}{\sqrt{a_0}}\Big), \tag{6.56}$$

which satisfies by (6.55),

$$\begin{cases} -\tilde{\varepsilon}^2 \Delta \hat{\vartheta}_\varepsilon \geq (a(x) - |\hat{\vartheta}_\varepsilon|^2)\hat{\vartheta}_\varepsilon & \text{in } \mathbb{R}^2, \\ \hat{\vartheta}_\varepsilon > 0 & \text{in } \mathbb{R}^2. \end{cases}$$

Therefore $\hat{\vartheta}_\varepsilon$ is a supersolution of (6.19) with $\tilde{\varepsilon}$ instead of ε. By Remark 6.2 we infer that $\hat{\vartheta}_\varepsilon \geq \eta_{\tilde{\varepsilon}}$ in $\mathbb{R}^2$. By (6.47) and (6.56), it leads to $\hat{\eta}_\varepsilon \geq \tilde{\eta}_\varepsilon$ in $\mathbb{R}^2$. Since $\|\hat{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = \|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we conclude that $\hat{\eta}_\varepsilon \equiv \tilde{\eta}_\varepsilon$ and hence $k_\varepsilon = \hat{k}_\varepsilon$, contradiction.

Step 4: Energy bound for small $\varepsilon > 0$. We now prove that for small $\varepsilon > 0$,

$$E_\varepsilon(\tilde{\eta}_\varepsilon) \leq C|\ln \varepsilon|. \tag{6.57}$$

Let φ be the test function constructed in the proof of *6.3.a)* in Proposition 6.3. Setting $\hat{\varphi} = \|\varphi\|^{-1}_{L^2(\mathbb{R}^2)}\varphi$, it suffices to check that $E_\varepsilon(\hat{\varphi}) \leq C|\ln \varepsilon|$ by the minimizing property of $\tilde{\eta}_\varepsilon$. First we show that $\|\varphi\|_{L^2(\mathbb{R}^2)}$ remains close to 1 as $\varepsilon \to 0$. Since $\int_{\mathbb{R}^2} a^+ = 1$, we have $\int_{\mathbb{R}^2} |\varphi|^2 = 1 + \int_{\mathcal{D}}(|\varphi|^2 - a^+(x))$ and by (6.31),

$$\int_{\mathcal{D}} \big||\varphi|^2 - a^+(x)\big| \leq C\left(\int_{\mathcal{D}}(|\varphi|^2 - a^+(x))^2\right)^{1/2} \leq C\varepsilon.$$

Hence $\|\varphi\|^2_{L^2(\mathbb{R}^2)} = 1 + \mathcal{O}(\varepsilon)$. Then we derive from (6.31),

$$\int_{\mathbb{R}^2} |\nabla \hat{\varphi}|^2 = \|\varphi\|^{-2}_{L^2(\mathbb{R}^2)} \int_{\mathbb{R}^2} |\nabla \varphi|^2 \leq \int_{\mathbb{R}^2} |\nabla \varphi|^2 + C\varepsilon|\ln \varepsilon| \leq C|\ln \varepsilon|$$

and

$$\begin{aligned} \frac{1}{\varepsilon^2}\int_{\mathcal{D}}(a(x) - |\hat{\varphi}|^2)^2 = &\frac{1}{\varepsilon^2}\int_{\mathcal{D}}(a(x) - |\varphi|^2)^2 + \frac{2(1 - \|\varphi\|^{-2}_{L^2(\mathbb{R}^2)})}{\varepsilon^2}\int_{\mathcal{D}}(a(x) - |\varphi|^2)|\varphi|^2 \\ &+ \frac{(1 - \|\varphi\|^{-2}_{L^2(\mathbb{R}^2)})^2}{\varepsilon^2}\int_{\mathcal{D}}|\varphi|^4 \\ \leq &C + C\left(\frac{1}{\varepsilon^2}\int_{\mathcal{D}}(a - |\varphi|^2)^2\right)^{1/2} \leq C. \end{aligned}$$

Therefore $E_\varepsilon(\hat{\varphi}) \leq C|\ln \varepsilon|$ and (6.57) holds.

Step 5: First bound on the Lagrange multiplier for small $\varepsilon > 0$. Let $\tilde{\eta}_\varepsilon$ be the positive solution of (6.8) and let $k_\varepsilon \in \mathbb{R}$ be such that $\tilde{\eta}_\varepsilon$ satisfies (6.46). Multiplying (6.46) by $\tilde{\eta}_\varepsilon$, integrating by parts and using that $\int_{\mathbb{R}^2} |\tilde{\eta}_\varepsilon|^2 = 1$, we obtain that

$$k_\varepsilon = \int_{\mathbb{R}^2} |\nabla \tilde{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\big(|\tilde{\eta}_\varepsilon|^2 - a(x)\big)|\tilde{\eta}_\varepsilon|^2. \tag{6.58}$$

From (6.57) we derive

$$\left|\int_{\mathbb{R}^2} |\nabla\tilde{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2\setminus\mathcal{D}} \big(|\tilde{\eta}_\varepsilon|^2 - a(x)\big)|\tilde{\eta}_\varepsilon|^2\right| \le C|\ln\varepsilon|$$

and

$$\left|\frac{1}{\varepsilon^2}\int_{\mathcal{D}}(|\tilde{\eta}_\varepsilon|^2 - a(x))|\tilde{\eta}_\varepsilon|^2\right| \le \frac{1}{\varepsilon^2}\int_{\mathcal{D}}(|\tilde{\eta}_\varepsilon|^2 - a(x))^2 + \frac{1}{\varepsilon^2}\int_{\mathcal{D}} a(x)\big||\tilde{\eta}_\varepsilon|^2 - a(x)\big|$$
$$\le C|\ln\varepsilon| + \frac{C}{\varepsilon^2}\left(\int_{\mathcal{D}}(|\tilde{\eta}_\varepsilon|^2 - a(x))^2\right)^{1/2} \le C\varepsilon^{-1}|\ln\varepsilon|^{1/2}.$$

Hence, by (6.58), we have

$$|k_\varepsilon| \le C\varepsilon^{-1}|\ln\varepsilon|^{1/2}. \tag{6.59}$$

Step 6: Proof of (6.48). We define the functional $\tilde{E}_\varepsilon : \mathcal{H} \to \mathbb{R}$ by

$$\tilde{E}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon(x) - |u|^2)^2 - (a_\varepsilon^-(x))^2 \tag{6.60}$$

where $a_\varepsilon(x)$ is given by (6.52). Then, by (6.47), we get

$$\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = \frac{a_0 + k_\varepsilon\varepsilon^2}{a_0} E_{\tilde{\varepsilon}}(\eta_{\tilde{\varepsilon}}) = \frac{a_0 + k_\varepsilon\varepsilon^2}{a_0} I(\tilde{\varepsilon}). \tag{6.61}$$

Since $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we have

$$\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon^+(x))^2 - (a^+(x))^2 \tag{6.62}$$

$$\ge I(\varepsilon) - \frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon^+(x))^2 - (a^+(x))^2. \tag{6.63}$$

Using the fact that $\int_{\mathbb{R}^2} a^+ = 1$, a simple computation leads to

$$-\frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon^+(x))^2 - (a^+(x))^2 = \frac{\pi a_0 k_\varepsilon^2\varepsilon^2}{4\Lambda} + \frac{\pi k_\varepsilon^3\varepsilon^4}{12\Lambda}. \tag{6.64}$$

Combining (6.61), (6.63) and (6.64), we infer that

$$\frac{\pi a_0 k_\varepsilon^2\varepsilon^2}{4\Lambda} \le |I(\tilde{\varepsilon}) - I(\varepsilon)| + \frac{|k_\varepsilon|\varepsilon^2}{a_0} I(\tilde{\varepsilon}) + \frac{\pi|k_\varepsilon|^3\varepsilon^4}{12\Lambda}. \tag{6.65}$$

For small $\varepsilon > 0$, we obtain using (6.39), (6.59) and *6.3.a)* in Proposition 6.3,

$$\big|I(\tilde{\varepsilon}) - I(\varepsilon)\big| \le C\varepsilon^{-1}|\ln\varepsilon||\tilde{\varepsilon} - \varepsilon| \le C|k_\varepsilon|\varepsilon^2|\ln\varepsilon| \tag{6.66}$$

and

$$\frac{|k_\varepsilon|\varepsilon^2}{a_0} I(\tilde{\varepsilon}) \le C|k_\varepsilon|\varepsilon^2|\ln\varepsilon|, \quad \frac{\pi|k_\varepsilon|^3\varepsilon^4}{12} \le C|k_\varepsilon|\varepsilon^2|\ln\varepsilon|.$$

Inserting this estimates in (6.65), we deduce that $|k_\varepsilon| \le C|\ln\varepsilon|$.

Step 7: Proof of (6.49). From (6.48), (6.61), (6.66) and *6.3.a)* in Proposition 6.3, we derive that $\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{O}(\varepsilon^2|\ln\varepsilon|^2)$. On the other hand, (6.48), (6.62) and (6.64) yield $\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \mathcal{O}(\varepsilon^2|\ln\varepsilon|^2)$ and (6.49) follows. □

6.3 Minimizing F_ε under the mass constraint

Our aim in this section is to make a first description of minimizers u_ε of F_ε under the mass constraint. We prove the existence of u_ε and some asymptotic properties of u_ε (in particular, we show that $|u_\varepsilon|$ is concentrated in $\mathcal{D}$). We also present some tools that we will require in the sequel, in particular the splitting of energy (6.9).

6.3.1 Existence and first properties of minimizers

First, we seek minimizers u_ε of F_ε under the constraint $\|u_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. We perform the minimization in $\mathcal{H}$ and we shall see that F_ε is well defined on $\mathcal{H}$:

Lemma 6.9 *For any $u \in \mathcal{H}$, $\sigma > 0$ and $R > \sqrt{a_0}$, we have*

$$\left|\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u)\right| \le \sigma \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\Omega^2 R^2}{8\Lambda^2 \sigma (R^2 - a_0)} \int_{\mathbb{R}^2} \left[(a(x) - |u|^2)^2 - (a^-(x))^2\right] + C_{R,\sigma}\,\Omega^2.$$

In particular, the functional F_ε is well defined on $\mathcal{H}$.

Proposition 6.10 *Assume that $\Omega < \Lambda\varepsilon^{-1}$. Then there exists at least one minimizer u_ε of F_ε in $\left\{u \in \mathcal{H} \,:\, \|u\|_{L^2(\mathbb{R}^2)} = 1\right\}$. Moreover, u_ε is smooth and there exists $\ell_\varepsilon \in \mathbb{R}$ such that u_ε satisfies*

$$-\Delta u_\varepsilon + 2i\Omega x^\perp \cdot \nabla u_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |u_\varepsilon|^2)u_\varepsilon + \ell_\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{6.67}$$

We emphasize that the result is stated for an angular velocity Ω strictly less than Λ/ε but we only consider in this chapter the case of an rotational speed Ω at most of order $|\ln\varepsilon|$, i.e.,

$$\Omega \le \omega_0 |\ln\varepsilon| \tag{6.68}$$

for some positive constant ω_0.

Before proving Lemma 6.9 and Proposition 6.10, we present some basic properties of any minimizer u_ε. We point out that the exponential decay of $|u_\varepsilon|$ outside the domain $\mathcal{D}$ (see *6.11.c)* in Proposition 6.11) shows that almost all the mass of u_ε is concentrated in $\mathcal{D}$.

Proposition 6.11 *Assume that* (6.68) *holds for some $\omega_0 > 0$. For ε sufficiently small, we have*

6.11.a) $E_\varepsilon(u_\varepsilon) \le C_{\omega_0} |\ln\varepsilon|^2$,

6.11.b) $|\ell_\varepsilon| \le C_{\omega_0}\, \varepsilon^{-1} |\ln\varepsilon|$,

6.11.c) $|u_\varepsilon(x)| \le C_{\omega_0}\, \varepsilon^{1/3} |\ln\varepsilon|^{1/2} \exp\left(\dfrac{a(x)}{4\varepsilon^{2/3}}\right)$ *for $x \in \mathbb{R}^2 \setminus \mathcal{D}$ with* $|x|_\Lambda \ge \sqrt{a_0 + 2\varepsilon^{1/3}}$,

6.11.d) $|u_\varepsilon(x)| \le \sqrt{a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2}$ *for $x \in \mathcal{D}$ with* $|x|_\Lambda \le \sqrt{a_0} - \varepsilon^{1/8}$,

6.11.e) $|u_\varepsilon| \le \sqrt{a_0} + C_{\omega_0}\, \varepsilon |\ln\varepsilon|$ *in* $\mathbb{R}^2$,

6.11.f) $\|\nabla u_\varepsilon\|_{L^\infty(K)} \le C_{\omega_0,K}\, \varepsilon^{-1}$ *for any compact set $K \subset \mathbb{R}^2$.*

Remark 6.4 *We observe that* 6.11.a) *in Proposition 6.11 implies*

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} \left(|u_\varepsilon|^4 + 2a^-(x)|u_\varepsilon|^2\right) + \int_{\mathcal{D}} (|u_\varepsilon|^2 - a(x))^2 \le C_{\omega_0}\, \varepsilon^2 |\ln \varepsilon|^2. \tag{6.69}$$

Proof of Lemma 6.9. Let $u \in \mathcal{H}$ and $\sigma \in (0,1)$. We have

$$4\sigma \left|\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u)\right| \le 4\sigma^2 \int_{\mathbb{R}^2} |\nabla u|^2 + \Omega^2 \int_{\mathbb{R}^2} |x|^2|u|^2 \le 4\sigma^2 \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\Omega^2}{\Lambda^2}\int_{\mathbb{R}^2} |x|^2_\Lambda |u|^2.$$

For $R > \sqrt{a_0}$, we easily check that $|x|^2_\Lambda \le -\frac{R^2}{R^2 - a_0}\, a(x)$ whenever $|x|_\Lambda \ge R$. Then we derive

$$4\sigma \left|\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u)\right| \le 4\sigma^2 \int_{\mathbb{R}^2} |\nabla u|^2 - \frac{\Omega^2 R^2}{2\Lambda^2(R^2 - a_0)} \int_{\mathbb{R}^2\setminus B^\Lambda_R} 2a(x)|u|^2 + \frac{\Omega^2}{\Lambda^2}\int_{B^\Lambda_R} |x|^2_\Lambda |u|^2. \tag{6.70}$$

Now we notice that

$$\begin{aligned}\int_{B^\Lambda_R} |x|^2_\Lambda |u|^2 &= \frac{R^2}{2(R^2-a_0)} \int_{B^\Lambda_R} -2a(x)|u|^2 - \frac{a_0}{R^2 - a_0}\int_{B^\Lambda_R} |x|^2_\Lambda|u|^2 + \frac{a_0 R^2}{R^2 - a_0}\int_{B^\Lambda_R} |u|^2 \\ &\le \frac{R^2}{2(R^2-a_0)} \int_{B^\Lambda_R} -2a(x)|u|^2 + \frac{R^2}{2(R^2-a_0)}\int_{B^\Lambda_R} |u|^4 + \frac{\pi R^4 a_0^2}{2\Lambda(R^2-a_0)}.\end{aligned}$$

Inserting this estimate in (6.70), we obtain

$$\begin{aligned}\left|\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u)\right| \le \sigma \int_{\mathbb{R}^2} |\nabla u|^2 &+ \frac{\Omega^2 R^2}{8\Lambda^2 \sigma (R^2 - a_0)} \int_{\mathbb{R}^2} \left[(a(x) - |u|^2)^2 - (a^-(x))^2\right] \\ &+ \frac{\pi \Omega^2 R^4 a_0^2}{8\Lambda^3\sigma(R^2 - a_0)}\end{aligned}$$

and the proof is complete. □

Proof of Proposition 6.10. Since $\Omega < \Lambda\varepsilon^{-1}$, we can find $0 < \delta < 1$ such that $\Omega \le \delta\Lambda\varepsilon^{-1}$. Taking in Lemma 6.9,

$$\sigma = \frac{\delta^2 + 1}{4} \quad \text{and} \quad R = \sqrt{\frac{2(1+\delta^2)a_0}{1 - \delta^2}},$$

we infer that for any $u \in \mathcal{H}$,

$$\frac{1-\delta^2}{4} E_\varepsilon(u) - C_\delta\, \Omega^2 \le F_\varepsilon(u) \le 2\, E_\varepsilon(u) + C_\delta\, \Omega^2. \tag{6.71}$$

We easily check that E_ε is coercive in $\mathcal{H}$ (i.e., there exists a positive constant C such that $E_\varepsilon(u) \ge C(\|u\|^2_{\mathcal{H}} - 1)$ for any $u \in \mathcal{H}$) and by (6.71), F_ε is coercive, too. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ be a minimizing sequence of F_ε in $\left\{u \in \mathcal{H} \,:\, \|u\|_{L^2(\mathbb{R}^2)} = 1\right\}$. From the coerciveness of F_ε, we get that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$ and therefore, there exists $u_\varepsilon \in \mathcal{H}$ such that up to a subsequence,

$$u_n \rightharpoonup u_\varepsilon \text{ weakly in } \mathcal{H} \text{ and } u_n \to u_\varepsilon \text{ in } L^4_{\text{loc}}(\mathbb{R}^2). \tag{6.72}$$

By Lemma 6.4, it results that $u_n \to u_\varepsilon$ in $L^2(\mathbb{R}^2)$ and consequently, $\|u_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. We write for $u \in \mathcal{H}$,

$$\begin{aligned}F_\varepsilon(u) =& \frac{1}{2}\int_{\mathbb{R}^2} \left|(\nabla - i\Omega x^\perp)u\right|^2 + \frac{1}{2\varepsilon^2}\int_{\{a^-(x)\ge\Omega^2\varepsilon^2|x|^2\}} \left[\frac{1}{2}|u|^4 + \left(a^-(x) - \varepsilon^2\Omega^2|x|^2\right)|u|^2\right] \\ &+ \frac{1}{4\varepsilon^2}\int_{\{a^-(x)\le\Omega^2\varepsilon^2|x|^2\}} \left[(a(x) - |u|^2)^2 - (a^-(x))^2 - 2\Omega^2\varepsilon^2|x|^2\,|u|^2\right].\end{aligned}$$

We observe that the functional

$$u \in \mathcal{H} \mapsto \frac{1}{2}\int_{\mathbb{R}^2} \left|(\nabla - i\Omega x^\perp)u\right|^2 + \frac{1}{2\varepsilon^2}\int_{\{a^-(x)\geq \Omega^2\varepsilon^2|x|^2\}} \left[\frac{1}{2}|u|^4 + (a^-(x) - \varepsilon^2\Omega^2|x|^2)|u|^2\right]$$

is convex continuous on $\mathcal{H}$ for the strong topology. Then from (6.72), it follows that $F_\varepsilon(u_\varepsilon) \leq \liminf_{n\to\infty} F_\varepsilon(u_n)$. Hence u_ε minimizes F_ε in $\{u \in \mathcal{H} : \|u\|_{L^2(\mathbb{R}^2)} = 1\}$ and by the Lagrange multiplier rule, there exists $\ell_\varepsilon \in \mathbb{R}$ such that (6.67) holds. By standard elliptic regularity, we deduce that u_ε is smooth in $\mathbb{R}^2$. □

Proof of Proposition 6.11. Proof of 6.11.a). Let $\tilde{\eta}_\varepsilon$ be the positive real minimizer of E_ε under the constraint $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. Since $\tilde{\eta}_\varepsilon$ is real valued, we have $(i\tilde{\eta}_\varepsilon, \nabla\tilde{\eta}_\varepsilon) \equiv 0$ and we derive from (6.57),

$$F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{\eta}_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) \leq C|\ln\varepsilon|. \tag{6.73}$$

By (6.71) (with $\delta = \frac{1}{\sqrt{2}}$), we infer that for ε small enough,

$$\frac{1}{8}E_\varepsilon(u_\varepsilon) - C\Omega^2 \leq F_\varepsilon(u_\varepsilon). \tag{6.74}$$

Combining (6.68), (6.73) and (6.74), we obtain *6.11.a)*.

Proof of 6.11.b). Multiplying equation (6.67) by u_ε and using $\int_{\mathbb{R}^2}|u_\varepsilon|^2 = 1$, we infer that

$$\ell_\varepsilon = \int_{\mathbb{R}^2}|\nabla u_\varepsilon|^2 - 2\Omega\int_{\mathbb{R}^2} x^\perp\cdot(iu_\varepsilon, \nabla u_\varepsilon) + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}(|u_\varepsilon|^2 - a(x))|u_\varepsilon|^2. \tag{6.75}$$

From *6.11.a)* and Lemma 6.9, we derive

$$\left|\int_{\mathbb{R}^2}|\nabla u_\varepsilon|^2 - 2\Omega\int_{\mathbb{R}^2} x^\perp\cdot(iu_\varepsilon, \nabla u_\varepsilon) + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2\setminus\mathcal{D}}(|u_\varepsilon|^2 - a(x))|u_\varepsilon|^2\right| \leq C_{\omega_0}|\ln\varepsilon|^2 \tag{6.76}$$

and arguing as in the proof of (6.59), we obtain by (6.69),

$$\left|\frac{1}{\varepsilon^2}\int_{\mathcal{D}}(|u_\varepsilon|^2 - a(x))|u_\varepsilon|^2\right| \leq C_{\omega_0}\,\varepsilon^{-1}|\ln\varepsilon|. \tag{6.77}$$

Using (6.75), (6.76) and (6.77), we conclude that $|\ell_\varepsilon| \leq C_{\omega_0}\varepsilon^{-1}|\ln\varepsilon|$.

Proof of 6.11.c). We argue as in [3], Proposition 2.5. Setting $U_\varepsilon := |u_\varepsilon|^2$, we deduce from equation (6.67),

$$\frac{1}{2}\Delta U_\varepsilon = |\nabla u_\varepsilon|^2 - 2\Omega\, x^\perp\cdot(iu_\varepsilon, \nabla u_\varepsilon) - \frac{1}{\varepsilon^2}(a(x) - U_\varepsilon)U_\varepsilon - \ell_\varepsilon U_\varepsilon$$

and hence

$$\Delta U_\varepsilon \geq \frac{2}{\varepsilon^2}\left(U_\varepsilon - (a(x) + \varepsilon^2|\ell_\varepsilon| + \varepsilon^2\Omega^2|x|^2)\right)U_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{6.78}$$

Let $\Theta_\varepsilon = \{x \in \mathbb{R}^2\setminus\mathcal{D} : a^-(x) > 2(\varepsilon^2|\ell_\varepsilon| + \varepsilon^2\Omega^2|x|^2)\}$. From (6.78), we infer that

$$\Delta U_\varepsilon \geq \frac{1}{\varepsilon^2}a^-(x)U_\varepsilon \geq 0 \quad \text{in } \Theta_\varepsilon \tag{6.79}$$

and thus U_ε is subharmonic in $\Theta_\varepsilon \subset \mathbb{R}^2 \setminus \mathcal{D}$. Note that by (6.69),

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} U_\varepsilon^2 \le C_{\omega_0}\varepsilon^2|\ln\varepsilon|^2. \tag{6.80}$$

By *6.11.b)*, for ε small enough we have $\partial\Theta_\varepsilon \subset \left\{x\in\mathbb{R}^2 \,:\, |x|_\Lambda^2 \le a_0 + \frac{\varepsilon^{1/3}}{2}\right\}$. Consider now for $r_\varepsilon = \sqrt{a_0+\varepsilon^{1/3}}$, the set $\Xi_\varepsilon = \mathbb{R}^2\setminus B^\Lambda_{r_\varepsilon} = \left\{x\in\mathbb{R}^2 \,:\, |x|_\Lambda^2 > a_0+\varepsilon^{1/3}\right\} \subset \Theta_\varepsilon$. Then for ε small and any $x_0\in\Xi_\varepsilon$, we have $B(x_0,\frac{\varepsilon^{1/3}}{2})\subset\Theta_\varepsilon$. We infer from the subharmonicity of U_ε in Θ_ε and (6.80),

$$0 \le U_\varepsilon(x_0) \le \frac{4}{\pi\varepsilon^{2/3}}\int_{B(x_0,\frac{\varepsilon^{1/3}}{2})} U_\varepsilon \le \frac{C}{\varepsilon^{1/3}}\left(\int_{B(x_0,\frac{\varepsilon^{1/3}}{2})} U_\varepsilon^2\right)^{1/2} \le C^\star_{\omega_0}\varepsilon^{2/3}|\ln\varepsilon| \quad \text{for } x_0\in\Xi_\varepsilon,$$

with a constant $C^\star_{\omega_0}$ independent of x_0. Hence we conclude that $U_\varepsilon \to 0$ locally uniformly in $\mathbb{R}^2\setminus\overline{\mathcal{D}}$ as $\varepsilon\to 0$. It also follows that $u_\varepsilon\in L^\infty(\mathbb{R}^2)$ and then $U_\varepsilon\in H^1(\mathbb{R}^2)$. By (6.79), U_ε is a subsolution of

$$\begin{cases} -\varepsilon^2\Delta w + a^-(x)w = 0 & \text{in } \Xi_\varepsilon,\\ w = C^\star_{\omega_0}\varepsilon^{2/3}|\ln\varepsilon| & \text{on } \partial\Xi_\varepsilon. \end{cases} \tag{6.81}$$

We easily check that for ε small enough,

$$v_{\text{out}}(x) = C^\star_{\omega_0}\varepsilon^{2/3}|\ln\varepsilon|\exp\left(\frac{a_0+\varepsilon^{1/3}-|x|_\Lambda^2}{\varepsilon^{2/3}}\right)$$

is a supersolution of (6.81). Therefore

$$U_\varepsilon(x) = |u_\varepsilon(x)|^2 \le v_{\text{out}}(x) \le C^\star_{\omega_0}\varepsilon^{2/3}|\ln\varepsilon|\exp\left(\frac{a_0-|x|_\Lambda^2}{2\varepsilon^{2/3}}\right) \quad \text{for } |x|_\Lambda^2 \ge a_0+2\varepsilon^{1/3}.$$

Proof of 6.11.d) and 6.11.e). We set $\tilde r_\varepsilon = \sqrt{a_0}-\varepsilon^{1/8}$ (recall that $r_\varepsilon = \sqrt{a_0+\varepsilon^{1/3}}$). We define in $B^\Lambda_{r_\varepsilon}$, the function

$$v_{\text{in}}(x) = \begin{cases} a(x) + |\ell_\varepsilon|\varepsilon^2 + \dfrac{\varepsilon^2\Omega^2}{\Lambda^2}|x|_\Lambda^2 & \text{if } |x|_\Lambda \le \tilde r_\varepsilon,\\[2ex] a_0 - (1-\dfrac{\varepsilon^2\Omega^2}{\Lambda^2})\tilde r_\varepsilon(2|x|_\Lambda - \tilde r_\varepsilon) + |\ell_\varepsilon|\varepsilon^2 & \text{if } \tilde r_\varepsilon \le |x|_\Lambda \le r_\varepsilon. \end{cases}$$

We easily verify that for ε sufficiently small, v_{in} satisfies

$$\begin{cases} -\varepsilon^2\Delta v_{\text{in}} \ge 2\left(a(x)+|\ell_\varepsilon|\varepsilon^2+\varepsilon^2\Omega^2|x|^2 - v_{\text{in}}\right)v_{\text{in}} & \text{in } B^\Lambda_{r_\varepsilon},\\ v_{\text{in}}(x) \ge C^\star_{\omega_0}\varepsilon^{2/3}|\ln\varepsilon| & \text{on } \partial B^\Lambda_{r_\varepsilon} \end{cases} \tag{6.82}$$

and

$$v_{\text{in}}(x) \ge a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2 > 0 \quad \text{in } B^\Lambda_{r_\varepsilon}.$$

Setting $V_\varepsilon = U_\varepsilon - v_{\text{in}}$, we deduce from (6.78) and (6.82),

$$\begin{cases} -\varepsilon^2\Delta V_\varepsilon + b(x)V_\varepsilon \le 0 & \text{in } B^\Lambda_{r_\varepsilon},\\ V_\varepsilon \le 0 & \text{on } \partial B^\Lambda_{r_\varepsilon}, \end{cases}$$

with

$$b(x) = 2\big(U_\varepsilon + v_{\rm in} - (a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2)\big) \geq 0 \quad \text{in } B^\Lambda_{r_\varepsilon}.$$

Hence $V_\varepsilon \leq 0$ which gives us *6.11.d)*. Then estimate *6.11.e)* directly follows from the construction of $v_{\rm in}$ and $v_{\rm out}$ and from *6.11.b)*.

Proof of 6.11.f). Without loss of generality, we may assume that $K = B_R$ with $R > 0$. Consider the re-scaled function $\tilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ defined for $x \in B_{3+R\varepsilon^{-1}}$. From (6.67), we obtain

$$-\Delta\tilde{u}_\varepsilon = (a(\varepsilon x) - |\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon - 2i\Omega\varepsilon^2 x^\perp \cdot \nabla\tilde{u}_\varepsilon + \ell_\varepsilon\varepsilon^2\tilde{u}_\varepsilon \quad \text{in} \quad B_{3+R\varepsilon^{-1}}.$$

Taking an arbitrary $x_0 \in B_{R\varepsilon^{-1}}$, it suffices to prove that exists a constant $C_R > 0$ independent of x_0 and ε such that

$$\|\nabla\tilde{u}_\varepsilon\|_{L^\infty(B(x_0,1))} \leq C_{\omega_0,R}. \tag{6.83}$$

By *6.11.c)*, we know that $a(x)u_\varepsilon$ is uniformly bounded in $\mathbb{R}^2$. Using *6.11.a)*, *6.11.b)* and *6.11.e)*, we derive that

$$\begin{aligned}\|\Delta\tilde{u}_\varepsilon\|_{L^2(B(x_0,3))} \leq & C\big(\|(a(x) + \ell_\varepsilon\varepsilon^2 - |u_\varepsilon|^2)u_\varepsilon\|_{L^\infty(\mathbb{R}^2)} + \Omega\varepsilon^2\|x^\perp \cdot \nabla\tilde{u}_\varepsilon\|_{L^2(B(x_0,3))}\big) \\ \leq & C_{\omega_0}(1 + \Omega\varepsilon\|x^\perp \cdot \nabla u_\varepsilon\|_{L^2(B_{R+1})}) \leq C_{\omega_0,R}.\end{aligned}$$

Since $\|\tilde{u}_\varepsilon\|_{L^\infty(B(x_0,3))} \leq C_{\omega_0}$ by *6.11.e)*, it follows that $\|\tilde{u}_\varepsilon\|_{H^2(B(x_0,2))} \leq C_{\omega_0,R}$. From Sobolev imbedding, we deduce that $\|\nabla\tilde{u}_\varepsilon\|_{L^4(B(x_0,2))} \leq C_{\omega_0,R}$. We now repeat the above argument and it follows $\|\Delta\tilde{u}_\varepsilon\|_{L^4(B(x_0,2))} \leq C_{\omega_0,R}(1 + \Omega\varepsilon^{3/2}\|\nabla\tilde{u}_\varepsilon\|_{L^4(B(x_0,2))}) \leq C_{\omega_0,R}$. It finally yields $\|\tilde{u}_\varepsilon\|_{W^{2,4}(B(x_0,1))} \leq C_{\omega_0,R}$ which implies (6.83) by Sobolev imbedding. □

6.3.2 Splitting the energy

In this section, we prove the splitting of the energy (6.9). The splitting technique has been introduced by Lassoued and Mironescu in [63]. The goal is to decouple the energy $F_\varepsilon(u)$ into two independent parts: the energy of the "vortex-free" profile $\tilde{\eta}_\varepsilon e^{i\Omega S}$ and the reduced energy of $u/(\tilde{\eta}_\varepsilon e^{i\Omega S})$ where the function S is defined in (6.5). For $\varepsilon > 0$, we introduce the class

$$\mathcal{G}_\varepsilon = \left\{v \in H^1_{\rm loc}(\mathbb{R}^2, \mathbb{C}) \,:\, \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla v|^2 + \tilde{\eta}_\varepsilon^4(1 - |v|^2)^2 < +\infty\right\}.$$

We have the following result (valid for any rotational speed Ω):

Lemma 6.12 *Let $u \in \mathcal{H}$ and $\varepsilon > 0$. Then $v = u/(\tilde{\eta}_\varepsilon e^{i\Omega S})$ is well defined, belongs to $\mathcal{G}_\varepsilon$ and*

$$F_\varepsilon(u) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v) + \tilde{\mathcal{T}}_\varepsilon(v) \tag{6.84}$$

where the functionals $\tilde{\mathcal{F}}_\varepsilon$ and $\tilde{\mathcal{T}}_\varepsilon$ are defined in (6.10) *and* (6.12).

Before proving Lemma 6.12, we are going to translate some of the properties of the map u_ε to $u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$. To this aim, we define the subclass $\tilde{\mathcal{G}}_\varepsilon \subset \mathcal{G}_\varepsilon$ by

$$\tilde{\mathcal{G}}_\varepsilon = \big\{v \in \mathcal{G}_\varepsilon \,:\, \tilde{\eta}_\varepsilon v \in \mathcal{H} \text{ and } \|\tilde{\eta}_\varepsilon v\|_{L^2(\mathbb{R}^2)} = 1\big\}.$$

Proposition 6.13 *Assume that* (6.68) *holds for some* $\omega_0 > 0$. *Let* u_ε *be a minimizer of* F_ε *in* $\{u \in \mathcal{H} : \|u\|_{L^2(\mathbb{R}^2)} = 1\}$. *Then* $v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$ *minimizes the functional* $\tilde{\mathcal{F}}_\varepsilon + \tilde{\mathcal{T}}_\varepsilon$ *in* $\tilde{\mathcal{G}}_\varepsilon$. *Moreover, for* $\varepsilon > 0$ *sufficiently small, we have*

6.13.a) $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le C_{\omega_0} |\ln \varepsilon|^2$,

6.13.b) $\left|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)\right| \le C_{\omega_0}\, \varepsilon |\ln \varepsilon|^3$,

6.13.c) $|v_\varepsilon(x)| \le 1 + C_{\omega_0}\, \varepsilon^{1/3}$ *for* $x \in \mathcal{D}$ *with* $|x|_\Lambda \le \sqrt{a_0} - \varepsilon^{1/8}$,

6.13.d) $\|\nabla v_\varepsilon\|_{L^\infty(K)} \le C_{\omega_0,K}\, \varepsilon^{-1}$ *for any compact subset* $K \subset \mathcal{D}$.

Proof of Lemma 6.12: Step 1. For $u \in \mathcal{H}$, we set $\tilde{v} = u/\tilde{\eta}_\varepsilon \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$. We want to prove that $\tilde{v} \in \mathcal{G}_\varepsilon$ and

$$E_\varepsilon(u) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \tilde{\mathcal{E}}_\varepsilon(\tilde{v}) + \frac{k_\varepsilon}{2} \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 (|\tilde{v}|^2 - 1). \tag{6.85}$$

We consider the sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ defined by $u_n(x) = \zeta\left(n^{-1}|x|\right) u(x)$ where ζ is the "cut-off" type function defined in (6.23). We easily check that $u_n \to u$ a.e. and $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^2$. Setting $\tilde{v}_n = u_n/\tilde{\eta}_\varepsilon$, then we have $\tilde{v}_n \to \tilde{v}$ a.e. and $\nabla \tilde{v}_n \to \nabla \tilde{v}$ a.e. in $\mathbb{R}^2$. Since u_n has a compact support, we get that $\tilde{v}_n \in \mathcal{G}_\varepsilon$ for any $n \in \mathbb{N}$. We have

$$|\nabla u_n|^2 = |\nabla \tilde{\eta}_\varepsilon|^2 + \tilde{\eta}_\varepsilon^2 |\nabla \tilde{v}_n|^2 + (|\tilde{v}_n|^2 - 1)|\nabla \tilde{\eta}_\varepsilon|^2 + \tilde{\eta}_\varepsilon \nabla \tilde{\eta}_\varepsilon \cdot \nabla(|\tilde{v}_n|^2 - 1),$$

and therefore,

$$\begin{aligned} E_\varepsilon(u_n) = E_\varepsilon(\tilde{\eta}_\varepsilon) &+ \frac{1}{2} \int_{\mathbb{R}^2} \left(\tilde{\eta}_\varepsilon^2 |\nabla \tilde{v}_n|^2 + \frac{\tilde{\eta}_\varepsilon^4}{2\varepsilon^2}(|\tilde{v}_n|^2 - 1)^2\right) \\ &+ \frac{1}{2} \int_{\mathbb{R}^2} \left((|\tilde{v}_n|^2 - 1)|\nabla \tilde{\eta}_\varepsilon|^2 + \tilde{\eta}_\varepsilon \nabla \tilde{\eta}_\varepsilon \cdot \nabla(|\tilde{v}_n|^2 - 1) + \frac{1}{\varepsilon^2} \tilde{\eta}_\varepsilon^2 (|\tilde{v}_n|^2 - 1)(\tilde{\eta}_\varepsilon^2 - a(x))\right). \end{aligned}$$

As in [63], the main idea is to multiply the equation (6.46) by $\tilde{\eta}_\varepsilon(|\tilde{v}_n|^2 - 1)$ and then to integrate by parts. It leads to

$$\int_{\mathbb{R}^2} \left\{ (|\tilde{v}_n|^2 - 1)|\nabla \tilde{\eta}_\varepsilon|^2 + \tilde{\eta}_\varepsilon \nabla \tilde{\eta}_\varepsilon \nabla(|\tilde{v}_n|^2 - 1) + \frac{\tilde{\eta}_\varepsilon^2}{\varepsilon^2}(|\tilde{v}_n|^2 - 1)(\tilde{\eta}_\varepsilon^2 - a(x)) \right\} = k_\varepsilon \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 (|\tilde{v}_n|^2 - 1)$$

and we conclude that for every $n \in \mathbb{N}$,

$$E_\varepsilon(u_n) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \tilde{\mathcal{E}}_\varepsilon(\tilde{v}_n) + \frac{k_\varepsilon}{2} \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 (|\tilde{v}_n|^2 - 1).$$

Now we observe that

$$|u_n| \le |u| \quad \text{and} \quad |\nabla u_n| \le |\nabla u| + |u| \quad \text{a.e. in } \mathbb{R}^2, \tag{6.86}$$

and by the dominated convergence theorem, it results that $E_\varepsilon(u_n) \to E_\varepsilon(u)$ and

$$\frac{k_\varepsilon}{2} \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 (|\tilde{v}_n|^2 - 1) = \frac{k_\varepsilon}{2} \int_{\mathbb{R}^2} (|u_n|^2 - \tilde{\eta}_\varepsilon^2) \longrightarrow \frac{k_\varepsilon}{2} \int_{\mathbb{R}^2} (|u|^2 - \tilde{\eta}_\varepsilon^2) = \frac{k_\varepsilon}{2} \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 (|\tilde{v}|^2 - 1).$$

Applying Fatou's lemma, we obtain

$$\tilde{\mathcal{E}}_\varepsilon(\tilde{v}) \leq \lim_{n\to+\infty} \tilde{\mathcal{E}}_\varepsilon(\tilde{v}_n) = \lim_{n\to+\infty} \left\{ E_\varepsilon(u_n) - E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2}(|u_n|^2 - \tilde{\eta}_\varepsilon^2)\right\}$$
$$= E_\varepsilon(u) - E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2}\tilde{\eta}_\varepsilon^2(|\tilde{v}|^2-1) < +\infty,$$

and we conclude that $\tilde{v} \in \mathcal{G}_\varepsilon$. Since $|\tilde{v}_n||\nabla\tilde{\eta}_\varepsilon| \leq |\nabla u| + \tilde{\eta}_\varepsilon|\nabla\tilde{v}|$, we infer from (6.86) that $\tilde{\eta}_\varepsilon^2|\nabla\tilde{v}_n|^2 \leq C(|\nabla u|^2 + |u|^2 + \tilde{\eta}_\varepsilon^2|\nabla\tilde{v}|^2)$ and $\tilde{\eta}_\varepsilon^4(|\tilde{v}_n|^2-1)^2 \leq 2(|u|^4 + \tilde{\eta}_\varepsilon^4)$. By the dominated convergence theorem, we finally get that

$$\tilde{\mathcal{E}}_\varepsilon(\tilde{v}) = \lim_{n\to+\infty}\tilde{\mathcal{E}}_\varepsilon(\tilde{v}_n) = E_\varepsilon(u) - E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2}\tilde{\eta}_\varepsilon^2(|\tilde{v}|^2-1).$$

Step 2. Consider now $\tilde{u} = u/e^{i\Omega S}$. Then $\tilde{u} \in \mathcal{H}$ and we have the decomposition

$$F_\varepsilon(u) = E_\varepsilon(\tilde{u}) + \frac{\Omega}{1+\Lambda^2}\int_{\mathbb{R}^2}\nabla^\perp a\cdot(i\tilde{u},\nabla\tilde{u}) + \frac{\Omega^2}{2}\int_{\mathbb{R}^2}\big(|\nabla S|^2 - 2x^\perp\cdot\nabla S\big)|\tilde{u}|^2. \tag{6.87}$$

Indeed, we use that

$$|\nabla u|^2 - 2\Omega x^\perp\cdot(iu,\nabla u) = |\nabla\tilde{u}|^2 + \frac{2\Omega}{1+\Lambda^2}\nabla^\perp a\cdot(i\tilde{u},\nabla\tilde{u}) + \Omega^2\big(|\nabla S|^2 - 2x^\perp\cdot\nabla S\big)|\tilde{u}|^2 \text{ a.e. in } \mathbb{R}^2.$$

Since $|\nabla S| \leq C|x|$, $|\nabla a| \leq C|x|$, we infer that (6.87) holds.
Step 3. We show that (6.84) takes place. Let $u \in \mathcal{H}$. Set $\tilde{u} = u/e^{i\Omega S}$ and $v = \tilde{u}/\tilde{\eta}_\varepsilon$. By Step 1 and Step 2, it results that $\tilde{u} \in \mathcal{H}$ and $v \in \mathcal{G}_\varepsilon$. By (6.85), we have

$$E_\varepsilon(\tilde{u}) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \tilde{\mathcal{E}}_\varepsilon(v) + \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2}\tilde{\eta}_\varepsilon^2(|v|^2-1). \tag{6.88}$$

Since $\nabla^\perp a\cdot(i\tilde{u},\nabla\tilde{u}) = \tilde{\eta}_\varepsilon^2\nabla^\perp a\cdot(iv,\nabla v)$ and $|\tilde{u}|^2 = \tilde{\eta}_\varepsilon^2|v|^2$ a.e. in $\mathbb{R}^2$, we infer from (6.87) and (6.88) that

$$F_\varepsilon(u) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \tilde{\mathcal{E}}_\varepsilon(v) + \tilde{\mathcal{R}}_\varepsilon(v) + \frac{\Omega^2}{2}\int_{\mathbb{R}^2}\big(|\nabla S|^2 - 2x^\perp\cdot\nabla S\big)\tilde{\eta}_\varepsilon^2|v|^2 + \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2}\tilde{\eta}_\varepsilon^2(|v|^2-1). \tag{6.89}$$

On the other hand, (6.87) yields

$$F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \frac{\Omega^2}{2}\int_{\mathbb{R}^2}\big(|\nabla S|^2 - 2x^\perp\cdot\nabla S\big)\tilde{\eta}_\varepsilon^2 \tag{6.90}$$

and the conclusion follows combining (6.89) and (6.90). □

Remark 6.5 *The energy of the "vortex-free" profile is given by*

$$F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) = E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{\pi a_0^3(1-\Lambda^2)^2}{24(1+\Lambda^2)\Lambda^3}\Omega^2 + o(1). \tag{6.91}$$

It directly follows from (6.90) *and Proposition 6.8.*

Proof of Proposition 6.13. The minimizing property of v_ε follows directly from Proposition 6.10 and Lemma 6.12.

Proof of 6.13.a) and 6.13.b). Since u_ε minimizes F_ε in $\{u \in \mathcal{H} : \|u\|_{L^2(\mathbb{R}^2)} = 1\}$, we have using Lemma 6.12,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \le F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}),$$

and it yields

$$\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le |\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)| + |\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)|. \tag{6.92}$$

Arguing as in the proof of Lemma 6.9 with $\sigma = 1/4$ and $R = \sqrt{2a_0}$, we infer from *6.11.e)* in Proposition 6.11 and (6.69),

$$\begin{aligned}
\left|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)\right| &\le \frac{1}{4}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\nabla v_\varepsilon|^2 + \frac{4\Omega^2}{(\Lambda^2+1)^2}\int_{\mathbb{R}^2} |x|_\Lambda^2 |u_\varepsilon|^2 \\
&\le \frac{1}{4}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\nabla v_\varepsilon|^2 + \frac{4\Omega^2}{(\Lambda^2+1)^2}\int_{\mathbb{R}^2\setminus B^\Lambda_{\sqrt{2a_0}}} 2a^-(x)|u_\varepsilon|^2 + \frac{8a_0\Omega^2}{(\Lambda^2+1)^2}\int_{B^\Lambda_{\sqrt{2a_0}}} |u_\varepsilon|^2 \\
&\le \frac{1}{2}\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) + C_{\omega_0}|\ln\varepsilon|^2.
\end{aligned} \tag{6.93}$$

We obtain from (6.48), (6.50) and (6.69) that

$$\begin{aligned}
|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)| &= \Big|\frac{1}{2}\int_{\mathbb{R}^2} \left(\Omega^2|\nabla S|^2 - 2\Omega^2 x^\perp\cdot\nabla S + k_\varepsilon\right)(|u_\varepsilon|^2 - \tilde{\eta}_\varepsilon^2)\Big| \\
&\le C_{\omega_0}|\ln\varepsilon|^2\left[\int_{\mathbb{R}^2\setminus B^\Lambda_{\sqrt{2a_0}}} 2a^-(x)(|u_\varepsilon|^2+\tilde{\eta}_\varepsilon^2) + \left(\int_{B^\Lambda_{\sqrt{2a_0}}} (|u_\varepsilon|^2 - a^+)^2 + (\tilde{\eta}_\varepsilon^2 - a^+)^2\right)^{1/2}\right] \\
&\le C_{\omega_0}\,\varepsilon|\ln\varepsilon|^3.
\end{aligned} \tag{6.94}$$

According to (6.92), (6.93) and (6.94), we conclude that $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le C_{\omega_0}|\ln\varepsilon|^2$.

Proof of 6.13.c). From *6.8.c)* in Proposition 6.8, *6.11.b)* and *6.11.d)*, we infer that

$$|v_\varepsilon(x)| = \frac{|u_\varepsilon(x)|}{\tilde{\eta}_\varepsilon(x)} \le \frac{\sqrt{a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2}}{(1 - C\varepsilon^{1/3})\sqrt{a(x)}} \le 1 + C_{\omega_0}\varepsilon^{1/3} \quad \text{for } x \in B^\Lambda_{\sqrt{a_0}-\varepsilon^{1/8}}.$$

Proof of 6.13.d). Let $K \subset B^\Lambda_{\sqrt{a_0}}$ be any compact set. We denote $\tilde{v}_\varepsilon = e^{i\Omega S}v_\varepsilon = \frac{u_\varepsilon}{\tilde{\eta}_\varepsilon}$. By *6.8.c)* in Proposition 6.8, we know that there exists $C_K > 0$ independent of ε such that $\tilde{\eta}_\varepsilon \ge (1 - C\varepsilon^{1/3})\sqrt{a} \ge C_K$ in K. Since $\nabla\tilde{v}_\varepsilon = \tilde{\eta}_\varepsilon^{-1}\nabla u_\varepsilon - (\tilde{\eta}_\varepsilon^{-2}\nabla\tilde{\eta}_\varepsilon)u_\varepsilon$, using Proposition 6.8 and Proposition 6.11, it follows $\|\nabla\tilde{v}_\varepsilon\|_{L^\infty(K)} \le C_{\omega_0,K}\varepsilon^{-1}$. Hence we deduce (using *6.13.c)*) that

$$\|\nabla v_\varepsilon\|_{L^\infty(K)} \le \|\nabla\tilde{v}_\varepsilon\|_{L^\infty(K)} + \Omega\|\tilde{v}_\varepsilon\nabla S\|_{L^\infty(K)} \le C_{\omega_0,K}\,\varepsilon^{-1}$$

and the proof is complete. □

6.3.3 Splitting the domain

The main goal in this section is to show that we can excise the region of $\mathbb{R}^2$ where the density $|u_\varepsilon|$ is very small (which corresponds to the exterior of $\mathcal{D}$) without modifying the relevant part in the energy.

Proposition 6.14 *Assume that* (6.68) *holds. For small* $\varepsilon > 0$ *and* $\nu \in [1,2]$, *we set* $\mathcal{D}_\varepsilon^\nu = \{x \in \mathbb{R}^2 : a(x) > \nu|\ln\varepsilon|^{-3/2}\}$. *We have*

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \le C_{\omega_0}|\ln\varepsilon|^{-1}.$$

Proof. Since u_ε minimizes F_ε on $\{u \in \mathcal{H} : \|u\|_{L^2(\mathbb{R}^2)} = 1\}$, we have for ε sufficiently small that $F_\varepsilon(u_\varepsilon) \le F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S})$. Then Lemma 6.12 yields $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \le 0$ and we derive from *6.13.b)* in Proposition 6.13,

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \le C_{\omega_0}\varepsilon|\ln\varepsilon|^3. \tag{6.95}$$

We now set $\mathcal{N}_\varepsilon^\nu = \mathbb{R}^2 \setminus \mathcal{D}_\varepsilon^\nu$. From the previous inequality, it suffices to prove that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{N}_\varepsilon^\nu) \ge -C_{\omega_0}|\ln\varepsilon|^{-1} \tag{6.96}$$

for a constant $C_{\omega_0} > 0$ independent of ε and ν. Arguing as in the proof of Lemma 6.9 with $\sigma = 1/4$ and $R = \sqrt{2a_0}$, we infer from (6.69),

$$\begin{aligned}
\Big|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon, \mathcal{N}_\varepsilon^\nu)\Big| \le & \frac{1}{4}\int_{\mathcal{N}_\varepsilon^\nu} \tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{4\Omega^2}{(1+\Lambda^2)^2}\int_{\mathcal{N}_\varepsilon^\nu} |x|_\Lambda^2|u_\varepsilon|^2\\
\le & \frac{1}{4}\int_{\mathcal{N}_\varepsilon^\nu} \tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{4\Omega^2}{(1+\Lambda^2)^2}\int_{\mathbb{R}^2\setminus B^\Lambda_{\sqrt{2a_0}}} 2a^-(x)|u_\varepsilon|^2 + \frac{8a_0\Omega^2}{(1+\Lambda^2)^2}\int_{B^\Lambda_{\sqrt{2a_0}}\setminus\mathcal{D}_\varepsilon^\nu} |u_\varepsilon|^2\\
\le & \frac{1}{4}\int_{\mathcal{N}_\varepsilon^\nu} \tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{8a_0\Omega^2}{(1+\Lambda^2)^2}\int_{B^\Lambda_{\sqrt{2a_0}}\setminus\mathcal{D}_\varepsilon^\nu} |u_\varepsilon|^2 + C_{\omega_0}\varepsilon^2|\ln\varepsilon|^4.
\end{aligned}$$

By (6.69), we may also estimate

$$\begin{aligned}
\int_{B^\Lambda_{\sqrt{2a_0}}\setminus\mathcal{D}_\varepsilon^\nu} |u_\varepsilon|^2 = & \int_{B^\Lambda_{\sqrt{2a_0}}\setminus B^\Lambda_{\sqrt{a_0}}} |u_\varepsilon|^2 + \int_{B^\Lambda_{\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} (|u_\varepsilon|^2 - a(x)) + \int_{B^\Lambda_{\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} a(x)\\
\le & C\Big(\int_{B^\Lambda_{\sqrt{2a_0}}\setminus B^\Lambda_{\sqrt{a_0}}} |u_\varepsilon|^4\Big)^{1/2} + C\Big(\int_{B^\Lambda_{\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} (|u_\varepsilon|^2 - a(x))^2\Big)^{1/2} + C|\ln\varepsilon|^{-3}\\
\le & C_{\omega_0}(|\ln\varepsilon|^{-3} + \varepsilon|\ln\varepsilon|).
\end{aligned}$$

Then it follows that

$$|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon, \mathcal{N}_\varepsilon^\nu)| \le \frac{1}{2}\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon, \mathcal{N}_\varepsilon^\nu) + C_{\omega_0}|\ln\varepsilon|^{-1} \tag{6.97}$$

which leads to (6.96). □

For some technical reasons, it will be easier to deal with a^+ instead of $\tilde{\eta}_\varepsilon^2$ in the energies. To replace $\tilde{\eta}_\varepsilon^2$ by a^+, we shall prove that the energy estimates inside $\mathcal{D}_\varepsilon^\nu$ remain unchanged.

Proposition 6.15 *Assume that* (6.68) *holds for some* $\omega_0 > 0$. *We have*

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \le C_{\omega_0} |\ln \varepsilon|^2 \quad \text{and} \quad \mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \le C_{\omega_0} |\ln \varepsilon|^{-1}$$

where $\mathcal{E}_\varepsilon$ *and* $\mathcal{F}_\varepsilon$ *are defined in* (6.18).

Proof. From *6.8.c)* in Proposition 6.8, we infer that

$$\left\| \frac{a - \tilde{\eta}_\varepsilon^2}{\tilde{\eta}_\varepsilon^2} \right\|_{L^\infty(\mathcal{D}_\varepsilon^\nu)} \le C\varepsilon^{1/3} \quad \text{and} \quad \left\| \frac{a^2 - \tilde{\eta}_\varepsilon^4}{\tilde{\eta}_\varepsilon^4} \right\|_{L^\infty(\mathcal{D}_\varepsilon^\nu)} \le C\varepsilon^{1/3}$$

and then *6.13.a)* in Proposition 6.13 yields

$$\left| \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) - \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \right| \le C\varepsilon^{1/3}\, \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \le C_{\omega_0} \varepsilon^{1/3} |\ln \varepsilon|^2. \tag{6.98}$$

Using *6.11.a)* and *6.11.e)* in Proposition 6.11, we derive

$$\left| \mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) - \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \right| \le \Omega \int_{\mathcal{D}_\varepsilon^\nu} \frac{a - \tilde{\eta}_\varepsilon^2}{\tilde{\eta}_\varepsilon^2} |u_\varepsilon|\, |\nabla u_\varepsilon| \le C\varepsilon^{1/3} \Omega (E_\varepsilon(u_\varepsilon, \mathcal{D}_\varepsilon^\nu))^{1/2} \le C_{\omega_0} \varepsilon^{1/3} |\ln \varepsilon|^2.$$

Therefore, it follows that

$$\left| \mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) - \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \right| \le C_{\omega_0} \varepsilon^{1/3} |\ln \varepsilon|^2. \tag{6.99}$$

Then the conclusion comes immediately from *6.13.a)* in Proposition 6.13 and Proposition 6.14. □

6.4 Energy and degree estimates

This section is devoted to the proof of Theorem 6.1. The method we use is inspired from [76, 78] and provides some information about the location and the number of vortices inside $\mathcal{D}$.

6.4.1 Construction of vortex balls and expansion of the rotation energy

We start with the construction of vortex balls by a method due to Sandier [75] and Sandier and Serfaty [77]; it permits to localize the vorticity set of v_ε.

Proposition 6.16 *Assume that* (6.68) *holds for some* $\omega_0 > 0$. *Then there exists a positive constant* $\mathcal{K}_{\omega_0}$ *such that for* ε *sufficiently small, there exist* $\nu_\varepsilon \in (1,2)$ *and a finite collection of disjoint balls* $\{B_i\}_{i \in I_\varepsilon} := \{B(p_i, r_i)\}_{i \in I_\varepsilon}$ *satisfying the conditions:*

(i) *for every* $i \in I_\varepsilon, B_i \subset\subset \mathcal{D}_\varepsilon = \{x \in \mathbb{R}^2 \,:\, a(x) > \nu_\varepsilon |\ln \varepsilon|^{-3/2}\}$,

(ii) $\{x \in \mathcal{D}_\varepsilon \,:\, |v_\varepsilon(x)| < 1 - |\ln \varepsilon|^{-5}\} \subset \cup_{i \in I_\varepsilon} B_i$,

(iii) $\displaystyle\sum_{i \in I_\varepsilon} r_i \le |\ln \varepsilon|^{-10}$,

(iv) $\displaystyle\frac{1}{2}\int_{B_i} a(x)|\nabla v_\varepsilon|^2 \ge \pi a(p_i)|d_i|\big(|\ln \varepsilon| - \mathcal{K}_{\omega_0} \ln |\ln \varepsilon|\big)$,

where $d_i = \deg\left(\dfrac{v_\varepsilon}{|v_\varepsilon|}, \partial B_i\right)$ *for every* $i \in I_\varepsilon$.

Proof. According to the technique presented in [75] and [77], we construct as in [3] (using Proposition 6.15 with $\nu = 1$) a finite collection of disjoint balls $\{B_i\}_{i\in\tilde{I}_\varepsilon} = \{B(p_i, r_i)\}_{i\in\tilde{I}_\varepsilon}$ such that

$$\{x \in \mathcal{D} \,:\, a(x) > |\ln\varepsilon|^{-3/2} \text{ and } |v_\varepsilon(x)| < 1 - |\ln\varepsilon|^{-5}\} \subset \cup_{i\in\tilde{I}_\varepsilon} B_i\,,$$

(iii) is fulfilled and

$$\int_{B_i} \frac{a(x)}{2}|(\nabla - i\Omega x^\perp)v_\varepsilon|^2 \geq \pi a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_{\omega_0}\ln|\ln\varepsilon|\big) \quad \text{for each } i \in \tilde{I}_\varepsilon.$$

By *(iii)*, we can find $\nu_\varepsilon \in (1,2)$ such that $\partial\{x \in \mathcal{D} \,:\, a(x) > \nu_\varepsilon|\ln\varepsilon|^{-3/2}\} \cap \cup_{i\in\tilde{I}_\varepsilon} B_i = \emptyset$. By cancelling the balls B_i that are not included in $\{x \in \mathcal{D} \,:\, a(x) > \nu_\varepsilon|\ln\varepsilon|^{-3/2}\}$, it remains a finite collection $\{B_i\}_{i\in I_\varepsilon}$ that satisfies *(i), (ii)* and *(iii)*. Notice now that *(iv)* takes place since we have

$$\Omega^2\int_{B_i}\frac{a(x)}{2}|x|^2|v_\varepsilon|^2 \leq \Omega^2\int_{B_i}|x|^2|u_\varepsilon|^2 \leq C_{\omega_0}|\ln\varepsilon|^2 r_i^2,$$

$$\Big|\Omega\int_{B_i} a(x)x^\perp\cdot(iv_\varepsilon,\nabla v_\varepsilon)\Big| \leq C\Omega\int_{B_i}\frac{a(x)}{\tilde{\eta}_\varepsilon}|u_\varepsilon|\,|\nabla v_\varepsilon| \leq C\Omega\|\sqrt{a}\nabla v_\varepsilon\|_{L^2(B_i)}r_i \leq C_{\omega_0}|\ln\varepsilon|^2 r_i \tag{6.100}$$

(here we used Proposition 6.15). Hence these terms can be absorbed by $\mathcal{K}_{\omega_0}\ln|\ln\varepsilon|$ (up to a different constant $\mathcal{K}_{\omega_0}+1$). □

We are now in a position to compute an asymptotic expansion of the rotation energy according to the center of each vortex ball B_i and the associated degree d_i:

Proposition 6.17 *Assume that* (6.68) *holds for some* $\omega_0 > 0$. *For* ε *sufficiently small, we have*

$$\mathcal{R}_\varepsilon\big(v_\varepsilon,\mathcal{D}_\varepsilon\big) = \frac{-\pi\Omega}{1+\Lambda^2}\sum_{i\in I_\varepsilon}(a^2(p_i) - \nu_\varepsilon^2|\ln\varepsilon|^{-3})\,d_i \;+\; o(|\ln\varepsilon|^{-5}).$$

Proof. By Proposition 6.16, $\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i \subset \mathcal{D}_\varepsilon\setminus\{|v_\varepsilon| < 1/2\}$ whenever ε is small enough. For $x \in \mathcal{D}_\varepsilon$ such that $|v_\varepsilon(x)| \geq 1/2$, we set

$$w_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}\,.$$

Since $(iv_\varepsilon,\nabla v_\varepsilon) = |v_\varepsilon|^2(iw_\varepsilon,\nabla w_\varepsilon)$ in $\mathcal{D}_\varepsilon\setminus\{|v_\varepsilon| < 1/2\}$, we have

$$\begin{aligned}\mathcal{R}_\varepsilon\left(v_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i\right) = {}& \frac{\Omega}{1+\Lambda^2}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)\nabla^\perp a\cdot(iw_\varepsilon,\nabla w_\varepsilon)\\ &+\frac{\Omega}{1+\Lambda^2}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)(|v_\varepsilon|^2-1)\nabla^\perp a\cdot(iw_\varepsilon,\nabla w_\varepsilon).\end{aligned} \tag{6.101}$$

Then we estimate using Proposition 6.15,

$$\begin{aligned}\left|\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)(|v_\varepsilon|^2-1)\,\nabla^\perp a\cdot(iw_\varepsilon,\nabla w_\varepsilon)\right| &\leq C\varepsilon\left(\mathcal{E}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)\right)^{1/2}\|\nabla w_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon\setminus\{|v_\varepsilon|<1/2\})}\\ &\leq C\varepsilon|\ln\varepsilon|\|\nabla w_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon\setminus\{|v_\varepsilon|<1/2\})}\,.\end{aligned} \tag{6.102}$$

In $\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon| < 1/2\}$, we have $|\nabla w_\varepsilon| \leq 2(|\nabla v_\varepsilon| + |\nabla |v_\varepsilon||) \leq 4|\nabla v_\varepsilon|$. We deduce that

$$\int_{\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon|<1/2\}} |\nabla w_\varepsilon|^2 \leq 16 \int_{\mathcal{D}_\varepsilon} |\nabla v_\varepsilon|^2 \leq 16 |\ln \varepsilon|^{3/2} \int_{\mathcal{D}_\varepsilon} a(x) |\nabla v_\varepsilon|^2 \leq C |\ln \varepsilon|^{7/2} \tag{6.103}$$

and hence we obtain combining (6.101), (6.102) and (6.103),

$$\mathcal{R}_\varepsilon \left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\right) = \frac{\Omega}{1+\Lambda^2} \int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} a(x) \nabla^\perp a \cdot (i w_\varepsilon, \nabla w_\varepsilon) + \mathcal{O}(\varepsilon |\ln \varepsilon|^4). \tag{6.104}$$

Since $(iw_\varepsilon, \nabla w_\varepsilon) = w_\varepsilon \wedge \nabla w_\varepsilon$ and $a(x)\nabla^\perp a = \nabla^\perp \mathcal{P}_\varepsilon(x)$ with

$$\mathcal{P}_\varepsilon(x) = \frac{a^2(x) - \nu_\varepsilon^2 |\ln \varepsilon|^{-3}}{2}, \tag{6.105}$$

we derive that

$$\begin{aligned}\int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} a(x) \nabla^\perp a \cdot (i w_\varepsilon, \nabla w_\varepsilon) &= \int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} \nabla^\perp \mathcal{P}_\varepsilon(x) \cdot (w_\varepsilon \wedge \nabla w_\varepsilon) \\ &= -\sum_{i\in I_\varepsilon} \int_{\partial B_i} \mathcal{P}_\varepsilon(x) \left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau}\right)\end{aligned}$$

where τ denotes the counterclockwise oriented unit tangent vector to ∂B_i. The smoothness of v_ε implies the existence of $\alpha_\varepsilon \in (\frac{1}{2}, \frac{2}{3})$ such that $\mathcal{U} = \left\{x \in \mathbb{R}^2 \,:\, |v_\varepsilon| < \alpha_\varepsilon\right\}$ is a smooth open set. Then we set for $i \in I_\varepsilon$, $\mathcal{U}_i = B_i \cap \mathcal{U}$ (notice that by Proposition 6.16, $\mathcal{U}_i \subset\subset B_i$ for small ε). Using (6.103), we derive

$$\begin{aligned}\left| \int_{\partial B_i} \mathcal{P}_\varepsilon(x) \left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau}\right) - \int_{\partial \mathcal{U}_i} \mathcal{P}_\varepsilon(x) \left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau}\right)\right| &= \left| \int_{B_i \setminus \mathcal{U}_i} \nabla^\perp \mathcal{P}_\varepsilon(x) \cdot (w_\varepsilon \wedge \nabla w_\varepsilon)\right| \\ &\leq C\, r_i \, \|\nabla w_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon|<1/2\})} \\ &\leq C\, r_i \, |\ln \varepsilon|^{7/4}\end{aligned}$$

and since $|v_\varepsilon| \leq \alpha_\varepsilon$ in $\mathcal{U}_i$ and $|\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i)| \leq r_i \|\nabla \mathcal{P}_\varepsilon\|_{L^\infty(\mathcal{D})}$, $\forall x \in B(p_i, r_i)$, it results from Proposition 6.15,

$$\begin{aligned}\left| \int_{\partial \mathcal{U}_i} (\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i)) \left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau}\right)\right| &= \alpha_\varepsilon^{-2} \left| \int_{\partial \mathcal{U}_i} (\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i)) \left(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau}\right)\right| \\ &\leq \alpha_\varepsilon^{-2} \left| \int_{\mathcal{U}_i} a(x) \nabla^\perp a \cdot (i v_\varepsilon, \nabla v_\varepsilon)\right| \\ &\quad + 2\alpha_\varepsilon^{-2} \left| \int_{\mathcal{U}_i} (\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i)) \det(\nabla v_\varepsilon)\right| \\ &\leq C \left(r_i \, \|\sqrt{a}\, \nabla v_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon)} + r_i \, |\ln \varepsilon|^{3/2} \, \|\sqrt{a}\, \nabla v_\varepsilon\|^2_{L^2(\mathcal{U}_i)}\right) \\ &\leq C\, r_i \, |\ln \varepsilon|^{7/2}\,.\end{aligned}$$

Therefore we conclude by *(iii)* in Proposition 6.16 that

$$\begin{aligned}\mathcal{R}_\varepsilon \left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\right) &= \frac{-\Omega}{1+\Lambda^2} \sum_{i\in I_\varepsilon} \mathcal{P}_\varepsilon(p_i) \int_{\partial \mathcal{U}_i} w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau} \; + \; o(|\ln \varepsilon|^{-5}) \\ &= \frac{-2\pi\Omega}{1+\Lambda^2} \sum_{i\in I_\varepsilon} \mathcal{P}_\varepsilon(p_i)\, d_i \; + \; o(|\ln \varepsilon|^{-5}).\end{aligned}$$

On the other hand, we infer from (6.100) and *(iii)* in Proposition 6.16 that

$$\left|\mathcal{R}_\varepsilon(v_\varepsilon, \cup_{i\in I_\varepsilon} B_i)\right| \leq C|\ln\varepsilon|^2 \sum_{i\in I_\varepsilon} r_i \leq C|\ln\varepsilon|^{-8}.$$

According to (6.105), the proof is completed. □

6.4.2 Asymptotic behavior for subcritical velocities. Proof of *(i)* in Theorem 6.1

In this section, we prove *(i)* in Theorem 6.1. We will distinguish different types of vortex balls through the partition $I_\varepsilon = I_0 \cup I_* \cup I_-$ where

$$\begin{aligned} I_0 =& \big\{i \in I_\varepsilon \, : \, d_i \geq 0 \text{ and } |p_i|_\Lambda < |\ln\varepsilon|^{-1/6}\big\}, \\ I_* =& \big\{i \in I_\varepsilon \, : \, d_i \geq 0 \text{ and } |p_i|_\Lambda \geq |\ln\varepsilon|^{-1/6}\big\}, \\ I_- =& \big\{i \in I_\varepsilon \, : \, d_i < 0\big\} \end{aligned}$$

in order to improve the lower bound for $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)$ (see (6.111)). In the sequel, we assume that

$$\Omega \leq \Omega_1 + \omega_1 \ln|\ln\varepsilon| \tag{6.106}$$

for some constant $\omega_1 \in \mathbb{R}$. Therefore, if ε is small, we have $\Omega \leq \frac{3}{a_0}|\ln\varepsilon|$ and we will use the constant $\mathcal{K}_{\frac{3}{a_0}}$ given by Proposition 6.16. In fact, one can choose instead of $\frac{3}{a_0}$ any other constant ω_0 such that $\omega_0 > \frac{1+\Lambda^2}{a_0}$. First, we show the following:

Proposition 6.18 *Assume that* (6.106) *holds with* $\omega_1 < \omega_1^\star := \frac{-(1+\Lambda^2)\mathcal{K}_{\frac{3}{a_0}}}{a_0}$. *Then for* ε *sufficiently small, we have* $\sum_{i\in I_\varepsilon} |d_i| = 0$ *and*

$$|v_\varepsilon| \to 1 \quad \text{in } L^\infty_{\text{loc}}(\mathcal{D}) \text{ as } \varepsilon \to 0. \tag{6.107}$$

Moreover,

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = o(1) \quad \text{and} \quad \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) = o(1). \tag{6.108}$$

Proof. From Proposition 6.15 and Proposition 6.16, we get that

$$\begin{aligned} \mathcal{O}(|\ln\varepsilon|^{-1}) \geq \mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq& \frac{1}{2}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon} B_i} a(x)|\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\mathcal{D}_\varepsilon} a^2(x)(1-|v_\varepsilon|^2)^2 \\ &+ \pi \sum_{i\in I_\varepsilon} a(p_i)|d_i| \left(|\ln\varepsilon| - \mathcal{K}_{\frac{3}{a_0}} \ln|\ln\varepsilon|\right) + \mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon). \end{aligned} \tag{6.109}$$

Combining Proposition 6.17 and (6.106), it results that

$$\begin{aligned} \mathcal{R}_\varepsilon\left(v_\varepsilon, \mathcal{D}_\varepsilon\right) \geq& \frac{-\pi a_0 \Omega}{1+\Lambda^2} \sum_{i\in I_0} a(p_i)|d_i| - \frac{\pi(a_0 - |\ln\varepsilon|^{-1/3})\Omega}{1+\Lambda^2} \sum_{i\in I_*} a(p_i)|d_i| + o(|\ln\varepsilon|^{-5}) \\ \geq& -\pi \sum_{i\in I_0\cup I_*} a(p_i)|d_i||\ln\varepsilon| - \frac{\pi a_0\omega_1}{1+\Lambda^2}\sum_{i\in I_0} a(p_i)|d_i|\ln|\ln\varepsilon| \\ &+ \frac{\pi}{2a_0}\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{2/3} + o(|\ln\varepsilon|^{-5}) \end{aligned} \tag{6.110}$$

(here we used that

$$\frac{(a_0 - |\ln\varepsilon|^{-1/3})\Omega}{1+\Lambda^2} \le |\ln\varepsilon| - \frac{1}{a_0}|\ln\varepsilon|^{2/3} + \frac{a_0\omega_1}{1+\Lambda^2}\ln|\ln\varepsilon| \le |\ln\varepsilon| - \frac{1}{2a_0}|\ln\varepsilon|^{2/3}$$

for ε small). Then we deduce from (6.109) and (6.110) that for ε small enough,

$$\frac{1}{2}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 + \int_{\mathcal{D}_\varepsilon}\frac{a^2(x)}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2 - \pi\big(\frac{a_0\omega_1}{1+\Lambda^2} + \mathcal{K}_{\frac{3}{a_0}}\big)\sum_{i\in I_0} a(p_i)|d_i|\ln|\ln\varepsilon|$$
$$+\frac{\pi}{4a_0}\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{2/3} + \frac{\pi}{2}\sum_{i\in I_-} a(p_i)|d_i||\ln\varepsilon| + o(|\ln\varepsilon|^{-5}) \le \mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) \le \mathcal{O}(|\ln\varepsilon|^{-1}). \tag{6.111}$$

Since $\frac{a_0\omega_1}{1+\Lambda^2} < -\mathcal{K}_{\frac{3}{a_0}}$ and $a(p_i) \ge a_0/2$ for $i \in I_0$, we derive from (6.111) that $\sum_{i\in I_0}|d_i| = o(|\ln\varepsilon|^{-1})$. Now since $a(p_i) \ge |\ln\varepsilon|^{-3/2}$ in $\mathcal{D}_\varepsilon$, we also obtain from (6.111) that $\sum_{i\in I_*}|d_i| = \mathcal{O}(|\ln\varepsilon|^{-1/6})$ and $\sum_{i\in I_-}|d_i| = \mathcal{O}(|\ln\varepsilon|^{-1/2})$. Hence $\sum_{i\in I_\varepsilon}|d_i| \equiv 0$ for ε sufficiently small. Coming back to (6.111), we infer that for any $0 < R < \sqrt{a_0}$,

$$\frac{1}{\varepsilon^2}\int_{B_R^\Lambda}(1-|v_\varepsilon|^2)^2 \le \frac{C_R}{\varepsilon^2}\int_{\mathcal{D}_\varepsilon} a^2(x)(1-|v_\varepsilon|^2)^2 \le o(1). \tag{6.112}$$

Then the proof of (6.107) follows as in [16] using the estimate *6.13.d)* in Proposition 6.13 on $|\nabla v_\varepsilon|$.

Since $\sum_{i\in I_\varepsilon}|d_i| = 0$, we derive from Proposition 6.17 that $\mathcal{R}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$. Using that $\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) \le o(1)$, we deduce that $\mathcal{E}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$ and hence we have $\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$. By (6.98) and (6.99), it leads to

$$\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1) \tag{6.113}$$

and $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$. Using (6.95) and (6.96), we get that

$$o(1) \le \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) \le -\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) + o(1) \le o(1) \tag{6.114}$$

and therefore $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = o(1)$. By (6.97), we have

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) = \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) + \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) \ge \frac{1}{2}\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) + o(1)$$

and it results from (6.114) that $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) = o(1)$. By (6.113), we conclude that $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) = o(1)$. □

Proof of (i) in Theorem 6.1. By *6.8.c)* in Proposition 6.8 and (6.107), it follows that $|u_\varepsilon| \to \sqrt{a^+}$ in $L^\infty_{\text{loc}}(\mathcal{D})$. According to *6.11.c)* in Proposition 6.11, it turns out that $|u_\varepsilon| \to \sqrt{a^+}$ in $L^\infty_{\text{loc}}(\mathbb{R}^2\setminus\partial\mathcal{D})$. Moreover, by (6.108), for any sequence $\varepsilon_n \to 0$ we can extract a subsequence (still denoted (ε_n)) such that $v_{\varepsilon_n} \to \alpha$ in $H^1_{\text{loc}}(\mathcal{D})$ for some constant $\alpha \in S^1$. We obtain that $u_{\varepsilon_n}e^{-i\Omega S} \to \alpha\sqrt{a^+}$ in $H^1_{\text{loc}}(\mathcal{D})$ by *6.8.e)* in Proposition 6.8. By Lemma 6.12, *6.13.b)* in Proposition 6.13 and (6.108), we conclude that (6.6) holds. □

6.4.3 Vortex existence near the critical velocity. Proof of *(ii)* in Theorem 6.1

We now prove *(ii)* in Theorem 6.1. We will use an appropriate test function in order to improve the upper bound of the energy $F_\varepsilon(u_\varepsilon)$.

Proof of (ii) in Theorem 6.1. Step1: Construction of a test function. Assume that $\Omega_1 + \delta \ln|\ln\varepsilon| \le \Omega \le \omega_0|\ln\varepsilon|$ for some positive constants δ and ω_0 (thus, $\omega_0 > \frac{\Lambda^2+1}{a_0}$). We consider the map $\tilde{v}_\varepsilon$ defined by

$$\tilde{v}_\varepsilon(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } |x| \ge \varepsilon, \\ \dfrac{x}{\varepsilon} & \text{otherwise} \end{cases}$$

and we set $\hat{u}_\varepsilon = \tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon$. We easily check that $\hat{u}_\varepsilon \in \mathcal{H}$. Lemma 6.12 yields

$$F_\varepsilon(\hat{u}_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon).$$

Then we estimate

$$\left|\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\right| \le \frac{1}{2}\int_{B_\varepsilon} \left|\Omega^2|\nabla S|^2 - 2\Omega^2 x^\perp \cdot \nabla S + k_\varepsilon\right| \tilde{\eta}_\varepsilon^2(1 - |\tilde{v}_\varepsilon|^2) = o(1).$$

A straightforward computation (using Proposition 6.7) leads to

$$\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) \le -\frac{\pi a_0^2 \delta}{1+\Lambda^2}\ln|\ln\varepsilon| + \mathcal{O}(1)$$

and consequently

$$F_\varepsilon(\hat{u}_\varepsilon) \le F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \frac{\pi a_0^2 \delta}{1+\Lambda^2}\ln|\ln\varepsilon| + \mathcal{O}(1). \tag{6.115}$$

We now set $\tilde{u}_\varepsilon = m_\varepsilon^{-1}\hat{u}_\varepsilon$ with $m_\varepsilon = \|\hat{u}_\varepsilon\|_{L^2(\mathbb{R}^2)}$ (so that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$). Since $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we have

$$m_\varepsilon^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\tilde{v}_\varepsilon|^2 = 1 + \int_{B_\varepsilon} \tilde{\eta}_\varepsilon^2(|\tilde{v}_\varepsilon|^2 - 1) = 1 + \mathcal{O}(\varepsilon^2).$$

From this estimate, we easily check that

$$F_\varepsilon(\tilde{u}_\varepsilon) = F_\varepsilon(\hat{u}_\varepsilon) + o(1). \tag{6.116}$$

Step 2. By the minimizing property of u_ε, we know that $F_\varepsilon(u_\varepsilon) \le F_\varepsilon(\tilde{u}_\varepsilon)$. In view of *6.13.b)* in Proposition 6.13, (6.115) and (6.116), it yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \le -\frac{\pi a_0^2 \delta}{1+\Lambda^2}\ln|\ln\varepsilon| + \mathcal{O}(1).$$

Using (6.96) and then (6.99), we derive that

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le -\frac{\pi a_0^2 \delta}{1+\Lambda^2}\ln|\ln\varepsilon| + \mathcal{O}(1). \tag{6.117}$$

On the other hand, by Proposition 6.17, we have

$$\begin{aligned}\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) &\ge -\frac{\pi\omega_0}{1+\Lambda^2}\sum_{i\in I_\varepsilon,\, d_i>0} a^2(p_i)\, d_i|\ln\varepsilon| \;+\; o(1) \\ &\ge -\frac{\pi\omega_0\, a_0}{1+\Lambda^2}\sum_{i\in \hat{I}_\varepsilon,\, d_i>0} a(p_i)\, d_i|\ln\varepsilon| - \frac{\pi}{2}\sum_{i\in I_\varepsilon\setminus\hat{I}_\varepsilon,\, d_i>0} a(p_i)\, d_i|\ln\varepsilon| \;+\; o(1)\end{aligned}$$

where we denoted

$$\hat{I}_\varepsilon = \{ i \in I_\varepsilon \,:\, a(p_i) \geq \frac{\Lambda^2+1}{2\omega_0} \}.$$

Then, by Proposition 6.16, we deduce that

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \mathcal{E}_\varepsilon(v_\varepsilon, \cup_{i\in I_\varepsilon} B_i) + \mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq -C_{\omega_0} \sum_{i\in \hat{I}_\varepsilon,\, d_i>0} a(p_i)\, d_i |\ln \varepsilon| \;+\; o(1)$$

for some constant $C_{\omega_0} > 0$. Therefore, by (6.117), it results that for small $\varepsilon > 0$,

$$\sum_{i\in \hat{I}_\varepsilon,\, d_i>0} d_i > 0.$$

We conclude that there exists $i_0 \in \hat{I}_\varepsilon$ such that $d_{i_0} > 0$, so that there exists at least one vortex inside the bulk $\mathcal{D}$ which remains at a positive distance (independent of ε) from $\partial\mathcal{D}$. If in addition, (6.106) holds, we claim that u_ε has at least one vortex close to the origin. Indeed, by (6.111) and (6.117), we obtain

$$-\pi\Big(\frac{a_0\omega_1}{1+\Lambda^2} + \mathcal{K}_{\frac{3}{a_0}}\Big) \sum_{i\in I_0} a(p_i)|d_i| \ln|\ln\varepsilon| \leq -\frac{\pi a_0^2 \delta}{1+\Lambda^2} \ln|\ln\varepsilon| + \mathcal{O}(1)$$

which implies for ε small enough that $\sum_{i\in I_0} |d_i| \geq C > 0$ for a constant C independent of ε. Hence, for ε small, there exists a ball B_{j_0} ($j_0 \in I_0$) that carries a vortex x^ε with $|x^\varepsilon| \leq \mathcal{O}(|\ln\varepsilon|^{-1/6})$. □

6.4.4 Energy estimates near the critical velocity. Proof of *(iii)* in Theorem 6.1

In this section, we prove the energy estimates stated in *(iii)* in Theorem 6.1 in the regime (6.106). First, we shall prove that the number of vortex balls with nonzero degree lying in a slightly smaller domain than $\mathcal{D}_\varepsilon$, is bounded.

Proposition 6.19 *Assume that* (6.106) *holds. Then*

$$N_0 := \sum_{i\in I_0} |d_i| \leq C_{\omega_1} \tag{6.118}$$

and setting $\mathcal{B}_\varepsilon = \{ x \in \mathbb{R}^2 \,:\, a(x) \geq |\ln\varepsilon|^{-1/2} \}$*, we have for* ε *sufficiently small,*

$$\sum_{i\in I_*\cup I_-,\, p_i\in\mathcal{B}_\varepsilon} |d_i| = 0. \tag{6.119}$$

Proof. Arguing as for (6.111), we derive that for ε small enough,

$$\begin{aligned}
\int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} a(x)|\nabla v_\varepsilon|^2 + \sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{2/3} + \sum_{i\in I_-} a(p_i)|d_i||\ln\varepsilon| \leq& \\
\leq C\Big|\frac{a_0\omega_1}{1+\Lambda^2} + \mathcal{K}_{\frac{3}{a_0}}\Big| \sum_{i\in I_0} a(p_i)|d_i| \ln|\ln\varepsilon| + \mathcal{O}(|\ln\varepsilon|^{-1})& \\
\leq C_0 N_0 \ln|\ln\varepsilon| + \mathcal{O}(|\ln\varepsilon|^{-1})&
\end{aligned} \tag{6.120}$$

for some positive constant C_0 independent of ε. We set

$$\tilde{I}_* = \{i \in I_* \, : \, p_i \in \mathcal{B}_\varepsilon\} \, , \quad N_* = \sum_{i \in \tilde{I}_*} |d_i|,$$

and

$$\tilde{I}_- = \{i \in I_- \, : \, p_i \in \mathcal{B}_\varepsilon\} \, , \quad N_- = \sum_{i \in \tilde{I}_-} |d_i|.$$

Since $a(p_i) \geq |\ln\varepsilon|^{-1/2}$ for any $i \in \tilde{I}_* \cup \tilde{I}_-$, we obtain from (6.120),

$$\int_{\mathcal{D}_\varepsilon \setminus \cup_{i \in I_\varepsilon} B_i} a(x)|\nabla v_\varepsilon|^2 + N_*|\ln\varepsilon|^{1/6} + N_-|\ln\varepsilon|^{1/2} \leq C_0 N_0 \ln|\ln\varepsilon| + \mathcal{O}(|\ln\varepsilon|^{-1}) \tag{6.121}$$

which implies in particular that

$$\max\{N_*, N_-\} \leq \frac{N_0}{2} \tag{6.122}$$

for ε sufficiently small. We now show that N_0 is uniformly bounded in ε. Consider the sets

$$\mathcal{I}_\varepsilon = \big[\, |\ln\varepsilon|^{-1/6}, \frac{\sqrt{a_0}}{2} \,\big] \quad \text{and} \quad \mathcal{J}_\varepsilon = \big\{ r \in \mathcal{I}_\varepsilon \, : \, \partial B_r^\Lambda \cap (\cup_{i \in I_\varepsilon} \overline{B}_i) = \emptyset \big\} \, .$$

Notice that $\mathcal{J}_\varepsilon$ is a finite union of intervals verifying $|\mathcal{I}_\varepsilon \setminus \mathcal{J}_\varepsilon| \leq |\ln\varepsilon|^{-10}$. For $r \in \mathcal{J}_\varepsilon$ and ε small, we have $|v_\varepsilon| \geq \frac{1}{2}$ on ∂B_r^Λ and therefore, we can define

$$D(r) = \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B_r^\Lambda\right).$$

By (6.122), we obtain that for small ε,

$$|D(r)| = \Big| \sum_{|p_i|_\Lambda < r} d_i \Big| \geq N_0 - N_- \geq \frac{N_0}{2} \quad \text{for any } r \in \mathcal{J}_\varepsilon.$$

We have (using elliptic coordinates $x_1 = r\cos\theta$, $x_2 = \Lambda^{-1} r \sin\theta$)

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \cup_{i \in I_\varepsilon} B_i} a(x)|\nabla v_\varepsilon|^2 \geq \frac{3a_0}{4\Lambda} \int_{\mathcal{J}_\varepsilon} \left(\int_0^{2\pi} |\nabla v_\varepsilon|^2 r \, d\theta \right) dr \geq C \int_{\mathcal{J}_\varepsilon} \frac{1}{r} \bigg(\int_0^{2\pi} \Big| v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau} \Big|^2 r^2 \, d\theta \bigg) \, dr.$$

We set $w_\varepsilon = \frac{v_\varepsilon}{|v_\varepsilon|}$ in $B^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \cup_{i \in I_\varepsilon} B_i$. Since $|v_\varepsilon \wedge \dfrac{\partial v_\varepsilon}{\partial \tau}| = |v_\varepsilon|^2 |w_\varepsilon \wedge \dfrac{\partial w_\varepsilon}{\partial \tau}| \geq \dfrac{1}{4} |w_\varepsilon \wedge \dfrac{\partial w_\varepsilon}{\partial \tau}|$ in $B^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \cup_{i \in I_\varepsilon} B_i$, we infer that

$$\begin{aligned}
\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \cup_{i \in I_\varepsilon} B_i} a(x)|\nabla v_\varepsilon|^2 &\geq C \int_{\mathcal{J}_\varepsilon} \frac{1}{r} \bigg(\int_0^{2\pi} \Big| w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau} \Big|^2 r^2 \, d\theta \bigg) dr \\
&\geq C \int_{\mathcal{J}_\varepsilon} \frac{1}{r} \bigg(\int_0^{2\pi} w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau} \, r \, d\theta \bigg)^2 dr \geq C \int_{\mathcal{J}_\varepsilon} \frac{D(r)^2}{r} \, dr \geq C N_0^2 \int_{\mathcal{J}_\varepsilon} \frac{dr}{r}.
\end{aligned}$$

Notice now that

$$\left| \int_{\mathcal{I}_\varepsilon} \frac{dr}{r} - \int_{\mathcal{J}_\varepsilon} \frac{dr}{r} \right| \leq |\ln\varepsilon|^{1/6} |\mathcal{I}_\varepsilon \setminus \mathcal{J}_\varepsilon| = o(1)$$

and since $\int_{\mathcal{I}_\varepsilon} \frac{dr}{r} = C \ln|\ln\varepsilon| + \mathcal{O}(1)$, we finally get that

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \cup_{i\in I_\varepsilon} B_i} a(x)|\nabla v_\varepsilon|^2 \geq C_1 \ln|\ln\varepsilon| N_0^2$$

for some positive constant C_1 independent of ε. From (6.121), we derive

$$\left(C_1 N_0^2 - C_0 N_0\right) \ln|\ln\varepsilon| \leq \mathcal{O}(|\ln\varepsilon|^{-1})$$

which implies that N_0 is uniformly bounded in ε. Then it follows by (6.121) that

$$N_* \leq \mathcal{O}(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^{1/6}}) \quad \text{and} \quad N_- \leq \mathcal{O}(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^{1/2}}).$$

Therefore, $N_- = N_* = 0$ for ε sufficiently small. □

Proof of (iii) in Theorem 6.1. From Proposition 6.17, (6.106) and (6.119), we infer that for ε small,

$$\begin{aligned}\mathcal{R}_\varepsilon\big(v_\varepsilon, \mathcal{D}_\varepsilon\big) &\geq \frac{-\pi a_0 \Omega}{1+\Lambda^2} \sum_{i\in I_0} a(p_i)|d_i| - \frac{\pi\Omega}{1+\Lambda^2} |\ln\varepsilon|^{-1/2} \sum_{i\in I_*\setminus \tilde{I}_*} a(p_i)|d_i| + o(|\ln\varepsilon|^{-5}) \\ &\geq -\pi \sum_{i\in I_0} a(p_i)|d_i|\big(|\ln\varepsilon| + \frac{a_0\omega_1}{1+\Lambda^2}\ln|\ln\varepsilon|\big) - \frac{2\pi}{a_0}\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{1/2} + o(|\ln\varepsilon|^{-5}).\end{aligned}$$

We now inject this estimate in (6.109) to derive that $\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon| \leq CN_0 \ln|\ln\varepsilon| + o(1)$ and hence, by (6.118), $\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{1/2} = o(1)$. It yields

$$\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{R}_\varepsilon\left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\right) + o(1) \geq -\pi \sum_{i\in I_0} a(p_i)|d_i|\big(|\ln\varepsilon| + \frac{a_0\omega_1}{1+\Lambda^2}\ln|\ln\varepsilon|\big) + o(1).$$

Since $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + \mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq \mathcal{O}(|\ln\varepsilon|^{-1})$, it follows

$$\begin{aligned}\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) &\leq \pi \sum_{i\in I_0} a(p_i)|d_i|\big(|\ln\varepsilon| + \frac{a_0\omega_1}{1+\Lambda^2}\ln|\ln\varepsilon|\big) + o(1) \\ &\leq C_{\omega_1} N_0 |\ln\varepsilon| + o(1) \leq C_{\omega_1}|\ln\varepsilon|.\end{aligned} \tag{6.123}$$

Set $\mathcal{A}_\varepsilon = \mathcal{D}_\varepsilon \setminus B^\Lambda_{2|\ln\varepsilon|^{-1/6}}$. Matching *(iv)* in Proposition 6.16 with (6.123), we finally obtain

$$\begin{aligned}\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{A}_\varepsilon) \leq \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_0} B_i) &\leq \pi(\frac{a_0\omega_1}{1+\Lambda^2} + \mathcal{K}_{\frac{3}{a_0}}) \sum_{i\in I_0} a(p_i)|d_i| \ln|\ln\varepsilon| + o(1) \\ &\leq C_{\omega_1} N_0 \ln|\ln\varepsilon| \leq C_{\omega_1}\ln|\ln\varepsilon|\end{aligned}$$

and the proof is complete. □

Remark 6.6 *For general potentials $a(x)$, the analysis becomes rather delicate when the set of maximum points of the quotient $\frac{\xi}{a}$ in $\mathcal{D} = \{x \in \mathbb{R}^2 : a(x) > 0\}$ is not finite. Recall that ξ is the* *solution of the problem* (6.16)*. An example is given by the following perturbation at the origin of the harmonic potential* $1-|x|^2$*:*

$$a(x) = \begin{cases} \frac{1}{1+|x|^2} & \text{if } |x| < 1, \\ \frac{2-|x|}{2} & \text{if } |x| \geq 1. \end{cases}$$

Here, the set of maximum points of the quotient $\frac{\xi}{a}$ *is a circle centered in the origin.*

Chapter 7

Energy expansion and vortex location for a two-dimensional rotating Bose-Einstein condensate

Abstract

We continue the analysis started in Chapter 6 on a model describing a two dimensional rotating Bose-Einstein condensate. This model consists in minimizing under the unit mass constraint, a Gross-Pitaevskii energy defined in $\mathbb{R}^2$. In this contribution, we estimate the critical rotational speeds Ω_d for having exactly d vortices in the bulk of the condensate and we determine their topological charge and their precise location. Our approach relies on asymptotic energy expansion techniques developed by Serfaty [80, 81, 82] for the Ginzburg-Landau energy of superconductivity in the high κ limit.

This chapter is written in collaboration with V. Millot; the original text is published in Rev. Math. Phys. **18** (2006), 119–162 (cf. [56]) and some of these results were announced in C. R. Math. Acad. Sci. Paris **340** (2005), 571–576 (cf. [54]).

7.1 Introduction

As in Chapter 6, we consider here a two dimensional model describing a condensate placed in a trap that strongly confines the atoms in the direction of the rotation axis. In the nondimensionalized form, the wave function minimizes the Gross-Pitaevskii (GP) energy

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right] - \Omega\, x^\perp \cdot (iu, \nabla u) \right\} dx \tag{7.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1 \tag{7.2}$$

where $\varepsilon > 0$ is small and describes the ratio of two characteristic lengths and $\Omega = \Omega(\varepsilon) \geq 0$ is the angular velocity. The function $a(x)$ in (7.1) comes from the existence of a potential trapping the atoms, and is normalized such that $\int_{\mathbb{R}^2} a^+(x) = 1$. We will restrict our attention to the specific case of a harmonic trapping, that is $a(x) = a_0 - x_1^2 - \Lambda^2 x_2^2$ with $a_0 = \sqrt{2\Lambda/\pi}$ for some constant $\Lambda \in (0,1]$, which corresponds to actual experiments (see [65, 66]).

Our goal is to compute an asymptotic expansion of the energy $F_\varepsilon(u_\varepsilon)$ and to determine the number and the location of vortices according to the value of the angular speed $\Omega(\varepsilon)$ in the limit $\varepsilon \to 0$. More precisely, we want to estimate the critical velocity Ω_d for which the dth vortex becomes energetically favorable and to derive a reduced energy governing the location of the vortices (the so-called "*renormalized energy*" by analogy with [17, 80, 81]).

We have started in Chapter 6 the analysis of minimizers u_ε of the functional F_ε under the constraint (7.2) and we have already determined the critical rotational speed $\Omega_1 = \frac{\sqrt{\pi}(1+\Lambda^2)}{\sqrt{2\Lambda}}|\ln\varepsilon|$ of nucleation of the first vortex inside the domain

$$\mathcal{D} = \left\{x \in \mathbb{R}^2 \,:\, a(x) > 0\right\}.$$

In the physical context, the set $\mathcal{D}$ represents the region occupied by the condensate since in the limit $\varepsilon \to 0$, the minimization of F_ε forces $|u_\varepsilon|^2$ to be close to the function $a^+(x)$ ($F_\varepsilon(u_\varepsilon)$ remaining small in front of $1/\varepsilon^2$). We proved that for subcritical velocities $\Omega \le \Omega_1 - \delta \ln|\ln\varepsilon|$ with $-\delta < \omega_1^\star < 0$ for some constant $\omega_1^\star$, there is no vortices in the region $\mathcal{D}$ and u_ε behaves as the *vortex-free* profile $\tilde{\eta}_\varepsilon e^{i\Omega S}$ where the phase function $S : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$S(x) = \frac{\Lambda^2 - 1}{\Lambda^2 + 1}\, x_1 x_2 \tag{7.3}$$

and $\tilde{\eta}_\varepsilon$ is the (unique) positive solution of the minimization problem

$$\text{Min}\,\left\{E_\varepsilon(u) \,:\, u \in \mathcal{H},\, \|u\|_{L^2(\mathbb{R}^2)} = 1\right\} \tag{7.4}$$

with

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right]$$

$$\text{and} \qquad \mathcal{H} = \left\{u \in H^1(\mathbb{R}^2, \mathbb{C}) \,:\, \int_{\mathbb{R}^2} |x|^2|u|^2 < \infty\right\}.$$

In this contribution which constitutes the sequel of Chapter 6, we push forward the study of minimizers u_ε. First, we prove the following estimate on the critical speed Ω_d for any integer $d \ge 1$ in the asymptotic $\varepsilon \to 0$,

$$\Omega_d = \frac{1+\Lambda^2}{a_0}(|\ln\varepsilon| + (d-1)\ln|\ln\varepsilon|) = \frac{\sqrt{\pi}(1+\Lambda^2)}{\sqrt{2\Lambda}}(|\ln\varepsilon| + (d-1)\ln|\ln\varepsilon|)\,.$$

Then we show that for velocities ranged between Ω_d and Ω_{d+1}, any minimizer has exactly d vortices of degree $+1$ inside $\mathcal{D}$. Establishing an asymptotic expansion of $F_\varepsilon(u_\varepsilon)$ as $\varepsilon \to 0$, we derive the distribution of vortices within $\mathcal{D}$ as a minimizing configuration of the reduced energy given by (7.5) below. We also improve the result stated in Chapter 6 for the nonexistence of vortices in the subcritical case by showing that the best constant is $\omega_1^\star = 0$, that is subcritical velocities go up to $\Omega_1 - \delta\ln|\ln\varepsilon|$ for any $\delta > 0$.

Our main theorem can be stated as follows:

Theorem 7.1 *Let u_ε be any minimizer of F_ε in $\mathcal{H}$ under the constraint* (7.2) *and let* $0<\delta\ll 1$ *be any small constant.*

(i) If $\Omega \le \Omega_1 - \delta\ln|\ln\varepsilon|$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_0=\varepsilon_0(R_0,\delta)>0$ such that for any $\varepsilon<\varepsilon_0$, u_ε is vortex free in $B^\Lambda_{R_0} = \{x\in\mathbb{R}^2 \,:\, |x|^2_\Lambda = x_1^2+\Lambda^2x_2^2 < R_0^2\}$, i.e., u_ε does not vanish in $B^\Lambda_{R_0}$. In addition,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde\eta_\varepsilon e^{i\Omega S}) + o(1).$$

(ii) If $\Omega_d + \delta\ln|\ln\varepsilon| \le \Omega \le \Omega_{d+1} - \delta\ln|\ln\varepsilon|$ for some integer $d\ge 1$, then for any $R_0<\sqrt{a_0}$, there exists $\varepsilon_1=\varepsilon_1(R_0,d,\delta)>0$ such that for any $\varepsilon<\varepsilon_1$, u_ε has exactly d vortices $x_1^\varepsilon,\dots,x_d^\varepsilon$ of degree one in $B^\Lambda_{R_0}$. Moreover,

$$|x_j^\varepsilon| \le C\,\Omega^{-1/2} \quad \text{for any } j=1,\dots,d\,, \quad \text{and} \quad |x_i^\varepsilon - x_j^\varepsilon| \ge C\,\Omega^{-1/2} \quad \text{for any } i\neq j$$

where $C>0$ denotes a constant independent of ε. Setting $\tilde x_j^\varepsilon = \sqrt{\Omega}\,x_j^\varepsilon$, the configuration $(\tilde x_1^\varepsilon,\dots,\tilde x_d^\varepsilon)$ tends to minimize as $\varepsilon\to 0$ the renormalized energy

$$w(b_1,\dots,b_d) = -\pi a_0 \sum_{i\neq j} \ln|b_i-b_j| + \frac{\pi a_0}{1+\Lambda^2}\sum_{j=1}^d |b_j|^2_\Lambda. \tag{7.5}$$

In addition,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde\eta_\varepsilon e^{i\Omega S}) - \frac{\pi a_0^2 d}{1+\Lambda^2}(\Omega-\Omega_1) + \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| + \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{d,\Lambda} + o(1) \tag{7.6}$$

where $Q_{d,\Lambda}$ is a constant depending only on d and Λ.

These results are in agreement with the study made by Castin and Dum [34] who have looked for minimizers in a reduced class of functions. More precisely, we find the same critical angular velocities Ω_d as well as a distribution of vortices around the origin at a scale $\Omega^{-1/2}$. The minimizing configurations for the renormalized energy $w(\cdot)$ have been studied in the radial case $\Lambda=1$ by Gueron and Shafrir in [49]. They prove that for $d\le 6$, regular polygons centered at the origin and *stars* are local minimizers. For larger d, they numerically found minimizers with a shape of concentric polygons and then triangular lattices as d increases. These figures are exactly the ones observed in physical experiments (see [65, 66]).

Our approach, suggested in [4] by Aftalion and Du, strongly relies on techniques developed by Serfaty [80, 81, 82] for the Ginzburg-Landau (GL) energy of superconductivity in the high κ limit. We point out that Serfaty has already applied the method to a simplified GP energy (the study is made in a ball instead of $\mathbb{R}^2$ with $a(x)\equiv 1$ and the minimization is performed without mass constraint) and has obtained in [83] a result analogue to Theorem 7.1 which shows that the simple model captures the main features of the full model concerning vortices. We emphasize once more that we treat here the exact physical model without any simplifying assumptions. The outline of our proof follows Serfaty's method but many technical difficulties arise from the specificities of the problem such as the unit mass constraint or the degenerate behavior of the function $a(x)$ near the boundary of $\mathcal{D}$. As we shall see, a very delicate analysis is required so that we prefer sometimes to write all the details even if some proofs follow closely other authors. More precisely, we also make use of the following results on the GL functional [7, 12, 13, 18, 63, 75, 77, 88], starting from the pioneering work of Béthuel, Brezis and Hélein [17]. We finally refer to Chapter 6 for additional references on mathematical studies of vortices in BECs.

For the convenience, we recall now some results already established in Chapter 6. First, we have proved the existence and smoothness of any minimizer u_ε of F_ε under the constraint (7.2) in the regime

$$\Omega \leq \frac{1+\Lambda^2}{a_0}\Big(|\ln\varepsilon| + \omega_1 \ln|\ln\varepsilon|\Big) \tag{7.7}$$

for some constant $\omega_1 \in \mathbb{R}$, as well as some qualitative properties: $E_\varepsilon(u_\varepsilon) \leq C|\ln\varepsilon|^2$, $|u_\varepsilon| \lesssim \sqrt{a^+}$ in any compact $K \subset \mathcal{D}$ and $|u_\varepsilon|$ decreases exponentially fast to 0 outside $\mathcal{D}$. We have also showed the existence and uniqueness of the positive minimizer $\tilde\eta_\varepsilon$ of E_ε under the mass constraint (7.2) for every $\varepsilon > 0$. Concerning the Lagrange multiplier $k_\varepsilon \in \mathbb{R}$ associated to $\tilde\eta_\varepsilon$ and the qualitative properties of $\tilde\eta_\varepsilon$, we have obtained:

$$|k_\varepsilon| \leq C|\ln\varepsilon|\,, \tag{7.8}$$

$E_\varepsilon(\tilde\eta_\varepsilon) \leq C|\ln\varepsilon|$ for ε small and $\tilde\eta_\varepsilon \to \sqrt{a^+}$ in $L^\infty(\mathbb{R}^2) \cap C^1_{\text{loc}}(\mathcal{D})$ as $\varepsilon \to 0$. Using a splitting technique introduced by Lassoued and Mironescu [63], we were able to decouple into two independent parts the energy $F_\varepsilon(u)$ for any $u \in \mathcal{H}$. The first part corresponds to the energy of the vortex-free profile $\tilde\eta_\varepsilon e^{i\Omega S}$ and the second part to a reduced energy of $v = u/(\tilde\eta_\varepsilon e^{i\Omega S})$, i.e.,

$$F_\varepsilon(u) = F_\varepsilon(\tilde\eta_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v) + \tilde{\mathcal{T}}_\varepsilon(v) \tag{7.9}$$

where the functionals $\tilde{\mathcal{F}}_\varepsilon$ and $\tilde{\mathcal{T}}_\varepsilon$ are defined by

$$\tilde{\mathcal{F}}_\varepsilon(v) = \tilde{\mathcal{E}}_\varepsilon(v) + \tilde{\mathcal{R}}_\varepsilon(v)\,, \tag{7.10}$$

$$\tilde{\mathcal{E}}_\varepsilon(v) = \int_{\mathbb{R}^2} \frac{\tilde\eta_\varepsilon^2}{2}|\nabla v|^2 + \frac{\tilde\eta_\varepsilon^4}{4\varepsilon^2}(|v|^2-1)^2\,, \quad \tilde{\mathcal{R}}_\varepsilon(v) = \frac{\Omega}{1+\Lambda^2}\int_{\mathbb{R}^2} \tilde\eta_\varepsilon^2 \nabla^\perp a \cdot (iv, \nabla v)\,, \tag{7.11}$$

$$\tilde{\mathcal{T}}_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^2} \big(\Omega^2|\nabla S|^2 - 2\Omega^2 x^\perp \cdot \nabla S + k_\varepsilon\big)\tilde\eta_\varepsilon^2(|v|^2-1). \tag{7.12}$$

Since the function $\tilde\eta_\varepsilon$ does not vanish, the vortex structure of any minimizer u_ε can be studied via the map

$$v_\varepsilon = u_\varepsilon/(\tilde\eta_\varepsilon e^{i\Omega S}),$$

applying the Ginzburg-Landau techniques to the weighted energy $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon)$. It is intuitively clear that difficulties will arise in the region where $\tilde\eta_\varepsilon$ is small and we will require the following properties of v_ε inherited from u_ε and $\tilde\eta_\varepsilon$: $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \leq C|\ln\varepsilon|^2$, $\big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)\big| \leq o(1)$, $|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)| \leq C|\ln\varepsilon|^2$, $|\nabla v_\varepsilon| \leq C_K\varepsilon^{-1}$ and $|v_\varepsilon| \lesssim 1$ in any compact $K \subset \mathcal{D}$. In the sequel, it will be more convenient to replace in the different functionals the function $\tilde\eta_\varepsilon^2$ by its limit $a^+(x)$. We denote by $\mathcal{F}_\varepsilon$, $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ the corresponding functionals (see Notations below). In the regime (7.7), we have computed in Chapter 6 some fundamental bounds for the energy of v_ε in a domain slightly smaller than $\mathcal{D}$:

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le o(1), \tag{7.13}$$

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le C_{\omega_1} |\ln \varepsilon|, \tag{7.14}$$

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \{|x|_\Lambda < 2|\ln\varepsilon|^{-1/6}\}) \le C_{\omega_1} \ln|\ln\varepsilon|, \tag{7.15}$$

where

$$\mathcal{D}_\varepsilon = \{x \in \mathcal{D} \,:\, a(x) > \nu_\varepsilon |\ln\varepsilon|^{-3/2}\} \tag{7.16}$$

and ν_ε is a chosen parameter in the interval $(1, 2)$ (see Proposition 6.16 in Chapter 6). These estimates represent the starting point of our analysis here.

The plan of the chapter is as follows. In Section 2, we prove that the subset of $\mathcal{D}$ where $|v_\varepsilon|$ is smaller than $1/2$ can be covered by a family of disjoint discs such that each radius vanishes as $\varepsilon \to 0$, the cardinal of this family is uniformly bounded with respect to ε and v_ε has a non vanishing degree around each disc of the family. We will call such a collection of discs a *fine structure of vortices* and a *vortex* one of these discs (identified with their center). In Section 3, we establish various lower energy estimates namely inside a vortex and away from the vortices. In Section 4, we prove Theorem 7.1 matching the lower energy estimates with upper estimates coming from the construction of trial functions. These constructions are presented in Section 5 which can be read independently of the rest of the chapter. Finally, we prove in the Appendix, an auxiliary result that we shall use in the proof of Theorem 7.1.

Notations. Throughout the chapter, we denote by C a positive constant independent of ε and we use the subscript to point out a possible dependence on the argument. For $x = (x_1, x_2) \in \mathbb{R}^2$, we write

$$|x|_\Lambda = \sqrt{x_1^2 + \Lambda^2 x_2^2} \quad \text{and} \quad B_R^\Lambda = \left\{x \in \mathbb{R}^2,\, |x|_\Lambda < R\right\}$$

and for $\mathcal{A} \subset \mathbb{R}^2$,

$$\tilde{\mathcal{E}}_\varepsilon(v, \mathcal{A}) = \int_{\mathcal{A}} \frac{1}{2}\tilde{\eta}^2 |\nabla v|^2 + \frac{\tilde{\eta}^4}{4\varepsilon^2}(1 - |v|^2)^2, \quad \mathcal{E}_\varepsilon(v, \mathcal{A}) = \int_{\mathcal{A}} \frac{1}{2} a|\nabla v|^2 + \frac{a^2}{4\varepsilon^2}(1-|v|^2)^2$$

$$\tilde{\mathcal{R}}_\varepsilon(v, \mathcal{A}) = \frac{\Omega}{1+\Lambda^2}\int_{\mathcal{A}} \tilde{\eta}^2 \nabla^\perp a \cdot (iv, \nabla v), \quad \mathcal{R}_\varepsilon(v, \mathcal{A}) = \frac{\Omega}{1+\Lambda^2}\int_{\mathcal{A}} a\nabla^\perp a \cdot (iv, \nabla v)$$

$$\tilde{\mathcal{F}}_\varepsilon(v, \mathcal{A}) = \tilde{\mathcal{E}}_\varepsilon(v, \mathcal{A}) + \tilde{\mathcal{R}}_\varepsilon(v, \mathcal{A}), \quad \mathcal{F}_\varepsilon(v, \mathcal{A}) = \mathcal{E}_\varepsilon(v, \mathcal{A}) + \mathcal{R}_\varepsilon(v, \mathcal{A}). \tag{7.17}$$

We do not write the dependence on $\mathcal{A}$ when $\mathcal{A} = \mathbb{R}^2$.

7.2 Fine structure of vortices

The main goal of this section is to construct a fine structure of vortices away from the boundary of $\mathcal{D}$. The analysis here follows the ideas in [17] and [18]. The main difficulty in our situation is due to the presence in the energy of the weight function $a(x)$ which vanishes on $\partial\mathcal{D}$ and it does not allow us to construct the structure up to the boundary because of the resulting degeneracy in the energy estimates. Throughout this chapter, we assume that Ω satisfies (7.7), so that (7.13), (7.14) and (7.15) hold. We will prove the following results for the map $v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$:

Theorem 7.2 1) *For any* $R \in (\frac{\sqrt{a_0}}{2}, \sqrt{a_0})$ *there exists* $\varepsilon_R > 0$ *such that for any* $\varepsilon < \varepsilon_R$,

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B_R^\Lambda \setminus B^\Lambda_{\frac{\sqrt{a_0}}{2}}.$$

2) *There exist some constants* $N \in \mathbb{N}$, $\lambda_0 > 0$ *and* $\varepsilon_0 > 0$ *(which only depend on* ω_1*) such that for any* $\varepsilon < \varepsilon_0$*, one can find a finite collection of points* $\{x_j^\varepsilon\}_{j\in J_\varepsilon} \subset B^\Lambda_{\frac{\sqrt{a_0}}{4}}$ *such that* $\mathrm{Card}(J_\varepsilon) \leq N$ *and*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } \overline{B}^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \big(\cup_{j\in J_\varepsilon} B(x_j^\varepsilon, \lambda_0\varepsilon)\big).$$

Remark 7.1 *The statement of Theorem 7.2 also holds if the radius* $\frac{\sqrt{a_0}}{2}$ *is replaced by an arbitrary* $r \in (0, R)$ *but then the constants in Theorem 7.2 depend on* r*. For the sake of simplicity, we prefer to fix* $r = \frac{\sqrt{a_0}}{2}$.

In the next proposition, we replace as in [80] the discs $\{B(x_j^\varepsilon, \lambda_0\varepsilon)\}_{j\in J_\varepsilon}$ obtained in Theorem 7.2 by slightly larger discs $B(x_j^\varepsilon, \rho)$ (deleting some of the points x_j^ε if necessary), in order to get a precise information on the behavior of v_ε on $\partial B(x_j^\varepsilon, \rho)$. The resulting family of discs will represent the vortices of the map v_ε (and hence the vortices of u_ε also).

Proposition 7.3 *Let* $0 < \beta < \mu < 1$ *be given constants such that* $\overline{\mu} := \mu^{N+1} > \beta$ *and let* $\{x_j^\varepsilon\}_{j\in J_\varepsilon}$ *be the collection of points given by 2) in Theorem 7.2. There exists* $0 < \varepsilon_1 < \varepsilon_0$ *such that for any* $\varepsilon < \varepsilon_1$*, we can find* $\tilde{J}_\varepsilon \subset J_\varepsilon$ *and* $\rho > 0$ *verifying*

(i) $\lambda_0\varepsilon \leq \varepsilon^\mu \leq \rho \leq \varepsilon^{\overline{\mu}} < \varepsilon^\beta$,

(ii) $|v_\varepsilon| \geq \frac{1}{2}$ *in* $\overline{B}^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \cup_{j\in\tilde{J}_\varepsilon} B(x_j^\varepsilon, \rho)$,

(iii) $|v_\varepsilon| \geq 1 - \dfrac{2}{|\ln\varepsilon|^2}$ *on* $\partial B(x_j^\varepsilon, \rho)$ *for every* $j \in \tilde{J}_\varepsilon$,

(iv) $\displaystyle\int_{\partial B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \leq \frac{C(\beta,\mu)}{\rho}$ *for every* $j \in \tilde{J}_\varepsilon$,

(v) $|x_i^\varepsilon - x_j^\varepsilon| \geq 8\rho$ *for every* $i, j \in \tilde{J}_\varepsilon$ *with* $i \neq j$.

Moreover, for each $j \in \tilde{J}_\varepsilon$*, we have*

$$D_j := \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_j^\varepsilon, \rho)\right) \neq 0 \quad \text{and} \quad |D_j| \leq C \tag{7.18}$$

for a constant C *independent of* ε.

Remark 7.2 *We point out that for every* $j \in \tilde{J}_\varepsilon$*, the disc* $B(x_j^\varepsilon, \rho)$ *carries at least one zero of* v_ε *since the degree* $D_j \neq 0$.

7.2.1 Some local estimates

We start with a fundamental lemma. It strongly relies on Pohozaev's identity and it will play a similar role as Theorem III.2 in [17]. In our situation, we only derive local estimates as in [7, 18, 88]. Some of the arguments used in the proof are taken from [7, 18].

Lemma 7.4 *For any $0 < R < \sqrt{a_0}$ and $\frac{2}{3} < \alpha < 1$, there exists a positive constant $C_{R,\alpha}$ such that*

$$\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(1-|v_\varepsilon|^2)^2 \le C_{R,\alpha} \quad \text{for any } x_0 \in B_R^\Lambda.$$

Proof. Step 1. Set $\tilde{u}_\varepsilon = u_\varepsilon e^{-i\Omega S}$. We claim that

$$E_\varepsilon(\tilde{u}_\varepsilon, \mathcal{D}_\varepsilon) \le C|\ln\varepsilon| \tag{7.19}$$

where $\mathcal{D}_\varepsilon$ is defined in (7.16). Indeed, since $\tilde{u}_\varepsilon = \tilde{\eta}_\varepsilon v_\varepsilon$, we get that

$$|\nabla\tilde{u}_\varepsilon|^2 \le C(\tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2|\nabla\tilde{\eta}_\varepsilon|^2)$$

By Proposition 6.8 and Proposition 6.13 in Chapter 6, $|v_\varepsilon| \le C$, $\tilde{\eta}_\varepsilon^2 \le Ca$ in $\mathcal{D}_\varepsilon$ and $E_\varepsilon(\tilde{\eta}_\varepsilon) \le C|\ln\varepsilon|$ and consequently,

$$\int_{\mathcal{D}_\varepsilon}|\nabla\tilde{u}_\varepsilon|^2 \le C\left(\int_{\mathcal{D}_\varepsilon} a(x)|\nabla v_\varepsilon|^2 + \int_{\mathcal{D}_\varepsilon}|\nabla\tilde{\eta}_\varepsilon|^2\right) \le C|\ln\varepsilon|$$

by (7.14). On the other hand, we also have

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathcal{D}_\varepsilon}(a(x)-|\tilde{u}_\varepsilon|^2)^2 &\le \frac{C}{\varepsilon^2}\int_{\mathcal{D}_\varepsilon}\left[(a(x)-\tilde{\eta}_\varepsilon^2)^2 + \tilde{\eta}_\varepsilon^4(1-|v_\varepsilon|^2)^2\right]\\ &\le \frac{C}{\varepsilon^2}\left(\int_{\mathcal{D}_\varepsilon}(a(x)-\tilde{\eta}_\varepsilon^2)^2 + \int_{\mathcal{D}_\varepsilon}a^2(x)(1-|v_\varepsilon|^2)^2\right) \le C|\ln\varepsilon|\end{aligned}$$

and therefore (7.19) follows.

Step 2. We are going to show that one can find a constant $C_{R,\alpha} > 0$, independent of ε, such that for any $x_0 \in B_R^\Lambda$, there is some $r_0 \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$ satisfying

$$E_\varepsilon\left(\tilde{u}_\varepsilon, \partial B(x_0, r_0)\right) \le \frac{C_{R,\alpha}}{r_0}.$$

We proceed by contradiction. Assume that for all $M > 0$, there is $x_M \in B_R^\Lambda$ such that

$$E_\varepsilon\left(\tilde{u}_\varepsilon, \partial B(x_M, r)\right) \ge \frac{M}{r}, \quad \text{for any } r \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3}). \tag{7.20}$$

Obviously, for ε small, $B(x_M, \varepsilon^{\alpha/2+1/3}) \subset \mathcal{D}_\varepsilon$. Integrating (7.20) for $r \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$, we derive that

$$E_\varepsilon\left(\tilde{u}_\varepsilon, \mathcal{D}_\varepsilon\right) \ge M\int_{\varepsilon^\alpha}^{\varepsilon^{\alpha/2+1/3}}\frac{dr}{r} = M(\alpha/2-1/3)|\ln\varepsilon|$$

which contradicts Step 1 for M large enough.

Step 3. Fix $x_0 \in B_R^\Lambda$ and let $r_0 \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$ be given by Step 2. We recall that any minimizer u_ε of F_ε in $\{u \in \mathcal{H}, \|u\|_{L^2(\mathbb{R}^2)} = 1\}$ satisfies

$$-\Delta u_\varepsilon + 2i\Omega x^\perp \cdot \nabla u_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |u_\varepsilon|^2)u_\varepsilon + \ell_\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^2$$

where ℓ_ε denotes the Lagrange multiplier. Therefore, we have

$$-\Delta \tilde{u}_\varepsilon = \frac{1}{\varepsilon^2}(a(x_0) - |\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon + \frac{1}{\varepsilon^2}(a(x) - a(x_0))\tilde{u}_\varepsilon + 2i\Omega(\nabla S - x^\perp) \cdot \nabla \tilde{u}_\varepsilon \\ + (\ell_\varepsilon + 2\Omega^2 x^\perp \cdot \nabla S - \Omega^2|\nabla S|^2)\tilde{u}_\varepsilon \quad \text{in } B(x_0, r_0). \tag{7.21}$$

As in the proof of the Pohozaev identity, we multiply (7.21) by $(x - x_0) \cdot \nabla \tilde{u}_\varepsilon$ and we integrate by parts in $B(x_0, r_0)$. We have

$$\int_{B(x_0,r_0)} -\Delta \tilde{u}_\varepsilon \cdot [(x - x_0) \cdot \nabla \tilde{u}_\varepsilon] = \frac{r_0}{2}\int_{\partial B(x_0,r_0)} |\nabla \tilde{u}_\varepsilon|^2 - r_0 \int_{\partial B(x_0,r_0)} \left|\frac{\partial \tilde{u}_\varepsilon}{\partial \nu}\right|^2 \tag{7.22}$$

and

$$\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon \cdot [(x - x_0) \cdot \nabla \tilde{u}_\varepsilon] = \\ = \frac{1}{2\varepsilon^2}\int_{B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 - \frac{r_0}{4\varepsilon^2}\int_{\partial B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 \tag{7.23}$$

(where ν is the outer normal vector to $\partial B(x_0, r_0)$). From (7.21), (7.22) and (7.23) we derive that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 \le C\Bigg(r_0\int_{\partial B(x_0,r_0)} |\nabla \tilde{u}_\varepsilon|^2 + r_0\varepsilon^{-2}\int_{\partial B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 \\ + r_0\varepsilon^{-2}\int_{B(x_0,r_0)} |a(x) - a(x_0)||\tilde{u}_\varepsilon||\nabla \tilde{u}_\varepsilon| + \Omega r_0 \int_{B(x_0,r_0)} |\nabla \tilde{u}_\varepsilon|^2 \\ + (\Omega^2 + |\ell_\varepsilon|)r_0 \int_{B(x_0,r_0)} |\tilde{u}_\varepsilon||\nabla \tilde{u}_\varepsilon|\Bigg).$$

Then we estimate each integral term in the right hand side of the previous inequality. By Proposition 6.11 in Chapter 6, we have $|\ell_\varepsilon| \le C\varepsilon^{-1}|\ln\varepsilon|$ and $|\tilde{u}_\varepsilon| \le C$ in $\mathbb{R}^2$. According to (7.19), we obtain

$$\varepsilon^{-2}\int_{\partial B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 \le C\varepsilon^{-2}\int_{\partial B(x_0,r_0)} \left[(a(x_0) - a(x))^2 + (a(x) - |\tilde{u}_\varepsilon|^2)^2\right] \\ \le C\varepsilon^{-2}\int_{\partial B(x_0,r_0)} (a(x) - |\tilde{u}_\varepsilon|^2)^2 + C_R\varepsilon^{\frac{3}{2}\alpha - 1},$$

and

$$\Omega r_0 \int_{B(x_0,r_0)} |\nabla \tilde{u}_\varepsilon|^2 \le \Omega r_0 E_\varepsilon(\tilde{u}_\varepsilon, \mathcal{D}_\varepsilon) \le C_R\,\varepsilon^{\alpha/2+1/3}|\ln\varepsilon|^2,$$

and

$$r_0\varepsilon^{-2}\int_{B(x_0,r_0)} |a(x) - a(x_0)||\tilde{u}_\varepsilon||\nabla \tilde{u}_\varepsilon| \le C_R\, r_0^2\,\varepsilon^{-2}\int_{B(x_0,r_0)} |\nabla \tilde{u}_\varepsilon| \\ \le C_R\, r_0^3\,\varepsilon^{-2}[E_\varepsilon(\tilde{u}_\varepsilon, \mathcal{D}_\varepsilon)]^{1/2} \le C_R\,\varepsilon^{\frac{3}{2}\alpha - 1}|\ln\varepsilon|^{1/2},$$

and

$$(\Omega^2+|\ell_\varepsilon|)r_0\int_{B(x_0,r_0)}|\tilde{u}_\varepsilon||\nabla\tilde{u}_\varepsilon|\le C_R\varepsilon^{-1}|\ln\varepsilon|\,r_0^2\,[E_\varepsilon(\tilde{u}_\varepsilon,\mathcal{D}_\varepsilon)]^{1/2}\le C_R\,\varepsilon^{\alpha-\frac{1}{3}}|\ln\varepsilon|^{3/2}$$

(here we use that $|a(x)-a(x_0)|\le C_R\,r_0$ for any $x\in B(x_0,r_0)$). We finally get that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)}(a(x_0)-|\tilde{u}_\varepsilon|^2)^2\le C_{R,\alpha}\big(1+r_0E_\varepsilon\left(\tilde{u}_\varepsilon,\partial B(x_0,r_0)\right)\big)$$

for some constant $C_{R,\alpha}$ independent of ε. By Step 2, we conclude that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x_0)-|\tilde{u}_\varepsilon|^2)^2\le C_{R,\alpha}. \tag{7.24}$$

Since $\|\tilde{\eta}_\varepsilon-\sqrt{a}\|_{C^1(B_R^\Lambda)}\le C_R\varepsilon^2|\ln\varepsilon|$ by Proposition 6.8 in Chapter 6, we have

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(1-|v_\varepsilon|^2)^2&\le\frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(\tilde{\eta}_\varepsilon^2-|\tilde{u}_\varepsilon|^2)^2\\&\le\frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x)-|\tilde{u}_\varepsilon|^2)^2+o(1)\\&\le\frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x_0)-|\tilde{u}_\varepsilon|^2)^2+o(1)\le C_{R,\alpha}\end{aligned}$$

and we conclude with (7.24). □

The next result will allow us to define the notion of a bad disc as in [17].

Proposition 7.5 *For any $0<R<\sqrt{a_0}$, there exist two positive constants λ_R and μ_R such that if*

$$\frac{1}{\varepsilon^2}\int_{B(x_0,2l)}(1-|v_\varepsilon|^2)^2\le\mu_R\quad\text{with } x_0\in B_R^\Lambda,\ \frac{l}{\varepsilon}\ge\lambda_R\ \text{ and } l\le\frac{\sqrt{a_0}-R}{2},$$

then $|v_\varepsilon|\ge 1/2$ in $B(x_0,l)$.

Proof. In Proposition 6.13 in Chapter 6, we proved the existence of a constant $C_R>0$ independent of ε such that

$$|\nabla v_\varepsilon|\le\frac{C_R}{\varepsilon}\quad\text{in } B^\Lambda_{\frac{\sqrt{a_0}+R}{2}}.$$

Then the result follows as in [17], Theorem III.3. □

Definition 7.6 For $0<R<\sqrt{a_0}$ and $x\in B_R^\Lambda$, we say that $B(x,\lambda_R\varepsilon)$ is a **bad disc** if

$$\frac{1}{\varepsilon^2}\int_{B(x,2\lambda_R\varepsilon)}(1-|v_\varepsilon|^2)^2\ge\mu_R.$$

Now we can give a local version of Theorem 7.2. We will see that Lemma 7.4 plays a crucial role in the proof.

Proposition 7.7 *For any* $0 < R < \sqrt{a_0}$ *and* $\frac{2}{3} < \alpha < 1$, *there exist positive constants* $N_{R,\alpha}$ *and* $\varepsilon_{R,\alpha}$ *such that for every* $\varepsilon < \varepsilon_{R,\alpha}$ *and* $x_0 \in B_R^\Lambda$, *one can find* $x_1, \dots, x_{N_\varepsilon} \in B(x_0, \varepsilon^\alpha)$ *with* $N_\varepsilon \le N_{R,\alpha}$ *verifying*

$$|v_\varepsilon| \ge \frac{1}{2} \quad in \quad B(x_0, \varepsilon^\alpha) \setminus \left(\cup_{k=1}^{N_\varepsilon} B(x_k, \lambda_R \varepsilon) \right).$$

Proof. We follow the ideas in [17], Chapter IV. Consider a family of discs $\{B(x_i, \lambda_R\varepsilon)\}_{i\in\mathcal{F}}$ such that

$$x_i \in B(x_0, \varepsilon^\alpha), \tag{7.25}$$

$$B\left(x_i, \frac{\lambda_R \varepsilon}{4}\right) \cap B\left(x_j, \frac{\lambda_R \varepsilon}{4}\right) = \emptyset \quad \text{for } i \ne j, \tag{7.26}$$

$$B(x_0, \varepsilon^\alpha) \subset \bigcup_{i \in \mathcal{F}} B(x_i, \lambda_R \varepsilon).$$

Obviously, the discs $\{B(x_i, 2\lambda_R\varepsilon)\}_{i\in\mathcal{F}}$ cannot intersect more that C times (where C is a universal constant) and

$$\bigcup_{i\in\mathcal{F}} B(x_i, 2\lambda_R\varepsilon) \subset B(x_0, \varepsilon^{\alpha'})$$

with $\alpha' = \frac{1}{2}(\alpha + \frac{2}{3})$. We denote by $\mathcal{F}'$ the set of indices $i \in \mathcal{F}$ such that $B(x_i, \lambda_R\varepsilon)$ is a bad disc. We derive from Definition 7.6 that

$$\mu_R \operatorname{Card}(\mathcal{F}') \le \sum_{i\in\mathcal{F}} \frac{1}{\varepsilon^2} \int_{B(x_i, 2\lambda_R\varepsilon)} (1 - |v_\varepsilon|^2)^2 \le \frac{C}{\varepsilon^2} \int_{B(x_0, \varepsilon^{\alpha'})} (1 - |v_\varepsilon|^2)^2.$$

The conclusion now follows by Lemma 7.4 and Proposition 7.5. □

Remark 7.3 *By the proof of Proposition 7.7, it follows that any family of discs* $\{B(x_i, \lambda_R\varepsilon)\}_{i\in\mathcal{F}}$ *satisfying* (7.25) *and* (7.26) *cannot contain more than* $N_{R,\alpha}$ *bad discs.*

In the sequel, we will require the following crucial lemma to prove that vortices of degree zero do not occur. This result has its source in [7, 18] and the proof is based on the construction of a suitable test function. Hence the main difference and difficulty in our case come from the mass constraint we have to take into account in the construction of test functions.

Lemma 7.8 *Let* $D > 0$, $0 < \beta < 1$ *and* $\gamma > 1$ *be given constants such that* $\gamma\beta < 1$. *Let* $0 < R < \sqrt{a_0}$ *and* $0 < \rho < \varepsilon^\beta$ *be such that* $\rho^\gamma > \lambda_R\varepsilon$. *We assume that for* $x_0 \in B_R^\Lambda$,

(i) $\displaystyle\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 < \frac{D}{\rho}$,

(ii) $|v_\varepsilon| \ge \frac{1}{2}$ *on* $\partial B(x_0, \rho)$,

(iii) $\deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_0, \rho)\right) = 0.$

Then we have

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B(x_0, \rho^\gamma).$$

Proof of Lemma 7.8. We are going to construct a comparison function as in [7] or [18] to obtain the following estimate:

$$\int_{B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq C_{\beta,R}. \tag{7.27}$$

Since the degree of v_ε restricted to $\partial B(x_0,\rho)$ is zero, we may write on $\partial B(x_0,\rho)$

$$v_\varepsilon = |v_\varepsilon| e^{i\phi_\varepsilon}$$

where ϕ_ε is a smooth map from $\partial B(x_0,\rho)$ into $\mathbb{R}$. Then we define $\hat{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$ by

$$\begin{cases} \hat{v}_\varepsilon = \chi_\varepsilon e^{i\psi_\varepsilon} & \text{in } B(x_0,\rho) \\ \hat{v}_\varepsilon = v_\varepsilon & \text{in } \mathbb{R}^2 \setminus B(x_0,\rho) \end{cases}$$

where ψ_ε is the solution of

$$\begin{cases} \Delta\psi_\varepsilon = 0 & \text{in } B(x_0,\rho) \\ \psi_\varepsilon = \phi_\varepsilon & \text{on } \partial B(x_0,\rho), \end{cases}$$

and χ_ε has the form, written in polar coordinates centered at x_0,

$$\chi_\varepsilon(r,\theta) = (|v_\varepsilon(\rho e^{i\theta})| - 1)\xi(r) + 1$$

and ξ is a smooth function taking values in $[0,1]$ with small support near ρ with $\xi(\rho) = 1$. By Proposition 6.13 in Chapter 6, we know that $|v_\varepsilon(x)| \leq 1 + C\,\varepsilon^{1/3}$ for $x \in \mathcal{D}$ with $|x|_\Lambda \geq \sqrt{a_0} - \varepsilon^{1/8}$ and we deduce that $0 \leq \chi_\varepsilon \leq 1 + C\varepsilon^{1/3}$. Arguing as in [16], proof of Theorem 2, we may prove that

$$\int_{B(x_0,\rho)} |\nabla\psi_\varepsilon|^2 \leq C\rho \int_{\partial B(x_0,\rho)} \left|\frac{\partial\phi_\varepsilon}{\partial\tau}\right|^2 \leq C\rho \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 \tag{7.28}$$

and

$$\int_{B(x_0,\rho)} |\nabla\chi_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-\chi_\varepsilon^2)^2 \leq C\rho \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 + O(\rho). \tag{7.29}$$

From (7.28), (7.29) and assumption *(i)*, we infer that

$$\int_{B(x_0,\rho)} |\nabla\hat{v}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 \leq C\,. \tag{7.30}$$

We set $\tilde{v}_\varepsilon = m_\varepsilon^{-1}\,\hat{v}_\varepsilon$ with $m_\varepsilon = \|\tilde{\eta}_\varepsilon\hat{v}_\varepsilon\|_{L^2(\mathbb{R}^2)}$. Clearly, $\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon \in \mathcal{H}$ and $\|\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. Since $u_\varepsilon = \tilde{\eta}_\varepsilon e^{i\Omega S} v_\varepsilon$ minimizes the functional F_ε under the constraint (7.2), we have $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon)$ and by (7.9), it yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \leq \tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon). \tag{7.31}$$

We claim that

$$\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) \le \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon) + C\rho|\ln\varepsilon|^2 \quad \text{and} \quad \big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) - \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\big| = O(\rho^2|\ln\varepsilon|^2). \tag{7.32}$$

Indeed, we have already established in the proof of Proposition 6.13 in Chapter 6 that

$$\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le C|\ln\varepsilon|^2 \quad \text{and} \quad \big|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)\big| \le C|\ln\varepsilon|^2 \tag{7.33}$$

so that, using (7.30), $\|\tilde{\eta}_\varepsilon v_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, $\hat{v}_\varepsilon = v_\varepsilon$ in $\mathbb{R}^2 \setminus B(x_0,\rho)$ and (7.33), we obtain

$$\begin{aligned} m_\varepsilon^2 &= 1 + \int_{B(x_0,\rho)} \tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1) + \int_{B(x_0,\rho)} \tilde{\eta}_\varepsilon^2(1 - |v_\varepsilon|^2) \\ &= 1 + O(\rho\,\varepsilon|\ln\varepsilon|). \end{aligned} \tag{7.34}$$

From (7.30), (7.33) and (7.34), we derive

$$\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\tilde{v}_\varepsilon|^2 = m_\varepsilon^{-2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 + O(\rho\varepsilon|\ln\varepsilon|^3) \tag{7.35}$$

and

$$\tilde{\mathcal{R}}_\varepsilon(\tilde{v}_\varepsilon) = m_\varepsilon^{-2}\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) = \tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) + O(\rho\varepsilon|\ln\varepsilon|^3). \tag{7.36}$$

Since u_ε remains bounded in $\mathbb{R}^2$ and $E_\varepsilon(u_\varepsilon) \le C|\ln\varepsilon|^2$ by Proposition 6.13 in Chapter 6, we infer from (7.33),

$$\begin{aligned} \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1 - |\tilde{v}_\varepsilon|^2)^2 &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1 - |\hat{v}_\varepsilon|^2)^2 + \frac{2(1 - m_\varepsilon^{-2})}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2(1 - |\hat{v}_\varepsilon|^2)|\tilde{\eta}_\varepsilon\hat{v}_\varepsilon|^2 \\ &\quad + \frac{(1 - m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2} |\tilde{\eta}_\varepsilon\hat{v}_\varepsilon|^4 \\ &\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1 - |\hat{v}_\varepsilon|^2)^2 \\ &\quad + C\rho|\ln\varepsilon|\left(\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2\setminus B(x_0,\rho)} \tilde{\eta}_\varepsilon^4(1 - |v_\varepsilon|^2)^2\right)^{1/2}\left(\int_{\mathbb{R}^2\setminus B(x_0,\rho)} |u_\varepsilon|^4\right)^{1/2} \\ &\quad + C\rho^2|\ln\varepsilon|^2 \\ &\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1 - |\hat{v}_\varepsilon|^2)^2 + C\rho|\ln\varepsilon|^2. \end{aligned} \tag{7.37}$$

Finally, we obtain in the same way,

$$\big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) - \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\big| \le \big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) - \tilde{\mathcal{T}}_\varepsilon(\hat{v}_\varepsilon)\big| + \big|\tilde{\mathcal{T}}_\varepsilon(\hat{v}_\varepsilon) - \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\big| \tag{7.38}$$

$$\begin{aligned} &\le C|\ln\varepsilon|^2\bigg(\int_{B(x_0,\rho)} (1 + |x|^2)\tilde{\eta}_\varepsilon^2 + |1 - m_\varepsilon^{-2}|\int_{\mathbb{R}^2} (1 + |x|^2)\tilde{\eta}_\varepsilon^2|\hat{v}_\varepsilon|^2\bigg) \\ &\le C\rho^2|\ln\varepsilon|^2. \end{aligned} \tag{7.39}$$

From (7.35), (7.36), (7.37) and (7.38), we conclude that (7.32) holds.

Since $\hat{v}_\varepsilon = v_\varepsilon$ in $\mathbb{R}^2 \setminus B(x_0,\rho)$, we get from (7.31) and (7.32) that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, B(x_0,\rho)) \le \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon, B(x_0,\rho)) + C\rho|\ln\varepsilon|^2.$$

By (7.30), we have $\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, B(x_0,\rho)) \leq C$ and therefore,

$$\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon, B(x_0,\rho))\right| \leq C\Omega \int_{B(x_0,\rho)} |\nabla \hat{v}_\varepsilon| \leq C\Omega\rho \|\nabla \hat{v}_\varepsilon\|_{L^2(B(x_0,\rho))} = O(\rho|\ln\varepsilon|). \tag{7.40}$$

Hence, $\tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon, B(x_0,\rho)) \leq C$ and we conclude that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, B(x_0,\rho)) \leq C_\beta.$$

As for (7.40), using (7.33) we easily derive that $|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon, B(x_0,\rho))| = O(\rho|\ln\varepsilon|^2)$ and we finally get that $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon, B(x_0,\rho)) \leq C_\beta$ which clearly implies (7.27) since $\tilde{\eta}_\varepsilon^2 \to a^+$ uniformly as $\varepsilon \to 0$ (see Proposition 6.8 in Chapter 6).

We deduce from (7.27) that

$$\int_{2\rho^\gamma}^{\rho} \left(\int_{\partial B(x_0,s)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \right) ds \leq C_{\beta,R}.$$

Since $\int_{2\rho^\gamma}^{\rho} \frac{ds}{s|\ln s|^{1/2}} \geq C_\gamma |\ln\varepsilon|^{1/2}$, we derive that for small ε there exists $s_0 \in [2\rho^\gamma, \rho]$ such that

$$\int_{\partial B(x_0,s_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq \frac{C_{\beta,R}}{s_0|\ln s_0|^{1/2}} .$$

Repeating the arguments used to prove (7.27), we find that

$$\int_{B(x_0,s_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq \frac{C_{\beta,R}}{|\ln s_0|^{1/2}}.$$

In particular, we have

$$\frac{1}{\varepsilon^2}\int_{B(x_0,2\rho^\gamma)} (1-|v_\varepsilon|^2)^2 = o(1)$$

and the conclusion follows by Proposition 7.5. □

We obtain as in [18] Proposition IV.3 the following result which gives us an estimate of the contribution in the energy of any vortex. We reproduce here the proof for completeness.

Proposition 7.9 *Let $0 < R < \sqrt{a_0}$ and $\frac{2}{3} < \alpha < 1$. Let $x_0 \in B_R^\Lambda$ and assume that $|v_\varepsilon(x_0)| < \frac{1}{2}$. Then there exists a positive constant $C_{R,\alpha}$ (which only depends on R, α and ω_1) such that*

$$\int_{B(x_0,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_{R,\alpha}|\ln\varepsilon|.$$

Proof. Let $N_{R,\alpha}$ and $x_1,\ldots,x_{N_\varepsilon} \in B(x_0,\varepsilon^\alpha)$ be as in Proposition 7.7. We set

$$\delta_\alpha = \frac{\alpha^{1/2}-\alpha}{3(N_{R,\alpha}+1)}$$

and for $k = 0,\ldots,3N_{R,\alpha}+2$, we consider

$$\alpha_k = \alpha^{1/2} - k\delta_\alpha\ , \quad \mathcal{I}_k = [\varepsilon^{\alpha_k}, \varepsilon^{\alpha_{k+1}}] \quad \text{and} \quad \mathcal{C}_k = B(x_0,\varepsilon^{\alpha_{k+1}}) \setminus B(x_0,\varepsilon^{\alpha_k}).$$

Then there is some $k_0 \in \{1, \dots, 3N_{R,\alpha} + 1\}$ such that

$$\mathcal{C}_{k_0} \cap \Big(\cup_{j=1}^{N_\varepsilon} B(x_j, \lambda_R \varepsilon) \Big) = \emptyset. \tag{7.41}$$

Indeed, since $N_\varepsilon \le N_{R,\alpha}$ and $2\lambda_R \varepsilon < |\mathcal{I}_k|$ for small ε, the union of N_ε intervals of length $2\lambda_R\varepsilon$

$$\bigcup_{j=1}^{N_\varepsilon} \big(|x_i - x_0| - \lambda_R \varepsilon, |x_i - x_0| + \lambda_R \varepsilon \big)$$

cannot intersect all the intervals $\mathcal{I}_k$ of disjoint interior, for $1 \le k \le 3N_{R,\alpha} + 1$. From (7.41) we deduce that

$$|v_\varepsilon(x)| \ge \frac{1}{2} \quad \text{for any } x \in \mathcal{C}_{k_0}.$$

Therefore, for every $\rho \in \mathcal{I}_{k_0}$,

$$d_{k_0} = \deg \left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_0, \rho) \right)$$

is well defined and does not depend on ρ. We claim that

$$d_{k_0} \neq 0. \tag{7.42}$$

By contradiction, we suppose that $d_{k_0} = 0$. According to (7.14), it results that

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}+R}{2}}} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \le C_R |\ln \varepsilon|.$$

Using the same argument as in Step 2 of the proof of Lemma 7.4, there is a constant $C_{R,\alpha}$ such that

$$\int_{\partial B(x_0,\rho_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \le \frac{C_{R,\alpha}}{\rho_0} \quad \text{for some } \rho_0 \in \mathcal{I}_{k_0}.$$

According to Lemma 7.8 (with $\beta = \alpha_{k_0+1}$ and $\gamma = \frac{\alpha_{k_0-1}}{\alpha_{k_0}}$), we should have $|v_\varepsilon(x_0)| \ge \frac{1}{2}$ which is a contradiction.

By (7.42), we obtain for every $\rho \in \mathcal{I}_{k_0}$,

$$1 \le |d_{k_0}| = \frac{1}{2\pi} \left| \int_{\partial B(x_0,\rho)} \frac{1}{|v_\varepsilon|^2} \big(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau} \big) \right| \le C \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|$$

(we use that $|v_\varepsilon| \ge \frac{1}{2}$ in $\mathcal{C}_{k_0}$). Then Cauchy-Schwarz inequality yields

$$\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 \ge \frac{C}{\rho} \quad \text{for any } \rho \in \mathcal{I}_{k_0}$$

and the conclusion follows integrating on $\mathcal{I}_{k_0}$. □

7.2.2 Proofs of Theorem 7.2 and Proposition 7.3

The part 1) in Theorem 7.2 follows directly from Lemma 7.10 below.

Lemma 7.10 *There exists a constant* $\varepsilon_R > 0$ *such that for any* $0 < \varepsilon < \varepsilon_R$,

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B^\Lambda_R \setminus B^\Lambda_{\frac{\sqrt{a_0}}{5}}.$$

Proof. First, we fix some $\alpha \in (\frac{2}{3}, 1)$. We proceed by contradiction. Suppose that there is some $x_0 \in B^\Lambda_R \setminus B^\Lambda_{\frac{\sqrt{a_0}}{5}}$ such that $|v_\varepsilon(x_0)| < 1/2$. Then for any ε sufficiently small, we have $B(x_0, \varepsilon^\alpha) \subset \mathcal{D}_\varepsilon \setminus \{|x|_\Lambda < 2|\ln \varepsilon|^{-1/6}\}$ and therefore, by (7.15), we get that

$$\int_{B(x_0,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \leq C_R\, \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \{|x|_\Lambda < 2|\ln\varepsilon|^{-1/6}\}) \leq C_R \ln|\ln\varepsilon|$$

which contradicts Proposition 7.9 for ε small enough. □

Proof of 2) in Theorem 7.2. We fix some $\frac{2}{3} < \alpha < 1$. As in the proof of Proposition 7.7, we consider a finite family of points $\{x_j\}_{j\in\mathcal{J}}$ satisfying

$$\begin{aligned} & x_j \in B^\Lambda_{\frac{\sqrt{a_0}}{2}} \\ & B\left(x_i, \frac{\lambda_0\varepsilon}{4}\right) \cap B\left(x_j, \frac{\lambda_0\varepsilon}{4}\right) = \emptyset \quad \text{for } i \neq j\,, \\ & B^\Lambda_{\frac{\sqrt{a_0}}{2}} \subset \bigcup_{j\in\mathcal{J}} B\left(x_j, \lambda_0\varepsilon\right), \end{aligned}$$

where $\lambda_0 := \lambda_{\frac{\sqrt{a_0}}{2}}$ (defined in Proposition 7.5 with $R = \frac{\sqrt{a_0}}{2}$) and we denote by J_ε the set of indices $j \in \mathcal{J}$ such that $B(x_j, \lambda_0\varepsilon)$ contains at least one point y_j verifying

$$|v_\varepsilon(y_j)| < \frac{1}{2}\,. \tag{7.43}$$

Obviously, $B(x_j, \lambda_0\varepsilon)$ is a bad disc for every $j \in J_\varepsilon$. Applying Lemma 7.10 (with $R = \frac{3\sqrt{a_0}}{4}$), we infer that there exists ε_0 such that for any $0 < \varepsilon < \varepsilon_0$,

$$B(x_j, \lambda_0\varepsilon) \subset B^\Lambda_{\frac{\sqrt{a_0}}{4}} \quad \text{for any } j \in J_\varepsilon. \tag{7.44}$$

Then it remains to prove that $\mathrm{Card}(J_\varepsilon)$ is bounded independently of ε. Using Proposition 7.9 (with $R = \frac{\sqrt{a_0}}{2}$), we derive that for any $j \in J_\varepsilon$ and any point y_j satisfying (7.43) in the ball $B(x_j, \lambda_0\varepsilon)$,

$$\int_{B(x_j,2\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq \int_{B(y_j,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_\alpha |\ln\varepsilon| \tag{7.45}$$

for some positive constant C_α which only depends on α. We set for ε small enough

$$W = \bigcup_{j\in J_\varepsilon} B(x_j, 2\varepsilon^\alpha) \subset B^\Lambda_{\frac{\sqrt{a_0}}{3}}\,.$$

We claim that there is a positive integer M_α independent of ε such that any $y \in W$ belongs to at most M_α balls in the collection $\{B(x_j, 2\varepsilon^\alpha)\}_{j\in J_\varepsilon}$. Indeed, for each $y \in W$, consider the subset $K_y \subset J_\varepsilon$ defined by

$$K_y = \big\{j \in J_\varepsilon \,:\, y \in B(x_j, 2\varepsilon^\alpha)\big\}.$$

We have for every $j \in K_y$,

$$x_j \in B(y, 2\varepsilon^\alpha) \subset B(y, \varepsilon^{\alpha'}) \subset B^\Lambda_{\frac{\sqrt{a_0}}{2}} \quad \text{with } \alpha' = \frac{1}{2}(\alpha + \frac{2}{3}). \tag{7.46}$$

Since the family of discs $\{B(x_j, \lambda_0\varepsilon)\}_{j\in K_y}$ is a subcover of $B(y, \varepsilon^{\alpha'})$ satisfying (7.25) and (7.26), we conclude from Remark 7.3 that

$$\mathrm{Card}(K_y) \le M_\alpha$$

with $M_\alpha = N_{\frac{\sqrt{a_0}}{2},\alpha'}$. From (7.45), we infer that

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}} |\nabla v_\varepsilon|^2 \ge \int_W |\nabla v_\varepsilon|^2 \ge \frac{1}{M_\alpha} \sum_{j\in J_\varepsilon} \int_{B(x_j,2\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \ge C_\alpha \mathrm{Card}(J_\varepsilon)|\ln\varepsilon|. \tag{7.47}$$

On the other hand, we know by (7.14),

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}} |\nabla v_\varepsilon|^2 \le C \int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}} a(x)|\nabla v_\varepsilon|^2 \le C|\ln\varepsilon| \tag{7.48}$$

for a constant C independent of ε. Matching (7.47) and (7.48), we conclude that $\mathrm{Card}(J_\varepsilon)$ is uniformly bounded. □

In the following, we will prove Proposition 7.3. We proceed exactly as in [80], using Theorem 7.2 and an adaptation of Theorem V.1 in [7]. We will use Proposition 6.16 in Chapter 6, that was shown by a method due to Sandier [75] and Sandier-Serfaty [77].

Proof of Proposition 7.3. By Theorem 7.2, we have for ε small enough,

$$\cup_{j\in J_\varepsilon} B(x^\varepsilon_j, \lambda_0\varepsilon) \subset B^\Lambda_{\frac{\sqrt{a_0}}{3}}.$$

From *(iii)* in Proposition 6.16 in Chapter 6, there exists a radius $r_\varepsilon \in (\frac{\sqrt{a_0}}{3}, \frac{\sqrt{a_0}}{2}]$ such that

$$\bar{B}_i \cap \partial B^\Lambda_{r_\varepsilon} = \emptyset \quad \text{for every } i \in I_\varepsilon. \tag{7.49}$$

Hence we have

$$|v_\varepsilon| \ge 1 - |\ln\varepsilon|^{-5} \quad \text{on } \partial B^\Lambda_{r_\varepsilon}.$$

The existence of a subset $\tilde{J}_\varepsilon \subset J_\varepsilon$ satisfying *(i)-(v)* can now be proved identically as Proposition 3.2 in [80] and it remains to prove (7.18). From the proof of Theorem 7.2, we know (by construction) that each disc $B(x^\varepsilon_k, \lambda_0\varepsilon)$, $k \in J_\varepsilon$, contains at least one point y_k such that $|v_\varepsilon(y_k)| < \frac{1}{2}$. Therefore each disc $B(x^\varepsilon_j, \rho)$, $j \in \tilde{J}_\varepsilon$, contains at least one of the y_k's with $|x^\varepsilon_j - y_k| < \lambda_0\varepsilon$. Assume now that $D_j = 0$. By Lemma 7.8 with $\gamma = \mu^{-1/2}$, it would lead to $|v_\varepsilon| \ge \frac{1}{2}$ in $B(x^\varepsilon_j, \rho^\gamma)$ and then $|v_\varepsilon(y_k)| \ge \frac{1}{2}$ for ε small enough, contradiction. We also find a bound on the degrees D_j:

$$|D_j| = \frac{1}{2\pi}\left|\int_{\partial B(x^\varepsilon_j,\rho)} \frac{1}{|v_\varepsilon|^2}\big(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial\tau}\big)\right| \le C\|\nabla v_\varepsilon\|_{L^2(\partial B(x^\varepsilon_j,\rho))}\sqrt{\rho} \le C$$

by *(iv)* in Proposition 7.3. □

7.3 Some lower energy estimates

In this section, we obtain various lower energy estimates for v_ε in terms of the vortex structure defined in Section 7.2, Proposition 7.3. We start by proving a lower bound on the kinetic energy away from the vortices which brings out the interaction between vortices. The method that we use is based on the techniques developed in [7], [17] and [80, 81]. As in the previous section, the main difficulty is due to the degenerate behavior near the boundary of $\mathcal{D}$ of the function $a(x)$ since the method involves in our case the operator $-\operatorname{div}(a^{-1}\nabla)$ which is not uniformly elliptic in $\mathcal{D}$. To avoid this problem, we shall establish our estimates in B_R^Λ for an arbitrary radius $R \in [\sqrt{a_0}/2, \sqrt{a_0}\,)$. The underlying idea here is to let $R \to \sqrt{a_0}$ at the end of the analysis. To emphasize the possible dependence on R in the "error term", we will denote by $O_R(1)$ (respectively $o_R(1)$) any quantity which remains uniformly bounded in ε for fixed R (respectively any quantity which tends to 0 as $\varepsilon \to 0$ for fixed R). In the sequel, we will also write $\tilde{J}_\varepsilon = \{1, \ldots, n_\varepsilon\}$.

Proposition 7.11 *For any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$, let $\Theta_\rho = B_R^\Lambda \setminus \cup_{j=1}^{n_\varepsilon} B(x_j^\varepsilon, \rho)$. We have*

$$\frac{1}{2}\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \pi \sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon)|\ln\rho| + W_{R,\varepsilon}\big((x_1^\varepsilon, D_1), \ldots, (x_{n_\varepsilon}^\varepsilon, D_{n_\varepsilon})\big) + O_R(1) \tag{7.50}$$

where

$$W_{R,\varepsilon}\big((x_1^\varepsilon, D_1), \ldots, (x_{n_\varepsilon}^\varepsilon, D_{n_\varepsilon})\big) = -\pi \sum_{i \neq j} D_i D_j\, a(x_j^\varepsilon) \ln|x_i^\varepsilon - x_j^\varepsilon| - \pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{R,\varepsilon}(x_j^\varepsilon)$$

and $\Psi_{R,\varepsilon}$ is the unique solution of

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla\Psi_{R,\varepsilon}\right) = -\displaystyle\sum_{j=1}^{n_\varepsilon} D_j\, a(x_j^\varepsilon)\, \nabla\left(\frac{1}{a}\right)\cdot\nabla\big(\ln|x - x_j^\varepsilon|\big) & \text{in } B_R^\Lambda, \\ \Psi_{R,\varepsilon} = -\displaystyle\sum_{j=1}^{n_\varepsilon} D_j\, a(x_j^\varepsilon)\ln|x - x_j^\varepsilon| & \text{on } \partial B_R^\Lambda. \end{cases} \tag{7.51}$$

Moreover, if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$ then the term $O_R(1)$ in (7.50) is in fact $o_R(1)$.

Remark 7.4 *We point out that the dependence on R in the interaction term $W_{R,\varepsilon}$ only appears in the function $\Psi_{R,\varepsilon}$. Moreover, for $\Psi_{R,\varepsilon}$ to be well defined, $1/a(x)$ has to be bounded inside B_R^Λ so that we can not pass to the limit $R \to \sqrt{a_0}$ in (7.50) without an* a priori *deterioration of the error term.*

Proof of Proposition 7.11. We consider the solution Φ_ρ of the linear problem

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla\Phi_\rho\right) = 0 & \text{in } \Theta_\rho, \\ \Phi_\rho = 0 & \text{on } \partial B_R^\Lambda, \\ \Phi_\rho = \text{const.} & \text{on } \partial B(x_j^\varepsilon, \rho), \\ \displaystyle\int_{\partial B(x_j^\varepsilon,\rho)} \frac{1}{a}\frac{\partial\Phi_\rho}{\partial\nu} = 2\pi D_j & \text{for } j = 1, \ldots, n_\varepsilon, \end{cases}$$

and $\Phi_{R,\varepsilon}$ the solution of

$$\begin{cases} \operatorname{div}\left(\frac{1}{a}\nabla\Phi_{R,\varepsilon}\right) = 2\pi\sum_{j=1}^{n_\varepsilon} D_j\, \delta_{x_j^\varepsilon} & \text{in } B_R^\Lambda \\ \Phi_{R,\varepsilon} = 0 & \text{on } \partial B_R^\Lambda \end{cases} \tag{7.52}$$

For $x \in \Theta_\rho$, we set $w_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}$ and

$$\mathcal{S} = \left(-w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_2} + \frac{1}{a}\frac{\partial \Phi_\rho}{\partial x_1}\, ,\, w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_1} + \frac{1}{a}\frac{\partial \Phi_\rho}{\partial x_2}\right).$$

We easily check that $\operatorname{div}\mathcal{S} = 0$ in Θ_ρ and $\int_{\partial B_R^\Lambda} \mathcal{S}\cdot\nu = \int_{\partial B(x_j^\varepsilon,\rho)} \mathcal{S}\cdot\nu = 0$. By Lemma I.1 in [17], there exists $H \in C^1(\overline{\Theta}_\rho)$ such that $\mathcal{S} = \nabla^\perp H$ and hence we can write the Hodge-de Rham type decomposition

$$w_\varepsilon \wedge \nabla w_\varepsilon = \frac{1}{a}\,\nabla^\perp \Phi_\rho + \nabla H.$$

Consequently,

$$\begin{aligned} \int_{\Theta_\rho} a(x)|\nabla w_\varepsilon|^2 &= \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + 2\int_{\Theta_\rho} \nabla^\perp\Phi_\rho\cdot\nabla H + \int_{\Theta_\rho} a(x)|\nabla H|^2 \\ &\geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + 2\int_{\Theta_\rho} \nabla^\perp\Phi_\rho\cdot\nabla H. \end{aligned}$$

We observe that the last term is in fact equal to zero since it is the integral of a Jacobian and Φ_ρ is constant on $\partial\Theta_\rho$. Hence

$$\int_{\Theta_\rho} a(x)|\nabla w_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2.$$

Since $|\nabla v_\varepsilon|^2 \geq |v_\varepsilon|^2|\nabla w_\varepsilon|^2$ in Θ_ρ, we derive that

$$\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + T_1 + 2T_2$$

with

$$T_1 = \int_{\Theta_\rho} \left(|v_\varepsilon|^2 - 1\right)\frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 \quad \text{and} \quad T_2 = \int_{\Theta_\rho} \left(|v_\varepsilon|^2 - 1\right)\nabla\Phi_\rho^\perp\cdot\nabla H.$$

Arguing as in [7] (see Step 4 in the proof of Theorem 6), it turns out that $T_1 = o_R(1)$ and $T_2 = o_R(1)$ and therefore

$$\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + o_R(1). \tag{7.53}$$

On the other hand, integrating by parts we obtain

$$\int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 = \int_{\partial\Theta_\rho} \frac{1}{a(x)}\frac{\partial\Phi_\rho}{\partial\nu}\Phi_\rho = -2\pi\sum_{j=1}^{n_\varepsilon} D_j\,\Phi_\rho(z_j)$$

for any point $z_j \in \partial B(x_j^\varepsilon, \rho)$. Since n_ε and each D_j remain uniformly bounded in ε by Proposition 7.3, we may rewrite this equality as

$$\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 = -2\pi \sum_{j=1}^{n_\varepsilon} D_j \, \Phi_{R,\varepsilon}(z_j) + O\big(\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)}\big). \tag{7.54}$$

Using an adaptation of Lemma I.4 in [17] (see e.g. [15], Lemma 3.5), we derive that

$$\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \leq \sum_{j=1}^{n_\varepsilon} \left(\sup_{\partial B(x_j^\varepsilon, \rho)} \Phi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon, \rho)} \Phi_{R,\varepsilon} \right). \tag{7.55}$$

To estimate the right-hand-side term in (7.55), we introduce for $x \in B_R^\Lambda$,

$$\Psi_{R,\varepsilon}(x) = \Phi_{R,\varepsilon}(x) - \sum_{j=1}^{n_\varepsilon} D_j \, a(x_j^\varepsilon) \ln |x - x_j^\varepsilon|.$$

Since $\Phi_{R,\varepsilon}$ solves (7.52), we deduce that $\Psi_{R,\varepsilon}$ may be characterized as the solution of equation (7.51). By elliptic regularity, we infer that $\|\Psi_{R,\varepsilon}\|_{W^{2,p}(B_R^\Lambda)} \leq C_{R,p}$ for any $1 \leq p < 2$ (here we used that $\{x_j^\varepsilon\}_{j=1}^{n_\varepsilon} \subset B^\Lambda_{\frac{\sqrt{a_0}}{4}}$ by Theorem 7.2). In particular, $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R^\Lambda)$ and hence

$$\sup_{\partial B(x_j^\varepsilon, \rho)} \Psi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon, \rho)} \Psi_{R,\varepsilon} \leq C_R \sqrt{\rho} = o_R(1).$$

Since $|x_j^\varepsilon - x_i^\varepsilon| \geq 8\rho$, we derive from (7.18),

$$\begin{aligned}
\sup_{\partial B(x_j^\varepsilon, \rho)} \left(\sum_{i=1}^{n_\varepsilon} D_i \, a(x_i^\varepsilon) \ln |x - x_i^\varepsilon| \right) - \inf_{\partial B(x_j^\varepsilon, \rho)} \left(\sum_{i=1}^{n_\varepsilon} D_i \, a(x_i^\varepsilon) \ln |x - x_i^\varepsilon| \right) \leq \\
\leq \rho \sum_{i=1,\, i \neq j}^{n_\varepsilon} a(x_i^\varepsilon) \sup_{\partial B(x_j^\varepsilon, \rho)} \frac{|D_i|}{|x - x_i^\varepsilon|} \leq O(1),
\end{aligned}$$

(respectively $\leq o(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$). Coming back to (7.55), we obtain that $\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \leq O_R(1)$ (respectively $\leq o_R(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$). Inserting this estimate in (7.54), we get that

$$\begin{aligned}
\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 &= -2\pi \sum_{j=1}^{n_\varepsilon} D_j \, \Phi_{R,\varepsilon}(z_j) + O_R(1) \\
&= -2\pi \sum_{j=1}^{n_\varepsilon} D_j \, \Psi_{R,\varepsilon}(z_j) - 2\pi \sum_{i \neq j} D_i D_j \, a(x_i^\varepsilon) \ln |z_j - x_i^\varepsilon| \\
&\quad + 2\pi \sum_{j=1}^{n_\varepsilon} D_j^2 \, a(x_j^\varepsilon) |\ln \rho| + O_R(1)
\end{aligned} \tag{7.56}$$

(respectively $+o_R(1)$ as $\varepsilon \to 0$). Since $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R^\Lambda)$, we have $|\Psi_{R,\varepsilon}(z_j) - \Psi_{R,\varepsilon}(x_j^\varepsilon)| \leq C_R \sqrt{\rho} = o_R(1)$. Moreover, using (7.18) and $|x_j^\varepsilon - x_i^\varepsilon| \geq 8\rho$, we derive that

$$\left|\sum_{i\neq j} D_i D_j\, a(x_i^\varepsilon)(\ln|z_j - x_i^\varepsilon| - \ln|x_j^\varepsilon - x_i^\varepsilon|)\right| \leq \sum_{i\neq j} |D_i|\,|D_j|\,\ln\left|1 + \frac{z_j - x_j^\varepsilon}{x_j^\varepsilon - x_i^\varepsilon}\right|$$
$$\leq \sum_{i\neq j} |D_i|\,|D_j|\,\frac{\rho}{|x_j^\varepsilon - x_i^\varepsilon|} \leq O(1)$$

(respectively $\leq o(1)$ as $\varepsilon \to 0$). Hence (7.56) yields

$$\int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 = -2\pi\sum_{j=1}^{n_\varepsilon} D_j\,\Psi_{R,\varepsilon}(x_j^\varepsilon) - 2\pi\sum_{i\neq j} D_i D_j\, a(x_i^\varepsilon)\ln|x_j^\varepsilon - x_i^\varepsilon|$$
$$+ 2\pi\sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon)|\ln\rho| + O_R(1)$$

(respectively $+o_R(1)$ as $\varepsilon \to 0$). Combining this estimate with (7.53), we obtain the announced result. □

Arguing as [80, 81], we estimate the contribution in the energy of each vortex which yields the following lower bounds for $\mathcal{E}_\varepsilon(v_\varepsilon)$:

Lemma 7.12 *For any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$, we have*

$$\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \geq \pi\sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon)|\ln\rho| + \pi\sum_{j=1}^{n_\varepsilon} |D_j|\, a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + W_{R,\varepsilon} + O_R(1) \tag{7.57}$$

and

$$\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \geq \pi\sum_{j=1}^{n_\varepsilon} |D_j|\, a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + O(1). \tag{7.58}$$

Proof. In view of Proposition 7.11, it suffices to show that

$$\mathcal{E}_\varepsilon(v_\varepsilon, B(x_j^\varepsilon, \rho)) \geq \pi|D_j|\, a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + O(1) \quad \text{for } j = 1, \ldots, n_\varepsilon,$$

which is equivalent to

$$\frac{1}{2}\int_{B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \geq \pi|D_j|\,\ln\frac{\rho}{\varepsilon} + O(1) \quad \text{for } j = 1, \ldots, n_\varepsilon \tag{7.59}$$

(we used that $|a(x) - a(x_j^\varepsilon)| \leq C\rho$ for $x \in B(x_j^\varepsilon, \rho)$ and $\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \leq C_R|\ln\varepsilon|$). Setting

$$\hat{v}(y) = v_\varepsilon(\rho y + x_j^\varepsilon) \quad \text{for } y \in B(0,1) \text{ and } \hat{\varepsilon} = \frac{\varepsilon}{\rho\sqrt{a(x_j^\varepsilon)}},$$

we infer from Proposition 7.3 that $|\hat{v}| \geq 1 - \frac{2}{|\ln\varepsilon|^2}$ on $\partial B(0,1)$,

$$\frac{1}{2}\int_{\partial B(0,1)} |\nabla\hat{v}|^2 + \frac{1}{2\hat{\varepsilon}^2}(1 - |\hat{v}|^2)^2 = \frac{\rho}{2}\int_{\partial B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \leq C \tag{7.60}$$

and

$$\frac{1}{2}\int_{B(0,1)}|\nabla\hat{v}|^2+\frac{1}{2\hat{\varepsilon}^2}(1-|\hat{v}|^2)^2=\frac{1}{2}\int_{B(x_j^\varepsilon,\rho)}|\nabla v_\varepsilon|^2+\frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2.$$

As in the proof of Lemma VI.1 in [7], (7.60) yields for ε small enough,

$$\frac{1}{2}\int_{B(0,1)}|\nabla\hat{v}|^2+\frac{1}{2\hat{\varepsilon}^2}(1-|\hat{v}|^2)^2\geq\pi|D_j|\,|\ln\hat{\varepsilon}|+O(1)=\pi|D_j|\,\ln\frac{\rho}{\varepsilon}+O(1)$$

and hence (7.59) holds. □

As in Proposition 6.17 in Chapter 6, we may compute an asymptotic expansion of $\mathcal{R}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)$ in terms of vortices which leads, in view of Lemma 7.12, to lower expansions of $\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)$:

Lemma 7.13 *For any* $R\in[\frac{\sqrt{a_0}}{2},\sqrt{a_0}\,)$, *we have*

$$\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)\geq\pi\sum_{j=1}^{n_\varepsilon}D_j^2\,a(x_j^\varepsilon)|\ln\rho|+\pi\sum_{j=1}^{n_\varepsilon}|D_j|\,a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon}-\frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}a^2(x_j^\varepsilon)\,D_j+W_{R,\varepsilon}+O_R(1)\tag{7.61}$$

and

$$\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)\geq\pi\sum_{j=1}^{n_\varepsilon}|D_j|\,a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon}-\frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}a^2(x_j^\varepsilon)\,D_j+O(1).\tag{7.62}$$

Proof. We consider the family of balls $\{B_i\}_{i\in I_\varepsilon}$ given in Proposition 6.16 in Chapter 6. As in the proof of Proposition 7.3, we can find $r_\varepsilon\in[R,(R+\sqrt{a_0}\,)/2]$ such that (7.49) holds. Setting

$$I_R^+=\big\{i\in I_\varepsilon,\,|p_i|_\Lambda>r_\varepsilon\text{ and }d_i\geq0\big\}\quad\text{and}\quad I_R^-=\big\{i\in I_\varepsilon,\,|p_i|_\Lambda>r_\varepsilon\text{ and }d_i<0\big\},\tag{7.63}$$

we have $\overline{B}_i\subset\mathcal{D}_\varepsilon\backslash\overline{B}_{r_\varepsilon}^\Lambda$ for any $i\in I_R^+\cup I_R^-$. By Theorem 7.2, Proposition 7.3 and Proposition 6.16 in Chapter 6, we infer that for ε small enough,

$$|v_\varepsilon|\geq\frac{1}{2}\quad\text{in }\Xi_\varepsilon:=\mathcal{D}_\varepsilon\setminus\Big(\bigcup_{i\in I_R^+\cup I_R^-}B_i\cup\bigcup_{j=1}^{n_\varepsilon}B(x_j^\varepsilon,\rho)\Big).$$

Arguing exactly as Proposition 6.17 in Chapter 6, we obtain that

$$\mathcal{R}_\varepsilon(v_\varepsilon,\Xi_\varepsilon)=\frac{-\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}a^2(x_j^\varepsilon)\,D_j-\frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I_R^+\cup I_R^-}\big(a^2(p_i)-\nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i+o_R(1).\tag{7.64}$$

We recall that we have showed in the proof of Proposition 6.17 in Chapter 6 that

$$\mathcal{R}_\varepsilon(v_\varepsilon,\cup_{i\in I_R^+\cup I_R^-}B_i)=o(1).$$

In the same way, we may prove that $\mathcal{R}_\varepsilon(v_\varepsilon, \cup_{j=1}^{n_\varepsilon} B(x_j^\varepsilon, \rho)) = o(1)$. From *(iv)* in Proposition 6.16 in Chapter 6 and (7.64), we deduce that

$$\begin{aligned}\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) &\geq \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_R^+\cup I_R^-} B_i) + \sum_{i\in I_R^+\cup I_R^-} \frac{1}{2}\int_{B_i} a(x)|\nabla v_\varepsilon|^2 + \mathcal{R}_\varepsilon(v_\varepsilon, \Xi_\varepsilon) + o_R(1)\\ &\geq \mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon)\, D_j + \pi \sum_{i\in I_R^+\cup I_R^-} a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0 \ln|\ln\varepsilon|\big)\\ &\quad - \frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I_R^+\cup I_R^-}\big(a^2(p_i) - \nu_\varepsilon^2 |\ln\varepsilon|^{-3}\big) d_i + o_R(1). \end{aligned} \tag{7.65}$$

Since $p_i \notin \overline{B}_{r_\varepsilon}^\Lambda$ for $i \in I_R^+ \cup I_R^-$, we have $a(p_i) \ll a_0$ and we deduce that for ε small enough,

$$\pi \sum_{i\in I_R^+\cup I_R^-} a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0\ln|\ln\varepsilon|\big) - \frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I_R^+\cup I_R^-}\big(a^2(p_i) - \nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i \geq 0$$

which leads to

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon)\, D_j + o_R(1). \tag{7.66}$$

Combining (7.57) and (7.66), we obtain (7.61). Similarly, the inequality (7.66) applied with $R = \sqrt{a_0}/2$, and (7.58) yield (7.62). □

7.4 Proof of Theorem 7.1

In this section, we are going to prove Theorem 7.1 in terms of the map v_ε. We start by showing that vortices must be of degree one. This yields a fundamental improvement of the estimates obtained in the previous section. Then we treat separately the points *(i)* and *(ii)* of Theorem 7.1.

7.4.1 Vortices have degree one

Lemma 7.14 *Whenever ε is small enough, $D_j = +1$ for $j = 1, \ldots, n_\varepsilon$.*

Proof. By Proposition 6.15 in Chapter 6, we have $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq o(1)$. According to (7.62), it yields

$$\pi\sum_{j=1}^{n_\varepsilon}|D_j|\, a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} - \frac{\pi a_0\Omega}{1+\Lambda^2}\sum_{D_j>0} a(x_j^\varepsilon)\, D_j \leq \pi\sum_{j=1}^{n_\varepsilon}|D_j|\, a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon)\, D_j \leq O(1).$$

From (7.7), we derive that

$$\sum_{j=1}^{n_\varepsilon}|D_j|\, a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} \leq \sum_{D_j>0} D_j\, a(x_j^\varepsilon)|\ln\varepsilon| + o(|\ln\varepsilon|).$$

Since $\rho \geq \varepsilon^\mu$, it leads to (we recall that $D_j \neq 0$)

$$(1-\mu) \sum_{D_j<0} |D_j|\, a(x_j^\varepsilon)|\ln\varepsilon| \leq \mu \sum_{D_j>0} |D_j|\, a(x_j^\varepsilon)|\ln\varepsilon| + o(|\ln\varepsilon|).$$

By Theorem 7.2, $a(x_j^\varepsilon) \geq a_0/2$ and consequently,

$$\sum_{D_j<0} |D_j| \leq \frac{2\mu}{1-\mu} \sum_{D_j>0} |D_j| + o(1) \leq \frac{C\mu}{1-\mu} + o(1).$$

Choosing μ sufficiently small, it yields $D_j > 0$ for $j = 1, \ldots, n_\varepsilon$ whenever ε is small enough. Since $|x_j^\varepsilon| \leq C$ and $D_j > 0$, we may now assert that

$$-\pi \sum_{i \neq j} D_i D_j\, a(x_j^\varepsilon) \ln |x_i^\varepsilon - x_j^\varepsilon| \geq O(1)$$

and thus $W_{\frac{\sqrt{a_0}}{2},\varepsilon} \geq -\pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{\frac{\sqrt{a_0}}{2},\varepsilon}(x_j^\varepsilon) = O(1)$. Hence the inequality (7.61) (applied with $R = \sqrt{a_0}/2$) together with $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq o(1)$ leads us to

$$\pi \sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon)|\ln\rho| + \pi \sum_{j=1}^{n_\varepsilon} D_j\, a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon)\, D_j \leq O(1).$$

As previously, we derive from (7.7), $\sum_{j=1}^{n_\varepsilon} (D_j^2 - D_j)\, a(x_j^\varepsilon)|\ln\rho| \leq o(|\ln\varepsilon|)$. Since $\rho \leq \varepsilon^{\overline{\mu}}$ and $a(x_j^\varepsilon) \geq a_0/2$, we conclude that

$$\frac{\overline{\mu}\, a_0}{2} \sum_{j=1}^{n_\varepsilon} (D_j^2 - D_j) \leq o(1)$$

which yields $D_j = +1$ whenever ε is small enough. □

As a direct consequence of Lemma 7.14, we obtain the following improvement of Lemma 7.13:

Corollary 7.15 *For any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$, we have*

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) + W_{R,\varepsilon}\big((x_1^\varepsilon, +1), \ldots, (x_{n_\varepsilon}^\varepsilon, +1)\big) + O_R(1).$$

Proof. It follows directly from (7.61) and Lemma 7.14 that for any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$,

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) + W_{R,\varepsilon}\big((x_1^\varepsilon, +1), \ldots, (x_{n_\varepsilon}^\varepsilon, +1)\big) + O_R(1).$$

On the other hand, we have proved in the proofs of Proposition 6.14 and Proposition 6.15 in Chapter 6, that $|\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) - \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)| = o(1)$ and $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathbb{R}^2 \setminus \mathcal{D}_\varepsilon) \geq o(1)$. Hence we have $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + o(1)$ and the conclusion follows. □

7.4.2 The subcritical case

We are now able to prove *(i)* in Theorem 7.1. Following the proof of Theorem 6.1 in Chapter 6, it suffices to show Proposition 7.16 below.

Proposition 7.16 *Assume that* (7.7) *holds with* $\omega_1 < 0$. *Then for* ε *sufficiently small, we have that*

$$|v_\varepsilon| \to 1 \quad \text{in } L^\infty_{\text{loc}}(\mathcal{D}) \text{ as } \varepsilon \to 0. \tag{7.67}$$

Moreover,

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = o(1) \quad \text{and} \quad \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) = o(1). \tag{7.68}$$

Proof. We fix some $\frac{\sqrt{a_0}}{2} < R_0 < \sqrt{a_0}$. In the proof of Proposition 6.14 in Chapter 6, we have proved that $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \le o(1)$ so that Corollary 7.15 applied with $R = \frac{\sqrt{a_0}}{2}$ leads to

$$\pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi a_0 \Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon) \le \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) \le O(1).$$

Since $a(x_j^\varepsilon) \ge a_0/2$ and $\omega_1 < 0$, we deduce that

$$\frac{a_0|\omega_1|\, n_\varepsilon}{2} \ln|\ln\varepsilon| \le -\omega_1 \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon) \ln|\ln\varepsilon| \le O(1)$$

and then $n_\varepsilon \le o(1)$ which implies that $n_\varepsilon \equiv 0$ whenever ε is small enough. Using the notation (7.63), we derive from (7.65) that

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \ge \pi \sum_{i\in I^+_{R_0}\cup I^-_{R_0}} a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0 \ln|\ln\varepsilon|\big) - \frac{\pi\Omega}{1+\Lambda^2} \sum_{i\in I^+_{R_0}\cup I^-_{R_0}} \big(a^2(p_i) - \nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i$$

By Proposition 6.15 in Chapter 6, we have $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le O(|\ln\varepsilon|^{-1})$. Since $a(p_i) \ll a_0$ for $i \in I^+_{R_0} \cup I^-_{R_0}$, we infer that exists $c > 0$ independent of ε such that

$$\begin{aligned} c \sum_{i\in I^+_{R_0}\cup I^-_{R_0}} a(p_i)|d_i||\ln\varepsilon| \le{}& \pi \sum_{i\in I^+_{R_0}\cup I^-_{R_0}} a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0\ln|\ln\varepsilon|\big) \\ &- \frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I^+_{R_0}\cup I^-_{R_0}} \big(a^2(p_i) - \nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i \le O(|\ln\varepsilon|^{-1}). \end{aligned}$$

Since $a(x) \ge |\ln\varepsilon|^{-3/2}$ in $\mathcal{D}_\varepsilon$, we finally obtain

$$\sum_{i\in I^+_{R_0}\cup I^-_{R_0}} |d_i| \le O(|\ln\varepsilon|^{-1/2}).$$

Hence $\sum_{i\in I^+_{R_0}\cup I^-_{R_0}} |d_i| = 0$ for ε sufficiently small and we conclude from (7.64),

$$\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I^+_{R_0}\cup I^-_{R_0}} B_i) = o(1).$$

By the proof of Proposition 6.17 in Chapter 6, we also have $\mathcal{R}_\varepsilon(v_\varepsilon, \cup_{i\in I^+_{R_0}\cup I^-_{R_0}} B_i) = o(1)$ so that $\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = o(1)$. Consequently,

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + o(1) \le o(1).$$

Then the rest of the proof follows as in Proposition 6.18 in Chapter 6. □

7.4.3 The supercritical case

In this section, we will prove *(ii)* in Theorem 7.1. Writing

$$\Omega = \frac{1+\Lambda^2}{a_0}\big(|\ln\varepsilon| + \omega(\varepsilon)\ln|\ln\varepsilon|\big), \tag{7.69}$$

we assume that

$$(d-1)+\delta \leq \omega(\varepsilon) \leq d-\delta \tag{7.70}$$

for some integer $d \geq 1$ and some positive number $\delta \ll 1$ independent of ε. We start by proving that, in this regime, v_ε has vortices whenever ε is small enough:

Proposition 7.17 *Assume that* (7.70) *holds. Then, for ε sufficiently small, v_ε has exactly d vortices of degree one, i.e. $n_\varepsilon \equiv d$, and*

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = -\pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| + O(1). \tag{7.71}$$

Proof. Step 1. We start by proving that $n_\varepsilon \geq 1$ for ε sufficiently small. By Theorem 7.21 in Section 7.5 (with $d=1$), there exists $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$F_\varepsilon(\tilde{u}_\varepsilon) \leq F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \pi a_0\omega(\varepsilon)\ln|\ln\varepsilon| + O(1).$$

By the minimizing property of u_ε and (7.9), we have

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\eta_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$$

and since $|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)| = o(1)$ (see Proposition 6.13 in Chapter 6), we deduce that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq -\pi a_0\omega(\varepsilon)\ln|\ln\varepsilon| + O(1).$$

From here, it turns out by Corollary 7.15 applied with $R = \frac{\sqrt{a_0}}{2}$ (recall that $W_{\frac{\sqrt{a_0}}{2},\varepsilon} \geq O(1)$),

$$\begin{aligned}
-\pi a_0\omega(\varepsilon)\ln|\ln\varepsilon| + O(1) \geq \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) &\geq \pi\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) + O(1)\\
&\geq \pi\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\left(-\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\Omega|x_j^\varepsilon|_\Lambda^2}{1+\Lambda^2}\right) + O(1)\\
&\geq -\pi a_0\omega(\varepsilon)n_\varepsilon\ln|\ln\varepsilon| + O(1).
\end{aligned}$$

Hence $n_\varepsilon \geq 1 + o(1)$ and the conclusion follows.

Step 2. Now we show that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq -\pi a_0\, n_\varepsilon\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(n_\varepsilon^2 - n_\varepsilon)\ln|\ln\varepsilon| + O(1). \tag{7.72}$$

In the case $n_\varepsilon = 1$, we have already proved the result in the previous step. Then we may assume that $n_\varepsilon \geq 2$. Since $\|\Psi_{\frac{\sqrt{a_0}}{2},\varepsilon}\|_\infty = O(1)$, we get from Corollary 7.15 applied with $R = \frac{\sqrt{a_0}}{2}$,

$$\begin{aligned}\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) &\geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\Bigg(|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon} \ln|x_i^\varepsilon - x_j^\varepsilon| - \frac{\Omega a(x_j^\varepsilon)}{1+\Lambda^2}\Bigg) + O(1) \\ &\geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\Bigg(-\omega(\varepsilon)\ln|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega |x_j^\varepsilon|_\Lambda^2}{1+\Lambda^2}\Bigg) + O(1) \end{aligned} \tag{7.73}$$

Since $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq o(1)$, we derive that

$$-\sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} |x_j^\varepsilon|_\Lambda^2 \leq C\ln|\ln\varepsilon|.$$

On the other hand $-\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| \geq O(1)$ so that $|x_j^\varepsilon|^2 \leq C\big(\ln|\ln\varepsilon|\big)|\ln\varepsilon|^{-1}$ and hence

$$\begin{aligned}\pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\Bigg(-\omega(\varepsilon)\ln|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega |x_j^\varepsilon|_\Lambda^2}{1+\Lambda^2}\Bigg) &= \\ = -\pi a_0\, n_\varepsilon \omega(\varepsilon)\ln|\ln\varepsilon| - \pi a_0 \sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi a_0 \Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}|x_j^\varepsilon|_\Lambda^2 + o(1)\end{aligned} \tag{7.74}$$

Setting $r = \max_j |x_j^\varepsilon|$, we remark that

$$-\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}|x_j^\varepsilon|_\Lambda^2 \geq -(n_\varepsilon^2 - n_\varepsilon)\ln 2r + \frac{\Omega\Lambda^2 r^2}{1+\Lambda^2} \geq \frac{n_\varepsilon^2 - n_\varepsilon}{2}\ln|\ln\varepsilon| + O(1). \tag{7.75}$$

Combining (7.73), (7.74) and (7.75), we obtain (7.72).

Step 3. We start by proving that $n_\varepsilon \geq d$. The case $d = 1$ is proved in Step 1 so that we may assume that $d \geq 2$. By Theorem 7.21 in Section 7.5, there exists for ε small enough, $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$F_\varepsilon(\tilde{u}_\varepsilon) \leq F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + O(1).$$

As in Step 1, $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$ yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq -\pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + O(1) \tag{7.76}$$

Matching (7.72) with (7.76), we deduce that

$$-\omega(\varepsilon)n_\varepsilon + \frac{n_\varepsilon^2 - n_\varepsilon}{2} \leq -\omega(\varepsilon)d + \frac{d^2 - d}{2} + o(1)$$

and it yields

$$\omega(\varepsilon)(d - n_\varepsilon) \leq \frac{(d - n_\varepsilon)(d + n_\varepsilon - 1)}{2} + o(1). \tag{7.77}$$

If assume that $n_\varepsilon \leq d-1$, it would lead to

$$(d-1)+\delta \leq \frac{d+n_\varepsilon-1}{2}+o(1) \leq d-1+o(1)$$

which is impossible for ε small enough.

Assume now that $n_\varepsilon \geq d+1$. As previously we infer that (7.77) holds and therefore

$$d-\delta \geq \frac{d+n_\varepsilon-1}{2}+o(1) \geq d+o(1)$$

which is also impossible for ε small. Hence $n_\varepsilon \equiv d$ whenever ε is small enough which leads to (7.71) by (7.72) and (7.76). □

By Proposition 7.17, we may now assume that v_ε has exactly d vortices. We move on a first information on their location:

Lemma 7.18 *We have*

$$|x_j^\varepsilon| \leq C|\ln\varepsilon|^{-1/2} \quad \textit{for } j=1,\dots,d \quad \textit{and if } d\geq 2, \quad |x_i^\varepsilon - x_j^\varepsilon| \geq C|\ln\varepsilon|^{-1/2} \quad \textit{for } i\neq j.$$

Proof. Matching (7.71) with (7.73) and (7.74) and using that $n_\varepsilon = d$, we deduce that

$$-\pi a_0 \sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi a_0 \Omega}{1+\Lambda^2}\sum_{j=1}^d |x_j^\varepsilon|_\Lambda^2 \leq \pi a_0 (d^2-d)\ln\left(|\ln\varepsilon|^{1/2}\right)+O(1).$$

Hence

$$\sum_{j=1}^d \left(-\sum_{i\neq j} \ln\left(\sqrt{|\ln\varepsilon|}\,|x_i^\varepsilon - x_j^\varepsilon|\right) + \frac{\Omega |x_j^\varepsilon|^2}{2}\right) \leq O(1)$$

and the conclusion follows. □

Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ by Lemma 7.18, we may now improve the lower estimates obtained in Lemma 7.12 following the method in [80, 81], proof of Proposition 5.2.

Lemma 7.19 *For any* $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$, *we have*

$$\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \geq \pi a_0 \sum_{j=1}^d a(x_j^\varepsilon)|\ln\varepsilon| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1)$$

where γ_0 *is an absolute constant.*

Proof. Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ and $D_j = 1$, Proposition 7.11 yields

$$\frac{1}{2}\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \pi \sum_{j=1}^d a(x_j^\varepsilon)|\ln\rho| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + o_R(1) \tag{7.78}$$

and it remains to estimate $\mathcal{E}_\varepsilon(v_\varepsilon, B(x_j^\varepsilon,\rho))$ for $j=1,\dots,d$. We proceed as follows. Since $D_j = 1$, we may write on $\partial B(x_j^\varepsilon,\rho)$ in polar coordinates with center x_j^ε,

$$v_\varepsilon(x) = |v_\varepsilon(x)|\, e^{i(\theta+\psi_j(\theta))}, \quad \theta\in[0,2\pi]$$

where $\psi_j \in H^1([0,2\pi],\mathbb{R})$ and $\psi_j(0) = \psi_j(2\pi) = 0$. Then in each disc $B(x_j^\varepsilon, 2\rho)$, we consider the map $\hat{v}_\varepsilon$ defined by

$$\hat{v}_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & \text{if } x \in B(x_j^\varepsilon, \rho), \\ \left(\frac{r-\rho}{\rho} + \frac{2\rho - r}{\rho}|v_\varepsilon(x_j^\varepsilon + \rho\, e^{i\theta})|\right) e^{i\left(\theta + \psi_j(\theta)\frac{2\rho-r}{\rho} + \psi_j(0)\frac{\rho - r}{\rho}\right)} & \text{if } x \in B(x_j^\varepsilon, 2\rho) \setminus B(x_j^\varepsilon, \rho). \end{cases}$$

Then $\hat{v}_\varepsilon = \exp\, i(\theta + \psi_j(0))$ on $\partial B(x_j^\varepsilon, 2\rho)$. Exactly as in the proof of Proposition 5.2 in [80, 81], we prove that

$$\left|\mathcal{E}_\varepsilon(\hat{v}_\varepsilon, B(x_j^\varepsilon, 2\rho) \setminus B(x_j^\varepsilon, \rho)) - \pi a(x_j^\varepsilon) \ln 2\right| = o(1). \tag{7.79}$$

Since $|a(x) - a(x_j^\varepsilon)| = O(\rho)$ in $B(x_j^\varepsilon, 2\rho)$, we may write

$$\mathcal{E}_\varepsilon(\hat{v}_\varepsilon, B(x_j^\varepsilon, 2\rho)) = \frac{a(x_j^\varepsilon)}{2} \int_{B(x_j^\varepsilon, 2\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1 - |\hat{v}_\varepsilon|^2)^2 + o(1). \tag{7.80}$$

Now we shall recall a result in [17]. For $\tilde{\varepsilon} > 0$, we consider

$$I(\tilde{\varepsilon}) = \operatorname*{Min}_{u \in \mathcal{C}} \frac{1}{2} \int_{B(0,1)} |\nabla u|^2 + \frac{1}{2\tilde{\varepsilon}^2}(1 - |u|^2)^2$$

where

$$\mathcal{C} = \left\{ u \in H^1(B(0,1), \mathbb{C}),\ u(x) = \frac{x}{|x|} \text{ on } \partial B(0,1) \right\}.$$

Then we have

$$\lim_{\tilde{\varepsilon} \to 0} \left(I(\tilde{\varepsilon}) + \pi \ln \tilde{\varepsilon} \right) = \gamma_0. \tag{7.81}$$

Since $\hat{v}_\varepsilon(x) = \frac{x - x_j^\varepsilon}{|x - x_j^\varepsilon|}\, e^{i\psi_j(0)}$ on $\partial B(x_j^\varepsilon, 2\rho)$, we obtain by scaling

$$\begin{aligned} \frac{1}{2} \int_{B(x_j^\varepsilon, 2\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1 - |\hat{v}_\varepsilon|^2)^2 &\geq I\left(\frac{\varepsilon}{2\rho\sqrt{a(x_j^\varepsilon)}} \right) \\ &= \pi \ln \frac{\rho}{\varepsilon} + \pi \ln 2 + \frac{\pi}{2} \ln a(x_j^\varepsilon) + \gamma_0 + o(1). \end{aligned}$$

With (7.79) and (7.80), we derive that for $j = 1, \ldots, d$,

$$\begin{aligned} \mathcal{E}_\varepsilon(v_\varepsilon, B(x_j^\varepsilon, \rho)) &\geq \pi a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} + \frac{\pi a(x_j^\varepsilon)}{2} \ln a(x_j^\varepsilon) + a(x_j^\varepsilon)\gamma_0 + o(1) \\ &\geq \pi a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} + \frac{\pi a_0}{2} \ln a_0 + a_0 \gamma_0 + o(1). \end{aligned}$$

Combining this estimate with (7.78), we get the result. □

We are now able to give the asymptotic expansion of $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon)$ which will allow us to locate precisely the vortices. This concludes the proof of Theorem 7.1.

Proposition 7.20 *Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\, x_j^\varepsilon$ for $j = 1, \ldots, d$, as $\varepsilon \to 0$ the $\tilde{x}_j^\varepsilon$'s tend to minimize the renormalized energy $w : \mathbb{R}^{2d} \to \mathbb{R}$ given by*

$$w(b_1, \ldots, b_d) = -\pi a_0 \sum_{i \neq j} \ln |b_i - b_j| + \frac{\pi a_0}{1 + \Lambda^2} \sum_{j=1}^{d} |b_j|_\Lambda^2.$$

Moreover, we have

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = -\pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| + \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + o(1) \tag{7.82}$$

where $Q_{\Lambda,d} = \frac{\pi a_0}{2}(d^2-d)\ln(1+\Lambda^2) + \pi a_0 d\ln a_0 - \frac{\pi a_0 d^2}{2}\ln a_0 + a_0 d\gamma_0 - \pi a_0 d^2 \ell(\Lambda)$ *and* $\ell(\Lambda)$ *is given by* (7.118).

Proof. From Lemma 7.19 and (7.66), we infer that for any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$,

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi\sum_{j=1}^d a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^d a^2(x_j^\varepsilon) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1).$$

As in the proof of Corollary 7.15, this estimate implies

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \pi\sum_{j=1}^d a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^d a^2(x_j^\varepsilon) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1).$$

Expanding Ω and $a(x_j^\varepsilon)$, we derive that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \pi\sum_{j=1}^d a(x_j^\varepsilon)\Big(-\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\Omega|x_j^\varepsilon|_\Lambda^2}{1+\Lambda^2}\Big) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1)$$

and by Lemma 7.18, it yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq -\pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{1+\Lambda^2}\sum_{j=1}^d \Omega|x_j^\varepsilon|_\Lambda^2 + W_{R,\varepsilon} + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1). \tag{7.83}$$

By Lemma 7.18, we also have

$$W_{R,\varepsilon} = -\pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| - \pi\sum_{j=1}^d \Psi_{R,\varepsilon}(x_j^\varepsilon) + o(1). \tag{7.84}$$

Since $D_j = 1$ for all j, the function $\Psi_{R,\varepsilon}$ satisfies the equation

$$\begin{cases} \operatorname{div}\left(\frac{1}{a}\nabla\Psi_{R,\varepsilon}\right) = -\sum_{j=1}^d a(x_j^\varepsilon)\nabla\left(\frac{1}{a}\right)\cdot\nabla\left(\ln|x-x_j^\varepsilon|\right) & \text{in } B_R^\Lambda, \\ \Psi_{R,\varepsilon} = -\sum_{j=1}^d a(x_j^\varepsilon)\ln|x-x_j^\varepsilon| & \text{on } \partial B_R^\Lambda. \end{cases} \tag{7.85}$$

We infer from Lemma 7.18 that for $j = 1,\dots,d$,

$$a(x_j^\varepsilon)\nabla\left(\frac{1}{a}\right)\cdot\nabla\left(\ln|x-x_j^\varepsilon|\right) = \frac{-2a_0|x|_\Lambda^2}{a^2(x)|x|^2} + f_\varepsilon^j(x).$$

where f_ε^j satisfies $\|f_\varepsilon^j\|_{L^p(B_R^\Lambda)} = o_R(1)$ for any $p\in[1,2)$ and $\big\|a_0\ln|x| - a(x_j^\varepsilon)\ln|x-x_j^\varepsilon|\big\|_{C^1(\partial B_R^\Lambda)} = o(1)$. Letting Ψ_R to be the solution of the equation

$$\begin{cases} \operatorname{div}\left(\frac{1}{a}\nabla\Psi_R\right) = \frac{-2|x|_\Lambda^2}{a^2(x)|x|^2} & \text{in } B_R^\Lambda, \\ \Psi_R = -\ln|x| & \text{on } \partial B_R^\Lambda, \end{cases} \tag{7.86}$$

it follows by classical results that $\|\Psi_{R,\varepsilon} - a_0 d\Psi_R\|_{L^\infty(B_R^\Lambda)} = o_R(1)$. Hence we obtain from (7.84),

$$\lim_{\varepsilon\to 0} \{W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + \pi a_0 \sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon|\} = -\pi a_0 d^2 \Psi_R(0). \tag{7.87}$$

Combining (7.83) and (7.87), we are led to

$$\liminf_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| + \pi a_0 \sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| - \frac{\pi a_0}{1+\Lambda^2}\sum_{j=1}^d \Omega|x_j^\varepsilon|_\Lambda^2 \right\} \geq$$
$$\geq \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 - \pi a_0 d^2 \Psi_R(0).$$

Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\,x_j^\varepsilon$, it yields

$$\liminf_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| - w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon)\right\} \geq$$
$$\geq \frac{\pi a_0}{2}(d^2-d)\ln(1+\Lambda^2) + \pi a_0 d\ln a_0 - \frac{\pi a_0 d^2}{2}\ln a_0 + a_0 d\gamma_0 - \pi a_0 d^2\Psi_R(0).$$

Since $\Psi_R(0)\to \ell(\Lambda)$ as $R\to\sqrt{a_0}$ by Lemma 7.23 in Appendix, we conclude that

$$\liminf_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 \omega(\varepsilon) d\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| - w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon)\right\} \geq Q_{\Lambda,d} \tag{7.88}$$

and hence

$$\liminf_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 \omega(\varepsilon) d\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \geq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d}. \tag{7.89}$$

By Theorem 7.21 in Section 5, for any $\delta' > 0$, there exists $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$\limsup_{\varepsilon\to 0} \left\{ F_\varepsilon(\tilde{u}_\varepsilon) - F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \leq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + \delta'$$

As in the proof of Proposition 7.17, $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$ implies

$$\limsup_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \leq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + \delta'. \tag{7.90}$$

Matching (7.89) with (7.90), we conclude that

$$\lim_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d}$$

since δ' is arbitrarily small. Coming back to (7.88), we are led to

$$\operatorname{Min} w(b) + Q_{\Lambda,d} - \limsup_{\varepsilon\to 0} w(x_1^\varepsilon,\dots,x_d^\varepsilon) \geq Q_{\Lambda,d}$$

and therefore $\lim_{\varepsilon\to 0} w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon) = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b)$ which ends the proof. □

Remark 7.5 *In the case $d = 1$, the expansion of the energy takes the simpler form*

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = -\pi a_0 \omega(\varepsilon)\ln|\ln\varepsilon| + Q_{\Lambda,1} + o(1)$$

and the renormalized energy $w(\cdot)$ reduces to $w(b) = (\pi a_0|b|_\Lambda^2)/(1+\Lambda^2)$. In particular, if x^ε denotes the single vortex of v_ε, we have $\sqrt{\Omega}\,x^\varepsilon \to 0$ as ε goes to 0.

7.5 Upper bound of the energy

Here, we give the construction of the test functions used in the previous sections. The difficulties are twofold: the mass constraint we have to take into account and the vanishing property of the function $a(x)$ on the boundary of $\mathcal{D}$. Hence the classical methods can not be applied directly. Concerning the mass constraint, we simply renormalize a suitable trial function. This procedure requires a high precision in the energy estimates and an almost optimal choice of the preliminary trial function. To overcome the degeneracy problem induced by the function $a(x)$, we proceed by upper approximation of $a(x)$. In the sequel, we assume that (7.7) holds. Using notation (7.69), the result can be stated as follows:

Theorem 7.21 *Let $d \geq 1$ be an integer. For any $\delta > 0$, there exists $(\tilde{u}_\varepsilon)_{\varepsilon>0} \subset \mathcal{H}$ verifying $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and*

$$\limsup_{\varepsilon\to 0} \left\{ F_\varepsilon(\tilde{u}_\varepsilon) - F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \leq \underset{b\in\mathbb{R}^{2d}}{\text{Min}}\ w(b) + Q_{\Lambda,d} + \delta$$

where the constant $Q_{\Lambda,d}$ is defined in Proposition 7.20.

As mentioned above, the proof of Theorem 7.21 is based on a first construction which is given by the following proposition. Here, some of the main ingredients are taken from a previous construction due to André and Shafrir [13].

Proposition 7.22 *Let $d \geq 1$ be an integer. For any $\delta > 0$, there exists $(\hat{v}_\varepsilon)_{\varepsilon>0}$ such that $\tilde{\eta}_\varepsilon \hat{v}_\varepsilon \in \mathcal{H}$ and*

$$\limsup_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \leq \underset{b\in\mathbb{R}^{2d}}{\text{Min}}\ w(b) + Q_{\Lambda,d} + \delta.$$

Proof. Step 1. Let $\sigma > 0$ and $\kappa > 0$ be two small parameters that we will choose later. We consider the function $a_\sigma : \overline{\mathcal{D}} \to \mathbb{R}$ given by

$$a_\sigma(x) = \begin{cases} a(x) & \text{if } |x|_\Lambda \leq \sqrt{a_0-\sigma}, \\ -2\sqrt{a_0-\sigma}\,|x|_\Lambda + 2a_0 - \sigma & \text{otherwise} \end{cases}$$

It turns out that $a_\sigma \in C^1(\overline{\mathcal{D}})$, $a_\sigma \geq a$ and $a_\sigma \geq C\sigma^2$ in $\overline{\mathcal{D}}$ for some positive constant C. Since a_σ does not vanish in $\overline{\mathcal{D}}$, we may define $\Phi_\sigma : \mathcal{D} \to \mathbb{R}$ the solution of the equation

$$\begin{cases} \text{div}(\frac{1}{a_\sigma}\nabla\Phi_\sigma) = 2\pi d\,\delta_0 & \text{in } \mathcal{D}, \\ \Phi_\sigma = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{7.91}$$

By the results in Chap. I of [17], we may find a map $v_0^\sigma \in C^2(\overline{\mathcal{D}} \setminus \{0\}, S^1)$ satisfying

$$v_0^\sigma \wedge \nabla v_0^\sigma = \frac{1}{a_\sigma}\nabla^\perp\Phi_\sigma \quad \text{in } \mathcal{D}\setminus\{0\}. \tag{7.92}$$

Set $\Theta_{\kappa,\varepsilon} = \mathcal{D} \setminus B(0, \kappa^{-1}\Omega^{-1/2})$. By (7.91) and (7.92), we have for ε small enough,

$$\int_{\Theta_{\kappa,\varepsilon}} a_\sigma |\nabla v_0^\sigma|^2 = \int_{\Theta_{\kappa,\varepsilon}} \frac{1}{a_\sigma} |\nabla \Phi_\sigma|^2 = -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{1}{a}\frac{\partial \Phi_\sigma}{\partial \nu}\Phi_\sigma$$
$$= -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{a_0^2 d^2}{a}\Big(\frac{\partial \Psi_\sigma}{\partial \nu} + \frac{1}{|x|}\Big)\big(\Psi_\sigma + \ln|x|\big) \tag{7.93}$$

where $\Psi_\sigma(x) = (a_0 d)^{-1}\Phi_\sigma(x) - \ln|x|$. Notice that $\Psi_\sigma \in C^{1,\alpha}(\overline{\mathcal{D}})$ for any $0 < \alpha < 1$, since it satisfies the equation

$$\begin{cases} \operatorname{div}\big(\frac{1}{a_\sigma}\nabla \Psi_\sigma\big) = f_\sigma(x) & \text{in } \mathcal{D}, \\ \Psi_\sigma = -\ln|x| & \text{on } \partial\mathcal{D} \end{cases} \tag{7.94}$$

with

$$f_\sigma(x) = -\nabla\Big(\frac{1}{a_\sigma(x)}\Big)\cdot\frac{x}{|x|^2} = \begin{cases} \dfrac{-2|x|_\Lambda^2}{a_\sigma^2(x)|x|^2} & \text{if } |x| \le \sqrt{a_0 - \sigma}, \\ \dfrac{-2\sqrt{a_0-\sigma}\,|x|_\Lambda}{a_\sigma^2(x)|x|^2} & \text{otherwise.} \end{cases}$$

From (7.93), we derive that

$$\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}} a|\nabla v_0^\sigma|^2 - \pi a_0 d^2 \ln(\kappa\Omega^{1/2})\right\} \le \lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}} a_\sigma|\nabla v_0^\sigma|^2 - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\}$$
$$\le -\pi a_0 d^2 \Psi_\sigma(0).$$

By Lemma 7.23 in Appendix, $\Psi_\sigma(0) \to \ell(\Lambda)$ as $\sigma \to 0$ where the constant $\ell(\Lambda)$ is defined in (7.118). Consequently, we may choose σ small such that

$$\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}} a|\nabla v_0^\sigma|^2 - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} \le -\pi a_0 d^2 \ell(\Lambda) + \frac{\delta}{2}. \tag{7.95}$$

In $\mathbb{R}^2 \setminus B(0,\kappa^{-1}\Omega^{-1/2})$, we define

$$\hat{v}_\varepsilon(x) = \begin{cases} v_0^\sigma(x) & \text{if } x \in \Theta_\kappa, \\ v_0^\sigma\big(\frac{\sqrt{a_0}\,x}{|x|_\Lambda}\big) & \text{if } x \in \mathbb{R}^2\setminus\mathcal{D}. \end{cases}$$

By Proposition 6.8 in Chapter 6, we have $\|\tilde{\eta}_\varepsilon^2\|_{L^\infty(\mathbb{R}^2\setminus\mathcal{D}_\varepsilon)} = o(1)$. Since $\hat{v}_\varepsilon$ does not depend on ε in $\mathbb{R}^2\setminus\mathcal{D}_\varepsilon$ and $|\hat{v}_\varepsilon| = 1$ in $\mathbb{R}^2\setminus\mathcal{D}_\varepsilon$, we derive that

$$\lim_{\varepsilon\to 0}\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, \mathbb{R}^2\setminus\mathcal{D}_\varepsilon) = 0. \tag{7.96}$$

From Proposition 6.8 in Chapter 6, we also know that

$$\left\|\frac{a-\tilde{\eta}_\varepsilon^2}{\tilde{\eta}_\varepsilon^2}\right\|_{L^\infty(\mathcal{D}_\varepsilon)} \le C\varepsilon^{1/3} \tag{7.97}$$

and hence (7.95) remains valid if one replaces a by $\tilde{\eta}_\varepsilon^2$ in the left hand side. Since v_0^σ is S^1-valued, we deduce that

$$\limsup_{\varepsilon\to 0}\left\{\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, \mathbb{R}^2\setminus B(0,\kappa^{-1}\Omega^{-1/2})) - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} \leq -\pi a_0 d^2 \ell(\Lambda) + \frac{\delta}{2}. \tag{7.98}$$

Step 2. We are going to extend $\hat{v}_\varepsilon$ to $B(0,\kappa^{-1}\Omega^{-1/2})$. As in [17], we may write in a neighborhood of 0 (using polar coordinates),

$$v_0^\sigma(x) = \exp\big(i(d\theta + \psi_\sigma(x))\big)$$

where ψ_σ is a smooth function in that neighborhood. Let $(b_1,\dots,b_d)\in\mathbb{R}^{2d}$ be a minimizing configuration for $w(\cdot)$, i.e.,

$$w(b_1,\dots,b_d) = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) \tag{7.99}$$

(note that we necesarily have $b_i\neq b_j$ for $i\neq j$). We choose κ sufficiently small such that $\max|b_j|\leq 1/4\kappa$ and we set $b_j^{(\varepsilon)} = \Omega^{-1/2}\,b_j$. Following the proof of Lemma 2.6 in [13], we write

$$e^{i\psi_\sigma(0)}\prod_{j=1}^d \frac{x-b_j^{(\varepsilon)}}{|x-b_j^{(\varepsilon)}|} = \exp\big(i(d\theta+\phi_\varepsilon(x))\big) \quad \text{for } x\in A_{\kappa,\varepsilon} = B(0,\kappa^{-1}\Omega^{-1/2})\setminus B(0,(2\kappa)^{-1}\Omega^{-1/2})$$

where ϕ_ε is a smooth function satisfying $|\nabla\phi_\varepsilon(x)|\leq C_\sigma\,\kappa^2\Omega^{1/2})$ and $|\phi_\varepsilon(x)-\psi_\sigma(0)| = C_\sigma\,\kappa^2$ for $x\in A_{\kappa,\varepsilon}$. We define in $A_{\kappa,\varepsilon}$,

$$\hat{v}_\varepsilon(x) = \exp\big(i(d\theta+\hat{\psi}_\varepsilon(x))\big)$$

with

$$\hat{\psi}_\varepsilon(x) = \big(2-2\kappa\Omega^{1/2}|x|\big)\phi_\varepsilon(x) + \big(2\kappa\Omega^{1/2}|x|-1\big)\psi_\sigma(x).$$

As in [13], we get that (using (7.97))

$$\limsup_{\varepsilon\to 0}\left\{\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, A_{\kappa,\varepsilon}) - \pi a_0 d^2\ln 2\right\} \leq \limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{A_{\kappa,\varepsilon}} a_\sigma|\nabla\hat{v}_\varepsilon|^2 - \pi a_0 d^2 \ln 2\right\} \leq C_\sigma\,\kappa^2. \tag{7.100}$$

Next we define $\hat{v}_\varepsilon$ in $\Xi_{\kappa,\varepsilon} = B(0,(2\kappa)^{-1}\Omega^{-1/2})\setminus\cup_{j=1}^d B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$ by

$$\hat{v}_\varepsilon(x) = e^{i\psi_\sigma(0)}\prod_{j=1}^d\frac{x-b_j^{(\varepsilon)}}{|x-b_j^{(\varepsilon)}|}.$$

Once more as in [13], we have (using (7.97))

$$\limsup_{\varepsilon\to 0}\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,\Xi_{\kappa,\varepsilon}) \leq \limsup_{\varepsilon\to 0}\frac{1}{2}\int_{\Xi_{\kappa,\varepsilon}} a_\sigma|\nabla\hat{v}_\varepsilon|^2 \leq \pi a_0(d^2+d)\ln\frac{1}{2\kappa} - \pi a_0\sum_{i\neq j}\ln|b_i-b_j| + C_\sigma\,\kappa. \tag{7.101}$$

Finally, in each $B_j^{(\varepsilon)} := B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$, we set

$$\hat{v}_\varepsilon(x) = e^{i\psi_\sigma(0)}\tilde{w}_\varepsilon^j\left(\frac{x-b_j^{(\varepsilon)}}{2\kappa\Omega^{-1/2}}\right) \tag{7.102}$$

where $\tilde{w}_\varepsilon^j$ realizes

$$\text{Min}\left\{\frac{1}{2}\int_{B(0,1)}|\nabla v|^2+\frac{1}{2\hat{\varepsilon}^2}(1-|v|^2)^2\,,\ v(y)=\prod_{i=1}^{d}\frac{2\kappa y+b_j-b_i}{|2\kappa y+b_j-b_i|}\ \text{ on }\partial B(0,1)\right\} \tag{7.103}$$

with

$$\hat{\varepsilon}=\frac{\varepsilon}{2\kappa\sqrt{a_0}\,\Omega^{-1/2}}.$$

As in the proof of Lemma 2.3 in [13], we derive

$$\lim_{\varepsilon\to0}\left\{\frac{1}{2}\int_{B(0,1)}|\nabla\tilde{w}_\varepsilon^j|^2+\frac{1}{2\hat{\varepsilon}^2}(1-|\tilde{w}_\varepsilon^j|^2)^2-\pi|\ln\hat{\varepsilon}|\right\}=\gamma_0+X(\kappa)$$

where γ_0 is defined in (7.81) and $X(\kappa)$ denotes a quantity satisfying $X(\kappa)\to0$ as $\kappa\to0$. By scaling, we obtain

$$\lim_{\varepsilon\to0}\left\{\frac{1}{2}\int_{B_j^{(\varepsilon)}}|\nabla\hat{v}_\varepsilon|^2+\frac{a_0}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2-\pi\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\}=\frac{\pi}{2}\ln a_0+\gamma_0+X(\kappa).$$

Notice that in $B_j^{(\varepsilon)}$,

$$a_\sigma(x)=a(x)\le a_0-(|\ln\varepsilon|+\omega_1\ln|\ln\varepsilon|)^{-1}\min_{y\in B(b_j,2\kappa)}\frac{a_0|y|_\Lambda^2}{1+\Lambda^2}$$

and consequently,

$$\limsup_{\varepsilon\to0}\left\{\frac{1}{2}\int_{B_j^{(\varepsilon)}}a_\sigma|\nabla\hat{v}_\varepsilon|^2+\frac{a_0a_\sigma}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2-\pi a_0\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\}\le$$
$$\le\frac{\pi a_0}{2}\ln a_0+a_0\gamma_0-\frac{\pi a_0|b_j|_\Lambda^2}{1+\Lambda^2}+X(\kappa).$$

By (7.97), it yields

$$\limsup_{\varepsilon\to0}\left\{\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,B_j^{(\varepsilon)})-\pi a_0\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\}\le\frac{\pi a_0}{2}\ln a_0+a_0\gamma_0-\frac{\pi a_0|b_j|_\Lambda^2}{1+\Lambda^2}+X(\kappa). \tag{7.104}$$

Combining (7.98), (7.100), (7.101) and (7.104), we conclude that for κ small enough,

$$\limsup_{\varepsilon\to0}\left\{\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon)-\pi a_0d|\ln\varepsilon|-\frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\right\}\le \tag{7.105}$$
$$\le-\pi a_0\sum_{i\neq j}\ln|b_i-b_j|-\frac{\pi a_0}{1+\Lambda^2}\sum_{j=1}^{d}|b_j|_\Lambda^2+Q_{\Lambda,d}+\delta.$$

Step 3. Now it remains to estimate $\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)$. Cauchy-Schwartz inequality yields

$$|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,\mathbb{R}^2\setminus\mathcal{D}_\varepsilon)|\le C\Omega\left(\int_{\mathbb{R}^2\setminus\mathcal{D}_\varepsilon}|x|^2\tilde{\eta}_\varepsilon^2\right)^{1/2}\left(\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,\mathbb{R}^2\setminus\mathcal{D}_\varepsilon)\right)^{1/2}. \tag{7.106}$$

By Proposition 6.8 in Chapter 6, $\Omega^2 \int_{\mathbb{R}^2\setminus\mathcal{D}_\varepsilon} |x|^2\tilde{\eta}_\varepsilon^2 \to 0$ as $\varepsilon \to 0$ and according to (7.96), it leads to

$$\lim_{\varepsilon\to 0} \left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) - \tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon, \mathcal{D}_\varepsilon)\right| = 0. \tag{7.107}$$

By the results in Chap. IX in [17], for $\hat{\varepsilon}$ sufficiently small and each $j = 1,\dots,d$, there exists exactly one disc $\hat{D}_\varepsilon^j \subset B(0,1)$ with $\operatorname{diam}(\hat{D}_\varepsilon^j) \le C\hat{\varepsilon}$ such that $|\tilde{w}_\varepsilon^j| \ge 1/2$ in $B(0,1) \setminus \hat{D}_\varepsilon^j$. By scaling, we infer that exist exactly d discs $D_\varepsilon^1,\dots,D_\varepsilon^d$ with $D_\varepsilon^j \subset B_j^{(\varepsilon)}$ and $\operatorname{diam}(D_\varepsilon^j) \le C\varepsilon$ such that

$$|\hat{v}_\varepsilon| \ge \frac{1}{2} \quad \text{in } \mathcal{D}_\varepsilon \setminus \cup_{j=1}^d D_\varepsilon^j.$$

We derive from (7.104) that

$$\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon, \cup_{j=1}^d D_\varepsilon^j)\right| \le C\Omega\,\varepsilon \sum_{j=1}^d \left(\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, B_j^{(\varepsilon)})\right)^{1/2} \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

and by (7.107), it leads to $\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) - \tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{j=1}^d D_\varepsilon^j)\right| = 0$. From (7.97), we infer that

$$\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{j=1}^d D_\varepsilon^j) - \mathcal{R}_\varepsilon(\hat{v}_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{j=1}^d D_\varepsilon^j)\right| = 0$$

and hence

$$\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) - \mathcal{R}_\varepsilon(\hat{v}_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{j=1}^d D_\varepsilon^j)\right| = 0. \tag{7.108}$$

To compute $\mathcal{R}_\varepsilon(\hat{v}_\varepsilon, \mathcal{D} \setminus \cup_{j=1}^d D_\varepsilon^j)$, we proceed as in Proposition 6.17 in Chapter 6 (here we use that $\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon) \le C|\ln\varepsilon|$ by (7.105)). It yields

$$\lim_{\varepsilon\to 0}\left(\mathcal{R}_\varepsilon(\hat{v}_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{j=1}^d D_\varepsilon^j) + \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^d a^2(b_j^{(\varepsilon)})\right) = 0$$

since $\deg(\hat{v}_\varepsilon/|\hat{v}_\varepsilon|, \partial D_\varepsilon^j) = +1$ for $j = 1,\dots,d$. Expanding $a^2(b_j^{(\varepsilon)})$ and Ω, we deduce from (7.108) that

$$\lim_{\varepsilon\to 0}\left(\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) + \pi a_0 d\,|\ln\varepsilon| + \pi a_0\omega(\varepsilon) d\ln|\ln\varepsilon|\right) = \frac{2\pi a_0}{1+\Lambda^2}\sum_{j=1}^d |b_j|_\Lambda^2. \tag{7.109}$$

Combining (7.99), (7.105) and (7.109), we obtain the announced result. □

Proof of Theorem 7.21. We consider the map $\hat{v}_\varepsilon$ given in Proposition 7.22 and we set

$$\tilde{v}_\varepsilon = m_\varepsilon^{-1}\hat{v}_\varepsilon \quad \text{and} \quad \tilde{u}_\varepsilon = \tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon \quad \text{with} \quad m_\varepsilon = \|\tilde{\eta}_\varepsilon\hat{v}_\varepsilon\|_{L^2(\mathbb{R}^2)}.$$

We are going to prove that the map $\tilde{u}_\varepsilon$ satisfies the required property. By Lemma 6.12 in Chapter 6, we have

$$F_\varepsilon(\tilde{u}_\varepsilon) = F(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon).$$

In view of Proposition 7.22, it suffices to prove that $\left|\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) - \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon)\right| \to 0$ and $\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon) \to 0$ as $\varepsilon \to 0$. We first estimate m_ε. Since $|\hat{v}_\varepsilon| = 1$ in $\mathbb{R}^2 \setminus \cup_{j=1}^d B_j^{(\varepsilon)}$ and $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we have

$$m_\varepsilon^2 = \int_{\mathbb{R}^2}\tilde{\eta}_\varepsilon^2 + \int_{\cup_{j=1}^d B_j^{(\varepsilon)}}\tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1) = 1 + \int_{\cup_{j=1}^d B_j^{(\varepsilon)}}\tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1).$$

Using Cauchy-Schwarz inequality, we derive from (7.102), (7.103) and Theorem III.2 in [17] that

$$\Big|\int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde\eta_\varepsilon^2(|\hat v_\varepsilon|^2-1)\Big| \le C|\ln\varepsilon|^{-1/2}\Big(\int_{\cup_{j=1}^d B_j^{(\varepsilon)}} (|\hat v_\varepsilon|^2-1)^2\Big)^{1/2} \le C\varepsilon|\ln\varepsilon|^{-1/2} \tag{7.110}$$

and thus

$$m_\varepsilon^2 = 1 + O(\varepsilon|\ln\varepsilon|^{-1/2}). \tag{7.111}$$

Using $|\hat v_\varepsilon| = 1$ in $\mathbb{R}^2\setminus\cup_{j=1}^d B_j^{(\varepsilon)}$, $|\nabla S|\le C|x|$, $|k_\varepsilon|\le C|\ln\varepsilon|$, (7.110) and (7.111), we derive that

$$\begin{aligned}\big|\tilde{\mathcal{T}}_\varepsilon(\tilde v_\varepsilon)\big| &\le C|\ln\varepsilon|^2\Big(|1-m_\varepsilon^{-2}|\int_{\mathbb{R}^2}(1+|x|^2)\tilde\eta_\varepsilon^2 + \int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde\eta_\varepsilon^2\big(|1-m_\varepsilon^{-2}||\hat v_\varepsilon|^2 + (1-|\hat v_\varepsilon|^2)\big)\Big)\\ &\le C\varepsilon|\ln\varepsilon|^{3/2}.\end{aligned}$$

Now we may estimate using (7.105), (7.109) and (7.111),

$$\int_{\mathbb{R}^2}\tilde\eta_\varepsilon^2|\nabla\tilde v_\varepsilon|^2 = m_\varepsilon^{-2}\int_{\mathbb{R}^2}\tilde\eta_\varepsilon^2|\nabla\hat v_\varepsilon|^2 = \int_{\mathbb{R}^2}\tilde\eta_\varepsilon^2|\nabla\hat v_\varepsilon|^2 + O(\varepsilon|\ln\varepsilon|^{1/2}), \tag{7.112}$$

and

$$\tilde{\mathcal{R}}_\varepsilon(\tilde v_\varepsilon) = m_\varepsilon^{-2}\tilde{\mathcal{R}}_\varepsilon(\hat v_\varepsilon) = \tilde{\mathcal{R}}_\varepsilon(\hat v_\varepsilon) + O(\varepsilon|\ln\varepsilon|^{1/2}). \tag{7.113}$$

We write

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\tilde\eta_\varepsilon^4(1-|\tilde v_\varepsilon|^2)^2 &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\tilde\eta_\varepsilon^4(1-|\hat v_\varepsilon|^2)^2 + \frac{2(1-m_\varepsilon^{-2})}{\varepsilon^2}\int_{\cup_{j=1}^d B_j^{(\varepsilon)}}\tilde\eta_\varepsilon^4(1-|\hat v_\varepsilon|^2)|\hat v_\varepsilon|^2\\ &\quad + \frac{(1-m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2}\tilde\eta_\varepsilon^4|\hat v_\varepsilon|^4.\end{aligned} \tag{7.114}$$

We infer from (7.105) and (7.111) that

$$\frac{(1-m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2}\tilde\eta_\varepsilon^4|\hat v_\varepsilon|^4 \le C|\ln\varepsilon|^{-1}, \tag{7.115}$$

and from (7.110) and (7.111),

$$\frac{|1-m_\varepsilon^{-2}|}{\varepsilon^2}\int_{\cup_{j=1}^d B_j^{(\varepsilon)}}\tilde\eta_\varepsilon^4|\hat v_\varepsilon|^2\big|1-|\hat v_\varepsilon|^2\big| \le C|\ln\varepsilon|^{-1}. \tag{7.116}$$

Combining (7.112), (7.113), (7.114), (7.115) and (7.116), we finally obtain that $\tilde{\mathcal{F}}_\varepsilon(\tilde v_\varepsilon) = \tilde{\mathcal{F}}_\varepsilon(\hat v_\varepsilon) + o(1)$ and the proof is complete. □

7.6 Appendix

In this appendix, we prove that the functions Ψ_R and Ψ_σ defined by (7.86) and respectively (7.94) converge to the same limiting function as $R\to\sqrt{a_0}$ and $\sigma\to 0$. The proof is based on the construction of suitable *barrier* functions.

Lemma 7.23 *For any $0 < R < \sqrt{a_0}$, respectively any $\sigma > 0$, let Ψ_R be the solution of equation (7.86), respectively Ψ_σ the solution of (7.94). Then $\Psi_R \to \Psi_\star$ as $R \to \sqrt{a_0}$, respectively $\Psi_\sigma \to \Psi_\star$ as $\sigma \to 0$, in $C^1_{\rm loc}(\mathcal{D})$ where $\Psi_\star$ is the unique solution in $C^0(\overline{\mathcal{D}})$ of*

$$\begin{cases} \operatorname{div}\big(\frac{1}{a}\nabla\Psi_\star\big) = \dfrac{-2|x|^2_\Lambda}{a^2(x)|x|^2} & \text{in } \mathcal{D}, \\ \Psi_\star = -\ln|x| & \text{on } \partial\mathcal{D}. \end{cases} \tag{7.117}$$

In particular,

$$\lim_{R\to\sqrt{a_0}} \Psi_R(0) = \lim_{\sigma\to 0}\Psi_\sigma(0) = \Psi_\star(0) =: \ell(\Lambda). \tag{7.118}$$

Proof. Step 1: Uniqueness of $\Psi_\star$. Assume that (7.117) admits two solutions $\Psi^1_\star$ and $\Psi^2_\star$ in $C^0(\overline{\mathcal{D}})$. Then the difference $\Psi^1_\star - \Psi^2_\star$ satisfies $\operatorname{div}(\frac{1}{a}\nabla(\Psi^1_\star - \Psi^2_\star)) = 0$ in $\mathcal{D}$ and $\Psi^1_\star - \Psi^2_\star = 0$ on $\partial\mathcal{D}$. By elliptic regularity, we infer that $\Psi^1_\star - \Psi^2_\star \in C^2(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$. Hence it follows $\Psi^1_\star - \Psi^2_\star \equiv 0$ by the classical maximum principle.

Step 2: Existence of $\Psi_\star$. We set for $y \in \mathcal{D}$,

$$\Upsilon_R(y) = \Psi_R\bigg(\frac{Ry}{\sqrt{a_0}}\bigg) - \zeta(y) + \ln(R/\sqrt{a_0})$$

where ζ is the solution of

$$\begin{cases} \Delta\zeta = 0 & \text{in } \mathcal{D}, \\ \zeta = -\ln|y| & \text{on } \partial\mathcal{D}. \end{cases}$$

Since Ψ_R solves (7.86), we deduce that Υ_R is the unique solution of

$$\begin{cases} -\operatorname{div}\bigg(\dfrac{1}{a_R(y)}\nabla\Upsilon_R\bigg) = \dfrac{f(y)}{a_R^2(y)} & \text{in } \mathcal{D}, \\ \Upsilon_R = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{7.119}$$

where $a_R(y) = a_0^2/R^2 - |y|^2_\Lambda$ and

$$f(y) = \frac{2|y|^2_\Lambda}{|y|^2} + 2(y_1, \Lambda^2 y_2)\cdot\nabla\zeta(y).$$

We easily check that $y \mapsto Ka_R(y)$, respectively $y \mapsto -Ka_R(y)$, defines a supersolution, resp. a subsolution, of (7.119) whenever the constant K satisfies $K \geq \|f\|_{L^\infty(\mathcal{D})}/(\Lambda^2 a_0)$. Hence

$$|\Upsilon_R| \leq Ca_R \quad \text{in } \mathcal{D} \tag{7.120}$$

for a constant C independent of R. By elliptic regularity, we deduce that Υ_R remains bounded in $W^{2,p}_{\rm loc}(\mathcal{D})$ as $R \to \sqrt{a_0}$ for any $1 \leq p < \infty$. Therefore, from any sequence $R_n \to \sqrt{a_0}$, we may extract a subsequence, still denoted by (R_n), such that $\Upsilon_{R_n} \to \Upsilon_\star$ in $C^1_{\rm loc}(\mathcal{D})$ where $\Upsilon_\star$ satisfies

$$-\operatorname{div}\bigg(\frac{1}{a(y)}\nabla\Upsilon_\star\bigg) = \frac{f}{a^2(y)} \quad \text{in } \mathcal{D}.$$

We infer from (7.120) that $|\Upsilon_\star(y)| \leq Ca(y)$ for any $y \in \mathcal{D}$ and hence $\Upsilon_\star \in C^0(\overline{\mathcal{D}})$ with $\Upsilon_{\star|\partial\mathcal{D}} = 0$. Consequently, the function $\Psi_\star := \Upsilon_\star + \zeta$ defines a solution of (7.117) which is continuous in $\overline{\mathcal{D}}$.

Step 3. By the uniqueness of $\Psi_\star$, we have that $\Upsilon_R \to \Psi_\star - \zeta$ in $C^1_{\mathrm{loc}}(\mathcal{D})$ as $R \to \sqrt{a_0}$ which clearly implies $\Psi_R \to \Psi_\star$ in $C^1_{\mathrm{loc}}(\mathcal{D})$ as $R \to \sqrt{a_0}$. To prove that $\Psi_\sigma \to \Psi_\star$ in $C^1_{\mathrm{loc}}(\mathcal{D})$ as $\sigma \to 0$, we may proceed as in Step 2. Indeed, we may show as in Step 2, that $|\Psi_\sigma - \zeta| \leq C a_\sigma$ in $\mathcal{D}$ for a constant C independent of σ. $\square$

Part III

Optimality of the Néel wall

Chapter 8

A compactness result in thin-film micromagnetics and the optimality of the Néel wall

Abstract

We study the asymptotics of a $2-d$ thin-film approximation energy where a transition angle is imposed on the admissible magnetizations. The goal is to show the optimality of the $1-d$ transition layers (*the Néel walls*) under $2-d$ perturbations. For that, we prove a compactness result for magnetizations in the energy regime corresponding to a finite number of Néel walls. The accumulation points are $2-d$ unit-valued divergence-free vector fields. In the case of zero-energy states, we show locally Lipschitz continuity and these limits classically satisfy the principle of characteristics. Then we conclude with the optimality of the straight walls in the regime of the specific line energy of the Néel wall.
This chapter is written in collaboration with F. Otto and it is published in *J. Eur. Math. Soc. (JEMS)* **10** (2008) (4), pp. 909–956 (cf. [57]).

8.1 Introduction

In this chapter we analyze a two-dimensional approximation of the micromagnetic energy of a thin-film in the absence of external field and crystalline anisotropy. Following [39, 41], the setting is determined by our goal to prove the optimality of Néel walls under 2−d variation. Let $\Omega' = (-1,1)\times\mathbb{R}$ be the transversal section of a thin infinitely extended cylinder (see Figure 8.1). The admissible magnetizations are smooth 2-d unit-length vector fields

$$m' = (m_1, m_2) : \mathbb{R}^2 \to S^1$$

that macroscopically act as an angle wall in Ω' (see Figure 8.2), i.e.,

$$m'(x') = \begin{pmatrix} m_{1,\infty} \\ \pm\sqrt{1-m_{1,\infty}{}^2} \end{pmatrix} \text{ for } \pm x_1 \geq 1,\ x_2 \in \mathbb{R}, \tag{8.1}$$

where $m_{1,\infty} \in [0,1)$ is some fixed number and we use the shorthand notation $x' = (x_1, x_2)$. Here and in the sequel, the prime always indicates an in-plane quantity. For each magnetization m' it corresponds a stray field $h = (h_1, h_2, h_3) : \mathbb{R}^3 \to \mathbb{R}^3$ which is a $3D$ vector field related to m' by the following variational formulation:

$$\int_{\mathbb{R}^3} h \cdot \nabla\zeta \, dx = \int_{\mathbb{R}^2} \zeta \, \nabla' \cdot m' \, dx', \quad \forall \zeta \in C_c^\infty(\mathbb{R}^3) \tag{8.2}$$

where we write $x = (x', x_3) \in \mathbb{R}^3$ and $\nabla' \cdot m'$ for the in-plane divergence of m'. Classically, this is,

$$\begin{cases} \nabla \cdot h = 0 & \text{in } \mathbb{R}^3 \setminus (\mathbb{R}^2 \times \{0\}), \\ [h_3] = -\nabla' \cdot m' & \text{in } \mathbb{R}^2 \times \{0\}, \end{cases}$$

where $[h_3]$ denotes the jump of the vertical component of h across the plane $\mathbb{R}^2 \times \{0\}$. The

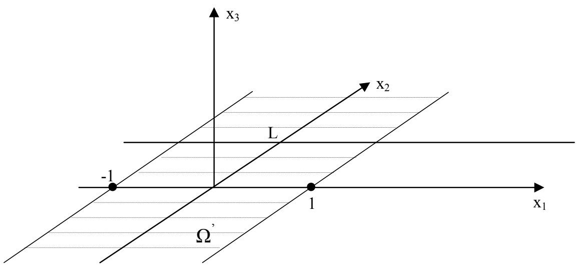


Figure 8.1: The infinite domain Ω'

magnetic field is uniquely determined by $\nabla' \cdot m'$ up to curl vector fields. The micromagnetic model states that the experimentally observed ground state for the magnetization m' and for the magnetostatic potential of the stray field is the minimizer of the micromagnetic energy. In order to assign the energy density for this configuration we assume that

$$m' \text{ and } h \text{ are } L\text{-periodic in the infinite } x_2\text{-direction,} \tag{8.3}$$

where L is an arbitrary positive number. In this chapter we focus on the following non-dimensionalized energy functional:

$$E_\varepsilon(m', h) = \varepsilon \int_{\mathbb{R} \times [0,L)} |\nabla' \cdot m'|^2 \, dx' + \int_{\mathbb{R} \times [0,L) \times \mathbb{R}} |h|^2 \, dx \tag{8.4}$$

where $\varepsilon > 0$ is a small length-scale. The first term in (8.4) plays the role of the exchange energy and the energy of the stray field is called the magnetostatic energy. The stray field equation yields that the minimal magnetostatic energy corresponds to the homogeneous $H^{-1/2}$-norm of $\nabla' \cdot m'$ and it is achieved for the curl-free stray field:

$$\min_{h \text{ with } (8.2)} \int_{\mathbb{R} \times [0,L) \times \mathbb{R}} |h|^2 \, dx = \frac{1}{2} \int_{\mathbb{R} \times [0,L)} \left| |\nabla'|^{-1/2} \nabla' \cdot m' \right|^2 dx'.$$

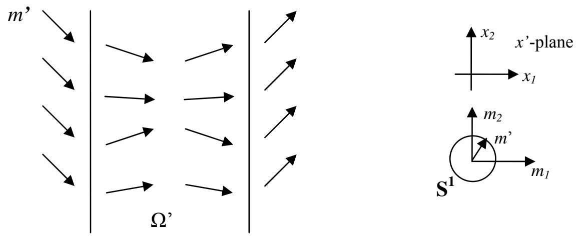


Figure 8.2: The admissible magnetization m'

Now we shall informally explain how the principle of pole avoidance leads to the formation of walls. For simplicity, we assume that the mesoscopic transition angle imposed by (8.1) on the boundary $\partial\Omega'$ is 180°, i.e., $m' \cdot \nu' = 0$ on $\partial\Omega'$. The boundary effects in the tangential direction are excluded by our choice of Ω' which is infinite in x_2-direction. The competition between the exchange and magnetostatic energy will try to enforce the divergence-free condition for m', i.e., $\nabla'\cdot m' = 0$ in Ω'. Therefore, we arrive at

$$|m'| = 1 \text{ and } \nabla'\cdot m' = 0 \text{ in } \Omega', \quad m' \cdot \nu' = 0 \text{ on } \partial\Omega'. \tag{8.5}$$

This mesoscopic thin-film description has been justified in [41] using the $\Gamma-$convergence method. We notice that the conditions in (8.5) are too rigid for smooth magnetization m'. This can be seen by writing $m' = \nabla'^{\perp}\psi$ with the help of a "stream function" ψ. Then (8.5) turns out that ψ is a solution of the Dirichlet problem for the eikonal equation:

$$|\nabla'^{\perp}\psi| = 1 \text{ in } \Omega', \quad \psi = 0 \text{ on } \partial\Omega'. \tag{8.6}$$

Using the characteristics method, it follows that there is no smooth solution of the equation (8.6). On the other hand, there are many continuous solutions that satisfy the first condition of (8.6) away from a set of vanishing Lebesgue measure. One of them is the "viscosity solution" given by the distance function

$$\psi(x') = \text{dist}\,(x', \partial\Omega')$$

that corresponds to the so-called Landau state for the magnetization m' (see Figure 8.3). Hence,

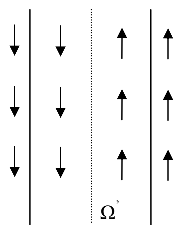


Figure 8.3: Landau state in Ω'

the divergence-free equation in (8.5) has to be interpreted in the distribution sense and it is expected to induce line-singularities for solutions m'. These ridges are an idealization of the wall formation in thin-film elements at the mesoscopic level. At the microscopic level, they are replaced by smooth transition layers where the magnetization varies very quickly. A final remark is that the normal component of m' does not jump across these discontinuity lines (because of (8.5)) and therefore, walls are determined by the angle between the mesoscopic levels in the adjacent domains. In the following we will concentrate on the Néel wall which is the favored

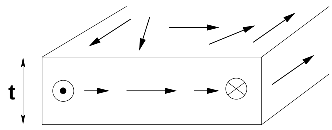


Figure 8.4: Néel wall in a $3D$ cylinder

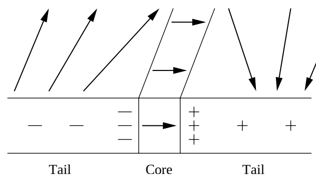


Figure 8.5: Charge distribution in core and tails for a Néel wall

wall type in very thin films (see Figure 8.4). It is characterized by a one-dimensional in-plane magnetization:

$$m' = (m_1(x_1), m_2(x_1)), \tag{8.7}$$

that avoids surface charges, but leads to volume charges (because of (8.1)), i.e.,

$$\nabla' \cdot m' = \frac{dm_1}{dx_1} \neq 0.$$

The prototype is the 180° Néel wall which corresponds to the boundary condition (8.1) for $m_{1,\infty} = 0$, i.e.,

$$m'(x_1) = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \text{ for } \pm x_1 \geq 1. \tag{8.8}$$

Let us now discuss about the scaling of the energy of the prototypical Néel wall. For magnetizations (8.7), the specific energy (8.4) reduces to

$$E_\varepsilon^{1d}(m') = \varepsilon \int_{\mathbb{R}} \left|\frac{dm_1}{dx_1}\right|^2 dx_1 + \frac{1}{2}\int_{\mathbb{R}} \left|\left|\frac{d}{dx_1}\right|^{1/2} m_1\right|^2 dx_1. \tag{8.9}$$

We define the Néel wall as the $1d$ minimizer of (8.9) under the boundary constraint (8.8). The Néel wall is a two length scale object: a small core ($|x_1| \lesssim w_{core}$) with fast varying rotation and a logarithmically decaying tail ($w_{core} \lesssim |x_1| \lesssim 1$) (see Figure 8.5). The finiteness of Ω' in x_1-direction in our setting serves as the confining mechanism for the Néel wall tail. This two-scale structure permits to the Néel wall to decrease the specific energy by a logarithmic factor. The prediction of the logaritmic decay was formally proved by Riedel and Seeger [73]; a

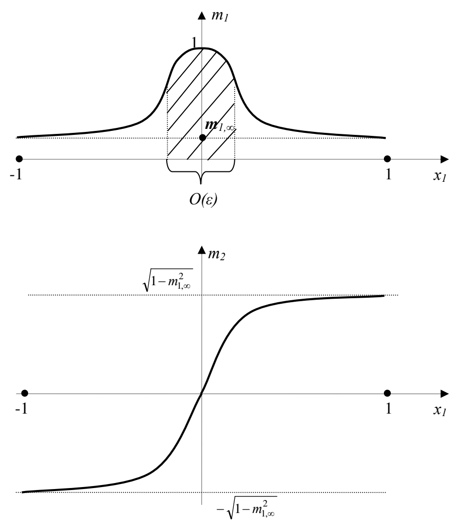


Figure 8.6: Qualitative behavior of the Néel wall

detailed mathematical discussion of their results was carried out by Garcia-Cervera [45]. Finally, Melcher rigorously established in [67, 68] the exact logarithmic scaling for the 180° Néel wall tails:

$$\min_{(8.7),(8.8)} E_\varepsilon^{1d}(m') \approx \frac{\pi}{2|\ln\varepsilon|} \quad \text{for } \varepsilon \ll 1$$

and the minimizer m_1 with $m_1(0) = 1$ is symmetric around 0 ($w_{core} \sim \varepsilon$) and satisfies

$$m_1(x_1) \sim \frac{\ln\frac{1}{|x_1|}}{|\ln\varepsilon|} \text{ for } \varepsilon \ll |x_1| \ll 1$$

(see Figure 8.6).

The stability of 180° Néel walls under arbitrary $2-d$ modulation was proved by DeSimone, Knüpfer and Otto in [39]:

$$\min_{\substack{m',h \\ m' \text{ with } (8.8)}} E_\varepsilon(m',h) \approx \min_{\substack{m',h \\ m'=m'(x_1) \text{ with } (8.8)}} E_\varepsilon(m',h) \approx \frac{\pi L}{2|\ln\varepsilon|} \quad \text{for } \varepsilon \ll 1.$$

This means that asymptotically, the minimal energy E_ε is assumed by a straight wall. More precisely, the variations of the optimal $1d$ transition layer in x_2-direction will not decrease the leading order term in the energy.

Our first result is a qualitative property of the optimal $1d$ transition layers: We prove that asymptotically, the minimal energy can be assumed *only* by the straight walls. This property holds for general boundary conditions (8.1). It is based on a compactness result for magnetizations $\{m'_\varepsilon\}$ with energies E_ε close to the minimal energy level: any accumulation limit m' has the singularities concentrated on a vertical line (see Figure 8.7).

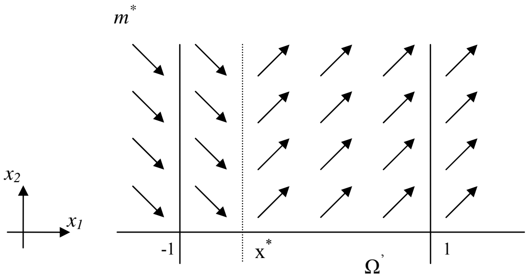


Figure 8.7: Straight wall

Theorem 8.1 *Let $m_{1,\infty} \in [0,1)$ and $L > 0$ be given. For any $\delta > 0$ there exists $\varepsilon_0 > 0$ with the following property: given $m' : \mathbb{R}^2 \to S^1$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ with*

$$\begin{aligned} &m' \text{ and } h \text{ are } L-\text{periodic in } x_2, \text{ i.e., (8.3) holds,} \\ &m' \text{ satisfies the boundary condition (8.1),} \\ &m' \text{ and } h' \text{ are related by (8.2),} \\ &|\ln \varepsilon| E_\varepsilon(m', h) \le L\frac{\pi}{2}(1 - m_{1,\infty})^2 + \varepsilon_0, \quad \text{for some } 0 < \varepsilon \le \varepsilon_0, \end{aligned} \tag{8.10}$$

then we have

$$\int_{\mathbb{R}\times[0,L)} |m' - m^*| \, dx' \le \delta, \tag{8.11}$$

where m^* is a straight wall given by

$$m^*(x_1, x_2) = \begin{pmatrix} m_{1,\infty} \\ \pm\sqrt{1 - m_{1,\infty}{}^2} \end{pmatrix} \text{ for } \pm x_1 > \pm x_1^*, \tag{8.12}$$

for some $x_1^* \in [-1, 1]$.

Remark: The estimate (8.11) holds in L^p for any $\delta_p > 0$ and $1 \le p < \infty$.

Let us first discuss the compactness result for the case of $1d$ magnetizations. We are interested in the asymptotics as $\varepsilon \to 0$ of families of $1d$ magnetizations in the more general context of an energy regime $O(\frac{1}{|\ln \varepsilon|})$. We show that such sequence of magnetizations is relatively compact in L^1_{loc} and the accumulation points in L^1_{loc} concentrate on a finite number of walls (see Figure 8.8). As a direct consequence, we obtain the optimality of the straight walls over $1d$ perturbations in the asymptotic regime of the minimal energy.

Theorem 8.2 *Let $m_{1,\infty} \in [0,1)$. Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$. For $k \in \mathbb{N}$, let $m'_k = (m_{1,k}, m_{2,k}) : \mathbb{R} \to S^1$ such that (8.1) holds and*

$$\limsup_{k\to\infty} |\ln \varepsilon_k| \left(\varepsilon_k \int_{\mathbb{R}} |\frac{dm_{1,k}}{dx_1}|^2 \, dx_1 + \int_{\mathbb{R}} \left| |\frac{d}{dx_1}|^{1/2} m_{1,k} \right|^2 dx_1 \right) < \infty. \tag{8.13}$$

Then $\{m'_k\}$ is relatively compact in $L^1_{loc}(\mathbb{R})$. Moreover, any accumulation point $m' : \mathbb{R} \to S^1$ of the sequence $\{m'_k\}_{k\uparrow\infty}$ in L^1_{loc} is of bounded variation and can be written as

$$m' = \sum_{n=1}^{2N} \begin{pmatrix} m_{1,\infty} \\ (-1)^n \sqrt{1 - m_{1,\infty}{}^2} \end{pmatrix} 1_{(b_{n-1}, b_n)},$$

where $-\infty = b_0 < b_1 < \cdots < b_{2N-1} < b_{2N} = +\infty$ and $b_n \in [-1, 1]$ for $n = 1, \ldots, 2N-1$.

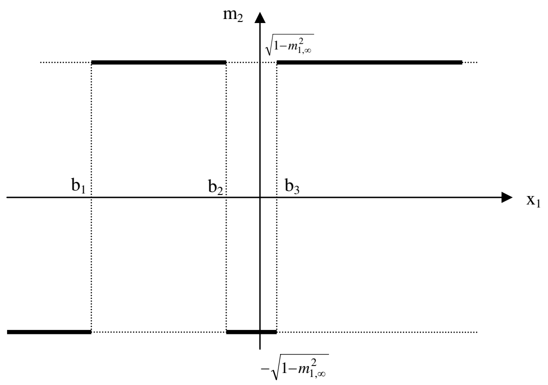


Figure 8.8: The m_2 component of a limit with three walls

One may ask whether the above sequences of $1d$ magnetizations are relatively compact in BV since their limit has bounded variation. The answer is negative in general. For that, we construct a family of $1d$ magnetizations with the energy level in the regime $O(\frac{1}{|\ln \varepsilon|})$ such that the sequence of total variations of $\{\frac{dm_{1,k}}{dx_1}\}$ blows-up as $k \to \infty$:

Theorem 8.3 *There exists a sequence $\{m'_k : \mathbb{R} \to S^1\}$ with the properties:*

$$\begin{aligned} &(8.1) \text{ holds for some } m_{1,\infty} \in [0,1), \\ &\lim_{k\to\infty} \int_{\mathbb{R}} \left|\frac{dm_{1,k}}{dx_1}\right| dx_1 = \infty, \\ &(8.13) \text{ holds for some } \{\varepsilon_k\}_{k\in\mathbb{N}} \text{ with } \varepsilon_k \to 0. \end{aligned}$$

Now we investigate the asymptotics as $\varepsilon \to 0$ of families of $2d$ magnetizations when the energy $E_\varepsilon(m'_\varepsilon, h_\varepsilon)$ is placed in the regime $O(\frac{1}{|\ln \varepsilon|})$. One of the issues we discuss here is the question of the L^1_{loc}-compactness of the magnetizations $\{m'_\varepsilon\}_{\varepsilon\downarrow 0}$ in the above energy regime, i.e., whether the topological constraint $|m'_\varepsilon| = 1$ passes to the limit. The difficulty arises from the fact that in general the sequence of divergences $\{\nabla' \cdot m'_\varepsilon\}$ is not uniformly bounded in L^1_{loc} (a counter-example is given in Theorem 8.3). This was one of the particularities used in the entropy methods for proving compactness results for the Modica-Mortola type problems; we refer to the studies of Jin and Kohn [62], Ambrosio, De Lellis and Mantegazza [10], DeSimone, Kohn, Müller and Otto [40], Rivière and Serfaty [74], Alouges, Rivière and Serfaty [8], Jabin, Otto and Perthame [60]. For our model, the idea is to use a duality argument in the spirit of

[39, 41] based on an ε-perturbation of a logarithmically failing Gagliardo-Nirenberg inequality (see Section 8.2). Since the compactness result is a local issue, we state it in the context of the unit ball $B_1 \subset \mathbb{R}^3$ with no imposed boundary conditions:

Theorem 8.4 *Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$. For $k \in \mathbb{N}$, let $m'_k : B'_1 \to S^1$ and $h_k : B_1 \to \mathbb{R}^3$ be related by*

$$\int_{B_1} h_k \cdot \nabla\zeta \, dx = \int_{B'_1} m'_k \cdot \nabla'\zeta \, dx', \forall \zeta \in C_c^\infty(B_1). \tag{8.14}$$

Suppose that

$$\limsup_{k\to\infty} |\ln \varepsilon_k| \left(\varepsilon_k \int_{B'_1} |\nabla' \cdot m'_k|^2 \, dx' + \int_{B_1} |h_k|^2 \, dx \right) < \infty. \tag{8.15}$$

Then $\{m'_k\}_{k\uparrow\infty}$ is relatively compact in $L^1(B'_1)$ and any accumulation point $m' : B'_1 \to \mathbb{R}^2$ satisfies

$$|m'| = 1 \text{ a.e. in } B'_1 \quad \text{and} \quad \nabla' \cdot m' = 0 \text{ distributionally in } B'_1. \tag{8.16}$$

We now focus on the behavior of the finite-energy states m'. As in (8.6), by (8.16), we formally have that $m' = \nabla'^{\perp}\phi$ where ϕ satisfies the eikonal equation $|\nabla'\phi| = 1$. We discuss the case of zero-energy states, i.e., m' is an accumulation point of sequences $\{m'_\varepsilon\}_{\varepsilon\downarrow 0}$ such that the limit in (8.15) vanishes for some stray potentials $\{h_\varepsilon\}$ (in the absence of any boundary condition). The main tool is the principle of characteristics for the eikonal equation. We show that every zero-energy state m' is locally Lipschitz continuous. The difference with respect to the zero-energy states for the Ginzburg-Landau models treated in [60] consists in the avoidance of vortices. Our result can be stated as follows:

Theorem 8.5 *Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$. For $k \in \mathbb{N}$, let $m'_k : B'_1 \to S^1$ and $h_k : B_1 \to \mathbb{R}^3$ be related by* (8.14). *Suppose that*

$$\lim_{k\to\infty} |\ln \varepsilon_k| \left(\varepsilon_k \int_{B'_1} |\nabla' \cdot m'_k|^2 \, dx' + \int_{B_1} |h_k|^2 \, dx \right) = 0. \tag{8.17}$$

Then any accumulation point $m' : B'_1 \to \mathbb{R}^2$ of $\{m'_k\}_{k\uparrow\infty}$ in $L^1(B'_1)$ satisfies

a) m' is locally Lipschitz in B'_1;

b) m' satisfies the principle of characteristics related to (8.16)*, i.e., for any $x'_0 \in B'_1$ we have that*

$$m'(x'_0 + t m'(x'_0)^\perp) = m'(x'_0) \text{ for any } t \in \mathbb{R} \text{ with } x'_0 + t m'(x'_0)^\perp \in B'_1$$

(see Figure 8.9).

Remark In general, a function m' satisfying *a)* and *b)* in Theorem 8.5 is not globally Lipschitz in B'_1; an example is given by

$$m'(x') = \left(\frac{x' - P}{|x' - P|} \right)^\perp \quad \text{for any } x' \in B'_1,$$

for some $P \in \partial B'_1$ (P plays the role of a vortex on the boundary).

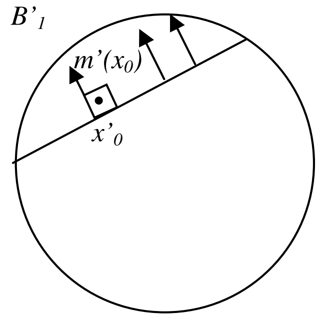


Figure 8.9: Principle of characteristics

The outline of the chapter is as follows. In Section 8.2, we give some fundamental estimates based on a duality argument and a logarithmically failing interpolation inequality. In Section 8.3, we prove Theorem 8.4. In Section 8.4, we focus on the zero-energy states: we establish a list of lemmas that lead to Theorem 8.5. In Section 8.5, we show the optimality of the straight walls in Theorem 8.1 as an application of Theorems 8.4 and 8.5. In Section 8.6 we discuss the behavior of $1d$ magnetizations by proving Theorems 8.2 and 8.3.

8.2 Some fundamental localized estimates

We present some inequalities in the spirit of [39, 41] that are to be used in the next sections. The idea is the following: in order to have the compactness of magnetizations $\{m'_\varepsilon\}$, we need to control in some sense their divergences $\{\sigma_\varepsilon = \nabla' \cdot m'_\varepsilon\}$. Since the second term in the energy corresponds to the homogeneous $H^{-1/2}$-norm of σ_ε, the energy regime $O(\frac{1}{|\ln \varepsilon|})$ induces a bound on σ_ε. By a duality argument, it is enough to study the rate of the failing Gagliardo-Nirenberg type inequality:

$$\int \big| |\nabla|^{1/2}\chi \big|^2 \, dx' \not\lesssim \sup |\chi| \int |\nabla \chi| \, dx'.$$

It is known that this rate is logarithmically slow for an $\varepsilon-$perturbation of the homogeneous $H^{1/2}-$norm. The optimal prefactor of the logarithmical failure is $\frac{2}{\pi}$ and was proved in [39]. This suggests the optimal leading term in the following localized estimates:

Proposition 8.6 *Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be related by*

$$\int_{\mathbb{R}^3} h \cdot \nabla\zeta \, dx = \int_{\mathbb{R}^2} \sigma\zeta \, dx', \quad \forall \zeta \in C_c^\infty(B_1) \tag{8.18}$$

where $x' = (x_1, x_2) \in \mathbb{R}^2$ and $x = (x', x_3) \in \mathbb{R}^3$. Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of locally bounded variation and $\eta \in C_c^\infty(\mathbb{R}^3)$ be such that

$$\operatorname{supp} \eta \subset B_1 \subset \mathbb{R}^3. \tag{8.19}$$

Then there exists a universal constant $C > 0$ such that for all $\varepsilon \in (0, 1]$,

$$\left| \int_{\mathbb{R}^2} \eta^2 \chi \sigma \, dx' \right| \leq \left(\frac{4}{\pi} |\ln \varepsilon| \sup_{\mathbb{R}^2} |\chi| \int_{\mathbb{R}^2} \eta^2 |D'\chi| \int_{\mathbb{R}^3} \eta^2 |h|^2 \, dx \right)^{1/2} \tag{8.20}$$

$$+ C \sup_{\mathbb{R}^3} |\eta| \left(\varepsilon \int_{B_1'} |\sigma|^2 \, dx' + \int_{B_1} |h|^2 \, dx \right)^{1/2} \times \left(\sup_{\mathbb{R}^3} |\eta| + \sup_{\mathbb{R}^3} |\nabla \eta| \right) \left(\sup_{\mathbb{R}^2} |\chi| + \int_{B_1'} |D'\chi| \right), \tag{8.21}$$

where D' denotes the in-plane derivatives (∂_1, ∂_2).

Proof. We introduce some notations:

- C denotes a generic universal constant;
- $\bar{\zeta} : \mathbb{R}^3 \to \mathbb{R}$ denotes the harmonic extension of $\zeta : \mathbb{R}^2 \to \mathbb{R}$, i.e.,

$$\begin{cases} \Delta \bar{\zeta} = 0 & \text{in} \quad \mathbb{R}^3 \setminus (\mathbb{R}^2 \times \{0\}), \\ \bar{\zeta}(\cdot, x_3) = \zeta & \text{on} \quad \mathbb{R}^2; \end{cases}$$

- $\zeta_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ denotes the convolution of $\zeta : \mathbb{R}^2 \to \mathbb{R}$ with a universal kernel ρ_ε of the form

$$\rho_\varepsilon(x') = \frac{1}{\varepsilon^2} \rho_1(\frac{x'}{\varepsilon}) \quad \text{where} \quad \rho_1 \in C_c^\infty(B_1'),\ \rho_1 \geq 0,\ \int_{B_1'} \rho_1(x') \, dx' = 1.$$

It is sufficient to prove the estimate for $\chi \in W^{1,1}_{loc} \cap L^\infty(\mathbb{R}^2)$; in the general case of a function $\chi \in BV_{loc} \cap L^\infty(\mathbb{R}^2)$, it will follow by a density argument, using a sequence $\{\chi_\delta\} \subset W^{1,1}_{loc} \cap L^\infty(\mathbb{R}^2)$ such that $\chi_\delta \to \chi$ a.e. in B_1', $\sup_{\mathbb{R}^2} |\chi_\delta| \leq \sup_{\mathbb{R}^2} |\chi|$ and $\int_{B_1'} |\nabla' \chi_\delta| \, dx' \to \int_{B_1'} |D'\chi|$ (hence, $|D'\chi_\delta| \overset{w^*}{\rightharpoonup} |D'\chi|$ weakly* as measures in B_1').

We rewrite the left-hand side of (8.20) of our estimate as follows:

$$\int_{\mathbb{R}^2} \eta^2 \chi \sigma \, dx' = \int_{\mathbb{R}^2} \eta \sigma \big(\eta \chi - (\eta \chi)_\varepsilon \big) \, dx' + \int_{\mathbb{R}^2} \eta \sigma \overline{(\eta \chi)_\varepsilon} \, dx'$$

and by (8.18) (where $\operatorname{supp} \eta \overline{(\eta \chi)_\varepsilon} \subset B_1$),

$$\int_{\mathbb{R}^2} \eta \sigma \overline{(\eta \chi)_\varepsilon} \, dx' = \int_{\mathbb{R}^3} h \cdot \nabla \big(\eta \overline{(\eta \chi)_\varepsilon} \big) \, dx = \int_{\mathbb{R}^3} \overline{(\eta \chi)_\varepsilon} h \cdot \nabla \eta \, dx + \int_{\mathbb{R}^3} \eta h \cdot \nabla \overline{(\eta \chi)_\varepsilon} \, dx.$$

Hence, we obtain the estimate

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^2} \eta^2\chi\sigma\,dx'\Big| &\le \Big(\int_{\mathbb{R}^2} \eta^2\sigma^2\,dx'\Big)^{1/2}\Big(\int_{\mathbb{R}^2} \big|\eta\chi-(\eta\chi)_\varepsilon\big|^2\,dx'\Big)^{1/2} \\
&\quad + \sup_{\mathbb{R}^3}|\overline{(\eta\chi)_\varepsilon}|\int_{\mathbb{R}^3}|h|\,|\nabla\eta|\,dx + \Big(\int_{\mathbb{R}^3}\eta^2|h|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^3}\big|\nabla\overline{(\eta\chi)_\varepsilon}\big|^2\,dx\Big)^{1/2} \\
&\overset{(8.19)}{\le} \sup_{\mathbb{R}^3}|\eta|\Big(\int_{B_1'}\sigma^2\,dx'\Big)^{1/2}\Big(\int_{\mathbb{R}^2}\big|\eta\chi-(\eta\chi)_\varepsilon\big|^2\,dx'\Big)^{1/2} \qquad (8.22)\\
&\quad + C\sup_{\mathbb{R}^3}|\nabla\eta|\cdot\sup_{\mathbb{R}^3}|\overline{(\eta\chi)_\varepsilon}|\Big(\int_{B_1}|h|^2\,dx\Big)^{1/2} \qquad (8.23)\\
&\quad + \Big(\int_{\mathbb{R}^3}\eta^2|h|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^3}\big|\nabla\overline{(\eta\chi)_\varepsilon}\big|^2\,dx\Big)^{1/2}. \qquad (8.24)
\end{aligned}
$$

As we shall see, only the term (8.24) contributes to the leading order term (8.20). We first address (8.22) and (8.23). For (8.23), we observe that by the maximum principle,

$$
\sup_{\mathbb{R}^3}|\overline{(\eta\chi)_\varepsilon}| \le \sup_{\mathbb{R}^2}|(\eta\chi)_\varepsilon| \le \sup_{\mathbb{R}^2}|\eta\chi| \le \sup_{\mathbb{R}^3}|\eta|\cdot\sup_{\mathbb{R}^2}|\chi|,
$$

so that (8.23) can indeed be absorbed into (8.21). For (8.22), we have

$$
\begin{aligned}
\int_{\mathbb{R}^2}\big|\eta\chi-(\eta\chi)_\varepsilon\big|^2\,dx' &\le (\sup_{\mathbb{R}^2}|(\eta\chi)_\varepsilon| + \sup_{\mathbb{R}^2}|\eta\chi|)\int_{\mathbb{R}^2}\big|\eta\chi-(\eta\chi)_\varepsilon\big|\,dx' \\
&\le 2\varepsilon\sup_{\mathbb{R}^2}|\eta\chi|\int_{\mathbb{R}^2}\big|\nabla'(\eta\chi)\big|\,dx' \\
&\le 2\varepsilon\sup_{\mathbb{R}^2}|\eta\chi|\int_{\mathbb{R}^2}\big(|\eta||\nabla'\chi| + |\chi||\nabla'\eta|\big)\,dx' \\
&\overset{(8.19)}{\le} C\varepsilon\sup_{\mathbb{R}^3}|\eta|\cdot\sup_{\mathbb{R}^2}|\chi| \\
&\quad\times\Big(\sup_{\mathbb{R}^3}|\eta|\int_{B_1'}|\nabla'\chi|\,dx' + \sup_{\mathbb{R}^3}|\nabla\eta|\cdot\sup_{\mathbb{R}^2}|\chi|\Big).
\end{aligned}
$$

Hence, (8.22) can be absorbed into (8.21). We now turn to (8.24). In order to have the desired inequality, it is sufficient to prove that

$$
\begin{aligned}
\int_{\mathbb{R}^3}\big|\nabla\overline{(\eta\chi)_\varepsilon}\big|^2\,dx &\le \frac{4}{\pi}|\ln\varepsilon|\sup_{\mathbb{R}^2}|\chi|\int_{\mathbb{R}^2}\eta^2|\nabla'\chi|\,dx' \qquad (8.25)\\
&\quad + C(\sup_{\mathbb{R}^3}|\eta| + \sup_{\mathbb{R}^3}|\nabla\eta|)^2\Big(\sup_{\mathbb{R}^2}|\chi| + \int_{B_1'}|\nabla'\chi|\,dx'\Big)^2. \qquad (8.26)
\end{aligned}
$$

We appeal to the following identity

$$
\int_{\mathbb{R}^3}|\nabla\bar{\phi}|^2\,dx = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|\phi(x'+z')-\phi(x')|^2\,dx'\,dz', \qquad (8.27)
$$

which we apply to $\phi = (\eta\chi)_\varepsilon$. Actually, (8.27) is easy to establish (see also [39]): first of all, by homogeneity and isotropy, it results that for every $\xi' \in \mathbb{R}^2$,

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|z'|^3}|1-e^{i\xi'\cdot z'}|^2\,dz' &= \frac{|\xi'|}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|\tilde z'|^3}|1-e^{i\tilde z_1}|^2\,d\tilde z'\\ &= \frac{2|\xi'|}{\pi}\int_0^{2\pi}\left(\int_0^\infty \frac{1}{r^2}\sin^2\big(\frac{r|\cos\theta|}{2}\big)\,dr\right)d\theta\\ &= \frac{|\xi'|}{\pi}\int_0^{2\pi}\int_0^\infty \frac{|\cos\theta|}{s^2}\sin^2 s\,ds\,d\theta\\ &= \frac{|\xi'|}{\pi}\int_0^{2\pi}|\cos\theta|\,d\theta\int_0^\infty\frac{\sin^2 s}{s^2}\,ds = 2|\xi'|. \end{aligned} \tag{8.28}$$

Then, it turns out in terms of the Fourier transform,

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla\bar\phi|^2\,dx &= \int_{\mathbb{R}^3}|\xi|^2\,|\mathcal{F}(\bar\phi)(\xi)|^2\,d\xi\\ &= 2\int_{\mathbb{R}^2}|\xi'|\,|\mathcal{F}'(\phi)(\xi')|^2\,d\xi'\\ &\overset{(8.28)}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|1-e^{i\xi'\cdot z'}|^2|\mathcal{F}'(\phi)(\xi')|^2\,d\xi'\,dz'\\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|\phi(x'+z')-\phi(x')|^2\,dx'\,dz',\end{aligned}$$

i.e., (8.27) holds. We split the $z'-$integral on the right-hand side of (8.27) into three different regions:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz'$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_1'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz' \tag{8.29}$$

$$+ \frac{1}{2\pi}\int_{B_\varepsilon'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz' \tag{8.30}$$

$$+ \frac{1}{2\pi}\int_{B_1'\setminus B_\varepsilon'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz'. \tag{8.31}$$

As we shall see, only (8.31) contributes to the leading order term (8.25). We first address (8.29) and (8.30). We start with the term (8.29) corresponding to the long wave length (i.e., $|z'|\geq 1$). Since

$$\begin{aligned}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx' &\leq 2\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon|^2\,dx'\\ &\leq 2\int_{\mathbb{R}^2}|\eta\chi|^2\,dx' \overset{(8.19)}{\leq} C\sup_{\mathbb{R}^3}|\eta|^2\cdot\sup_{\mathbb{R}^2}|\chi|^2,\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_1'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz'\\ &\qquad\leq C\sup_{\mathbb{R}^3}|\eta|^2\cdot\sup_{\mathbb{R}^2}|\chi|^2\int_{\mathbb{R}^2\setminus B_1'}\frac{1}{|z'|^3}\,dz' \leq C\sup_{\mathbb{R}^3}|\eta|^2\cdot\sup_{\mathbb{R}^2}|\chi|^2,\end{aligned} \tag{8.32}$$

i.e., (8.29) is absorbed by (8.26). We now tackle the short wave length term (8.30). We have

$$\begin{aligned}\int_{\mathbb{R}^2} |(\eta\chi)_\varepsilon(x'+z') - (\eta\chi)_\varepsilon(x')|^2\,dx' &\le |z'|^2 \int_{\mathbb{R}^2} |\nabla'(\eta\chi)_\varepsilon|^2\,dx' \\ &\le |z'|^2 \sup_{\mathbb{R}^2} |\nabla'(\eta\chi)_\varepsilon| \int_{\mathbb{R}^2} |\nabla'(\eta\chi)_\varepsilon|\,dx' \\ &\le \frac{C}{\varepsilon}|z'|^2 \sup_{\mathbb{R}^2}|\eta\chi| \int_{\mathbb{R}^2} |\nabla'(\eta\chi)|\,dx'\end{aligned}$$

and thus,

$$\begin{aligned}&\frac{1}{2\pi}\int_{B'_\varepsilon} \frac{1}{|z'|^3}\int_{\mathbb{R}^2} |(\eta\chi)_\varepsilon(x'+z') - (\eta\chi)_\varepsilon(x')|^2\,dx'\,dz' \\ &\qquad \overset{(8.19)}{\le} C(\sup_{\mathbb{R}^3}|\eta| + \sup_{\mathbb{R}^3}|\nabla\eta|)^2 \Big(\sup_{\mathbb{R}^2}|\chi| + \int_{B'_1} |\nabla'\chi|\,dx'\Big)^2 \cdot \frac{1}{\varepsilon}\int_{B'_\varepsilon} \frac{dz'}{|z'|},\end{aligned} \tag{8.33}$$

i.e., (8.30) can also be absorbed by (8.26).

We finally address the medium wave length term (8.31). We start by observing that

$$\int_{\mathbb{R}^2} |(\eta\chi)_\varepsilon(x'+z') - (\eta\chi)_\varepsilon(x')|^2\,dx' \le \int_{\mathbb{R}^2} |(\eta\chi)(x'+z') - (\eta\chi)(x')|^2\,dx'.$$

We consider the integrand, which we shall rewrite in form of

$$\begin{aligned}|(\eta\chi)(x'+z') - (\eta\chi)(x')|^2 = (\chi(x'+z') - \chi(x'))\int_0^1 \eta^2(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \\ + \textit{remainder}.\end{aligned}$$

To do so, we proceed as follows

$$\begin{aligned}(\eta\chi)(x'+z') - (\eta\chi)(x') &= \int_0^1 \nabla'(\eta\chi)(x'+tz')\cdot z'\,dt \\ &= \int_0^1 \eta(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt + \int_0^1 \chi(x'+tz')\nabla'\eta(x'+tz')\cdot z'\,dt,\end{aligned}$$

and thus,

$$\begin{aligned}|(\eta\chi)(x'+z') - (\eta\chi)(x')|^2 &= (\chi(x'+z') - \chi(x'))\int_0^1 \eta^2(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \\ &\quad + \chi(x'+z')\int_0^1 (\eta(x'+z') - \eta(x'+tz'))\eta(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \\ &\quad - \chi(x')\int_0^1 (\eta(x') - \eta(x'+tz'))\eta(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \\ &\quad + \int_0^1 \eta(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \int_0^1 \chi(x'+tz')\nabla'\eta(x'+tz')\cdot z'\,dt \\ &\quad + \Big(\int_0^1 \chi(x'+tz')\nabla'\eta(x'+tz')\cdot z'\,dt\Big)^2.\end{aligned}$$

This yields the estimate

$$
\begin{aligned}
|(\eta\chi)(x'+z') - (\eta\chi)(x')|^2 \le 2\sup_{\mathbb{R}^2}|\chi| \int_0^1 \eta^2(x'+tz')|\nabla'\chi(x'+tz')\cdot z'|\,dt \\
+ 3|z'|^2 \sup_{\mathbb{R}^2}|\chi|\cdot\sup_{\mathbb{R}^3}|\nabla\eta| \int_0^1 |\eta(x'+tz')||\nabla'\chi(x'+tz')|\,dt \\
+ |z'|^2 \sup_{\mathbb{R}^2}|\chi|^2\cdot\sup_{\mathbb{R}^3}|\nabla\eta| \int_0^1 |\nabla'\eta(x'+tz')|\,dt.
\end{aligned}
$$

Integration in x' gives

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |(\eta\chi)(x'+z') - (\eta\chi)(x')|^2\,dx' \\
&\le 2\sup_{\mathbb{R}^2}|\chi| \int_{\mathbb{R}^2} \eta^2|\nabla'\chi\cdot z'|\,dx' + 3|z'|^2 \sup_{\mathbb{R}^2}|\chi|\cdot\sup_{\mathbb{R}^3}|\nabla\eta| \int_{\mathbb{R}^2}|\eta||\nabla'\chi|\,dx' \\
&\quad + |z'|^2 \sup_{\mathbb{R}^2}|\chi|^2\cdot\sup_{\mathbb{R}^3}|\nabla\eta| \int_{\mathbb{R}^2}|\nabla'\eta|\,dx' \\
&\le 2\sup_{\mathbb{R}^2}|\chi| \int_{\mathbb{R}^2} \eta^2|\nabla'\chi\cdot z'|\,dx' \\
&\quad + C|z'|^2(\sup_{\mathbb{R}^3}|\eta| + \sup_{\mathbb{R}^3}|\nabla\eta|)^2\left(\sup_{\mathbb{R}^2}|\chi| + \int_{B_1'}|\nabla'\chi|\,dx'\right)^2.
\end{aligned}
$$

Integration in z' yields

$$
\begin{aligned}
&\int_{B_1'\setminus B_\varepsilon'} \frac{1}{|z'|^3}\int_{\mathbb{R}^2} |(\eta\chi)(x'+z') - (\eta\chi)(x')|^2\,dx'\,dz' \\
&\le 2\sup_{\mathbb{R}^2}|\chi| \int_{\mathbb{R}^2}\eta^2(x')\int_{B_1'\setminus B_\varepsilon'}\frac{1}{|z'|^3}|\nabla'\chi(x')\cdot z'|\,dz'\,dx' \\
&\quad + C(\sup_{\mathbb{R}^3}|\eta| + \sup_{\mathbb{R}^3}|\nabla\eta|)^2\left(\sup_{\mathbb{R}^2}|\chi| + \int_{B_1'}|\nabla'\chi|\,dx'\right)^2 \int_{B_1'\setminus B_\varepsilon'}\frac{dz'}{|z'|}.
\end{aligned}
\tag{8.34}
$$

Notice that for any $v' \in \mathbb{R}^2$,

$$
\begin{aligned}
\int_{B_1'\setminus B_\varepsilon'} \frac{1}{|z'|^3}|v'\cdot z'|\,dz' &= \int_0^{2\pi}\int_\varepsilon^1 \frac{1}{r^3}\left|v'\cdot\begin{pmatrix} r\cos\theta \\ r\sin\theta\end{pmatrix}\right| r\,dr\,d\theta \\
&= |v'|\int_0^{2\pi}|\cos\theta|\,d\theta\int_\varepsilon^1\frac{1}{r}\,dr = 4|\ln\varepsilon|\,|v'|.
\end{aligned}
$$

Hence (8.34) turns into

$$
\begin{aligned}
\frac{1}{2\pi}\int_{B_1'\setminus B_\varepsilon'} \frac{1}{|z'|^3}\int_{\mathbb{R}^2} |(\eta\chi)(x'+z') - (\eta\chi)(x')|^2\,dx'\,dz' \le \frac{4}{\pi}|\ln\varepsilon|\sup_{\mathbb{R}^2}|\chi|\int_{\mathbb{R}^2}\eta^2|\nabla'\chi|\,dx' \\
+ C(\sup_{\mathbb{R}^3}|\eta| + \sup_{\mathbb{R}^3}|\nabla\eta|)^2\left(\sup_{\mathbb{R}^2}|\chi| + \int_{B_1'}|\nabla'\chi|\,dx'\right)^2.
\end{aligned}
\tag{8.35}
$$

Combining identity (8.27) with the estimates (8.32), (8.33) and (8.35), we conclude that (8.25) holds. □

By rescaling length in Proposition 8.6 from unity to some $R > 0$, we obtain:

Corollary 8.7 *Let $R > 0$ and $x_0 = (x'_0, 0) \in \mathbb{R}^2 \times \{0\}$. Consider $h : \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be related by*

$$\int_{\mathbb{R}^3} h \cdot \nabla\zeta \, dx = \int_{\mathbb{R}^2} \sigma\zeta \, dx', \quad \forall \zeta \in C_c^\infty(B(x_0, R)).$$

Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of locally bounded variation and $\eta \in C_c^\infty(\mathbb{R}^3)$ be such that

$$\operatorname{supp} \eta \subset B(x_0, R) \subset \mathbb{R}^3. \tag{8.36}$$

Then there exists a universal constant $C > 0$ such that for all $\varepsilon \in (0, R]$,

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \eta^2 \chi\sigma \, dx' \right| \leq & \left(\frac{4}{\pi} |\ln \varepsilon| \sup_{\mathbb{R}^2} |\chi| \int_{\mathbb{R}^2} \eta^2 |D'\chi| \int_{\mathbb{R}^3} \eta^2 |h|^2 \, dx \right)^{1/2} \\ & + C(1 + \sqrt{|\ln R|}) \sup_{\mathbb{R}^3} |\eta| \left(\varepsilon \int_{B'(x'_0, R)} |\sigma|^2 \, dx' + \int_{B(x_0,R)} |h|^2 \, dx \right)^{1/2} \\ & \times \left(\sup_{\mathbb{R}^3} |\eta| + R \sup_{\mathbb{R}^3} |\nabla\eta| \right) \left(\sqrt{R} \sup_{\mathbb{R}^2} |\chi| + \frac{1}{\sqrt{R}} \int_{B'(x'_0,R)} |D'\chi| \right). \end{aligned} \tag{8.37}$$

Proof. The change of variables $x = R\hat{x} + x_0$ (and $\varepsilon = R\hat{\varepsilon}$) preserves (8.18) and turns (8.36) into $\operatorname{supp} \eta \subset \hat{B}_1$, so that we may apply Proposition 8.6. It yields in the original variables:

$$\begin{aligned} \left| R^{-2} \int_{\mathbb{R}^2} \eta^2 \chi\sigma \, dx' \right| \leq & \left(\frac{4}{\pi} |\ln \frac{\varepsilon}{R}| \sup_{\mathbb{R}^2} |\chi| \, R^{-1} \int_{\mathbb{R}^2} \eta^2 |D'\chi| \, R^{-3} \int_{\mathbb{R}^3} \eta^2 |h|^2 \, dx \right)^{1/2} \\ & + C \sup_{\mathbb{R}^3} |\eta| \left(\frac{\varepsilon}{R} R^{-2} \int_{B'(x'_0,R)} |\sigma|^2 \, dx' + R^{-3} \int_{B(x_0,R)} |h|^2 \, dx \right)^{1/2} \\ & \times (\sup_{\mathbb{R}^3} |\eta| + R \sup_{\mathbb{R}^3} |\nabla\eta|) \left(\sup_{\mathbb{R}^2} |\chi| + R^{-1} \int_{B'(x'_0,R)} |D'\chi| \right), \end{aligned}$$

that is,

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \eta^2 \chi\sigma \, dx' \right| \leq & \left(\frac{4}{\pi} (|\ln \varepsilon| + |\ln R|) \sup_{\mathbb{R}^2} |\chi| \int_{\mathbb{R}^2} \eta^2 |D'\chi| \int_{\mathbb{R}^3} \eta^2 |h|^2 \, dx \right)^{1/2} \\ & + C \sup_{\mathbb{R}^3} |\eta| \left(\varepsilon \int_{B'(x'_0,R)} |\sigma|^2 \, dx' + \int_{B(x_0,R)} |h|^2 \, dx \right)^{1/2} \\ & \times (\sup_{\mathbb{R}^3} |\eta| + R \sup_{\mathbb{R}^3} |\nabla\eta|) \left(R^{1/2} \sup_{\mathbb{R}^2} |\chi| + R^{-1/2} \int_{B'(x'_0,R)} |D'\chi| \right). \end{aligned}$$

The conclusion is now straightforward. □

If one drops the test function η and localizes the function χ in Corollary 8.7, the following result comes out:

Corollary 8.8 *Let $d, R > 0$ and $x_0 = (x'_0, 0) \in \mathbb{R}^2 \times \{0\}$. Consider $h : \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be related by*

$$\int_{\mathbb{R}^3} h \cdot \nabla\zeta \, dx = \int_{\mathbb{R}^2} \sigma\zeta \, dx', \quad \forall \zeta \in C_c^\infty(B(x_0, R + d)).$$

Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of bounded variation such that

$$\operatorname{supp}\chi \subseteq \bar{B}'(x_0', R) \subset \mathbb{R}^2.$$

Then there exists a universal constant $C > 0$ such that for all $\varepsilon \in (0, R+d]$,

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \chi\sigma\, dx' \right| \le & \left(\frac{4}{\pi} |\ln\varepsilon| \sup_{\mathbb{R}^2} |\chi| \int_{\mathbb{R}^2} |D'\chi| \int_{B(x_0,R+d)} |h|^2\, dx \right)^{1/2} \\ & + C\frac{(1+R+d)^2}{d} \left(\varepsilon \int_{B'(x_0',R+d)} |\sigma|^2\, dx' + \int_{B(x_0,R+d)} |h|^2\, dx \right)^{1/2} \\ & \times \left(\sup_{\mathbb{R}^2} |\chi| + \int_{\mathbb{R}^2} |D'\chi| \right). \end{aligned}$$

Proof. Let $\eta \in C_c^\infty(B(x_0, R+d))$ be such that

$$\eta = 1 \text{ in } B'(x_0', R) \times \{0\},\ |\eta| \le 1 \text{ and } |\nabla\eta| \le \frac{C}{d} \text{ in } B(x_0, R+d). \tag{8.38}$$

We apply Corollary 8.7:

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \eta^2 \chi\sigma dx' \right| \overset{(8.38)}{\le} & \left(\frac{4}{\pi} |\ln\varepsilon| \sup_{\mathbb{R}^2} |\chi| \int_{\mathbb{R}^2} |D'\chi| \int_{B(x_0,R+d)} |h|^2\, dx \right)^{1/2} \\ & + C(1+\sqrt{|\ln(R+d)|}) \left(\varepsilon \int_{B'(x_0',R+d)} |\sigma|^2\, dx' + \int_{B(x_0,R+d)} |h|^2\, dx \right)^{1/2} \\ & \times (1 + \frac{R+d}{d}) \left(\sqrt{R+d} \sup_{\mathbb{R}^2} |\chi| + \frac{1}{\sqrt{R+d}} \int_{\mathbb{R}^2} |D'\chi| \right), \end{aligned}$$

and the conclusion is straightforward. □

A periodic version of Proposition 8.6 is the following:

Corollary 8.9 *Let $L > 0$ be a positive number. Consider $h : \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be related by*

$$\int_{\mathbb{R}^3} h \cdot \nabla\zeta\, dx = \int_{\mathbb{R}^2} \sigma\zeta\, dx', \quad \forall \zeta \in C_c^\infty(\mathbb{R}^3).$$

Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of bounded variation in $\mathbb{R} \times [0, L)$ and $\eta \in C^\infty(\mathbb{R}^3)$ be such that

$$\operatorname{supp}\eta \subset (-2, 2) \times \mathbb{R} \times (-1, 1). \tag{8.39}$$

Assume that the functions

$$h, \sigma, \chi \text{ and } \eta \text{ are } L-\text{periodic in } x_2. \tag{8.40}$$

Then there exists a universal constant $C > 0$ such that for all $\varepsilon \in (0, L]$,

$$\begin{aligned} \left| \int_{\mathbb{R}\times[0,L)} \eta^2 \chi\sigma\, dx' \right| \le & \left(\frac{4}{\pi} |\ln\varepsilon| \sup_{\mathbb{R}^2} |\chi| \int_{\mathbb{R}\times[0,L)} \eta^2 |D'\chi| \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2 |h|^2\, dx \right)^{1/2} \\ & + C\frac{\tilde{L}^3}{L^2} \sup_{\mathbb{R}^3} |\eta| \left(\varepsilon \int_{\mathbb{R}\times[0,L)} |\sigma|^2\, dx' + \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} |h|^2\, dx \right)^{1/2} \\ & \times (\sup_{\mathbb{R}^3} |\eta| + L \sup_{\mathbb{R}^3} |\nabla\eta|) \left(\sqrt{L} \sup_{\mathbb{R}^2} |\chi| + \frac{1}{\sqrt{L}} \int_{\mathbb{R}\times[0,L)} |D'\chi| \right), \end{aligned} \tag{8.41}$$

where $\tilde{L} = \max\{2, L\}$.

Proof. Select a universal $\zeta \in C_c^\infty(\mathbb{R})$ such that

$$\operatorname{supp}\zeta \subset (-1,1),\ |\zeta| \le 1,\ \sum_{k\in\mathbb{Z}} \zeta^2(x_2+k) = 1,\ \forall x_2 \in \mathbb{R} \tag{8.42}$$

and set

$$\tilde{\eta}(x_1,x_2,x_3) = \zeta(\frac{x_2}{L})\eta(x_1,x_2,x_3),\ \forall (x_1,x_2,x_3) \in \mathbb{R}^3. \tag{8.43}$$

In view of (8.39) and (8.42) we have that

$$\operatorname{supp}\tilde{\eta} \subset B_R$$

for some radius

$$\tilde{L} \le R \le 2\tilde{L}. \tag{8.44}$$

Hence, we may apply (8.37) to σ, h and $\tilde{\eta}$. Notice that because of (8.40) and (8.42),

$$\begin{aligned}
\int_{\mathbb{R}^2} \tilde{\eta}^2 \chi\sigma\, dx' &= \int_{\mathbb{R}\times[0,L)} \eta^2 \chi\sigma\, dx', \\
\int_{\mathbb{R}^3} \tilde{\eta}^2 |h|^2\, dx &= \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2 |h|^2\, dx, \\
\int_{\mathbb{R}^2} \tilde{\eta}^2 |D'\chi| &= \int_{\mathbb{R}\times[0,L)} \eta^2 |D'\chi|.
\end{aligned}$$

Furthermore, we have because of (8.40) and (8.44),

$$\begin{aligned}
\int_{B'_R} |\sigma|^2\, dx' &\le C\frac{\tilde{L}}{L} \int_{\mathbb{R}\times[0,L)} |\sigma|^2\, dx', \\
\int_{B_R} |h|^2\, dx &\le C\frac{\tilde{L}}{L} \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} |h|^2\, dx, \\
\int_{B'_R} |D'\chi| &\le C\frac{\tilde{L}}{L} \int_{\mathbb{R}\times[0,L)} |D'\chi|.
\end{aligned}$$

Finally, it follows from (8.42) and (8.43),

$$\begin{aligned}
\sup_{\mathbb{R}^3} |\tilde{\eta}| &\le \sup_{\mathbb{R}^3} |\eta| \\
\sup_{\mathbb{R}^3} |\nabla\tilde{\eta}| &\le \frac{C}{L} \sup_{\mathbb{R}^3} |\eta| + \sup_{\mathbb{R}^3} |\nabla\eta|.
\end{aligned}$$

Hence, (8.37) yields (8.41). □

Remark: The conclusion of Corollary 8.9 holds true for a more general support of η than (8.39) (for example, $(-a,a)\times\mathbb{R}\times(a,a)$ for every $a>0$). The choice of the interval $(-2,2)$ in (8.39) (as support in x_1 variable) is needed in the proof of Theorem 8.1 due to the choice of the boundary data (8.1).

8.3 Compactness of the Néel wall

This section is devoted to the proof of the compactness result for magnetizations in the energy regime $O(\frac{1}{|\ln\varepsilon|})$:

Proof of Theorem 8.4. Since $|m'_k| = 1$ in B'_1, it results that the sequence $\{\|m'_k\|_{L^\infty(B'_1)}\}$ is bounded and therefore, there exists $m' \in L^\infty(B'_1, \mathbb{R}^2)$ such that up to a subsequence,

$$m'_k \overset{w^*}{\rightharpoonup} m' \quad \text{weakly}^* \text{ in } L^\infty. \tag{8.45}$$

In particular,

$$|m'|^2 \leq 1 \text{ a.e. in } B'_1. \tag{8.46}$$

In order to have the strong convergence in some L^p with $1 \leq p < \infty$, we need to show that $|m'| = 1$ a.e. in B'_1. Indeed, that will imply $\|m'_k\|_{L^2(B'_1)} \to \|m'\|_{L^2(B'_1)}$ and by the weak convergence in L^2, it will lead to the strong convergence in L^2 and then, in any other L^p, $1 \leq p < \infty$. We define the finite positive measures $\{\mu_k\} \subset \mathcal{M}(B'_1)$ as

$$\mu_k(A') = |\ln\varepsilon_k| \int_{B_1\cap(A'\times\mathbb{R})} |h_k|^2\, dx$$

for every Borel set $A' \subset B'_1$. Then by (8.15), the family of positive measures $\{\mu_k\}$ is bounded in $\mathcal{M}(B'_1)$ and hence, there exists a positive measure $\mu \in \mathcal{M}(B'_1)$ such that

$$\mu_k \overset{w^*}{\rightharpoonup} \mu \quad \text{weakly}^* \text{ in } \mathcal{M}(B'_1).$$

Let $x'_0 \in B'_1$ be a Lebesgue point of m' and of vanishing $\mathcal{H}^1$-density of μ, i.e.,

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B'(x'_0,r)} |m'(x') - m'(x'_0)|\, dx' = 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B'(x'_0,r))}{r} = 0 \tag{8.47}$$

(by Lebesgue decomposition theorem and Vitali covering lemma, almost every point in B'_1 has the above properties). We want to show that $|m'(x'_0)| = 1$. As in [39], we identify a curved center line of the transition layer: let X_k be the orbit of the vector field ${m'_k}^\perp$ passing by x'_0 (see Figure 8.10), i.e.,

$$\begin{cases} \dot{X}_k(t) = {m'_k}^\perp(X_k(t)), \\ X_k(0) = x'_0. \end{cases}$$

The orbit X_k does not have cycles and it separates the ball B'_1 into a right side G'_k (where m'_k is the inner normal vector to $\partial G'_k$) and a left side $B'_1 \setminus G'_k$. We define

$$\chi_k = \begin{cases} \frac{1}{2} & \text{in} \quad G'_k, \\ -\frac{1}{2} & \text{in} \quad B'_1 \setminus G'_k. \end{cases} \tag{8.48}$$

Then $\chi_k \in BV_{loc}(B'_1)$ with $D'\chi_k = m'_k\, \mathcal{H}^1 \llcorner X_k$. Moreover, in the ball $B'(x'_0, 1-|x'_0|) \subset B'_1$ we have that for every $r \in (0, 1-|x'_0|)$,

$$\int_{B'(x'_0,r)} |D'\chi_k| = \mathcal{H}^1(\{X_k \in B'(x'_0,r)\}) \geq 2r \tag{8.49}$$

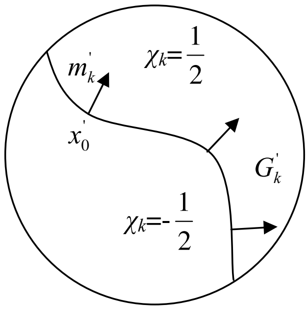


Figure 8.10: The orbit X_k of the vector field $m_k'^{\perp}$ passing by x_0' in the ball B_1'

and the integration by parts yields

$$\int_{\partial B'(x_0',r)} \chi_k^- m_k' \cdot \nu \, d\mathcal{H}^1 = \int_{B'(x_0',r)} \chi_k \, \nabla' \cdot m_k' \, dx' + \int_{B'(x_0',r)} |D'\chi_k| \tag{8.50}$$

where χ_k^- denotes the interior trace of $\chi_k\big|_{\partial B'(x_0',r)}$ and ν is the unit outer normal vector on $\partial B'(x_0', r)$.

We proceed in three steps:

Step 1. *The sequence $\{\chi_k\}$ is uniformly locally bounded in $BV(B_1')$ and any accumulation point χ of $\{\chi_k\}$ in $L^1_{loc}(B_1')$ belongs to $BV_{loc}(B_1', \{-\frac{1}{2}, \frac{1}{2}\})$.* For that, it is enough to prove that $\{\chi_k\}$ is bounded in $BV(B'(x_0', r))$ for any ball $B'(x_0', r+d) \subset B_1'$ where r and d are arbitrary positive numbers and $x_0' \in B_1'$. We apply Corollary 8.8 in the ball $B(x_0, r+d)$ for the restriction of $\chi_k\big|_{B'(x_0',r)}$ where $x_0 = (x_0', 0) \in B_1$:

$$\begin{aligned}\left|\int_{B'(x_0',r)} \chi_k \nabla' \cdot m_k' \, dx'\right| &\overset{(8.48)}{\le} \left(\frac{2}{\pi}|\ln \varepsilon_k| \int_{\bar{B}'(x_0',r)} |D'\chi_k| \int_{B(x_0,r+d)} |h_k|^2 \, dx\right)^{1/2} \\ &\quad + \frac{C}{d}\left(\varepsilon_k \int_{B_1'} |\nabla' \cdot m_k'|^2 \, dx' + \int_{B_1} |h_k|^2 \, dx\right)^{1/2} \\ &\quad \times \left(1 + \int_{\bar{B}'(x_0',r)} |D'\chi_k|\right),\end{aligned}$$

which implies by Young's inequality,

$$\begin{aligned}&\left|\int_{B'(x_0',r)} \chi_k \nabla' \cdot m_k' \, dx'\right| \\ &\quad \overset{(8.15)}{\le} \delta \int_{\bar{B}'(x_0',r)} |D'\chi_k| + \frac{C}{\delta}\mu_k(B'(x_0', r+d)) + \frac{C}{d\sqrt{|\ln \varepsilon_k|}}\left(1 + \int_{\bar{B}'(x_0',r)} |D'\chi_k|\right) \\ &\quad \le (\delta + \frac{C}{d\sqrt{|\ln \varepsilon_k|}}) \int_{\bar{B}'(x_0',r)} |D'\chi_k| + \frac{C}{\delta}\mu_k(B'(x_0', r+d)) + \frac{C}{d\sqrt{|\ln \varepsilon_k|}}\end{aligned} \tag{8.51}$$

for some small $\delta > 0$. Here we wrote

$$\int_{\bar{B}'(x_0',r)} |D'\chi_k| = \int_{B'(x_0',r)} |D'\chi_k| + \int_{\partial B'(x_0',r)} |\chi_k^-| \, d\mathcal{H}^1 \overset{(8.48)}{\le} \int_{B'(x_0',r)} |D'\chi_k| + \pi r. \tag{8.52}$$

By (8.50), (8.51) and (8.52), we deduce that

$$\int_{\partial B'(x_0',r)} \chi_k^- m_k' \cdot \nu \, d\mathcal{H}^1 \geq (1-\delta-\frac{C}{d\sqrt{|\ln\varepsilon_k|}})\int_{B'(x_0',r)} |D'\chi_k| - \frac{C}{\delta}\mu_k(B'(x_0',r+d)) \\ - \frac{C}{d\sqrt{|\ln\varepsilon_k|}}(1+r\pi) - \pi\delta r. \tag{8.53}$$

It results that $\int_{B'(x_0',r)} |D'\chi_k| \leq C_d$ and thus, up to a subsequence, there exists a function $\chi \in BV(B'(x_0',r), \{-\frac{1}{2}, \frac{1}{2}\})$ such that

$$\chi_k \to \chi \quad \text{in} \quad L^1(B'(x_0',r)). \tag{8.54}$$

Step 2. *We show that* $|m'(x_0')| = 1$. We restrict the analysis in a ball $B'(x_0', 2R) \subset B_1'$. To this purpose, we apply (8.53) for every ball $B'(x_0',r)$ where $r \in (0,R)$ and $d = R$: by (8.49), we deduce that

$$\int_{\partial B'(x_0',r)} \chi_k^- m_k' \cdot \nu \, d\mathcal{H}^1 + \frac{C}{\delta}\mu_k(B'(x_0',2R)) + \frac{C}{R\sqrt{|\ln\varepsilon_k|}}(1+r\pi) + \pi\delta r \geq 2(1-\delta-\frac{C}{R\sqrt{|\ln\varepsilon_k|}})r.$$

Integrating for $r \in (0,R)$ and dividing by R^2, it leads to

$$\frac{1}{R^2}\int_{B'(x_0',R)} \chi_k^- m_k' \cdot \nu \, dx' + \frac{C}{\delta}\frac{\mu_k(B'(x_0',2R))}{R} + \frac{C}{R^2\sqrt{|\ln\varepsilon_k|}}(1+R\pi) + \pi\delta \geq 1-\delta-\frac{C}{R\sqrt{|\ln\varepsilon_k|}}.$$

We know that $\chi_k^- = \chi_k$ $\mathcal{L}^2-$a.e. in $B'(x_0',R)$ and for almost every $R>0$, $\mu(\partial B'(x_0',R)) = 0$. Passing to the limit as $k \to \infty$, it follows from (8.45) and (8.54) that

$$\frac{1}{R^2}\int_{B'(x_0',R)} \chi m' \cdot \nu \, dx' + \frac{C}{\delta}\frac{\mu(B'(x_0',2R))}{R} \geq 1-(1+\pi)\delta, \tag{8.55}$$

for a.e. $R \in (0, \frac{1-|x_0'|}{2})$. By (8.48) we notice that

$$\frac{1}{R^2}\int_{B'(x_0',R)} \chi m' \cdot \nu \, dx' \leq \frac{1}{2R^2}\int_{B'(x_0',R)} |m'(x') - m'(x_0')| \, dx' + |m'(x_0')| \, \big|\frac{1}{R^2}\int_{B'(x_0',R)} \chi\nu \, dx'\big|$$

and we estimate the modulus of the vector $w = \frac{1}{R^2}\int_{B'(x_0',R)} \chi\nu$ as

$$|w| = \frac{1}{R^2}\int_{B'(x_0',R)} \chi\Big(\nu \cdot \frac{w}{|w|}\Big) \, dx' \leq \frac{1}{2R^2}\Big(\int_{B_R'} (x_1)^+ + \int_{B_R'} (x_1)^-\Big) = 1.$$

Letting now $R \to 0$, we conclude by (8.47) and (8.55) that $|m'(x_0')| \geq 1-(1+\pi)\delta$ for every small δ, and hence, by (8.46), $|m'(x_0')| = 1$.

Step 3.*End of proof.* Let now m' be an accumulation point of the sequence $\{m_k'\}$. Since $|m_k'| = 1$, we deduce that $|m'| = 1$ a.e. in B_1'. By (8.15), we have that $\int_{B_1} |h_k|^2 \, dx \to 0$ as $k \to \infty$ and therefore, (8.14) yields that

$$\lim_{k\to\infty}\int_{B_1'} \zeta \, \nabla' \cdot m_k' \, dx' = 0, \forall \zeta \in C_c^\infty(B_1').$$

Thus, $\nabla' \cdot m' = 0$ distributionally in B_1'. □

8.4 Zero-energy states

In order to prove Theorem 8.5, we proceed in several steps. A key ingredient to Theorem 8.5 is the following additional property of limits m':

Lemma 8.10 *Next to* (8.16), *any accumulation point* $m' : B_1' \to \mathbb{R}^2$ *of* $\{m_k'\}_{k\uparrow\infty}$ *in* $L^1(B_1')$ *has the following property: for all* $x_0' \in B_1'$ *there exists* $\chi : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ *such that*

$$\nabla' \cdot (\chi m') = |D'\chi| \quad \textit{distributionally in} \;\; B_1', \tag{8.56}$$

$$\int_{B'(x_0',r)} |D'\chi| \geq 2r, \quad \textit{for all} \;\; 0 < r < 1 - |x_0'|. \tag{8.57}$$

Proof of Lemma 8.10. Let $x_0' \in B_1'$ be given. Let $\{\chi_k\}$ be defined in B_1' as in the proof of Theorem 8.4 (see (8.48)). By Step 1 in the proof of Theorem 8.4, we know that the sequence

$$\left\{ \int_{B_r'} |D'\chi_k| \right\}_{k\uparrow\infty} \text{is bounded for all } 0 < r < 1. \tag{8.58}$$

Hence, after passage to a subsequence, we may assume that there exists $\chi : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ of locally bounded variation such that

$$\chi_k \to \chi \text{ in } L^1(B_1'). \tag{8.59}$$

It remains to argue that χ satisfies (8.56) and (8.57). For a given $\zeta \in C_c^\infty(B_1')$, we shall establish the following four statements:

$$-\int_{B_1'} \chi_k \nabla'\zeta \cdot m_k' \, dx' - \int_{B_1'} \zeta |D'\chi_k| \to 0, \tag{8.60}$$

$$-\int_{B_1'} \chi \nabla'\zeta \cdot m' \, dx' - \int_{B_1'} \zeta |D'\chi| \geq 0 \quad \text{if } \zeta \geq 0, \tag{8.61}$$

$$-\int_{B_1'} \chi \nabla'\zeta \cdot m' \, dx' - \int_{B_1'} \zeta |D'\chi| \leq 0 \quad \text{if } \zeta \geq 0, \tag{8.62}$$

$$\int_{B_1'} \zeta |D'\chi_k| \to \int_{B_1'} \zeta |D'\chi| \quad \text{if } \zeta \geq 0. \tag{8.63}$$

In order to establish (8.60), we will use again the identity based on the construction of χ_k, i.e., $m_k' \cdot D'\chi_k = |D'\chi_k|$; namely,

$$\begin{aligned} -\int_{B_1'} \nabla'\zeta \cdot m_k' \chi_k \, dx' - \int_{B_1'} \zeta |D'\chi_k| &= \int_{B_1'} \zeta \chi_k \, \nabla' \cdot m_k' \, dx' + \int_{B_1'} \zeta m_k' \cdot D'\chi_k - \int_{B_1'} \zeta |D'\chi_k| \\ &= \int_{B_1'} \zeta \chi_k \, \nabla' \cdot m_k' \, dx'. \end{aligned} \tag{8.64}$$

The second ingredient is Corollary 8.8, applied for the function $\zeta\chi_k$ in the ball B_1' and $d = \operatorname{dist}(\operatorname{supp}\zeta, \partial B_1) > 0$. Because of $\sup|\chi_k| = \frac{1}{2}$, we obtain

$$\left|\int_{\mathbb{R}^2}(\zeta\chi_k)\nabla'\cdot m_k'\,dx'\right| \le \left(\frac{2}{\pi}|\ln\varepsilon_k|\sup|\zeta|\int_{B_1'}|D'(\zeta\chi_k)|\int_{B_1}|h_k|^2\,dx\right)^{1/2} + \frac{C}{d}\left(\varepsilon_k\int_{B_1'}|\nabla'\cdot m_k'|^2\,dx' + \int_{B_1}|h_k|^2\,dx\right)^{1/2} \times\left(1+\int_{B_1'}|D'(\zeta\chi_k)|\right).$$

Since $|D'(\zeta\chi_k)| \le \frac{1}{2}|\nabla'\zeta| + |\zeta||D'\chi_k|$, by (8.58) we deduce that the sequence $\left\{\int_{B_1'}|D'(\zeta\chi_k)|\right\}$ is bounded and by (8.17), it follows that

$$\int_{B_1'}\zeta\chi_k\,\nabla'\cdot m_k'\,dx' \to 0 \text{ as } k\to\infty. \tag{8.65}$$

Now (8.60) follows from (8.64) and (8.65). Statement (8.61) follows easily from (8.60). Indeed, because of (8.59) and $m_k' \to m'$ in $L^1(B_1')$, we have

$$\int_{B_1'}\chi_k\nabla'\zeta\cdot m_k'\,dx' \to \int_{B_1'}\chi\nabla'\zeta\cdot m'\,dx'; \tag{8.66}$$

on the other hand, the lower semicontinuity of $|D'\chi_k|$ under (8.59) implies

$$\int_{B_1'}\zeta|D'\chi| \le \liminf_{k\to\infty}\int_{B_1'}\zeta|D'\chi_k| \quad \text{if } \zeta\ge 0 \text{ in } B_1'.$$

Statement (8.62) is a general fact which follows from (8.16). Indeed, let $\{m_\delta'\}_{\delta\downarrow 0}$ denote the mollification of m' by convolution. For any $r<1$ and sufficiently small δ, we then have in a classical sense:

$$\nabla'\cdot m_\delta' = 0 \text{ and } |m_\delta'|^2 \le 1 \text{ in } B_r'. \tag{8.67}$$

Therefore,

$$\int_{B_1'}\chi\nabla'\zeta\cdot m_\delta'\,dx' \overset{(8.67)}{=} \int_{B_1'}\chi\nabla'\cdot(\zeta m_\delta')\,dx' = -\int_{B_1'}\zeta m_\delta'\cdot D'\chi \overset{(8.67)}{\le} \int_{B_1'}\zeta|D'\chi| \quad \text{if } \zeta\ge 0.$$

Statement (8.63) is a straightforward consequence of the previous ones:

$$\lim_{k\to\infty}\int_{B_1'}\zeta|D'\chi_k| \overset{(8.60)}{=} -\lim_{k\to\infty}\int_{B_1'}\chi_k\nabla'\zeta\cdot m_k'\,dx' \overset{(8.66)}{=} -\int_{B_1'}\chi\nabla'\zeta\cdot m'\,dx' \overset{(8.61),(8.62)}{=} \int_{B_1'}\zeta|D'\chi|,$$

if $\zeta \ge 0$.

We now argue that (8.56) and (8.57) are true. We start with (8.56). From (8.61) and (8.62), we already know that

$$-\int_{B_1'}\chi\nabla'\zeta\cdot m'\,dx' = \int_{B_1'}\zeta|D'\chi| \quad \text{for all } \zeta\in C_c^\infty(B_1') \text{ with } \zeta\ge 0. \tag{8.68}$$

Since any $\zeta \in C_c^\infty(B_1')$ can be approximated both in $H^1(B_1')$ and $C_c(B_1')$ by ζ_δ's of the form

$$\zeta_\delta = \zeta_\delta^+ - \zeta_\delta^- \text{ with } \zeta_\delta^+, \zeta_\delta^- \in C_c^\infty(B_1'), \tag{8.69}$$

(8.68) implies (8.56). An approximation of the form (8.69) can be constructed as follows

$$\zeta_\delta = \phi_\delta(\zeta),$$

where $\{\phi_\delta\}_{\delta\downarrow 0} \subset C^\infty(\mathbb{R})$ is an approximation of the identity with the following properties:

$$\phi_\delta(t) = 0 \text{ for } |t| \leq \delta, \ \frac{d\phi_\delta}{dt}(t) \to 1 \text{ for } t \neq 0, \ \left|\frac{d\phi_\delta}{dt}(t)\right| \leq 1 \text{ for all } t.$$

We now address (8.57). Let $0 < r < 1 - |x_0'|$ be given. We will derive (8.57) from the corresponding property of χ_k (see (8.49)) and (8.63). Let $\{\eta_\delta\}_{\delta\downarrow 0} \subset C_c^\infty(B_1')$ be an approximation of the characteristic function $1_{B'(x_0',r)}$ in the following sense

$$\eta_\delta(x') = 0 \text{ for } x' \notin B'(x_0', r),\ \eta_\delta(x') = 1 \text{ for } x' \in B'(x_0', r-\delta),\ 0 \leq \eta_\delta(x') \leq 1 \text{ for } x' \in B_1'.$$

Then

$$\int_{B'(x_0',r)} |D'\chi| \geq \int_{B_1'} \eta_\delta |D'\chi| \overset{(8.63)}{=} \lim_{k\to\infty} \int_{B_1'} \eta_\delta |D'\chi_k| \geq \liminf_{k\to\infty} \int_{B'(x_0',r-\delta)} |D'\chi_k| \overset{(8.49)}{\geq} 2(r-\delta).$$

□

The next lemma establishes that the χ's from Lemma 8.10 are minimal (perimeter minimizing). It is a well-known general fact that sets whose normal can be extended to a divergence-free unit-length vector field are minimal.

Lemma 8.11 *Let $\chi : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ have the property* (8.56) *for some $m' : B_1' \to S^1$ with*

$$\nabla' \cdot m' = 0 \quad \textit{distributionally in } B_1'.$$

Then χ is minimal in B_1' in the sense that for any function $\tilde{\chi} : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ with $\operatorname{supp}(\tilde{\chi} - \chi) \subset\subset B_1'$*, we have*

$$|D'\chi|(B_1') \leq |D'\tilde{\chi}|(B_1').$$

Proof of Lemma 8.11 . Let $0 < r < 1$ be such that $\operatorname{supp}(\tilde{\chi} - \chi) \subset B_r'$. Select an $\zeta \in C_c^\infty(B_1')$ with $\zeta = 1$ in B_r' and $\zeta \geq 0$ in B_1'. Then we have

$$\begin{aligned} |D'\chi|(B_r') - |D'\tilde{\chi}|(B_r') &= \int_{B_1'} \zeta |D'\chi| - \int_{B_1'} \zeta |D'\tilde{\chi}| \\ &\overset{(8.56)}{=} -\int_{B_1'} \chi \nabla'\zeta \cdot m'\, dx' - \int_{B_1'} \zeta |D'\tilde{\chi}| \\ &= -\int_{B_1'} \tilde{\chi} \nabla'\zeta \cdot m'\, dx' - \int_{B_1'} \zeta |D'\tilde{\chi}|. \end{aligned}$$

The argument used to establish the inequality (8.62) in the proof of Lemma 8.10 also yields this lemma (with χ replaced by $\tilde{\chi}$). □

For convenience of the reader, the following lemma gives an elementary proof for the fact that minimal sets in two dimensions are locally half-spaces.

Lemma 8.12 *Let $\chi : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ satisfy*

$$\chi \text{ is minimal in } B_1', \tag{8.70}$$

$$\int_{B_r'} |D'\chi| \geq 2r \text{ for all } r \in (0,1). \tag{8.71}$$

Then χ is the characteristic function of a centered half-space in $B'_{1-\frac{\pi}{4}}$ (see Figure 8.11), i.e., there exists $\nu \in S^1$ such that

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \mathcal{L}^2\text{-a.e. in } B'_{1-\frac{\pi}{4}}.$$

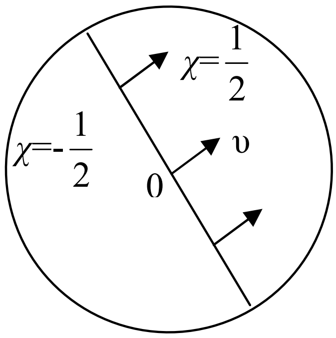


Figure 8.11: The characteristic χ in the ball $B'_{1-\frac{\pi}{4}}$

Proof of Lemma 8.12. We start by arguing that

$$|D'\chi|(B_1') \leq \pi. \tag{8.72}$$

Let $0 < r < 1$ be arbitrary. We compare χ to $\tilde{\chi}_+$, $\tilde{\chi}_-$ given by

$$\tilde{\chi}_+ = \left\{ \begin{array}{cc} \frac{1}{2} & \text{in } B_r' \\ \chi & \text{else} \end{array} \right\}, \quad \tilde{\chi}_- = \left\{ \begin{array}{cc} -\frac{1}{2} & \text{in } B_r' \\ \chi & \text{else} \end{array} \right\}.$$

By assumption (8.70), we obtain that

$$\begin{aligned} |D'\chi|(B_r') \leq \min\Bigg\{ & |D'\tilde{\chi}_-|(B_r') + \int_{\partial B_r'} |\chi^- - \tilde{\chi}_-^-| \, d\mathcal{H}^1, \\ & |D'\tilde{\chi}_+|(B_r') + \int_{\partial B_r'} |\chi^- - \tilde{\chi}_+^-| \, d\mathcal{H}^1 \Bigg\}, \end{aligned}$$

where χ^-, $\tilde{\chi}_-^-$ and $\tilde{\chi}_+^-$ denote the interior traces of $\chi\big|_{\partial B_r'}$, $\tilde{\chi}_-\big|_{\partial B_r'}$ and $\tilde{\chi}_+\big|_{\partial B_r'}$ respectively. In view of the form of $\tilde{\chi}_-$, $\tilde{\chi}_+$, this turns into

$$\begin{aligned} |D'\chi|(B_r') &\leq \min\left\{ \int_{\partial B_r'} |\chi^- + \frac{1}{2}| \, d\mathcal{H}^1, \int_{\partial B_r'} |\chi^- - \frac{1}{2}| \, d\mathcal{H}^1 \right\} \\ &= \min\left\{ \pi r + \int_{\partial B_r'} \chi^- \, d\mathcal{H}^1, \pi r - \int_{\partial B_r'} \chi^- \, d\mathcal{H}^1 \right\} \\ &\leq \pi r. \end{aligned}$$

From this, we deduce (8.72) by monotone convergence under $r \uparrow 1$. We now argue that there exists an $r \in [1-\frac{\pi}{4}, 1)$ such that

$$\int_{\partial B_r'} |D_\theta \chi^-| \in \{0, 2\} \tag{8.73}$$

where $\int_{\partial B_r'} |D_\theta \chi^-|$ denotes the total variation of the trace χ^- on $\partial B_r'$. Indeed, we have

$$\begin{aligned} \mathcal{L}^1(\{r \in (0,1) \,:\, \int_{\partial B_r'} |D_\theta \chi^-| \geq 4\}) &\leq \frac{1}{4}\int_0^1 \Big(\int_{\partial B_r'} |D_\theta \chi^-|\Big)\, dr \\ &\leq \frac{1}{4}|D'\chi|(B_1') \overset{(8.72)}{\leq} \frac{\pi}{4}. \end{aligned}$$

Hence, there exists $1-\frac{\pi}{4} \leq r < 1$ such that

$$\int_{\partial B_r'} |D_\theta \chi^-| < 4. \tag{8.74}$$

But since $\chi^- \in \{-\frac{1}{2}, \frac{1}{2}\}$, we have that $\int_{\partial B_r'} |D_\theta \chi^-| \in \{0, 2, 4, \dots\}$, so that (8.74) entails (8.73). We now argue that there exists $\nu \in S^1$ such that

$$\chi^- = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \mathcal{H}^1\text{-a.e. on } \partial B_r', \tag{8.75}$$

where r is as in (8.73). Indeed, because of (8.73), there exist $\nu \in S^1$ and $\alpha \in \mathbb{R}$ such that

$$\chi^- = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \text{else} \end{array} \right\} \mathcal{H}^1\text{-a.e. on } \partial B_r'. \tag{8.76}$$

We compare χ with $\tilde{\chi}$ given by

$$\tilde{\chi} = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > \alpha \text{ and } x' \in B_r', \\ -\frac{1}{2} & \text{for } x' \cdot \nu \leq \alpha \text{ and } x' \in B_r', \\ \chi & \text{else.} \end{array} \right\}$$

Because of (8.76), the traces of $\chi\big|_{\partial B_r'}$ and $\tilde{\chi}\big|_{\partial B_r'}$ coincide. Hence we obtain by the assumption (8.70),

$$|D'\chi|(B_r') \leq |D'\tilde{\chi}|(B_r'). \tag{8.77}$$

Because of assumption (8.71) this yields

$$2r \leq \mathcal{H}^1(\{x' \cdot \nu = \alpha\} \cap B_r'),$$

which enforces $\alpha = 0$ so that (8.76) turns into (8.75). We finally argue that

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \mathcal{L}^2\text{-a.e. in } B_r', \tag{8.78}$$

where ν is as in (8.75). Indeed, (8.75) implies that

$$\int_{B_r'} \nu \cdot D'\chi = \int_{\partial B_r'} \nu \cdot \frac{x'}{r}\chi^- \, d\mathcal{H}^1 = 2r,$$

whereas (8.77) yields

$$|D'\chi|(B'_r) \leq \mathcal{H}^1(\{x' \cdot \nu = 0\} \cap B'_r) \leq 2r.$$

Hence we necessarily have

$$D'\chi = \nu|D'\chi| \quad |D'\chi|\text{-a.e. in } B'_r.$$

Since $\chi \in \{-\frac{1}{2}, \frac{1}{2}\}$, this implies that

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \text{else} \end{array} \right\} \mathcal{L}^2\text{-a.e. in } B'_r,$$

for some $\alpha \in \mathbb{R}$. Since its trace χ^- is given by (8.75), χ must indeed be of form (8.78). □

The next lemma establishes that the characteristic functions from Lemma 8.10 are locally ordered.

Lemma 8.13 *Let* $m' : B'_1 \to \mathbb{R}^2$ *satisfy* (8.16). *Let* $\chi_0 : B'_1 \to \{-\frac{1}{2}, \frac{1}{2}\}$ *have the properties:*

- χ_0 *is the characteristic function of a centered half-space, i.e., there exists* $\nu_0 \in S^1$ *such that*

$$\chi_0 = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu_0 > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \quad \text{in } B'_1;$$

- χ_0 *satisfies* (8.56).

Let $\chi : B'_1 \to \{-\frac{1}{2}, \frac{1}{2}\}$ *have the properties:*

- χ *is the characteristic function of a half-space, i.e., there exist* $\nu \in S^1$ *and* $\alpha \in \mathbb{R}$ *such that*

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \text{else} \end{array} \right\} \quad \text{in } B'_1;$$

- χ *satisfies* (8.56).

Then $\chi \leq \chi_0$ *in* $B'_{1-\frac{\pi}{4}}$ *or* $\chi \geq \chi_0$ *in* $B'_{1-\frac{\pi}{4}}$.

Proof of Lemma 8.13. We distinguish three cases.

Case 1: $\mathcal{H}^0(\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\}) \leq 1$ and $\alpha \leq 0$. In this case, we consider $\tilde{\chi}$ given by

$$\tilde{\chi} = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu_0 > 0 \text{ and } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \text{else} \end{array} \right\} \quad \text{in } B'_1$$

(see Figure 8.12). We argue that

$$\nabla' \cdot (\tilde{\chi} m') = |D'\tilde{\chi}| \text{ distributionally in } B'_1, \tag{8.79}$$

$$\int_{B'_r} |D'\tilde{\chi}| \geq 2r \text{ for all } r \in (0,1). \tag{8.80}$$

Indeed, (8.79) holds distributionally in

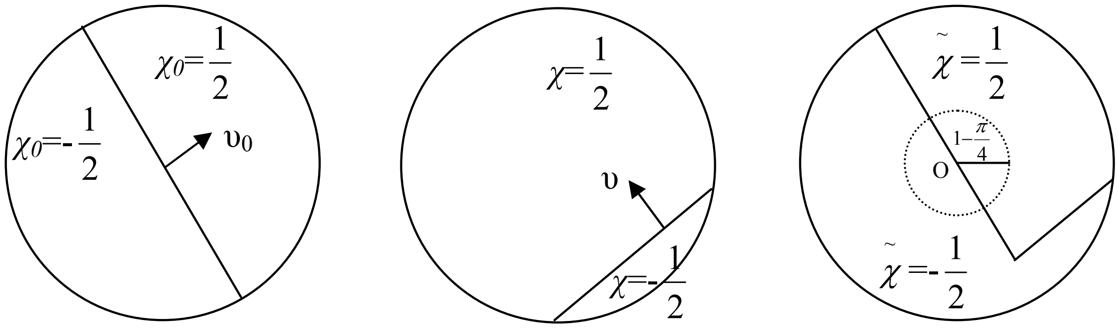


Figure 8.12: The characteristics χ_0, χ and $\tilde{\chi}$ in the ball B'_1

- $B'_1 \cap \{x' \cdot \nu_0 > 0\}$, since there $\tilde{\chi} = \chi$, so that (8.79) follows from the property (8.56) of χ;
- $B'_1 \cap \{x' \cdot \nu_0 < 0\}$, since there $\tilde{\chi} = -\frac{1}{2}$, so that (8.79) follows from (8.16);
- $B'_1 \cap \{x' \cdot \nu > \alpha\}$, since there $\tilde{\chi} = \chi_0$, so that (8.79) follows from the property (8.56) of χ_0;
- $B'_1 \cap \{x' \cdot \nu < \alpha\}$, since there $\tilde{\chi} = -\frac{1}{2}$, so that (8.79) follows from (8.16).

Hence, (8.79) holds distributionally in $B'_1 \setminus (\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\})$. By assumption, $\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\}$ consists of at most a single point. But (8.79) is oblivious to sets of vanishing $\mathcal{H}^1$-measure. This establishes (8.79). (8.80) follows from the fact that $0 \in \partial(\{x' \cdot \nu_0 > 0\} \cap \{x' \cdot \nu > \alpha\})$, which is a consequence of our assumption $\alpha \leq 0$. According to Lemma 8.11, (8.16) and (8.79) imply that $\tilde{\chi}$ is minimal in B'_1. According to Lemma 8.12, this and (8.80) imply that $\tilde{\chi}$ is the characteristic function of a centered half-space in $B'_{1-\frac{\pi}{4}}$. Hence $\{x' \cdot \nu_0 > 0\} \cap \{x' \cdot \nu > \alpha\}$ is a centered half-space in $B'_{1-\frac{\pi}{4}}$. In view of $\alpha \leq 0$, this yields

$$\{x' \cdot \nu_0 > 0\} \cap \{x' \cdot \nu > \alpha\} \cap B'_{1-\frac{\pi}{4}} = \{x' \cdot \nu_0 > 0\} \cap B'_{1-\frac{\pi}{4}},$$

that is

$$x' \cdot \nu > \alpha \text{ in } \{x' \cdot \nu_0 > 0\} \cap B'_{1-\frac{\pi}{4}},$$

whence $\chi \geq \chi_0$ in $B'_{1-\frac{\pi}{4}}$.

Case 2: $\mathcal{H}^0(\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\}) \leq 1$ and $\alpha \geq 0$. In this case, we consider $\tilde{\chi}$ given by

$$\tilde{\chi} = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu_0 > 0 \text{ or } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \text{else} \end{array} \right\} \text{ in } B'_1$$

and we argue as before to arrive at $\chi \leq \chi_0$ in $B'_{1-\frac{\pi}{4}}$.

Case 3: $\mathcal{H}^0(\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\}) > 1$. In this case, we necessarily have

$$\alpha = 0 \quad \text{and} \quad (\nu = \nu_0 \text{ or } \nu = -\nu_0).$$

In the case of $\nu = \nu_0$, we have $\chi = \chi_0$. The case of $\nu = -\nu_0$ cannot occur since then

$$\chi_0 + \chi = 0 \;\; \mathcal{L}^2\text{-a.e. in } B'_1$$

so that (8.56) could yield

$$|D'\chi_0| + |D'\chi| = \nabla'\cdot(\chi_0 m') + \nabla'\cdot(\chi m') = 0,$$

in particular, $D'\chi_0 = 0$ which is a contradiction. □

The next lemma establishes Lipschitz continuity of m' locally in B'_1. Because of translation and scaling invariance, it suffices to prove the following:

Lemma 8.14 *Let m' be as in Lemma 8.10. Let 0 and $y' \in B'_1$ be Lebesgue points of m'. Then*

$$|m'(y') - m'(0)| \leq \frac{2\sqrt{2}}{(1-\frac{\pi}{4})^2}|y'| \quad \textit{for all } y' \in B'_{\frac{1}{2}(1-\frac{\pi}{4})^2}.$$

Proof of Lemma 8.14. Let χ_0 and χ denote the characteristic functions associated to 0 and y' respectively, according to Lemma 8.10. According to Lemmas 8.11 and 8.12, there exist ν_0, $\nu \in S^1$ such that

$$\chi_0 = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x'\cdot\nu_0 > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \quad \text{in } B'_{1-\frac{\pi}{4}}, \tag{8.81}$$

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } (x'-y')\cdot\nu > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \quad \text{in } B'(y', (1-\frac{\pi}{4})(1-|y'|)). \tag{8.82}$$

Since

$$|y'| \leq \frac{1}{2}(1-\frac{\pi}{4})^2 \leq \frac{\frac{1}{2}(1-\frac{\pi}{4})}{2-\frac{\pi}{4}},$$

we have

$$B'(y', (1-\frac{\pi}{4})(1-|y'|)) \supset B'(0, (1-\frac{\pi}{4})(1-|y'|) - |y'|) \supset B'_{\frac{1}{2}(1-\frac{\pi}{4})},$$

so that both (8.81) and (8.82) hold in $B'_{\frac{1}{2}(1-\frac{\pi}{4})}$. Thus an application of Lemma 8.13 yields

$$\chi \leq \chi_0 \text{ in } B'_{\frac{1}{2}(1-\frac{\pi}{4})^2} \quad \text{or} \quad \chi \geq \chi_0 \text{ in } B'_{\frac{1}{2}(1-\frac{\pi}{4})^2}.$$

W.l.o.g. we consider only the first alternative, that is,

$$\{x'\cdot\nu_0 \leq 0\} \cap B'_{\frac{1}{2}(1-\frac{\pi}{4})^2} \subset \{(x'-y')\cdot\nu \leq 0\}.$$

Thus, $\nu\cdot\nu_0 > 0$. We introduce the abbreviations

$$\delta := \frac{y'\cdot\nu}{r}, \; r := \frac{1}{2}(1-\frac{\pi}{4})^2.$$

By elementary geometry (see Figure 8.13), this implies

$$|\nu - \nu_0|^2 \leq 2\delta^2. \tag{8.83}$$

Indeed, if $\nu = \nu_0$, then (8.83) is obvious. Otherwise, $\nu \neq \nu_0$ and then the point of intersection z' of the two lines respectively orthogonal to ν_0 and ν and passing through 0 and y', lies outside the ball B'_r; denoting by $\theta = \angle(\nu, \nu_0) \in (0, \frac{\pi}{2}]$ the angle between ν and ν_0, it follows that

$$\frac{y'\cdot\nu}{\sin\theta} = |z'| \geq r,$$

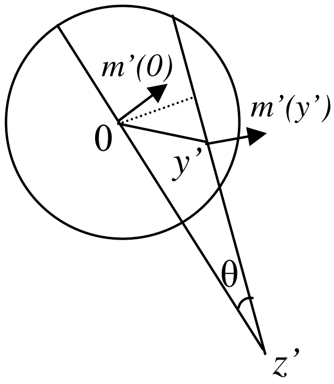


Figure 8.13: Geometry of characteristics

that is,

$$\delta \geq \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \geq |\nu - \nu_0|\frac{1}{\sqrt{2}}.$$

Hence,

$$|\nu - \nu_0| \leq \frac{2\sqrt{2}}{(1-\frac{\pi}{4})^2}|y'|.$$

It remains to prove that (8.56) implies

$$\nu = m'(y') \quad \text{and} \quad \nu_0 = m'(0). \tag{8.84}$$

W.l.o.g. we establish $\nu_0 = m'(0)$. Indeed, in the view of (8.81), (8.56) takes the form

$$\frac{1}{2}\int_{\{x'\cdot\nu_0<0\}} m'\cdot\nabla'\zeta\,dx' - \frac{1}{2}\int_{\{x'\cdot\nu_0>0\}} m'\cdot\nabla'\zeta\,dx' = \int_{\{x'\cdot\nu_0=0\}}\zeta\,d\mathcal{H}^1, \tag{8.85}$$

for all $\zeta \in C_c^\infty(B'_{1-\frac{\pi}{4}})$. We now fix a $\zeta_1 \in C_c^\infty(B'_{1-\frac{\pi}{4}})$ such that $\int_{\{x'\cdot\nu_0=0\}}\zeta_1\,d\mathcal{H}^1 = 1$ and for $r<1$, consider $\zeta_r \in C_c^\infty(B'_{r(1-\frac{\pi}{4})})$ given by

$$\zeta_r(x') = \frac{1}{r}\zeta_1(\frac{x'}{r}).$$

Since

$$\int_{\mathbb{R}^2}|\nabla'\zeta_r|\,dx' = \int_{\mathbb{R}^2}|\nabla'\zeta_1|\,dx'$$

and 0 is a Lebesgue point of m', we have

$$\begin{aligned}
&\lim_{r\to0}\Big(\frac{1}{2}\int_{\{x'\cdot\nu_0<0\}} m'\cdot\nabla'\zeta_r\,dx' - \frac{1}{2}\int_{\{x'\cdot\nu_0>0\}} m'\cdot\nabla'\zeta_r\,dx'\Big)\\
&\quad = m'(0)\cdot\lim_{r\to0}\Big(\frac{1}{2}\int_{\{x'\cdot\nu_0<0\}}\nabla'\zeta_r\,dx' - \frac{1}{2}\int_{\{x'\cdot\nu_0>0\}}\nabla'\zeta_r\,dx'\Big)\\
&\quad = (m'(0)\cdot\nu_0)\lim_{r\to0}\int_{\{x'\cdot\nu_0=0\}}\zeta_r\,d\mathcal{H}^1.
\end{aligned} \tag{8.86}$$

Since
$$\int_{\{x'\cdot\nu_0=0\}} \zeta_r \, d\mathcal{H}^1 = \int_{\{x'\cdot\nu_0=0\}} \zeta_1 \, d\mathcal{H}^1 = 1,$$
we obtain from (8.85) and (8.86), $m'(0)\cdot\nu_0 = 1$, which implies (8.84) because of $|m'(0)| = 1$. □

The last lemma establishes the principle of characteristics for m' in B_1'. Because of translation and scaling invariance and a continuity argument, it suffices to prove the following:

Lemma 8.15 *Let m' be as in Lemma 8.10 and Lipschitz continuous. Then*
$$m'(tm'(0)^{\perp}) = m'(0) \qquad \text{for all } |t| < 1 - \frac{\pi}{4}. \tag{8.87}$$

Proof of Lemma 8.15. Let χ be the characteristic function associated to 0 according to Lemma 8.10. From Lemmas 8.11 and 8.12 we gather that there exists $\nu \in S^1$ such that
$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x'\cdot\nu > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \text{ in } B'_{1-\frac{\pi}{4}}.$$
As in Lemma 8.14 we deduce from (8.56) and the continuity of m':
$$m' = \nu \quad \text{on } \{x'\cdot\nu = 0\} \cap B'_{1-\frac{\pi}{4}}.$$
This is a reformulation of (8.87). □

8.5 Optimality of the straight walls

In this section, we prove Theorem 8.1:

Proof of Theorem 8.1. Let $m' : \mathbb{R}^2 \to S^1$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy the hypothesis of Theorem 8.1. Using the same argument as in the proof of Theorem 8.4 and because of (8.1), there exists a set $G' \subset \mathbb{R}^2$ with the outer normal ν' such that
$$\begin{aligned} &G' \text{ is } L\text{-periodic in } x_2, \\ &(1,+\infty)\times\mathbb{R} \subset G',\ (-\infty,-1)\times\mathbb{R} \subset \mathbb{R}^2 \setminus G' \\ &m' = \nu' \text{ on } \partial G' \end{aligned} \tag{8.88}$$
(see Figure 8.14). This set was introduced in [39]. We consider the related characteristic function
$$\chi = \begin{cases} \frac{-1}{2} & \text{in } \ \mathbb{R}^2 \setminus G', \\ \frac{1}{2} & \text{in } \ G'. \end{cases} \tag{8.89}$$

Then (8.88) translates into
$$\chi \text{ is } L\text{-periodic in } x_2, \tag{8.90}$$
$$\chi = \pm\frac{1}{2} \text{ for } \pm x_1 \geq 1, \tag{8.91}$$
$$\int_{\mathbb{R}\times[0,L)} \eta^2 \chi \nabla'\cdot m' \, dx' = -\int_{\mathbb{R}\times[0,L)} \nabla'(\eta^2)\cdot m'\chi \, dx' - \int_{\mathbb{R}\times[0,L)} \eta^2 |D'\chi|, \tag{8.92}$$

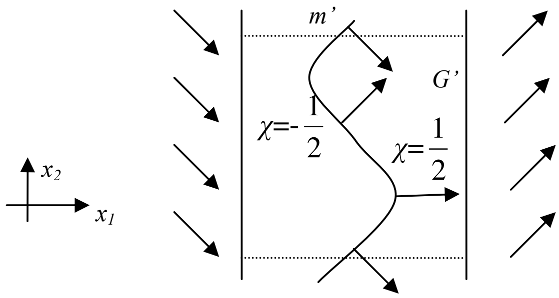


Figure 8.14: Center line of the wall

where $\eta \in C^\infty(\mathbb{R}^3)$ is a $L-$periodic function in x_2 that satisfies (8.39). We also introduce the energy density e as a non-negative measure on $\mathbb{R}^3$ via

$$\int_{\mathbb{R}^3} \zeta\, de = \frac{2}{\pi}|\ln\varepsilon|\Big(\varepsilon\int_{\mathbb{R}^2}\zeta|\nabla'\cdot m'|^2\,dx' + \int_{\mathbb{R}^3}\zeta|h|^2\,dx\Big),\ \forall\zeta\in C_c^\infty(\mathbb{R}^3). \tag{8.93}$$

Step 1. *We have an a priori bound on $L^{-1}\int_{\mathbb{R}\times[0,L)}|D'\chi|$ in terms of $L^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})$: for any $\alpha\in(0,1)$,*

$$\begin{aligned}(1-\alpha)L^{-1}\int_{\mathbb{R}\times[0,L)}|D'\chi| \le m_{1,\infty} + \frac{1}{4\alpha}L^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})\\ + \frac{C\tilde{L}^4}{L^2\sqrt{|\ln\varepsilon|}}\Big(L^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})\Big)^{1/2}\Big(1+L^{-1}\int_{\mathbb{R}\times[0,L)}|D'\chi|\Big),\end{aligned} \tag{8.94}$$

where $\tilde{L}=\max\{2,L\}$. Indeed, with the above choices and notations, (8.41) turns into

$$\begin{aligned}\Big|\int_{\mathbb{R}\times[0,L)}\eta^2\chi\nabla'\cdot m'\,dx'\Big| \overset{(8.91)}{\le} \Big(\int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\eta^2\,de\int_{\mathbb{R}\times[0,L)}\eta^2|D'\chi|\Big)^{1/2}\\ + \frac{C\tilde{L}^3}{L^2}\sup_{\mathbb{R}^3}|\eta|\Big(|\ln\varepsilon|^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})\Big)^{1/2}\\ \times(\sup_{\mathbb{R}^3}|\eta|+L\sup_{\mathbb{R}^3}|\nabla\eta|)\cdot\Big(L^{1/2}+L^{-1/2}\int_{\mathbb{R}\times[0,L)}|D'\chi|\Big).\end{aligned}$$

Using (8.92) on the left-hand side and Young's inequality on the first term of the right-hand side yields for any $\alpha\in(0,1)$,

$$\begin{aligned}(1-\alpha)\int_{\mathbb{R}\times[0,L)}\eta^2|D'\chi| \le -\int_{\mathbb{R}\times[0,L)}\nabla'(\eta^2)\cdot m'\chi\,dx' + \frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\eta^2\,de\\ + \frac{C\tilde{L}^3}{L^2}\sup_{\mathbb{R}^3}|\eta|\Big(|\ln\varepsilon|^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})\Big)^{1/2}\\ \times(\sup_{\mathbb{R}^3}|\eta|+L\sup_{\mathbb{R}^3}|\nabla\eta|)\cdot\Big(L^{1/2}+L^{-1/2}\int_{\mathbb{R}\times[0,L)}|D'\chi|\Big).\end{aligned} \tag{8.95}$$

We select $\eta:\mathbb{R}^3\to\mathbb{R}$ such that

$$\begin{aligned}&\eta=\eta(x_1,x_3),\ \eta=1\ \text{ on } (-1,1)\times\mathbb{R}\times\{0\},\\ &\operatorname{supp}\eta\subset(-2,2)\times\mathbb{R}\times(-1,1),\ |\eta|\le 1,\ |\nabla\eta|\le C.\end{aligned} \tag{8.96}$$

We consider the terms in (8.95) one-by-one:

$$\begin{aligned}
\int_{\mathbb{R}\times[0,L)} \eta^2|D'\chi| &\overset{(8.91,8.96)}{=} \int_{\mathbb{R}\times[0,L)} |D'\chi|,\\
-\int_{\mathbb{R}\times[0,L)} \nabla'(\eta^2)\cdot m'\chi\,dx' &\overset{(8.1,8.91,8.96)}{=} -\int_{(-\infty,-1)\times[0,L)} \begin{pmatrix}\partial_1\eta^2\\ 0\end{pmatrix}\cdot\begin{pmatrix} m_{1,\infty}\\ -\sqrt{1-{m_{1,\infty}}^2}\end{pmatrix}\frac{-1}{2}\,dx'\\
&\qquad -\int_{(1,+\infty)\times[0,L)} \begin{pmatrix}\partial_1\eta^2\\ 0\end{pmatrix}\cdot\begin{pmatrix} m_{1,\infty}\\ \sqrt{1-{m_{1,\infty}}^2}\end{pmatrix}\frac{1}{2}\,dx'\\
&= Lm_{1,\infty}, \qquad (8.97)\\
\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\,de &\le e(\mathbb{R}\times[0,L)\times\mathbb{R}).
\end{aligned}$$

Using (8.96) to estimate the $\eta-$terms in (8.95), we then obtain

$$\begin{aligned}
(1-\alpha)\int_{\mathbb{R}\times[0,L)} |D'\chi| \le{}& Lm_{1,\infty} + \frac{1}{4\alpha}e(\mathbb{R}\times[0,L)\times\mathbb{R})\\
&+\frac{C\tilde{L}^4}{L^2\sqrt{|\ln\varepsilon|}}e(\mathbb{R}\times[0,L)\times\mathbb{R})^{1/2}\Big(\sqrt{L}+\frac{1}{\sqrt{L}}\int_{\mathbb{R}\times[0,L)}|D'\chi|\Big).
\end{aligned}$$

Dividing by L yields (8.94).

Step 2. *Sketch of the proof of Theorem 8.1.* We give an argument by contradiction. To this purpose, we consider sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}\subset(0,\infty)$ with $\varepsilon_k\downarrow 0$, $\{m'_k:\mathbb{R}^2\to S^1\}_{k\uparrow\infty}$ and $\{h_k:\mathbb{R}^3\to\mathbb{R}^3\}_{k\uparrow\infty}$ that satisfy the first three hypothesis in Theorem 8.1 and

$$\limsup_{k\to\infty} L^{-1}e_k(\mathbb{R}\times[0,L)\times\mathbb{R}) \le (1-m_{1,\infty})^2, \qquad (8.98)$$

which corresponds to (8.10) (here, e_k is the energy density (8.93) associated to m'_k and h_k). Because of periodicity of e_k, (8.98) implies that the energy is locally bounded, so that we may apply Theorem 8.4. Hence there exists a measurable $m':\mathbb{R}^2\to S^1$ with

$$m'_k \to m' \quad \text{in } L^1_{loc}(\mathbb{R}^2), \qquad (8.99)$$

after passage to a subsequence. Properties (8.1) and (8.3) are preserved under (8.99) while in addition (see Theorem 8.4),

$$\nabla'\cdot m' = 0 \text{ distributionally in } \mathbb{R}^2.$$

Because of (8.1) and (8.3), (8.99) yields

$$\int_{\mathbb{R}\times[0,L)} |m'_k - m'|\,dx' \to 0.$$

We thus have to argue that m' has the form (8.12). Because of periodicity of e, (8.98) implies that exists a non-negative measure e on $\mathbb{R}^3$ such that

$$e_k \overset{w^*}{\rightharpoonup} e \quad \text{weakly}^* \text{ in } \mathcal{M}(\mathbb{R}^3), \qquad (8.100)$$

after passage to a subsequence. Notice that (8.98) is preserved under (8.100):

$$L^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R}) \le (1-m_{1,\infty})^2. \tag{8.101}$$

We shall argue that there exists an $x_1^* \in [-1,1]$ such that

$$\operatorname{supp} e \cap (-2,2)\times\mathbb{R}\times(-1,1) \subset \{x_1^*\}\times\mathbb{R}\times\{0\}. \tag{8.102}$$

We then apply Theorem 8.5 on balls in $(-2,x_1^*)\times\mathbb{R}\times(-1,1)$ and $(x_1^*,2)\times\mathbb{R}\times(-1,1)$ respectively. This yields that m' is locally Lipschitz and satisfies the principle of characteristics in both $(-2,x_1^*)\times\mathbb{R}\times(-1,1)$ and $(x_1^*,2)\times\mathbb{R}\times(-1,1)$. In view of the form (8.1), this indeed implies that m' is of the form (8.12). Hence it suffices to show (8.102).

Step 3. *Proof of* (8.102). We first address the function χ_k defined as in (8.89) for m_k'. In view of (8.94) (applied to χ_k and e_k) and (8.98), we have

$$\left\{L^{-1}\int_{\mathbb{R}\times[0,L)}|D'\chi_k|\right\}_{k\uparrow\infty} \text{ is bounded.} \tag{8.103}$$

Because of periodicity (8.90), there exists a measurable function $\chi:\mathbb{R}^2\to\{-\frac{1}{2},\frac{1}{2}\}$ of locally bounded variation such that

$$\chi_k \to \chi \text{ in } L^1_{loc}(\mathbb{R}^2). \tag{8.104}$$

Notice that periodicity (8.90) and the boundary conditions (8.91) are preserved by (8.104). We shall argue in Step 4 that χ is of the form

$$\chi = \pm\frac{1}{2} \text{ for } \pm x_1 > \pm x_1^*, \tag{8.105}$$

for some $x_1^*\in[-1,1]$. Now we give the argument how (8.105) implies (8.102). For this we turn back to (8.95). Again, because of the convergences (8.99), (8.100), (8.104) and the boundedness expressed in (8.98) and (8.103), (8.95) (applied for χ_k, m_k' and e_k) yields in the limit as $k\to\infty$

$$(1-\alpha)\int_{\mathbb{R}\times[0,L)}\eta^2|D'\chi| \le -\int_{\mathbb{R}\times[0,L)}\nabla'(\eta^2)\cdot m'\chi\,dx' + \frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\eta^2\,de \tag{8.106}$$

for any $\eta\in C^\infty(\mathbb{R}^3)$ that is $L-$ periodic in x_2 and satisfies (8.39). We choose

$$\alpha = \frac{(1-m_{1,\infty})}{2}.$$

In view of (8.106),

$$\int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\zeta\,d\lambda = \frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\zeta\,de - \int_{\mathbb{R}\times[0,L)}\nabla'\zeta\cdot m'\chi\,dx' - (1-\alpha)\int_{\mathbb{R}\times[0,L)}\zeta|D'\chi|$$

defines a non-negative distribution in $(-2,2)\times\mathbb{R}\times(-1,1)$ for functions $\zeta:\mathbb{R}^3\to\mathbb{R}$ which are $L-$periodic in x_2 and satisfy (8.39). Because of (8.105), λ simplifies to

$$\begin{aligned}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\zeta\,d\lambda = &\frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\zeta\,de\\ &+\frac{1}{2}\int_{(-\infty,x_1^*)\times[0,L)}\nabla'\zeta\cdot m'\,dx' - \frac{1}{2}\int_{(x_1^*,\infty)\times[0,L)}\nabla'\zeta\cdot m'\,dx'\\ &-(1-\alpha)\int_{[0,L)}\zeta(x_1^*,x_2,0)\,dx_2.\end{aligned} \tag{8.107}$$

In fact, λ is a non-negative measure: because of $|m'| = 1$ and the divergence-free property (see (8.16)), we have

$$\left|\frac{1}{2}\int_{(-\infty,x_1^*)\times[0,L)} \nabla'\zeta\cdot m'\,dx' - \frac{1}{2}\int_{(x_1^*,\infty)\times[0,L)} \nabla'\zeta\cdot m'\,dx'\right| \le \int_{[0,L)} |\zeta(x_1^*,x_2,0)|\,dx_2. \quad (8.108)$$

Estimate (8.108) formally follows from integration by parts and can be rigorously established by approximating m' with smooth m''s while preserving $|m'|\le 1$, $\nabla'\cdot m' = 0$ and the periodicity in x_2. We now consider $\zeta = \eta^2$ in (8.107) such that (8.39) holds and

$$\eta = \eta(x_1,x_3),\ \eta = 1 \text{ on } (-1,1)\times\mathbb{R}\times\{0\},\ |\eta|\le 1.$$

Using the same arguments as in (8.97), we learn that (8.107) turns into

$$\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\,d\lambda = \frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\,de + Lm_{1,\infty} - L(1-\alpha).$$

Since (8.101) implies that $\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\,de \le e(\mathbb{R}\times[0,L)\times\mathbb{R}) \le L(1-m_{1,\infty})^2$, this yields

$$\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\,d\lambda \le L\left[\frac{1}{4\alpha}(1-m_{1,\infty})^2 + m_{1,\infty} - (1-\alpha)\right]=0.$$

We let η^2 converge monotonically to one in $(-2,2)\times\mathbb{R}\times(-1,1)$ and obtain $\lambda((-2,2)\times[0,L)\times(-1,1)) \le 0$ and thus, $\lambda\equiv 0$ in $(-2,2)\times[0,L)\times(-1,1)$. Hence, (8.107) simplifies to

$$\begin{aligned}\frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \zeta\,de &= (1-\alpha)\int_{[0,L)} \zeta(x_1^*,x_2,0)\,dx_2 \\ &\quad -\frac{1}{2}\int_{(-\infty,x_1^*)\times[0,L)} \nabla'\zeta\cdot m'\,dx' + \frac{1}{2}\int_{(x_1^*,\infty)\times[0,L)} \nabla'\zeta\cdot m'\,dx' \\ &\overset{(8.108)}{\le} (1-\alpha)\int_{[0,L)} \zeta(x_1^*,x_2,0)\,dx_2 + \int_{[0,L)} |\zeta(x_1^*,x_2,0)|\,dx_2,\end{aligned}$$

for every $\zeta\in C^\infty(\mathbb{R}^3)$ that is $L-$periodic in x_2 and satisfies (8.39). This implies (8.102) by periodicity of e. Thus, it remains to prove (8.105).

Step 4. *Proof of* (8.105). We first notice that because of (8.98), (8.103) and the lower semicontinuity of $\int_{\mathbb{R}\times[0,L)} |D'\chi_k|$ under (8.104), (8.94) (applied for χ_k and e_k) yields in the limit as $k\to\infty$,

$$(1-\alpha)L^{-1}\int_{\mathbb{R}\times[0,L)} |D'\chi| \le m_{1,\infty} + \frac{(1-m_{1,\infty})^2}{4\alpha}.$$

As before, the choice of $\alpha = \frac{(1-m_{1,\infty})}{2}$ gives

$$L^{-1}\int_{\mathbb{R}\times[0,L)} |D'\chi| \le 1. \quad (8.109)$$

Now the boundary conditions (8.91) and the inequality (8.109) enforce the form (8.105). For the convenience of the reader, we display this standard argument. Let μ and ν' be the measure-theoretic *line measure* $|D'\chi|$ and *normal* $\dfrac{D'\chi}{|D'\chi|}$ related to the function χ of bounded variation.

Both inherit the periodicity of χ and are characterized by

$$-\int_{\mathbb{R}\times[0,L)} \nabla'\cdot\zeta'\chi\,dx' = \int_{\mathbb{R}\times[0,L)} \nu'\cdot\zeta'\,d\mu \tag{8.110}$$

for all $\zeta':\mathbb{R}^2\to\mathbb{R}^2$ which are $L-$periodic in x_2 and compactly supported in x_1. Now we show that (8.91) yields

$$\int_{\mathbb{R}\times[0,L)} \nu_1\,d\mu = L. \tag{8.111}$$

Indeed, (8.111) can be seen by selecting a function $\eta=\eta(x_1)$ with $\eta=1$ for $|x_1|\le 1$ and $\operatorname{supp}\eta\subset(-2,2)$ so that

$$\begin{aligned}\int_{\mathbb{R}\times[0,L)} \nu_1\,d\mu = \int_{\mathbb{R}\times[0,L)} \eta^2\nu_1\,d\mu &\overset{(8.110)}{=} -\int_{\mathbb{R}\times[0,L)} \frac{d\eta^2}{dx_1}\chi\,dx' \\ &= -\int_{(-\infty,-1)\times[0,L)} \frac{-1}{2}\frac{d\eta^2}{dx_1}\,dx' - \int_{(1,\infty)\times[0,L)} \frac{1}{2}\frac{d\eta^2}{dx_1}\,dx' = L.\end{aligned}$$

Now (8.109) (i.e., $\displaystyle\int_{\mathbb{R}\times[0,L)} d\mu\le L$) and (8.111) combine to $\int_{\mathbb{R}\times[0,L)}(1-\nu_1)\,d\mu\le 0$. But since $1-\nu_1\ge 0$ we must have $1-\nu_1=0$ $\mu-$a.e., that is, $\nu=\begin{pmatrix}1\\0\end{pmatrix}$ $\mu-$a.e. Hence (8.110) turns into

$$-\int_{\mathbb{R}\times[0,L)} \nabla'\cdot\zeta'\chi\,dx' = \int_{\mathbb{R}\times[0,L)} \zeta_1\,d\mu. \tag{8.112}$$

Choosing ζ' with $\zeta_1\equiv 0$, we deduce that χ has a representative with $\chi=\chi(x_1)$. In particular, (8.112) then yields

$$-\int_{\mathbb{R}} \frac{d\eta^2}{dx_1}\chi\,dx_1\ge 0,$$

for all $\eta=\eta(x_1)$ with compact support. Hence χ has a representative with $\chi=\chi(x_1)$ that is monotone non-decreasing. Since $\chi\in\{-\frac{1}{2},\frac{1}{2}\}$, this yields (8.105). Now the proof of the theorem is completed. □

Remark: One can improve (8.102) to $\operatorname{supp} e\subset\{x_1^*\}\times\mathbb{R}\times\{0\}$ using Corollary 8.9 for trial functions η with support in $(-a,a)\times\mathbb{R}\times(-a,a)$, where a is arbitrarily large.

8.6 The case of $1d$ magnetizations

In the framework of Theorem 8.1, we focus here on $1d$ magnetizations $m'=(m_1(x_1),m_2(x_1))$. As in [39], we consider the minimal stray field corresponding to m' in the strip $\mathbb{R}\times[0,1)$. For that, let $U\in H_0^1(\mathbb{R}\times(0,1)\times\mathbb{R})$ be the unique $1-$periodic function in x_2-direction that satisfies

$$\int_{\mathbb{R}\times(0,1)\times\mathbb{R}} \nabla U\cdot\nabla\zeta\,dx = -\int_{\mathbb{R}\times(0,1)} \zeta\,\nabla'\cdot m'\,dx',\quad \forall\zeta\in C_c^\infty(\mathbb{R}\times(0,1)\times\mathbb{R}). \tag{8.113}$$

(That is a direct application of the Lax-Milgram Theorem.) The function U is the unique symmetric harmonic map in $H_0^1(\mathbb{R}\times(0,1)\times\mathbb{R})$ with the trace of the normal derivative given by

$\nabla' \cdot m'$, i.e.,

$$\begin{cases} \Delta U = 0 & \text{in} \quad \mathbb{R} \times (0,1) \times (\mathbb{R} \setminus \{0\}), \\ \left[\frac{\partial U}{\partial x_3}\right] = \nabla' \cdot m' & \text{on} \quad \mathbb{R} \times (0,1), \end{cases}$$

where $[\xi]$ denotes the jump of a quantity ξ across the plane $\mathbb{R}^2 \times \{0\}$. Then an elementary computation yields that the stray field energy is given by the homogeneous $H^{-1/2}$ norm of the divergence of m':

$$\int_{\mathbb{R}\times(0,1)\times\mathbb{R}} |\nabla U|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}\times(0,1)} \left| |\nabla'|^{-1/2} \nabla' \cdot m' \right|^2 dx'. \tag{8.114}$$

Since m' is one-dimensional, then

$$\int_{\mathbb{R}\times(0,1)} \left| |\nabla'|^{-1/2} \nabla' \cdot m' \right|^2 dx' = \int_{\mathbb{R}} \left| |\frac{d}{dx_1}|^{1/2} m' \right|^2 dx_1$$

and therefore, (8.114) explains the expression of the energy $E_\varepsilon^{1d}(m')$ given in (8.9). Also observe that the chosen stray field energy is minimal because for any $h : \mathbb{R}^3 \to \mathbb{R}^3$ that is $1-$periodic in x_2 and satisfies (8.2) for $\nabla' \cdot m'$, we have

$$\int_{\mathbb{R}\times(0,1)\times\mathbb{R}} |\nabla U|^2 \, dx \leq \int_{\mathbb{R}\times(0,1)\times\mathbb{R}} |h|^2 \, dx.$$

We now present the proof of Theorems 8.2 and 8.3:

Proof of Theorem 8.2. We proceed in several steps:
Step 1. *We show that*

$$m_{1,k} - m_{1,\infty} \to 0 \quad \textit{in} \quad L^1(\mathbb{R}) \quad \textit{as} \quad k \to \infty.$$

Indeed, by (8.1) and (8.13), we deduce that

$$\begin{aligned} \int_{\mathbb{R}} |m_{1,k} - m_{1,\infty}|^2 \, dt = \int_{-1}^{1} |m_{1,k} - m_{1,\infty}|^2 \, dt &= \int_{-1}^{1} \int_{2}^{3} |m_{1,k}(t) - m_{1,k}(t+s)|^2 \, dt \, ds \\ &\leq 9 \int_{-1}^{1} \int_{2}^{3} \frac{|m_{1,k}(t) - m_{1,k}(t+s)|^2}{s^2} \, dt \, ds \\ &\leq 9 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|m_{1,k}(t) - m_{1,k}(s)|^2}{|t-s|^2} \, dt \, ds \ \to 0 \quad \text{as} \quad k \to \infty \end{aligned}$$

and the conclusion follows by (8.1).
Step 2. *We locate the regions where $m_{1,k}$ (and $m_{2,k}$) have large variations.* For that, we choose the intervals (a_n^k, b_n^k), $n = 1, \ldots, N_k$ in the following way (see Figure 8.15): we set $b_0^k = -\infty$ and we recursively define for $n = 1, \ldots, N_k$, $b_n^k \in (b_{n-1}^k, 1]$ to be the smallest number such that

$$m_{2,k}(b_n^k) = \frac{(-1)^{n-1}\sqrt{1 - {m_{1,\infty}}^2}}{2}$$

and respectively, $a_n^k \in [b_{n-1}^k, b_n^k]$ be the biggest number such that

$$m_{2,k}(a_n^k) = \frac{(-1)^{n}\sqrt{1 - {m_{1,\infty}}^2}}{2}.$$

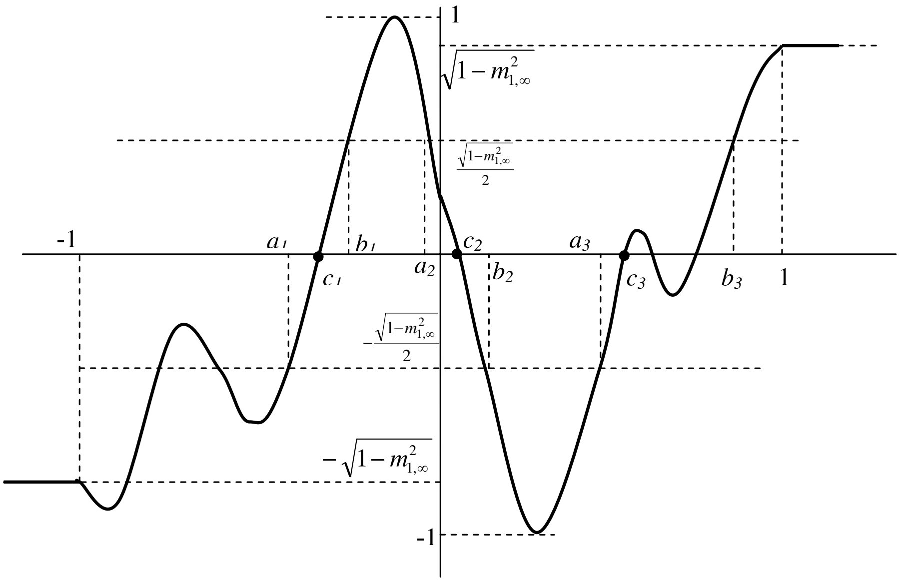


Figure 8.15: The variations of m_2

By (8.1), we have that

$$-1 < a_1^k < b_1^k \le a_2^k < b_2^k \le \cdots \le a_{N_k}^k < b_{N_k}^k < 1 \quad \text{and} \quad N_k \le \frac{2}{1-{m_{1,\infty}}^2}\int_{\mathbb{R}} \Big|\frac{dm_{2,k}}{dt}\Big|^2\, dt$$

since

$$\frac{1-{m_{1,\infty}}^2}{b_n^k - a_n^k} = \frac{1}{b_n^k - a_n^k}\left(\int_{a_n^k}^{b_n^k} \frac{dm_{2,k}}{dt}\, dt\right)^2 \le \int_{a_n^k}^{b_n^k} \Big|\frac{dm_{2,k}}{dt}\Big|^2\, dt \le \int_{\mathbb{R}} \Big|\frac{dm_{2,k}}{dt}\Big|^2\, dt.$$

We also notice that N_k is an odd integer (because of (8.1)),

$$|m_{2,k}| \le \frac{\sqrt{1-{m_{1,\infty}}^2}}{2} \text{ in any interval } (a_n^k, b_n^k) \tag{8.115}$$

$$\text{and} \quad (-1)^{n-1} m_{2,k} \le \frac{\sqrt{1-{m_{1,\infty}}^2}}{2} \text{ in } (b_{n-1}^k, b_n^k)\,,\ n = 1, \ldots, N_k. \tag{8.116}$$

Step 3. *We prove that the sequence $\{N_k\}$ is bounded.* The idea is to define a good step function with $2N_k$ jumps and to apply Corollary 8.9. Set

$$\chi_k = \begin{cases} \operatorname{sgn}(m_{1,k}) & \text{in} \quad (a_n^k, c_n^k) \quad \text{for } n = 1, \ldots, N_k, \\ 0 & \text{elsewhere,} \end{cases}$$

where $c_n^k \in [a_n^k, b_n^k]$ is the smallest number such that $m_{2,k}(c_n^k) = 0$. Since (8.115) implies that

$m_{1,k}$ does not change sign in (a_n^k, c_n^k), we obtain:

$$\int_{\mathbb{R}} |\frac{d\chi_k}{dt}| = 2N_k$$

$$\begin{aligned}\int_{-1}^{1} \chi_k \frac{dm_{1,k}}{dt}\, dt &= \sum_{n=1}^{N_k} \int_{a_n^k}^{c_n^k} \operatorname{sgn}(m_{1,k}) \frac{dm_{1,k}}{dt}\, dt \\ &= \sum_{n=1}^{N_k} \big(|m_{1,k}|(c_n^k) - |m_{1,k}|(a_n^k)\big) = N_k\big(1 - \frac{\sqrt{3 + {m_{1,\infty}}^2}}{2}\big).\end{aligned} \tag{8.117}$$

Now we apply Corollary 8.9 for the harmonic extension U_k of m'_k given by (8.113) where we choose $L = 1$ and for the test function $\eta = \eta(x_1, x_3) : \mathbb{R}^3 \to [-1, 1]$ with $\eta = 1$ in $(-1, 1)\times\mathbb{R}\times\{0\}$ and $\operatorname{supp} \eta \subset (-2, 2) \times \mathbb{R} \times (-1, 1)$,

$$\begin{aligned}\Big|\int_{\mathbb{R}\times[0,1)} \eta^2 \chi_k \frac{dm_{1,k}}{dx_1}\, dx'\Big| \leq &\Big(\frac{4}{\pi} |\ln \varepsilon_k| \int_{\mathbb{R}\times[0,1)} \eta^2 |D'\chi_k| \int_{\mathbb{R}\times[0,1)\times\mathbb{R}} \eta^2 |\nabla U_k|^2\, dx\Big)^{1/2} \\ &+ C\Big(\varepsilon_k \int_{\mathbb{R}\times[0,1)} |\frac{dm_{1,k}}{dx_1}|^2\, dx' + \int_{\mathbb{R}\times[0,1)\times\mathbb{R}} |\nabla U_k|^2\, dx\Big)^{1/2} \\ &\times \Big(1 + \int_{\mathbb{R}\times[0,1)} |D'\chi_k|\Big),\end{aligned}$$

that is,

$$\begin{aligned}\Big|\int_{-1}^{1} \chi_k \frac{dm_{1,k}}{dt}\, dt\Big| \overset{(8.114)}{\leq} &C\Big(|\ln \varepsilon_k|\, E^{1d}_{\varepsilon_k}(m'_k) \int_{\mathbb{R}} |\frac{d\chi_k}{dt}|\Big)^{1/2} \\ &+ \frac{C}{\sqrt{|\ln \varepsilon_k|}} \Big(|\ln \varepsilon_k| E^{1d}_{\varepsilon_k}(m'_k)\Big)^{1/2} \times \Big(1 + \int_{\mathbb{R}} |\frac{d\chi_k}{dt}|\Big).\end{aligned}$$

Therefore, by (8.13) and (8.117), we deduce that $N_k \leq C$ for some absolute constant $C > 0$.

Step 4. *We show that the sequence $\{m_{2,k}\}$ is relatively compact in L^1_{loc}.* We consider the step function

$$\psi_k = \sum_{n=1}^{N_k+1} (-1)^n \sqrt{1 - {m_{1,\infty}}^2}\ 1_{(b^k_{n-1}, b^k_n)},$$

where $b^k_{N_k+1} = +\infty$. Observe that

$$\int_{\mathbb{R}} |\frac{d\psi_k}{dt}| = 2N_k \sqrt{1 - {m_{1,\infty}}^2}.$$

It follows by Step 3 that the sequence $\{\psi_k\}$ is bounded in $BV_{loc}(\mathbb{R})$. Therefore, any accumulation point $\psi : \mathbb{R} \to \{\pm\sqrt{1 - {m_{1,\infty}}^2}\}$ of $\{\psi_k\}$ in L^1_{loc} is of bounded variation and has the form

$$\psi = \sum_{n=1}^{2N} (-1)^n \sqrt{1 - {m_{1,\infty}}^2}\ 1_{(b_{n-1}, b_n)},$$

where $-\infty = b_0 < b_1 < \cdots < b_{2N-1} < b_{2N} = +\infty$ and $b_n \in [-1,1]$ for $n = 1, \dots, 2N-1$. Finally, by (8.116), we have that $|\psi_k + m_{2,k}| \geq \frac{\sqrt{1-{m_{1,\infty}}^2}}{2}$ in $\mathbb{R}$ and therefore,

$$\begin{aligned}\int_{\mathbb{R}} |\psi_k - m_{2,k}|\, dt = \int_{-1}^{1} |\psi_k - m_{2,k}|\, dt &\leq \frac{2}{\sqrt{1-{m_{1,\infty}}^2}} \int_{-1}^{1} |\psi_k^2 - m_{2,k}^2|\, dt \\ &\leq \frac{2}{\sqrt{1-{m_{1,\infty}}^2}} \int_{-1}^{1} |(1-{m_{1,\infty}}^2) - m_{2,k}^2|\, dt \\ &\leq \frac{4}{\sqrt{1-{m_{1,\infty}}^2}} \int_{-1}^{1} |m_{1,k} - m_{1,\infty}|\, dt.\end{aligned}$$

We conclude by Step 1 that up to a subsequence, $m_{2,k} - \psi \to 0$ in $L^1(\mathbb{R})$, i.e.,

$$m_k' - \begin{pmatrix} m_{1,\infty} \\ \psi \end{pmatrix} \to 0 \quad \text{in} \quad L^1(\mathbb{R})$$

as $k \to \infty$. □

Since the asymptotic limit of the sequence $\{m_k'\}$ belongs to BV, one may ask whether the sequence $\{m_k'\}$ is bounded in BV. The answer is negative according to Theorem 8.3. The idea is that m_k' may have small variations on a large number of intervals (that have not been taken into account in the construction of the trial functions χ_k in the previous proof).

Proof of Theorem 8.3. For simplicity, we assume that $m_{1,\infty} = 0$. Set $\delta = \varepsilon^{1/4}$, $\omega = \varepsilon^{1/2}$ and $\eta = \varepsilon|\ln\varepsilon|$. For small $\varepsilon > 0$, we consider the following sample in $(-\omega, \omega)$:

$$f_\varepsilon(t) = \begin{cases} \frac{\delta}{|\ln\varepsilon|} \ln \frac{\omega}{\sqrt{t^2+\varepsilon^2}} & \text{if} \quad |t| \leq \sqrt{\omega^2 - \varepsilon^2}\,, \\ 0 & \text{if} \quad t \in (-\omega,\omega) \setminus (-\sqrt{\omega^2-\varepsilon^2}, \sqrt{\omega^2-\varepsilon^2}). \end{cases}$$

We define $m_{1,\varepsilon}$ as follows: we fill in the intervals $(-1, -\frac{1}{2})$ and $(\frac{1}{2}, 1)$ by at most $\frac{1}{2\omega}$ samples of length 2ω where $m_{1,\varepsilon}$ is given via f_ε. In the interval $(-\sqrt{\frac{1}{2}-\eta^2}, \sqrt{\frac{1}{2}-\eta^2})$, set

$$m_{1,\varepsilon}(t) = \frac{1}{|\ln(\sqrt{2}\eta)|} \ln \frac{1}{\sqrt{2(t^2+\eta^2)}}.$$

Otherwise, we set $m_{1,\varepsilon} = 0$. Hence, $m_{1,\varepsilon}$ is an H^1-function, $|m_{1,\varepsilon}| \leq \delta/2$ in $\mathbb{R} \setminus (-\frac{1}{2}, \frac{1}{2})$ and $m_{1,\varepsilon}(0) = 1$. We then define

$$m_{2,\varepsilon}(t) = \pm\sqrt{1 - m_{1,\varepsilon}^2(t)} \quad \text{if} \quad \pm\, t \geq 0;$$

hence, $m_{2,\varepsilon}$ is an H^1-function and (8.1) is satisfied. We compute the energy $E_\varepsilon^{1d}(\,(m_{1,\varepsilon}, m_{2,\varepsilon})\,)$. We have for $\varepsilon \ll 1$,

$$\begin{aligned}\int_{(-1,-\frac{1}{2})\cup(\frac{1}{2},1)} |\frac{dm_{1,\varepsilon}}{dt}|^2\, dt &\leq \frac{C}{\omega} \int_{-\omega}^{\omega} \frac{\delta^2}{|\ln\varepsilon|^2} \frac{t^2}{(t^2+\varepsilon^2)^2}\, dt \\ &\leq \frac{C}{\varepsilon|\ln\varepsilon|^2} \int_0^{\frac{\omega}{\varepsilon}} \frac{y^2}{(y^2+1)^2}\, dy \leq \frac{C}{\varepsilon|\ln\varepsilon|^2} \big(1 + \int_1^{\varepsilon^{-1/2}} \frac{dy}{y}\big) \leq \frac{C}{\varepsilon|\ln\varepsilon|}.\end{aligned} \tag{8.118}$$

Similarly, we compute that
$$\int_{(-\frac{1}{2},\frac{1}{2})} |\frac{dm_{1,\varepsilon}}{dt}|^2\,dt \leq \frac{C}{\varepsilon|\ln\varepsilon|}.$$
Now we compute the homogeneous $H^{1/2}-$norm of $m_{1,\varepsilon}$. For that, we extend the function $m_{1,\varepsilon}$ to the entire plane by
$$\tilde{m}_{1,\varepsilon}(t,s) = m_{1,\varepsilon}(\sqrt{t^2+s^2})\,,\ \forall (t,s)\in\mathbb{R}^2.$$
According to the trace estimate in $H^{1/2}$, it follows by the same argument as in (8.118),
$$\begin{aligned}\int_{\mathbb{R}} \Big| |\frac{d}{dt}|^{1/2} m_{1,\varepsilon}\Big|^2\,dt &\leq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \tilde{m}_{1,\varepsilon}(t,s)|^2\,dt\,ds \\ &\leq \frac{C}{\omega}\int_0^{\omega} \frac{\delta^2}{|\ln\varepsilon|^2}\frac{t^3}{(t^2+\varepsilon^2)^2}\,dt + \frac{C}{|\ln\eta|^2}\int_0^{1/2}\frac{t^3}{(t^2+\varepsilon^2)^2}\,dt \leq \frac{C}{|\ln\varepsilon|}.\end{aligned}$$
Hence, $|\ln\varepsilon| E_{\varepsilon}^{1d}(\,m'_{\varepsilon}\,) \leq C$ where $C>0$ is a universal constant. On the other hand, we have
$$\int_{\mathbb{R}} |\frac{dm_{1,\varepsilon}}{dt}|\,dt \geq \int_{(-1,-\frac{1}{2})\cup(\frac{1}{2},1)} |\frac{dm_{1,\varepsilon}}{dt}|\,dt \geq \frac{C\delta}{\omega|\ln\varepsilon|}\int_0^{\omega}\frac{t}{(t^2+\varepsilon^2)}\,dt \geq \frac{C}{\varepsilon^{1/4}} \to \infty \quad \text{as } \varepsilon\to 0.$$
□

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Abstract

In this dissertation, we first study the problem of lifting for functions $u \in BV(\Omega, S^1)$. We prove the existence of a BV lifting with an optimal control on the total variation. Then we compute the minimal variation of a lifting and construct an optimal lifting in the case of $\Omega \in \{S^1, S^2\}$; if $\Omega = S^2$, that relies on the study of topological singularities of u. We also show the connection between optimal liftings and minimizers of a $\Gamma-$limit energy.

In the second part, we study the vortex structure of a rotating Bose-Einstein condensate. We estimate the critical rotational speeds Ω_d for having exactly d vortices inside the bulk of the condensate and we determine their topological charge and their precise location.

Next we are interested in $1d$ transition layers which connect two opposite magnetisations (so called Néel walls) in a thin-film sample in micromagnetism. We prove the optimality of the Néel wall under $2d$ perturbations.

Keywords: BV functions, lifting, minimal connection, $\Gamma-$limit energy, Bose-Einstein condensate, vortices, renormalized energy, micromagnetism, Néel wall.