Energy expansion and vortex location for a two dimensional rotating Bose-Einstein condensate

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Abstract

We continue the analysis started in [14] on a two dimensional rotating Bose-Einstein condensate where we minimize a Gross-Pitaevskii energy defined in $\mathbb{R}^2$ under the unit mass constraint. We estimate the critical rotational speeds Ω_d for having d vortices in the bulk of the condensate and we determine precisely their location. Our approach relies on an asymptotic expansion of the energy.

1 Introduction

Since its first experimental achievement in dilute alkali gases, the phenomenon of Bose-Einstein condensation has given rise to a very active area of research in condensed matter physics. A Bose-Einstein condensate (BEC) is a quantum object in which every atom is in the lowest quantum state, so that it can be described by a single wave function. One of the most interesting feature of these systems is the superfluid behavior (see [10]): above some critical velocity, a BEC rotates through the existence of vortices, i.e., zeroes of the wave function around which there is a circulation of phase. When the angular speed gets larger, the number of vortices increases and they arrange themselves in a regular pattern around the center of the condensate. This has been observed experimentally by the ENS group [16, 17] and by the MIT group [1].

We consider here a two dimensional model describing a condensate placed in a trap that strongly confines the atoms in the direction of the rotation axis. In the nondimensionalized form (see [2]), the wave function u_ε minimizes the Gross-Pitaevskii energy

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right] - \Omega\, x^\perp \cdot (iu, \nabla u) \right\} dx \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1 \tag{1.2}$$

where $\varepsilon > 0$ is small and describes the ratio of two characteristic lengths and $\Omega = \Omega(\varepsilon) \geq 0$ is the angular velocity. The function $a(x)$ in (1.1) comes from the existence of a potential trapping the atoms, and is normalized such that $\int_{\mathbb{R}^2} a^+(x) = 1$. We will restrict our attention to the specific case of a harmonic trapping, that is $a(x) = a_0 - x_1^2 - \Lambda^2 x_2^2$ with $a_0 = \sqrt{2\Lambda/\pi}$ for some constant $\Lambda \in (0,1]$, which corresponds to actual experiments (see [16, 17]).

Our goal is to compute an asymptotic expansion of the energy $F_\varepsilon(u_\varepsilon)$ and to determine the number and the location of vortices of the wave function u_ε according to the value of the angular speed $\Omega(\varepsilon)$ in the limit $\varepsilon \to 0$. More precisely, we want to estimate the critical velocity Ω_d for which the dth vortex becomes energetically favorable and to express the part of the energy governing the location of the vortices (the "*renormalized energy*").

We have started in [14] the analysis of minimizers of the functional F_ε under the constraint (1.2) and we have determined the critical rotational speed $\Omega_1 = \frac{\sqrt{\pi}(1+\Lambda^2)}{\sqrt{2\Lambda}} |\ln\varepsilon|$ of nucleation of a first vortex inside the domain

$$\mathcal{D} = \big\{ x \in \mathbb{R}^2 \,:\, a(x) > 0 \big\}.$$

Actually, the set $\mathcal{D}$ represents the region occupied by the condensate since in the limit $\varepsilon \to 0$, the minimization of F_ε forces $|u_\varepsilon|^2$ to be close to a^+. We proved that for subcritical velocities $\Omega \leq \Omega_1 - \delta \ln|\ln\varepsilon|$ with $-\delta < \omega_1^\star < 0$ for some constant $\omega_1^\star$, there is no vortex in $\mathcal{D}$ and u_ε behaves as the "vortex-free profile" $\tilde{\eta}_\varepsilon e^{i\Omega S}$. Here, the phase function $S : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$S(x) = \frac{\Lambda^2 - 1}{\Lambda^2 + 1}\, x_1 \, x_2 \tag{1.3}$$

and $\tilde{\eta}_\varepsilon$ is the positive solution of the minimization problem

$$\operatorname{Min} \big\{ E_\varepsilon(u) \,:\, u \in \mathcal{H},\ \|u\|_{L^2(\mathbb{R}^2)} = 1 \big\} \tag{1.4}$$

where

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\big[(|u|^2 - a(x))^2 - (a^-(x))^2 \big] \quad \text{and} \quad \mathcal{H} = \big\{ u \in H^1(\mathbb{R}^2, \mathbb{C}) \,:\, \int_{\mathbb{R}^2} |x|^2 |u|^2 < \infty \big\}.$$

In this paper, we push forward the study of the shape of minimizers u_ε started in [14]. We find the following estimate of the critical speed Ω_d as $\varepsilon \to 0$,

$$\Omega_d = \frac{1+\Lambda^2}{a_0}(|\ln\varepsilon| + (d-1)\ln|\ln\varepsilon|) = \frac{\sqrt{\pi}(1+\Lambda^2)}{\sqrt{2\Lambda}}(|\ln\varepsilon| + (d-1)\ln|\ln\varepsilon|)$$

for any integer $d \geq 1$. We prove that for velocities ranged between Ω_d and Ω_{d+1}, the wave function has exactly d vortices of degree $+1$ inside $\mathcal{D}$ and we obtain the asymptotic expansion of $F_\varepsilon(u_\varepsilon)$ as $\varepsilon \to 0$. The vortices are distributed near the origin in a regular configuration in order to minimize the renormalized energy given by (1.5) below. We also improve the result stated in [14] for the nonexistence of vortices in the subcritical case by showing that the best constant is $\omega_1^\star = 0$, that is subcritical velocities go up to $\Omega_1 - \delta \ln|\ln\varepsilon|$ for any $\delta > 0$.

Our main theorem can be stated as follows:

Theorem 1.1. *Let u_ε be any minimizer of F_ε in $\mathcal{H}$ under the constraint* (1.2) *and let* $0 < \delta \ll 1$ *be any small constant.*

(i) If $\Omega \leq \Omega_1 - \delta \ln|\ln\varepsilon|$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_{R_0} > 0$ such that for any $\varepsilon < \varepsilon_{R_0}$, u_ε is vortex free in $B^\Lambda_{R_0} = \{x \in \mathbb{R}^2 : |x|^2_\Lambda = x_1^2 + \Lambda^2 x_2^2 \leq R_0^2\}$, i.e., u_ε does not vanish in $B^\Lambda_{R_0}$. In addition,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + o(1).$$

(ii) If $\Omega_d + \delta\ln|\ln\varepsilon| \leq \Omega \leq \Omega_{d+1} - \delta\ln|\ln\varepsilon|$ for some integer $d \geq 1$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_{R_0} > 0$ such that for any $\varepsilon < \varepsilon_{R_0}$, u_ε has exactly d vortices $x_1^\varepsilon, \ldots, x_d^\varepsilon$ of degree one in $B^\Lambda_{R_0}$. Moreover,

$$|x_j^\varepsilon| \leq C\,\Omega^{-1/2} \quad \textit{for any } j = 1, \ldots, d\,, \quad \textit{and} \quad |x_i^\varepsilon - x_j^\varepsilon| \geq C\,\Omega^{-1/2} \quad \textit{for any } i \neq j$$

where $C > 0$ denotes a constant independent of ε. Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\, x_j^\varepsilon$, the configuration $(\tilde{x}_1^\varepsilon, \ldots, \tilde{x}_d^\varepsilon)$ tends to minimize as $\varepsilon \to 0$ the renormalized energy

$$w(b_1, \ldots, b_d) = -\pi a_0 \sum_{i\neq j} \ln|b_i - b_j| + \frac{\pi a_0}{1+\Lambda^2} \sum_{j=1}^{d} |b_j|^2_\Lambda. \tag{1.5}$$

In addition,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \frac{\pi a_0^2 d}{1+\Lambda^2}(\Omega - \Omega_1) + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{d,\Lambda} + o(1) \tag{1.6}$$

where $Q_{d,\Lambda}$ is a constant depending only on d and Λ.

These results are in agreement with the study made by Castin and Dum [11] which have looked for minimizers in a reduced class of functions. More precisely, we find the same critical angular velocity Ω_1 as well as the regular distribution of vortices around the origin at a scale $\sqrt{\Omega}$. Our approach relies on the mathematical framework proposed by Aftalion and Du [2].

The minimizing configurations for the renormalized energy $w(\cdot)$ in the radial case $\Lambda = 1$ has been studied by Gueron and Shafrir in [12]. They prove that for $d \leq 6$, regular polygons centered at the origin and "stars" are local minimizers. For larger d, they numerically found minimizers with a shape of concentric polygons and then triangular lattices as d increases. These figures are exactly the ones observed in physical experiments (see [16, 17]).

Before describing the main ideas of the proof of Theorem 1.1, we shall recall some properties of any minimizer u_ε and of the profile $\tilde{\eta}_\varepsilon$ proved in [14]. In the regime

$$\Omega \leq \frac{1+\Lambda^2}{a_0}\Big(|\ln\varepsilon| + \omega_1 \ln|\ln\varepsilon|\Big) \tag{1.7}$$

for some constant $\omega_1 \in \mathbb{R}$, we proved the existence and smoothness of any minimizer u_ε of F_ε under the constraint (1.2) and some preliminary results about their behavior: $E_\varepsilon(u_\varepsilon) \leq C|\ln\varepsilon|^2$, $|u_\varepsilon| \lesssim \sqrt{a^+}$

in any compact $K \subset \mathcal{D}$ and $|u_\varepsilon|$ decreases exponentially quickly to 0 outside $\mathcal{D}$. For every $\varepsilon > 0$, we showed the existence and uniqueness of the positive minimizer $\tilde{\eta}_\varepsilon$ of E_ε under the mass constraint (1.2). Moreover, the corresponding Lagrange multiplier $k_\varepsilon \in \mathbb{R}$ satisfies

$$|k_\varepsilon| \leq C|\ln \varepsilon| \tag{1.8}$$

and we have $E_\varepsilon(\tilde{\eta}_\varepsilon) \leq C|\ln \varepsilon|$ for ε small and $\tilde{\eta}_\varepsilon \to \sqrt{a^+}$ in $L^\infty(\mathbb{R}^2) \cap C^1_{\text{loc}}(\mathcal{D})$ as $\varepsilon \to 0$. Using a splitting technique introduced by Lassoued and Mironescu [15], we obtained that for any $u \in \mathcal{H}$, the energy $F_\varepsilon(u)$ decouples into two independent parts: the energy of the "vortex-free profile" $\tilde{\eta}_\varepsilon e^{i\Omega S}$ and a reduced energy of $v = u/(\tilde{\eta}_\varepsilon e^{i\Omega S})$, i.e.,

$$F_\varepsilon(u) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v) + \tilde{\mathcal{T}}_\varepsilon(v) \tag{1.9}$$

where the functionals $\tilde{\mathcal{F}}_\varepsilon$ and $\tilde{\mathcal{T}}_\varepsilon$ are defined as follows

$$\tilde{\mathcal{F}}_\varepsilon(v) = \tilde{\mathcal{E}}_\varepsilon(v) + \tilde{\mathcal{R}}_\varepsilon(v), \tag{1.10}$$

$$\tilde{\mathcal{E}}_\varepsilon(v) = \int_{\mathbb{R}^2} \frac{\tilde{\eta}_\varepsilon^2}{2}|\nabla v|^2 + \frac{\tilde{\eta}_\varepsilon^4}{4\varepsilon^2}(|v|^2-1)^2\,, \quad \tilde{\mathcal{R}}_\varepsilon(v) = \frac{\Omega}{1+\Lambda^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 \nabla^\perp a \cdot (iv, \nabla v)\,, \tag{1.11}$$

$$\tilde{\mathcal{T}}_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^2} \big(\Omega^2|\nabla S|^2 - 2\Omega^2 x^\perp \cdot \nabla S + k_\varepsilon\big)\tilde{\eta}_\varepsilon^2(|v|^2-1). \tag{1.12}$$

Now the vortex structure of a minimizer u_ε can be studied via the map

$$v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S}),$$

applying the Ginzburg-Landau techniques to the weighted energy $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon)$. The difficulty will arise in the region where $\tilde{\eta}_\varepsilon$ is small and we will require the following properties of v_ε inherited from u_ε and $\tilde{\eta}_\varepsilon$ (see [14]): $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \leq C|\ln\varepsilon|^2$, $\big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)\big| \leq o(1)$, $|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)| \leq C|\ln\varepsilon|^2$, $|\nabla v_\varepsilon| \leq C_K \varepsilon^{-1}$ and $|v_\varepsilon| \lesssim 1$ in any compact $K \subset \mathcal{D}$. Since $\tilde{\eta}_\varepsilon$ is close to $\sqrt{a^+}$, it is more convenient to estimate the energies $\mathcal{F}_\varepsilon$, $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ (see Notations below) of v_ε inside $\mathcal{D}$. In the regime (1.7), we computed in [14] some fundamental bounds for the energy of v_ε in a domain slightly smaller than $\mathcal{D}$:

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq o(1), \tag{1.13}$$

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq C_{\omega_1}|\ln\varepsilon|, \tag{1.14}$$

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \{|x|_\Lambda < 2|\ln\varepsilon|^{-1/6}\}) \leq C_{\omega_1}\ln|\ln\varepsilon|, \tag{1.15}$$

where we denoted

$$\mathcal{D}_\varepsilon = \{x \in \mathcal{D} \,:\, a(x) > \nu_\varepsilon|\ln\varepsilon|^{-3/2}\} \tag{1.16}$$

and ν_ε is a chosen parameter in the interval $(1,2)$ (see Proposition 2.5). These estimates represent the starting point of our analysis here. They allow us to characterize a fine vortex structure for v_ε inside $\mathcal{D}$.

Now we proceed with the description of the proof of Theorem 1.1 while indicating the outline of the paper. In Section 2, we describes the vortex structure of v_ε in $B_R^\Lambda \subset\subset \mathcal{D}$ using the method of "bad discs" introduced by Bethuel, Brezis and Hélein [8]. We find that the number of bad discs is uniformly bounded, all of them remaining close to the origin (see Theorem 2.1). The main ingredients are the energy estimates (1.14) and (1.15) and a local version of the Pohozaev identity. Using the "clustering" method of Almeida and Bethuel [3] (see also Bethuel and Rivière [9], Serfaty [20, 21, 22]), we obtain a new family of disjoint discs $\big\{B(x_j^\varepsilon,\rho)\big\}_{j\in\tilde J_\varepsilon}$ with $\rho\sim\varepsilon^\alpha$ for some $\alpha\in(0,1)$ such that $|v_\varepsilon|\geq 1/2$ outside these discs and v_ε has a nonzero degree D_j on $\partial B(x_j^\varepsilon,\rho)$ (see Proposition 2.1). We identify *vortices* with the points x_j^ε. In Section 3, we find lower estimates of the energy taking into account the interaction between vortices. Following similar methods to [8], we evaluate the energy carried by each vortex (see Lemma 3.1)

$$\mathcal{E}_\varepsilon(v_\varepsilon,B(x_j^\varepsilon,\rho))\geq\pi a(x_j^\varepsilon)|D_j|\,\ln\frac{\rho}{\varepsilon}+\mathcal{O}(1) \tag{1.17}$$

and the energy away from the vortices (see Proposition 3.1)

$$\mathcal{E}_\varepsilon\big(v_\varepsilon,B_R^\Lambda\setminus\cup_{j\in\tilde J_\varepsilon}B(x_j^\varepsilon,\rho)\big)\geq\pi\sum_{j\in\tilde J_\varepsilon}D_j^2\,a(x_j^\varepsilon)|\ln\rho|+W_{R,\varepsilon}\big((x_j^\varepsilon,D_j)_{j\in\tilde J_\varepsilon}\big)+\mathcal{O}_R(1). \tag{1.18}$$

Here, the radius $R\in(\frac{\sqrt{a_0}}{2},\sqrt{a_0})$ is fixed and the error term $\mathcal{O}_R(1)$ is computed as a function of R. The quantity $W_{R,\varepsilon}$ is similar to the renormalized energy in [8] and involves the interaction between the vortices (see Proposition 3.1). An asymptotic expansion of $\mathcal{R}_\varepsilon(v_\varepsilon)$ away from the modified discs (see (3.15)) yields (see Lemma 3.2)

$$\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)\geq\mathcal{E}_\varepsilon(v_\varepsilon,B_R^\Lambda)-\frac{\pi\Omega}{1+\Lambda^2}\sum_{j\in\tilde J_\varepsilon}a^2(x_j^\varepsilon)\,D_j+o_R(1). \tag{1.19}$$

Section 4 is dedicated to the proof of Theorem 1.1. Combining (1.13), (1.17), (1.18) and (1.19), we deduce that every vortex is of degree 1, i.e., $D_j=1$ (see Lemma 4.1), which allows us to improve the above estimates and to obtain the result in the subcritical case *(i)* in Theorem 1.1. If $\Omega_d+\delta\ln|\ln\varepsilon|\leq\Omega\leq\Omega_{d+1}-\delta\ln|\ln\varepsilon|$ for any small $\delta>0$, we are led by the upper bound computed in Section 5 to the exact number of vortices Card $\tilde J_\varepsilon=d$ and to the following expansion of the energy (see Proposition 4.2)

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon)=-\frac{\pi a_0^2 d}{1+\Lambda^2}\,(\Omega-\Omega_1)+\frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|+\mathcal{O}(1).$$

Moreover, we find that the vortices are uniformly distributed at a scale $\Omega^{-1/2}$ around the origin (see Lemma 4.2). Then we compute an asymptotic formula of $W_{R,\varepsilon}$ as $\varepsilon\to 0$ and $R\to\sqrt{a_0}$ (see (4.21))

$$\lim_{\varepsilon\to 0}\Big\{W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon)+\pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon-x_j^\varepsilon|\Big\}=-\pi a_0 d^2\ell(\Lambda)+o(1)\text{ as }R\to\sqrt{a_0}$$

where $\ell(\Lambda)$ is a constant defined in Appendix which only depends on Λ. Using again the upper bound given by the test functions, we conclude that the rescaled configuration $(\tilde x_1^\varepsilon,\dots,\tilde x_d^\varepsilon)$ tends to minimize the renormalized energy w and we also find the complete expansion of the energy (1.6) (see Proposition 4.3). In Section 5, we construct appropriate test functions using a method due to André and Shafrir [5] and we obtain the upper bound of the energy announced in Section 4.

We emphasizes that our study concentrates on the vortex structure inside the domain $\mathcal{D}$. An interesting problem would be to analyze the vortices in the region where $|u_\varepsilon|$ is small, which surely requires other methods than energy estimates.

Notations. Throughout the paper, we denote by C a positive constant independent of ε and we use the subscript to point out a possible dependence on the argument. For $x = (x_1, x_2) \in \mathbb{R}^2$, we write

$$|x|_\Lambda = \sqrt{x_1^2 + \Lambda^2 x_2^2} \quad \text{and} \quad B_R^\Lambda = \big\{x \in \mathbb{R}^2,\, |x|_\Lambda < R\big\}$$

and for $\mathcal{A} \subset \mathbb{R}^2$,

$$\tilde{\mathcal{E}}_\varepsilon(v, \mathcal{A}) = \int_{\mathcal{A}} \frac{1}{2}\,\tilde{\eta}^2 |\nabla v|^2 + \frac{\tilde{\eta}^4}{4\varepsilon^2}(1 - |v|^2)^2\,, \quad \mathcal{E}_\varepsilon(v, \mathcal{A}) = \int_{\mathcal{A}} \frac{1}{2}\,a|\nabla v|^2 + \frac{a^2}{4\varepsilon^2}(1 - |v|^2)^2$$

$$\tilde{\mathcal{R}}_\varepsilon(v, \mathcal{A}) = \frac{\Omega}{1 + \Lambda^2} \int_{\mathcal{A}} \tilde{\eta}^2 \nabla^\perp a \cdot (iv, \nabla v)\,, \quad \mathcal{R}_\varepsilon(v, \mathcal{A}) = \frac{\Omega}{1 + \Lambda^2} \int_{\mathcal{A}} a \nabla^\perp a \cdot (iv, \nabla v)$$

$$\tilde{\mathcal{F}}_\varepsilon(v, \mathcal{A}) = \tilde{\mathcal{E}}_\varepsilon(v, \mathcal{A}) + \tilde{\mathcal{R}}_\varepsilon(v, \mathcal{A})\,, \quad \mathcal{F}_\varepsilon(v, \mathcal{A}) = \mathcal{E}_\varepsilon(v, \mathcal{A}) + \mathcal{R}_\varepsilon(v, \mathcal{A}). \tag{1.20}$$

We do not write the dependence on $\mathcal{A}$ when $\mathcal{A} = \mathbb{R}^2$.

2 Fine structure of vortices

The main goal of this section is to define a fine structure of vortices away from the boundary of $\mathcal{D}$. The analysis here follows the ideas in [8] and [9]. The main difficulty in our situation is due to the presence in the energy of the weight function $a(x)$ which vanishes on $\partial\mathcal{D}$ and it does not allow us to construct the structure up to the boundary. Throughout this paper, we assume that Ω satisfies (1.7), so that (1.13), (1.14) and (1.15) hold. We will prove the following results for $v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$:

Theorem 2.1. 1) *For any $R \in (\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$ there exists $\varepsilon_R > 0$ such that for any $\varepsilon < \varepsilon_R$,*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \textit{in } B_R^\Lambda \setminus B^\Lambda_{\frac{\sqrt{a_0}}{2}}\,.$$

2) *There exist some constants $N \in \mathbb{N}$, $\lambda_0 > 0$ and $\varepsilon_0 > 0$ (which only depend on ω_1) such that for any $\varepsilon < \varepsilon_0$, one can find a finite collection of points $\{x_j^\varepsilon\}_{j \in J_\varepsilon} \subset B^\Lambda_{\frac{\sqrt{a_0}}{4}}$ such that* $\mathrm{Card}(J_\varepsilon) \leq N$ *and*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \textit{in } \overline{B}^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \big(\cup_{j \in J_\varepsilon} B(x_j^\varepsilon, \lambda_0\varepsilon)\big).$$

Remark 2.1. The statement of Theorem 2.1 also holds if the radius $\frac{\sqrt{a_0}}{2}$ is replaced by an arbitrary $r \in (0, R)$ but then the constants in Theorem 2.1 depend on r. For the sake of simplicity, we prefer to fix $r = \frac{\sqrt{a_0}}{2}$.

In the next Proposition, we replace as in [20] the discs $\{B(x_j^\varepsilon, \lambda_0\varepsilon)\}_{j \in J_\varepsilon}$ obtained in Theorem 2.1 by slightly larger discs: $B(x_j^\varepsilon, \rho)$ (deleting some of the points x_j^ε if necessary), in order to get a precise information on the behavior of v_ε on $\partial B(x_j^\varepsilon, \rho)$. The centers of the resulting family of discs will represent the vortices of the map v_ε.

Proposition 2.1. *Let $0 < \beta < \mu < 1$ be given constants such that $\overline{\mu} := \mu^{N+1} > \beta$ and let $\{x_j^\varepsilon\}_{j \in J_\varepsilon}$ be the collection of points given by 2) in Theorem 2.1. There exists $0 < \varepsilon_1 < \varepsilon_0$ such that for any $\varepsilon < \varepsilon_1$, we can find $\tilde{J}_\varepsilon \subset J_\varepsilon$ and $\rho > 0$ verifying*

(i) $\lambda_0 \varepsilon \leq \varepsilon^\mu \leq \rho \leq \varepsilon^{\overline{\mu}} < \varepsilon^\beta$,

(ii) $|v_\varepsilon| \geq \dfrac{1}{2}$ *in* $\overline{B}^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \cup_{j \in \tilde{J}_\varepsilon} B(x_j^\varepsilon, \rho)$,

(iii) $|v_\varepsilon| \geq 1 - \dfrac{2}{|\ln \varepsilon|^2}$ *on* $\partial B(x_j^\varepsilon, \rho)$ *for every* $j \in \tilde{J}_\varepsilon$,

(iv) $\displaystyle\int_{\partial B(x_j^\varepsilon, \rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \leq \frac{C(\beta, \mu)}{\rho}$ *for every* $j \in \tilde{J}_\varepsilon$,

(v) $|x_i^\varepsilon - x_j^\varepsilon| \geq 8\rho$ *for every* $i, j \in \tilde{J}_\varepsilon$ *with* $i \neq j$.

Moreover, for each $j \in \tilde{J}_\varepsilon$, we have

$$D_j := \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_j^\varepsilon, \rho)\right) \neq 0 \quad \text{and} \quad |D_j| \leq C \tag{2.1}$$

for a constant C independent of ε.

Remark 2.2. We point out that each disc $B(x_j^\varepsilon, \rho)$ carries at least one zero of v_ε since $D_j \neq 0$ for any $j \in \tilde{J}_\varepsilon$.

2.1 Some local estimates

We start with a fundamental lemma. It strongly relies on Pohozaev's identity and it will play a similar role as Theorem III.2 in [8]. In our situation, we only derive local estimates as in [3, 9, 23]. Some of the arguments used in the proof are taken from [3, 9].

Lemma 2.1. *For any $0 < R < \sqrt{a_0}$ and $\frac{2}{3} < \alpha < 1$, there exists a positive constant $C_{R,\alpha}$ such that*

$$\frac{1}{\varepsilon^2} \int_{B(x_0, \varepsilon^\alpha)} (1 - |v_\varepsilon|^2)^2 \leq C_{R,\alpha} \quad \textit{for any } x_0 \in B_R^\Lambda.$$

Proof. Step 1. Set $\tilde{u}_\varepsilon = u_\varepsilon e^{-i\Omega S}$. We claim that

$$E_\varepsilon(\tilde{u}_\varepsilon, \mathcal{D}_\varepsilon) \leq C |\ln \varepsilon| \tag{2.2}$$

where $\mathcal{D}_\varepsilon$ is defined in (1.16). Indeed, since $\tilde{u}_\varepsilon = \tilde{\eta}_\varepsilon v_\varepsilon$, we get that

$$|\nabla \tilde{u}_\varepsilon|^2 \leq C(\tilde{\eta}_\varepsilon^2 |\nabla v_\varepsilon| + |v_\varepsilon|^2 |\nabla \tilde{\eta}_\varepsilon|^2)$$

By the results in [14], we know that $|v_\varepsilon| \leq C$, $\tilde{\eta}_\varepsilon^2 \leq Ca$ in $\mathcal{D}_\varepsilon$ and $E_\varepsilon(\tilde{\eta}_\varepsilon) \leq C|\ln \varepsilon|$ and consequently,

$$\int_{\mathcal{D}_\varepsilon} |\nabla \tilde{u}_\varepsilon|^2 \leq C\left(\int_{\mathcal{D}_\varepsilon} a(x) |\nabla v_\varepsilon|^2 + \int_{\mathcal{D}_\varepsilon} |\nabla \tilde{\eta}_\varepsilon|^2\right) \leq C |\ln \varepsilon|$$

by (1.14). On the other hand, we also have

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathcal{D}_\varepsilon}(a(x)-|\tilde u_\varepsilon|^2)^2 &\le \frac{C}{\varepsilon^2}\int_{\mathcal{D}_\varepsilon}\big[(a(x)-\tilde\eta_\varepsilon^2)^2+\tilde\eta_\varepsilon^4(1-|v_\varepsilon|^2)^2\big]\\ &\le \frac{C}{\varepsilon^2}\bigg(\int_{\mathcal{D}_\varepsilon}(a(x)-\tilde\eta_\varepsilon^2)^2+\int_{\mathcal{D}_\varepsilon}a^2(x)(1-|v_\varepsilon|^2)^2\bigg)\le C|\ln\varepsilon|\end{aligned}$$

and therefore (2.2) follows.

Step 2. We are going to show that one can find a constant $C_{R,\alpha}>0$, independent of ε, such that for any $x_0\in B_R^\Lambda$, there is some $r_0\in(\varepsilon^\alpha,\varepsilon^{\alpha/2+1/3})$ satisfying

$$E_\varepsilon\left(\tilde u_\varepsilon,\partial B(x_0,r_0)\right)\le\frac{C_{R,\alpha}}{r_0}.$$

We proceed by contradiction. Assume that for all $M>0$, there is $x_M\in B_R^\Lambda$ such that

$$E_\varepsilon\left(\tilde u_\varepsilon,\partial B(x_M,r)\right)\ge\frac{M}{r},\quad \text{for any } r\in\,(\varepsilon^\alpha,\varepsilon^{\alpha/2+1/3}). \tag{2.3}$$

Obviously, for ε small, $B(x_M,\varepsilon^{\alpha/2+1/3})\subset\mathcal{D}_\varepsilon$. Integrating (2.3) for $r\in\,(\varepsilon^\alpha,\varepsilon^{\alpha/2+1/3})$, we derive that

$$E_\varepsilon\left(\tilde u_\varepsilon,\mathcal{D}_\varepsilon\right)\ge M\int_{\varepsilon^\alpha}^{\varepsilon^{\alpha/2+1/3}}\frac{dr}{r}=M(\alpha/2-1/3)|\ln\varepsilon|$$

which contradicts Step 1 for M large enough.

Step 3. Fix $x_0\in B_R^\Lambda$ and let $r_0\in(\varepsilon^\alpha,\varepsilon^{\alpha/2+1/3})$ be given by Step 2. We recall that any minimizer u_ε of F_ε in $\big\{u\in\mathcal{H},\,\|u\|_{L^2(\mathbb{R}^2)}=1\big\}$ satisfies

$$-\Delta u_\varepsilon+2i\Omega x^\perp\cdot\nabla u_\varepsilon=\frac{1}{\varepsilon^2}(a(x)-|u_\varepsilon|^2)u_\varepsilon+\ell_\varepsilon u_\varepsilon\quad\text{in }\mathbb{R}^2$$

where ℓ_ε denotes the Lagrange multiplier. Therefore, we have

$$\begin{aligned}-\Delta\tilde u_\varepsilon=\frac{1}{\varepsilon^2}(a(x_0)-|\tilde u_\varepsilon|^2)\tilde u_\varepsilon+\frac{1}{\varepsilon^2}(a(x)-a(x_0))\tilde u_\varepsilon+2i\Omega(\nabla S-x^\perp)\cdot\nabla\tilde u_\varepsilon\\ +(\ell_\varepsilon+2\Omega^2x^\perp\cdot\nabla S-\Omega^2|\nabla S|^2)\tilde u_\varepsilon\quad\text{in }B(x_0,r_0).\end{aligned} \tag{2.4}$$

As in the proof of the Pohozaev identity, we multiply (2.4) by $(x-x_0)\cdot\nabla\tilde u_\varepsilon$ and we integrate by parts in $B(x_0,r_0)$. We have

$$\int_{B(x_0,r_0)}-\Delta\tilde u_\varepsilon\cdot[(x-x_0)\cdot\nabla\tilde u_\varepsilon]=\frac{r_0}{2}\int_{\partial B(x_0,r_0)}|\nabla\tilde u_\varepsilon|^2-r_0\int_{\partial B(x_0,r_0)}\left|\frac{\partial\tilde u_\varepsilon}{\partial\nu}\right|^2 \tag{2.5}$$

and

$$\begin{aligned}&\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)}(a(x_0)-|\tilde u_\varepsilon|^2)\tilde u_\varepsilon\cdot[(x-x_0)\cdot\nabla\tilde u_\varepsilon]=\\ &\quad=\frac{1}{2\varepsilon^2}\int_{B(x_0,r_0)}(a(x_0)-|\tilde u_\varepsilon|^2)^2-\frac{r_0}{4\varepsilon^2}\int_{\partial B(x_0,r_0)}(a(x_0)-|\tilde u_\varepsilon|^2)^2\end{aligned} \tag{2.6}$$

(where ν is the outer normal vector to $\partial B(x_0, r_0)$). From (2.4), (2.5) and (2.6) we derive that

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)}(a(x_0)-|\tilde u_\varepsilon|^2)^2 \le C\bigg(r_0\int_{\partial B(x_0,r_0)}|\nabla\tilde u_\varepsilon|^2 + r_0\varepsilon^{-2}\int_{\partial B(x_0,r_0)}(a(x_0)-|\tilde u_\varepsilon|^2)^2 \\ + r_0\varepsilon^{-2}\int_{B(x_0,r_0)}|a(x)-a(x_0)||\tilde u_\varepsilon||\nabla\tilde u_\varepsilon| + \Omega r_0\int_{B(x_0,r_0)}|\nabla\tilde u_\varepsilon|^2 \\ + (\Omega^2+|\ell_\varepsilon|)r_0\int_{B(x_0,r_0)}|\tilde u_\varepsilon||\nabla\tilde u_\varepsilon|\bigg).\end{aligned}$$

Then we estimate each integral term in the right hand side of the previous inequality. By the results in [14], we have $|\ell_\varepsilon| \le C\varepsilon^{-1}|\ln\varepsilon|$ and $|\tilde u_\varepsilon| \le C$ in $\mathbb{R}^2$. According to (2.2), we obtain

$$\begin{aligned}\varepsilon^{-2}\int_{\partial B(x_0,r_0)}(a(x_0)-|\tilde u_\varepsilon|^2)^2 &\le C\varepsilon^{-2}\int_{\partial B(x_0,r_0)}\big[(a(x_0)-a(x))^2+(a(x)-|\tilde u_\varepsilon|^2)^2\big] \\ &\le C\varepsilon^{-2}\int_{\partial B(x_0,r_0)}(a(x)-|\tilde u_\varepsilon|^2)^2 + C_R\varepsilon^{\frac{3}{2}\alpha-1},\end{aligned}$$

and

$$\Omega r_0\int_{B(x_0,r_0)}|\nabla\tilde u_\varepsilon|^2 \le \Omega r_0 E_\varepsilon(\tilde u_\varepsilon,\mathcal{D}_\varepsilon) \le C_R\,\varepsilon^{\alpha/2+1/3}|\ln\varepsilon|^2,$$

and

$$\begin{aligned}r_0\varepsilon^{-2}\int_{B(x_0,r_0)}|a(x)-a(x_0)||\tilde u_\varepsilon||\nabla\tilde u_\varepsilon| &\le C_R\,r_0^2\,\varepsilon^{-2}\int_{B(x_0,r_0)}|\nabla\tilde u_\varepsilon| \\ &\le C_R\,r_0^3\,\varepsilon^{-2}[E_\varepsilon(\tilde u_\varepsilon,\mathcal{D}_\varepsilon)]^{1/2} \le C_R\,\varepsilon^{\frac{3}{2}\alpha-1}|\ln\varepsilon|^{1/2},\end{aligned}$$

and

$$(\Omega^2+|\ell_\varepsilon|)r_0\int_{B(x_0,r_0)}|\tilde u_\varepsilon||\nabla\tilde u_\varepsilon| \le C_R\varepsilon^{-1}|\ln\varepsilon|\,r_0^2\,[E_\varepsilon(\tilde u_\varepsilon,\mathcal{D}_\varepsilon)]^{1/2} \le C_R\,\varepsilon^{\alpha-\frac{1}{3}}|\ln\varepsilon|^{3/2}$$

(here we use that $|a(x)-a(x_0)| \le C_R\,r_0$ for any $x \in B(x_0,r_0)$). We finally get that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)}(a(x_0)-|\tilde u_\varepsilon|^2)^2 \le C_{R,\alpha}\big(1+r_0E_\varepsilon\left(\tilde u_\varepsilon,\partial B(x_0,r_0)\right)\big)$$

for some constant $C_{R,\alpha}$ independent of ε. By Step 2, we conclude that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x_0)-|\tilde u_\varepsilon|^2)^2 \le C_{R,\alpha}. \tag{2.7}$$

Since $\|\tilde\eta_\varepsilon-\sqrt{a}\|_{C^1(B_R^\Lambda)} \le C_R\varepsilon^2|\ln\varepsilon|$ by [14] , we have

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(1-|v_\varepsilon|^2)^2 &\le \frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(\tilde\eta_\varepsilon^2-|\tilde u_\varepsilon|^2)^2 \\ &\le \frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x)-|\tilde u_\varepsilon|^2)^2+o(1) \\ &\le \frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x_0)-|\tilde u_\varepsilon|^2)^2+o(1) \le C_{R,\alpha}\end{aligned}$$

and we conclude with (2.7). ■

The next result will allow us to define the notion of a bad disc as in [8].

Proposition 2.2. *For any $0 < R < \sqrt{a_0}$, there exist two positive constants λ_R and μ_R such that if*

$$\frac{1}{\varepsilon^2}\int_{B(x_0,2l)}(1-|v_\varepsilon|^2)^2 \le \mu_R \quad \textit{with } x_0 \in B_R^\Lambda\,,\ \frac{l}{\varepsilon} \ge \lambda_R \ \textit{ and } l \le \frac{\sqrt{a_0}-R}{2},$$

then $|v_\varepsilon| \ge 1/2$ in $B(x_0,l)$.

Proof. In [14], we proved the existence of a constant $C_R > 0$ independent of ε such that

$$|\nabla v_\varepsilon| \le \frac{C_R}{\varepsilon} \quad \text{in } B^\Lambda_{\frac{\sqrt{a_0}+R}{2}}.$$

Then the result follows as in [8], Theorem III.3. ■

Definition 2.1. For $0 < R < \sqrt{a_0}$ and $x \in B_R^\Lambda$, we say that $B(x,\lambda_R\varepsilon)$ is a **bad disc** if

$$\frac{1}{\varepsilon^2}\int_{B(x,2\lambda_R\varepsilon)}(1-|v_\varepsilon|^2)^2 \ge \mu_R.$$

Now we can give a local version of Theorem 2.1. We will see that Lemma 2.1 plays a crucial role in the proof.

Proposition 2.3. *For any $0 < R < \sqrt{a_0}$ and $\frac{2}{3} < \alpha < 1$, there exist positive constants $N_{R,\alpha}$ and $\varepsilon_{R,\alpha}$ such that for every $\varepsilon < \varepsilon_{R,\alpha}$ and $x_0 \in B_R^\Lambda$, one can find $x_1,\dots,x_{N_\varepsilon} \in B(x_0,\varepsilon^\alpha)$ with $N_\varepsilon \le N_{R,\alpha}$ verifying*

$$|v_\varepsilon| \ge \frac{1}{2} \quad \textit{in} \quad B(x_0,\varepsilon^\alpha)\setminus\left(\cup_{k=1}^{N_\varepsilon}B(x_k,\lambda_R\varepsilon)\right).$$

Proof. We follow the ideas in [8], Chapter IV. Consider a family of discs $\{B(x_i,\lambda_R\varepsilon)\}_{i\in\mathcal{F}}$ such that

$$x_i \in B(x_0,\varepsilon^\alpha), \tag{2.8}$$

$$B\left(x_i,\frac{\lambda_R\varepsilon}{4}\right)\cap B\left(x_j,\frac{\lambda_R\varepsilon}{4}\right)=\emptyset \quad \text{for } i\ne j, \tag{2.9}$$

$$B(x_0,\varepsilon^\alpha)\subset\bigcup_{i\in\mathcal{F}}B(x_i,\lambda_R\varepsilon).$$

Obviously, the discs $\{B(x_i,2\lambda_R\varepsilon)\}_{i\in\mathcal{F}}$ cannot intersect more that C times (where C is a universal constant) and

$$\bigcup_{i\in\mathcal{F}}B(x_i,2\lambda_R\varepsilon)\subset B(x_0,\varepsilon^{\alpha'})$$

with $\alpha' = \frac{1}{2}(\alpha+\frac{2}{3})$. We denote by $\mathcal{F}'$ the set of indices $i\in\mathcal{F}$ such that $B(x_i,\lambda_R\varepsilon)$ is a bad disc. We derive from Definition 2.1 that

$$\mu_R\,\mathrm{Card}(\mathcal{F}') \le \sum_{i\in\mathcal{F}}\frac{1}{\varepsilon^2}\int_{B(x_i,2\lambda_R\varepsilon)}(1-|v_\varepsilon|^2)^2 \le \frac{C}{\varepsilon^2}\int_{B(x_0,\varepsilon^{\alpha'})}(1-|v_\varepsilon|^2)^2.$$

The conclusion now follows by Lemma 2.1 and Proposition 2.2. ■

Remark 2.3. By the proof of Proposition 2.3, it follows that any family of discs $\{B(x_i, \lambda_R\varepsilon)\}_{i\in\mathcal{F}}$ satisfying (2.8) and (2.9) cannot contain more than $N_{R,\alpha}$ bad discs.

We will need the following lemma to prove that vortices of degree zero do not occur. The main ingredients in the proof come from [3, 9].

Lemma 2.2. *Let $D > 0$, $0 < \beta < 1$ and $\gamma > 1$ be given constants such that $\gamma\beta < 1$. Let $0 < R < \sqrt{a_0}$ and $0 < \rho < \varepsilon^\beta$ be such that $\rho^\gamma > \lambda_R\varepsilon$. We assume that for $x_0 \in B_R^\Lambda$,*

(i) $\displaystyle\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 < \frac{D}{\rho}$,

(ii) $|v_\varepsilon| \geq \dfrac{1}{2}$ *on* $\partial B(x_0,\rho)$,

(iii) $\deg\left(\dfrac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_0,\rho)\right) = 0$.

Then we have
$$|v_\varepsilon| \geq \frac{1}{2} \quad \textit{in } B(x_0,\rho^\gamma).$$

Proof of Lemma 2.2. We are going to construct a comparison function as in [3] or [9] to obtain the following estimate:
$$\int_{B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq C_{\beta,R}. \tag{2.10}$$
Since the degree of v_ε restricted to $\partial B(x_0,\rho)$ is zero, we may write on $\partial B(x_0,\rho)$
$$v_\varepsilon = |v_\varepsilon| e^{i\phi_\varepsilon}$$
where ϕ_ε is a smooth map from $\partial B(x_0,\rho)$ into $\mathbb{R}$. Then we define $\hat{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$ by
$$\begin{cases} \hat{v}_\varepsilon = \chi_\varepsilon e^{i\psi_\varepsilon} & \text{in } B(x_0,\rho) \\ \hat{v}_\varepsilon = v_\varepsilon & \text{in } \mathbb{R}^2 \setminus B(x_0,\rho) \end{cases}$$
where ψ_ε is the solution of
$$\begin{cases} \Delta\psi_\varepsilon = 0 & \text{in } B(x_0,\rho) \\ \psi_\varepsilon = \phi_\varepsilon & \text{on } \partial B(x_0,\rho), \end{cases}$$
and χ_ε has the form, written in polar coordinates centered at x_0,
$$\chi_\varepsilon(r,\theta) = (|v_\varepsilon(\rho e^{i\theta})| - 1)\xi(r) + 1$$
and ξ is a smooth function taking values in $[0,1]$ with small support near ρ with $\xi(\rho) = 1$. By the results in [14], we know that $|v_\varepsilon(x)| \leq 1 + C\,\varepsilon^{1/3}$ for $x \in \mathcal{D}$ with $|x|_\Lambda \geq \sqrt{a_0} - \varepsilon^{1/8}$ and we deduce that $0 \leq \chi_\varepsilon \leq 1 + C\varepsilon^{1/3}$. Arguing as in [7], proof of Theorem 2, we may prove that
$$\int_{B(x_0,\rho)} |\nabla\psi_\varepsilon|^2 \leq C\rho \int_{\partial B(x_0,\rho)} \left|\frac{\partial\phi_\varepsilon}{\partial\tau}\right|^2 \leq C\rho \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 \tag{2.11}$$

and

$$\int_{B(x_0,\rho)} |\nabla \chi_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-\chi_\varepsilon^2)^2 \le C\rho \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 + O(\rho). \tag{2.12}$$

From (2.11), (2.12) and assumption *(i)*, we infer that

$$\int_{B(x_0,\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 \le C\,. \tag{2.13}$$

We set $\tilde{v}_\varepsilon = m_\varepsilon^{-1}\,\hat{v}_\varepsilon$ with $m_\varepsilon = \|\tilde{\eta}_\varepsilon \hat{v}_\varepsilon\|_{L^2(\mathbb{R}^2)}$. Clearly, $\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon \in \mathcal{H}$ and $\|\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. Since $u_\varepsilon = \tilde{\eta}_\varepsilon e^{i\Omega S} v_\varepsilon$ minimizes the functional F_ε under the constraint (1.2), we have $F_\varepsilon(u_\varepsilon) \le F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon)$ and by (1.9), it yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \le \tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon). \tag{2.14}$$

We claim that

$$\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) \le \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon) + C\rho|\ln\varepsilon|^2 \quad \text{and} \quad \big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) - \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\big| = O(\rho^2|\ln\varepsilon|^2). \tag{2.15}$$

Indeed, we proved in [14]

$$\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le C|\ln\varepsilon|^2 \quad \text{and} \quad \big|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)\big| \le C|\ln\varepsilon|^2 \tag{2.16}$$

so that, using (2.13), $\|\tilde{\eta}_\varepsilon v_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, $\hat{v}_\varepsilon = v_\varepsilon$ in $\mathbb{R}^2 \setminus B(x_0,\rho)$ and (2.16), we obtain

$$\begin{aligned} m_\varepsilon^2 &= 1 + \int_{B(x_0,\rho)} \tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1) + \int_{B(x_0,\rho)} \tilde{\eta}_\varepsilon^2(1-|v_\varepsilon|^2) \\ &= 1 + O(\rho\,\varepsilon|\ln\varepsilon|). \end{aligned} \tag{2.17}$$

From (2.13), (2.16) and (2.17), we derive

$$\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\nabla \tilde{v}_\varepsilon|^2 = m_\varepsilon^{-2} \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\nabla \hat{v}_\varepsilon|^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\nabla \hat{v}_\varepsilon|^2 + O(\rho\varepsilon|\ln\varepsilon|^3) \tag{2.18}$$

and

$$\tilde{\mathcal{R}}_\varepsilon(\tilde{v}_\varepsilon) = m_\varepsilon^{-2}\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) = \tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) + O(\rho\varepsilon|\ln\varepsilon|^3). \tag{2.19}$$

Since u_ε remains bounded in $\mathbb{R}^2$ and $E_\varepsilon(u_\varepsilon) \le C|\ln\varepsilon|^2$ by the results in [14], we infer from (2.16),

$$\begin{aligned} \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4 (1-|\tilde{v}_\varepsilon|^2)^2 &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1-|\hat{v}_\varepsilon|^2)^2 + \frac{2(1-m_\varepsilon^{-2})}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2(1-|\hat{v}_\varepsilon|^2)|\tilde{\eta}_\varepsilon\hat{v}_\varepsilon|^2 \\ &\quad + \frac{(1-m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2} |\tilde{\eta}_\varepsilon \hat{v}_\varepsilon|^4 \\ &\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1-|\hat{v}_\varepsilon|^2)^2 \\ &\quad + C\rho|\ln\varepsilon| \left(\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2\setminus B(x_0,\rho)} \tilde{\eta}_\varepsilon^4(1-|v_\varepsilon|^2)^2\right)^{1/2} \left(\int_{\mathbb{R}^2\setminus B(x_0,\rho)} |u_\varepsilon|^4\right)^{1/2} \\ &\quad + C\rho^2|\ln\varepsilon|^2 \\ &\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1-|\hat{v}_\varepsilon|^2)^2 + C\rho|\ln\varepsilon|^2. \end{aligned} \tag{2.20}$$

Finally, we obtain in the same way,

$$\begin{aligned}\left|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)-\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\right| &\le \left|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)-\tilde{\mathcal{T}}_\varepsilon(\hat{v}_\varepsilon)\right|+\left|\tilde{\mathcal{T}}_\varepsilon(\hat{v}_\varepsilon)-\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\right| && (2.21)\\ &\le C|\ln\varepsilon|^2\left(\int_{B(x_0,\rho)}(1+|x|^2)\tilde{\eta}_\varepsilon^2+|1-m_\varepsilon^{-2}|\int_{\mathbb{R}^2}(1+|x|^2)\tilde{\eta}_\varepsilon^2|\hat{v}_\varepsilon|^2\right)\\ &\le C\rho^2|\ln\varepsilon|^2. && (2.22)\end{aligned}$$

From (2.18), (2.19), (2.20) and (2.21), we conclude that (2.15) holds.

Since $\hat{v}_\varepsilon=v_\varepsilon$ in $\mathbb{R}^2\setminus B(x_0,\rho)$, we get from (2.14) and (2.15) that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,B(x_0,\rho))\le\tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon,B(x_0,\rho))+C\rho|\ln\varepsilon|^2.$$

By (2.13), we have $\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,B(x_0,\rho))\le C$ and therefore,

$$\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,B(x_0,\rho))\right|\le C\Omega\int_{B(x_0,\rho)}|\nabla\hat{v}_\varepsilon|\le C\Omega\rho\|\nabla\hat{v}_\varepsilon\|_{L^2(B(x_0,\rho))}=O(\rho|\ln\varepsilon|). \tag{2.23}$$

Hence, $\tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon,B(x_0,\rho))\le C$ and we conclude that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,B(x_0,\rho))\le C_\beta.$$

As for (2.23), using (2.16) we easily derive that $|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon,B(x_0,\rho))|=O(\rho|\ln\varepsilon|^2)$ and we finally get that $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon,B(x_0,\rho))\le C_\beta$ which clearly implies (2.10) since $\tilde{\eta}_\varepsilon^2\to a^+$ uniformly as $\varepsilon\to0$ (see [14]).

We deduce from (2.10) that

$$\int_{2\rho^\gamma}^{\rho}\left(\int_{\partial B(x_0,s)}|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right)ds\le C_{\beta,R}.$$

Since $\int_{2\rho^\gamma}^{\rho}\frac{ds}{s|\ln s|^{1/2}}\ge C_\gamma|\ln\varepsilon|^{1/2}$, we derive that for small ε there exists $s_0\in[2\rho^\gamma,\rho]$ such that

$$\int_{\partial B(x_0,s_0)}|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\le\frac{C_{\beta,R}}{s_0|\ln s_0|^{1/2}}\,.$$

Repeating the arguments used to prove (2.10), we find that

$$\int_{B(x_0,s_0)}|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\le\frac{C_{\beta,R}}{|\ln s_0|^{1/2}}.$$

In particular, we have

$$\frac{1}{\varepsilon^2}\int_{B(x_0,2\rho^\gamma)}(1-|v_\varepsilon|^2)^2=o(1)$$

and the conclusion follows by Proposition 2.2. ∎

We now obtain as in [9] Proposition IV.3 the following result which gives us an estimate of the contribution in the energy of any vortex.

Proposition 2.4. *Let $0 < R < \sqrt{a_0}$ and $\frac{2}{3} < \alpha < 1$. Let $x_0 \in B_R^\Lambda$ and assume that $|v_\varepsilon(x_0)| < \frac{1}{2}$. Then there exists a positive constant $C_{R,\alpha}$ (which only depends on R, α and ω_1) such that*

$$\int_{B(x_0,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_{R,\alpha} |\ln \varepsilon|.$$

Proof. Let $N_{R,\alpha}$ and $x_1, \ldots, x_{N_\varepsilon} \in B(x_0, \varepsilon^\alpha)$ be as in Proposition 2.3. We set

$$\delta_\alpha = \frac{\alpha^{1/2} - \alpha}{3(N_{R,\alpha} + 1)}$$

and for $k = 0, \ldots, 3N_{R,\alpha} + 2$, we consider

$$\alpha_k = \alpha^{1/2} - k\delta_\alpha \ , \quad \mathcal{I}_k = [\varepsilon^{\alpha_k}, \varepsilon^{\alpha_{k+1}}] \quad \text{and} \quad \mathcal{C}_k = B(x_0, \varepsilon^{\alpha_{k+1}}) \setminus B(x_0, \varepsilon^{\alpha_k}).$$

Then there is some $k_0 \in \{1, \ldots, 3N_{R,\alpha} + 1\}$ such that

$$\mathcal{C}_{k_0} \cap \left(\cup_{j=1}^{N_\varepsilon} B(x_j, \lambda_R \varepsilon) \right) = \emptyset. \tag{2.24}$$

Indeed, since $N_\varepsilon \leq N_{R,\alpha}$ and $2\lambda_R\varepsilon < |\mathcal{I}_k|$ for small ε, the union of N_ε intervals of length $2\lambda_R\varepsilon$

$$\bigcup_{j=1}^{N_\varepsilon} \big(|x_i - x_0| - \lambda_R\varepsilon, |x_i - x_0| + \lambda_R\varepsilon \big)$$

cannot intersect all the intervals $\mathcal{I}_k$ of disjoint interior, for $1 \leq k \leq 3N_{R,\alpha} + 1$. From (2.24) we deduce that

$$|v_\varepsilon(x)| \geq \frac{1}{2} \quad \text{for any } x \in \mathcal{C}_{k_0}.$$

Therefore, for every $\rho \in \mathcal{I}_{k_0}$,

$$d_{k_0} = \deg \left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_0, \rho) \right)$$

is well defined and does not depend on ρ. We claim that

$$d_{k_0} \neq 0. \tag{2.25}$$

By contradiction, we suppose that $d_{k_0} = 0$. According to (1.14), it results that

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}+R}{2}}} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \leq C_R |\ln \varepsilon|.$$

Using the same argument as in Step 2 of the proof of Lemma 2.1, there is a constant $C_{R,\alpha}$ such that

$$\int_{\partial B(x_0,\rho_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \leq \frac{C_{R,\alpha}}{\rho_0} \quad \text{for some } \rho_0 \in \mathcal{I}_{k_0}.$$

According to Lemma 2.2 (with $\beta = \alpha_{k_0+1}$ and $\gamma = \frac{\alpha_{k_0-1}}{\alpha_{k_0}}$), we should have $|v_\varepsilon(x_0)| \geq \frac{1}{2}$ which is a contradiction.

By (2.25), we obtain for every $\rho \in \mathcal{I}_{k_0}$,

$$1 \leq |d_{k_0}| = \frac{1}{2\pi} \left| \int_{\partial B(x_0,\rho)} \frac{1}{|v_\varepsilon|^2} \big(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau}\big) \right| \leq C \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|$$

(we use that $|v_\varepsilon| \geq \frac{1}{2}$ in $\mathcal{C}_{k_0}$). Then Cauchy-Schwarz inequality yields

$$\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 \geq \frac{C}{\rho} \quad \text{for any } \rho \in \mathcal{I}_{k_0}$$

and the conclusion follows integrating on $\mathcal{I}_{k_0}$. ∎

2.2 Proofs of Theorem 2.1 and Proposition 2.1

The part 1) in Theorem 2.1 follows directly from Lemma 2.3 below.

Lemma 2.3. *There exists a constant $\varepsilon_R > 0$ such that for any $0 < \varepsilon < \varepsilon_R$,*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B_R^\Lambda \setminus B^\Lambda_{\frac{\sqrt{a_0}}{5}}.$$

Proof. First, we fix some $\alpha \in (\frac{2}{3}, 1)$. We proceed by contradiction. Suppose that there is some $x_0 \in B_R^\Lambda \setminus B^\Lambda_{\frac{\sqrt{a_0}}{5}}$ such that $|v_\varepsilon(x_0)| < 1/2$. Then for any ε sufficiently small, we have $B(x_0, \varepsilon^\alpha) \subset \mathcal{D}_\varepsilon \setminus \{|x|_\Lambda < 2|\ln\varepsilon|^{-1/6}\}$ and therefore, by (1.15), we get that

$$\int_{B(x_0,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \leq C_R\, \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \{|x|_\Lambda < 2|\ln\varepsilon|^{-1/6}\}) \leq C_R \ln|\ln\varepsilon|$$

which contradicts Proposition 2.4 for ε small enough. ∎

Proof of 2) in Theorem 2.1. We fix some $\frac{2}{3} < \alpha < 1$. As in the proof of Proposition 2.3, we consider a finite family of points $\{x_j\}_{j\in\mathcal{J}}$ satisfying

$$\begin{aligned}
&x_j \in B^\Lambda_{\frac{\sqrt{a_0}}{2}} \\
&B\left(x_i, \frac{\lambda_0\varepsilon}{4}\right) \cap B\left(x_j, \frac{\lambda_0\varepsilon}{4}\right) = \emptyset \quad \text{for } i \neq j\,, \\
&B^\Lambda_{\frac{\sqrt{a_0}}{2}} \subset \bigcup_{j\in\mathcal{J}} B\left(x_j, \lambda_0\varepsilon\right),
\end{aligned}$$

where $\lambda_0 := \lambda_{\frac{\sqrt{a_0}}{2}}$ (defined in Proposition 2.2 with $R = \frac{\sqrt{a_0}}{2}$) and we denote by J_ε the set of indices $j \in \mathcal{J}$ such that $B(x_j, \lambda_0\varepsilon)$ contains at least one point y_j verifying

$$|v_\varepsilon(y_j)| < \frac{1}{2}\,. \tag{2.26}$$

Obviously, $B(x_j,\lambda_0\varepsilon)$ is a bad disc for every $j\in J_\varepsilon$. Applying Lemma 2.3 (with $R=\frac{3\sqrt{a_0}}{4}$), we infer that there exists ε_0 such that for any $0<\varepsilon<\varepsilon_0$,

$$B(x_j,\lambda_0\varepsilon)\subset B^\Lambda_{\frac{\sqrt{a_0}}{4}}\quad\text{for any } j\in J_\varepsilon. \tag{2.27}$$

Then it remains to prove that $\mathrm{Card}(J_\varepsilon)$ is bounded independently of ε. Using Proposition 2.4 (with $R=\frac{\sqrt{a_0}}{2}$), we derive that for any $j\in J_\varepsilon$ and any point y_j satisfying (2.26) in the ball $B(x_j,\lambda_0\varepsilon)$,

$$\int_{B(x_j,2\varepsilon^\alpha)}|\nabla v_\varepsilon|^2\geq\int_{B(y_j,\varepsilon^\alpha)}|\nabla v_\varepsilon|^2\geq C_\alpha|\ln\varepsilon| \tag{2.28}$$

for some positive constant C_α which only depends on α. We set for ε small enough

$$W=\bigcup_{j\in J_\varepsilon}B(x_j,2\varepsilon^\alpha)\subset B^\Lambda_{\frac{\sqrt{a_0}}{3}}\,.$$

We claim that there is a positive integer M_α independent of ε such that any $y\in W$ belongs to at most M_α balls in the collection $\{B(x_j,2\varepsilon^\alpha)\}_{j\in J_\varepsilon}$. Indeed, for each $y\in W$, consider the subset $K_y\subset J_\varepsilon$ defined by

$$K_y=\big\{j\in J_\varepsilon\,:\,y\in B(x_j,2\varepsilon^\alpha)\big\}.$$

We have for every $j\in K_y$,

$$x_j\in B(y,2\varepsilon^\alpha)\subset B(y,\varepsilon^{\alpha'})\subset B^\Lambda_{\frac{\sqrt{a_0}}{2}}\quad\text{with }\alpha'=\frac{1}{2}(\alpha+\frac{2}{3}). \tag{2.29}$$

Since the family of discs $\{B(x_j,\lambda_0\varepsilon)\}_{j\in K_y}$ is a subcover of $B(y,\varepsilon^{\alpha'})$ satisfying (2.8) and (2.9), we conclude from Remark 2.3 that

$$\mathrm{Card}(K_y)\leq M_\alpha$$

with $M_\alpha=N_{\frac{\sqrt{a_0}}{2},\alpha'}$. From (2.28), we infer that

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}}|\nabla v_\varepsilon|^2\geq\int_W|\nabla v_\varepsilon|^2\geq\frac{1}{M_\alpha}\sum_{j\in J_\varepsilon}\int_{B(x_j,2\varepsilon^\alpha)}|\nabla v_\varepsilon|^2\geq C_\alpha\mathrm{Card}(J_\varepsilon)|\ln\varepsilon|. \tag{2.30}$$

On the other hand, we know by (1.14),

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}}|\nabla v_\varepsilon|^2\leq C\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}}a(x)|\nabla v_\varepsilon|^2\leq C|\ln\varepsilon| \tag{2.31}$$

for a constant C independent of ε. Matching (2.30) and (2.31), we conclude that $\mathrm{Card}(J_\varepsilon)$ is uniformly bounded. ∎

In the following, we will prove Proposition 2.1. We proceed exactly as in [20], using Theorem 2.1 and an adaptation of Theorem V.1 in [3]. Before starting our proof, we need to recall the following result obtained in [14] by a method due to Sandier [18] and Sandier and Serfaty [19]:

Proposition 2.5. **([14])** *There exists a positive constant $\mathcal{K}_0$ such that for ε sufficiently small, there exist $\nu_\varepsilon \in (1,2)$ and a finite collection of disjoint balls $\{B_i\}_{i\in I_\varepsilon} := \{B(p_i, r_i)\}_{i\in I_\varepsilon}$ satisfying:*

(i) *for every $i \in I_\varepsilon$,* $\quad B_i \subset\subset \mathcal{D}_\varepsilon = \{x \in \mathbb{R}^2,\, a(x) > \nu_\varepsilon |\ln\varepsilon|^{-3/2}\}$,

(ii) $\{x \in \mathcal{D}_\varepsilon,\, |v_\varepsilon(x)| < 1 - |\ln\varepsilon|^{-5}\} \subset \cup_{i\in I_\varepsilon} B_i$,

(iii) $\displaystyle\sum_{i\in I_\varepsilon} r_i \le |\ln\varepsilon|^{-10}$,

(iv) $\displaystyle\frac{1}{2}\int_{B_i} a(x)|\nabla v_\varepsilon|^2 \ge \pi a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0 \ln|\ln\varepsilon|\big)$,

where $d_i = \deg\left(\dfrac{v_\varepsilon}{|v_\varepsilon|}, \partial B_i\right)$ *for every $i \in I_\varepsilon$.*

Proof of Proposition 2.1. By Theorem 2.1, we have for ε small enough,

$$\cup_{j\in J_\varepsilon} B(x_j^\varepsilon, \lambda_0\varepsilon) \subset B^\Lambda_{\frac{\sqrt{a_0}}{3}}.$$

From *(iii)* in Proposition 2.5, there exists a radius $r_\varepsilon \in (\frac{\sqrt{a_0}}{3}, \frac{\sqrt{a_0}}{2}]$ such that

$$\bar{B}_i \cap \partial B^\Lambda_{r_\varepsilon} = \emptyset \quad \text{for every } i \in I_\varepsilon. \tag{2.32}$$

Hence we have

$$|v_\varepsilon| \ge 1 - |\ln\varepsilon|^{-5} \quad \text{on } \partial B^\Lambda_{r_\varepsilon}.$$

The existence of a subset $\tilde{J}_\varepsilon \subset J_\varepsilon$ satisfying *(i)-(v)* can now be proved identically as Proposition 3.2 in [20] and it remains to prove (2.1). From the proof of Theorem 2.1, we know (by construction) that each disc $B(x_k^\varepsilon, \lambda_0\varepsilon)$, $k \in J_\varepsilon$, contains at least one point y_k such that $|v_\varepsilon(y_k)| < \frac{1}{2}$. Therefore each disc $B(x_j^\varepsilon, \rho)$, $j \in \tilde{J}_\varepsilon$, contains at least one of the y_k's with $|x_j^\varepsilon - y_k| < \lambda_0\varepsilon$. Assume now that $D_j = 0$. By Lemma 2.2 with $\gamma = \mu^{-1/2}$, it would lead to $|v_\varepsilon| \ge \frac{1}{2}$ in $B(x_j^\varepsilon, \rho^\gamma)$ and then $|v_\varepsilon(y_k)| \ge \frac{1}{2}$ for ε small enough, contradiction. We also find a bound on the degrees D_j:

$$|D_j| = \frac{1}{2\pi}\left|\int_{\partial B(x_j^\varepsilon,\rho)} \frac{1}{|v_\varepsilon|^2}\Big(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial\tau}\Big)\right| \le C\|\nabla v_\varepsilon\|_{L^2(\partial B(x_j^\varepsilon,\rho))}\sqrt{\rho} \le C$$

by *(iv)* in Proposition 2.1. ■

3 Some lower energy estimates

In this section, we obtain various lower energy estimates for v_ε in terms of the vortex structure defined in Section 2, Proposition 2.1. We start by proving a lower bound on the kinetic energy away from the vortices which brings out the interaction between vortices. The method that we use is based on the techniques developped in [3], [8] and [20]. As in the previous section, the main difficulty is due to the degenerate behavior near the boundary of $\mathcal{D}$ of the weight function $a(x)$. To avoid this problem, we shall establish our estimates in B_R^Λ for an arbitrary radius $R \in [\sqrt{a_0}/2, \sqrt{a_0}\,)$. To emphasize the possible dependence on R in the "error term", we will denote by $O_R(1)$ (respectively $o_R(1)$) any quantity which remains uniformly bounded in ε for fixed R (respectively any quantity which tends to 0 as $\varepsilon \to 0$ for fixed R). In the sequel, we also write $\tilde{J}_\varepsilon = \{1, \dots, n_\varepsilon\}$.

Proposition 3.1. *For any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$, let $\Theta_\rho = B_R^\Lambda \setminus \cup_{j=1}^{n_\varepsilon} B(x_j^\varepsilon, \rho)$. We have*

$$\frac{1}{2}\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \pi \sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon)|\ln \rho| + W_{R,\varepsilon}\big((x_1^\varepsilon, D_1), \dots, (x_{n_\varepsilon}^\varepsilon, D_{n_\varepsilon})\big) + O_R(1) \tag{3.1}$$

where

$$W_{R,\varepsilon}\big((x_1^\varepsilon, D_1), \dots, (x_{n_\varepsilon}^\varepsilon, D_{n_\varepsilon})\big) = -\pi \sum_{i \neq j} D_i D_j\, a(x_j^\varepsilon) \ln |x_i^\varepsilon - x_j^\varepsilon| - \pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{R,\varepsilon}(x_j^\varepsilon)$$

and $\Psi_{R,\varepsilon}$ is the unique solution of

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla\Psi_{R,\varepsilon}\right) = -\displaystyle\sum_{j=1}^{n_\varepsilon} D_j\, a(x_j^\varepsilon)\, \nabla\left(\dfrac{1}{a}\right) \cdot \nabla\big(\ln|x - x_j^\varepsilon|\big) & \text{in } B_R^\Lambda, \\ \Psi_{R,\varepsilon} = -\displaystyle\sum_{j=1}^{n_\varepsilon} D_j\, a(x_j^\varepsilon) \ln|x - x_j^\varepsilon| & \text{on } \partial B_R^\Lambda. \end{cases} \tag{3.2}$$

Moreover, if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$ then the term $O_R(1)$ in (3.1) is in fact $o_R(1)$.

Remark 3.1. We point out that the dependence on R in the interaction term $W_{R,\varepsilon}$ only appears in the function $\Psi_{R,\varepsilon}$. Moreover, for $\Psi_{R,\varepsilon}$ to be well defined, $1/a(x)$ has to be bounded inside B_R^Λ so that we can not pass to the limit $R \to \sqrt{a_0}$ in (3.1) without an *a priori* deterioration of the error term.

Proof of Proposition 3.1. We consider the solution Φ_ρ of the linear problem

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla\Phi_\rho\right) = 0 & \text{in } \Theta_\rho, \\ \Phi_\rho = 0 & \text{on } \partial B_R^\Lambda, \\ \Phi_\rho = \text{const.} & \text{on } \partial B(x_j^\varepsilon, \rho), \\ \displaystyle\int_{\partial B(x_j^\varepsilon,\rho)} \dfrac{1}{a}\dfrac{\partial\Phi_\rho}{\partial\nu} = 2\pi D_j & \text{for } j = 1, \dots, n_\varepsilon, \end{cases}$$

and $\Phi_{R,\varepsilon}$ the solution of

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla\Phi_{R,\varepsilon}\right) = 2\pi\displaystyle\sum_{j=1}^{n_\varepsilon} D_j\, \delta_{x_j^\varepsilon} & \text{in } B_R^\Lambda \\ \Phi_{R,\varepsilon} = 0 & \text{on } \partial B_R^\Lambda \end{cases} \tag{3.3}$$

For $x \in \Theta_\rho$, we set $w_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}$ and

$$\mathcal{S} = \left(-w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_2} + \frac{1}{a} \frac{\partial \Phi_\rho}{\partial x_1} \, , \, w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_1} + \frac{1}{a} \frac{\partial \Phi_\rho}{\partial x_2} \right).$$

We easily check that $\operatorname{div} \mathcal{S} = 0$ in Θ_ρ and $\int_{\partial B_R^\Lambda} \mathcal{S} \cdot \nu = \int_{\partial B(x_j^\varepsilon, \rho)} \mathcal{S} \cdot \nu = 0$. By Lemma I.1 in [8], there exists $H \in C^1(\overline{\Theta}_\rho)$ such that $\mathcal{S} = \nabla^\perp H$ and hence we can write the Hodge-de Rham type decomposition

$$w_\varepsilon \wedge \nabla w_\varepsilon = \frac{1}{a} \nabla^\perp \Phi_\rho + \nabla H.$$

Consequently,

$$\begin{aligned} \int_{\Theta_\rho} a(x) |\nabla w_\varepsilon|^2 &= \int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 + 2 \int_{\Theta_\rho} \nabla^\perp \Phi_\rho \cdot \nabla H + \int_{\Theta_\rho} a(x) |\nabla H|^2 \\ &\geq \int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 + 2 \int_{\Theta_\rho} \nabla^\perp \Phi_\rho \cdot \nabla H. \end{aligned}$$

We observe that the last term is in fact equal to zero since it is the integral of a Jacobian and Φ_ρ is constant on $\partial \Theta_\rho$. Hence

$$\int_{\Theta_\rho} a(x) |\nabla w_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2.$$

Since $|\nabla v_\varepsilon|^2 \geq |v_\varepsilon|^2 |\nabla w_\varepsilon|^2$ in Θ_ρ, we derive that

$$\int_{\Theta_\rho} a(x) |\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 + T_1 + 2T_2$$

with

$$T_1 = \int_{\Theta_\rho} \big(|v_\varepsilon|^2 - 1 \big) \frac{1}{a(x)} |\nabla \Phi_\rho|^2 \quad \text{and} \quad T_2 = \int_{\Theta_\rho} \big(|v_\varepsilon|^2 - 1 \big) \nabla \Phi_\rho^\perp \cdot \nabla H.$$

Arguing as in [3] (see Step 4 in the proof of Theorem 6), it turns out that $T_1 = o_R(1)$ and $T_2 = o_R(1)$ and therefore

$$\int_{\Theta_\rho} a(x) |\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 + o_R(1). \tag{3.4}$$

On the other hand, integrating by parts we obtain

$$\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 = \int_{\partial \Theta_\rho} \frac{1}{a(x)} \frac{\partial \Phi_\rho}{\partial \nu} \Phi_\rho = -2\pi \sum_{j=1}^{n_\varepsilon} D_j \, \Phi_\rho(z_j)$$

for any point $z_j \in \partial B(x_j^\varepsilon, \rho)$. Since n_ε and each D_j remain uniformly bounded in ε by Proposition 2.1, we may rewrite this equality as

$$\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 = -2\pi \sum_{j=1}^{n_\varepsilon} D_j \, \Phi_{R,\varepsilon}(z_j) + O\big(\| \Phi_{R,\varepsilon} - \Phi_\rho \|_{L^\infty(\Theta_\rho)} \big). \tag{3.5}$$

Using an adaptation of Lemma I.4 in [8] (see e.g. [6], Lemma 3.5), we derive that

$$\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \leq \sum_{j=1}^{n_\varepsilon} \left(\sup_{\partial B(x_j^\varepsilon,\rho)} \Phi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon,\rho)} \Phi_{R,\varepsilon} \right). \tag{3.6}$$

To estimate the right-hand-side term in (3.6), we introduce for $x \in B_R^\Lambda$,

$$\Psi_{R,\varepsilon}(x) = \Phi_{R,\varepsilon}(x) - \sum_{j=1}^{n_\varepsilon} D_j\, a(x_j^\varepsilon) \ln|x - x_j^\varepsilon|.$$

Since $\Phi_{R,\varepsilon}$ solves (3.3), we deduce that $\Psi_{R,\varepsilon}$ may be characterized as the solution of equation (3.2). By elliptic regularity, we infer that $\|\Psi_{R,\varepsilon}\|_{W^{2,p}(B_R^\Lambda)} \leq C_{R,p}$ for any $1 \leq p < 2$ (here we used that $\{x_j^\varepsilon\}_{j=1}^{n_\varepsilon} \subset B^\Lambda_{\frac{\sqrt{a_0}}{4}}$ by Theorem 2.1). In particular, $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R^\Lambda)$ and hence

$$\sup_{\partial B(x_j^\varepsilon,\rho)} \Psi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon,\rho)} \Psi_{R,\varepsilon} \leq C_R\sqrt{\rho} = o_R(1).$$

Since $|x_j^\varepsilon - x_i^\varepsilon| \geq 8\rho$, we derive from (2.1),

$$\sup_{\partial B(x_j^\varepsilon,\rho)} \left(\sum_{i=1}^{n_\varepsilon} D_i\, a(x_i^\varepsilon) \ln|x - x_i^\varepsilon| \right) - \inf_{\partial B(x_j^\varepsilon,\rho)} \left(\sum_{i=1}^{n_\varepsilon} D_i\, a(x_i^\varepsilon) \ln|x - x_i^\varepsilon| \right) \leq$$
$$\leq \rho \sum_{i=1,\, i\neq j}^{n_\varepsilon} a(x_i^\varepsilon) \sup_{\partial B(x_j^\varepsilon,\rho)} \frac{|D_i|}{|x - x_i^\varepsilon|} \leq O(1),$$

(respectively $\leq o(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$). Coming back to (3.6), we obtain that $\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \leq O_R(1)$ (respectively $\leq o_R(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$). Inserting this estimate in (3.5), we get that

$$\begin{aligned}
\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla\Phi_\rho|^2 &= -\, 2\pi \sum_{j=1}^{n_\varepsilon} D_j\, \Phi_{R,\varepsilon}(z_j) + O_R(1) \\
&= -\, 2\pi \sum_{j=1}^{n_\varepsilon} D_j\, \Psi_{R,\varepsilon}(z_j) - 2\pi \sum_{i\neq j} D_i D_j\, a(x_i^\varepsilon) \ln|z_j - x_i^\varepsilon| \\
&\quad + 2\pi \sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon) |\ln\rho| + O_R(1)
\end{aligned} \tag{3.7}$$

(respectively $+o_R(1)$ as $\varepsilon \to 0$). Since $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R^\Lambda)$, we have $|\Psi_{R,\varepsilon}(z_j) - \Psi_{R,\varepsilon}(x_j^\varepsilon)| \leq C_R\sqrt{\rho} = o_R(1)$. Moreover, using (2.1) and $|x_j^\varepsilon - x_i^\varepsilon| \geq 8\rho$, we derive that

$$\begin{aligned}
\left| \sum_{i\neq j} D_i D_j\, a(x_i^\varepsilon)(\ln|z_j - x_i^\varepsilon| - \ln|x_j^\varepsilon - x_i^\varepsilon|) \right| &\leq \sum_{i\neq j} |D_i|\,|D_j|\, \ln\left|1 + \frac{z_j - x_j^\varepsilon}{x_j^\varepsilon - x_i^\varepsilon}\right| \\
&\leq \sum_{i\neq j} |D_i|\,|D_j|\, \frac{\rho}{|x_j^\varepsilon - x_i^\varepsilon|} \leq O(1)
\end{aligned}$$

(respectively $\leq o(1)$ as $\varepsilon \to 0$). Hence (3.7) yields

$$\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 = -2\pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{R,\varepsilon}(x_j^\varepsilon) - 2\pi \sum_{i \neq j} D_i D_j a(x_i^\varepsilon) \ln |x_j^\varepsilon - x_i^\varepsilon| + 2\pi \sum_{j=1}^{n_\varepsilon} D_j^2 a(x_j^\varepsilon) |\ln \rho| + O_R(1)$$

(respectively $+o_R(1)$ as $\varepsilon \to 0$). Combining this estimate with (3.4), we obtain the announced result.■

Estimating the contribution in the energy of each vortex, we may easily deduce the following lower bounds for $\mathcal{E}_\varepsilon(v_\varepsilon)$:

Lemma 3.1. *For any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0})$, we have*

$$\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \geq \pi \sum_{j=1}^{n_\varepsilon} D_j^2 a(x_j^\varepsilon) |\ln \rho| + \pi \sum_{j=1}^{n_\varepsilon} |D_j| a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} + W_{R,\varepsilon} + O_R(1) \tag{3.8}$$

and

$$\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \geq \pi \sum_{j=1}^{n_\varepsilon} |D_j| a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} + O(1). \tag{3.9}$$

Proof. In view of Proposition 3.1, it suffices to show that

$$\mathcal{E}_\varepsilon(v_\varepsilon, B(x_j^\varepsilon, \rho)) \geq \pi |D_j| a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} + O(1) \quad \text{for } j = 1, \dots, n_\varepsilon,$$

which is equivalent to

$$\frac{1}{2} \int_{B(x_j^\varepsilon, \rho)} |\nabla v_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2} (1 - |v_\varepsilon|^2)^2 \geq \pi |D_j| \ln \frac{\rho}{\varepsilon} + O(1) \quad \text{for } j = 1, \dots, n_\varepsilon \tag{3.10}$$

(we used that $|a(x) - a(x_j^\varepsilon)| \leq C\rho$ for $x \in B(x_j^\varepsilon, \rho)$ and $\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \leq C_R |\ln \varepsilon|$). Setting

$$\hat{v}(y) = v_\varepsilon(\rho y + x_j^\varepsilon) \quad \text{for } y \in B(0,1) \text{ and } \quad \hat{\varepsilon} = \frac{\varepsilon}{\rho \sqrt{a(x_j^\varepsilon)}},$$

we infer from Proposition 2.1 that $\hat{v} \geq 1 - \frac{2}{|\ln \varepsilon|}$ on $\partial B(0,1)$,

$$\int_{\partial B(0,1)} \frac{|\nabla \hat{v}|^2}{2} + \frac{1}{4\hat{\varepsilon}^2} (1 - |\hat{v}|^2)^2 = \rho \int_{\partial B(x_j^\varepsilon, \rho)} \frac{|\nabla v_\varepsilon|^2}{2} + \frac{a(x_j^\varepsilon)}{4\varepsilon^2} (1 - |v_\varepsilon|^2)^2 \leq C \tag{3.11}$$

and

$$\frac{1}{2} \int_{B(0,1)} |\nabla \hat{v}|^2 + \frac{1}{2\hat{\varepsilon}^2} (1 - |\hat{v}|^2)^2 = \frac{1}{2} \int_{B(x_j^\varepsilon, \rho)} |\nabla v_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2} (1 - |v_\varepsilon|^2)^2.$$

As in the proof of Lemma VI.1 in [3], (3.11) yields for ε small enough,

$$\frac{1}{2} \int_{B(0,1)} |\nabla \hat{v}|^2 + \frac{1}{2\hat{\varepsilon}^2} (1 - |\hat{v}|^2)^2 \geq \pi |D_j| |\ln \hat{\varepsilon}| + O(1) = \pi |D_j| \ln \frac{\rho}{\varepsilon} + O(1)$$

and hence (3.10) holds. ■

As in [14], we may compute an asymptotic expansion of $\mathcal{R}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)$ in terms of vortices which leads, in view of Lemma 3.1, to lower expansions of $\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)$:

Lemma 3.2. *For any* $R\in[\frac{\sqrt{a_0}}{2},\sqrt{a_0}\,)$, *we have*

$$\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)\geq \pi\sum_{j=1}^{n_\varepsilon} D_j^2\,a(x_j^\varepsilon)|\ln\rho|+\pi\sum_{j=1}^{n_\varepsilon}|D_j|\,a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon}-\frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}a^2(x_j^\varepsilon)\,D_j+W_{R,\varepsilon}+O_R(1) \tag{3.12}$$

and

$$\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)\geq \pi\sum_{j=1}^{n_\varepsilon}|D_j|\,a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon}-\frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}a^2(x_j^\varepsilon)\,D_j+O(1). \tag{3.13}$$

Proof. We consider the family of balls $\{B_i\}_{i\in I_\varepsilon}$ given in Proposition 2.5. As in the proof of Proposition 2.1, we can find $r_\varepsilon\in[R,(R+\sqrt{a_0}\,)/2]$ such that (2.32) holds. Setting

$$I_R^+=\big\{i\in I_\varepsilon,\ |p_i|_\Lambda>r_\varepsilon \text{ and } d_i\geq 0\big\}\quad\text{and}\quad I_R^-=\big\{i\in I_\varepsilon,\ |p_i|_\Lambda>r_\varepsilon \text{ and } d_i<0\big\}, \tag{3.14}$$

we have $\overline{B}_i\subset\mathcal{D}_\varepsilon\setminus\overline{B}^\Lambda_{r_\varepsilon}$ for any $i\in I_R^+\cup I_R^-$. By Theorem 2.1, Proposition 2.1 and Proposition 2.5, we infer that for ε small enough,

$$|v_\varepsilon|\geq\frac12\quad\text{in }\Xi_\varepsilon:=\mathcal{D}_\varepsilon\setminus\Big(\bigcup_{i\in I_R^+\cup I_R^-}B_i\cup\bigcup_{j=1}^{n_\varepsilon}B(x_j^\varepsilon,\rho)\Big).$$

Arguing exactly as in [14], we obtain that

$$\mathcal{R}_\varepsilon(v_\varepsilon,\Xi_\varepsilon)=\frac{-\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}a^2(x_j^\varepsilon)\,D_j-\frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I_R^+\cup I_R^-}\big(a^2(p_i)-\nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i+o_R(1). \tag{3.15}$$

We recall that we have showed in [14] that $\mathcal{R}_\varepsilon(v_\varepsilon,\cup_{i\in I_R^+\cup I_R^-}B_i)=o(1)$. In the same way, we may prove that $\mathcal{R}_\varepsilon(v_\varepsilon,\cup_{j=1}^{n_\varepsilon}B(x_j^\varepsilon,\rho))=o(1)$. From *(iv)* in Proposition 2.5 and (3.15), we deduce that

$$\begin{aligned}\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)&\geq\mathcal{E}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{i\in I_R^+\cup I_R^-}B_i)+\sum_{i\in I_R^+\cup I_R^-}\frac12\int_{B_i}a(x)|\nabla v_\varepsilon|^2+\mathcal{R}_\varepsilon(v_\varepsilon,\Xi_\varepsilon)+o_R(1)\\&\geq\mathcal{E}_\varepsilon(v_\varepsilon,B_R^\Lambda)-\frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}a^2(x_j^\varepsilon)\,D_j+\pi\sum_{i\in I_R^+\cup I_R^-}a(p_i)|d_i|\big(|\ln\varepsilon|-\mathcal{K}_0\ln|\ln\varepsilon|\big)\\&\quad-\frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I_R^+\cup I_R^-}\big(a^2(p_i)-\nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i+o_R(1).\end{aligned} \tag{3.16}$$

Since $p_i\notin\overline{B}^\Lambda_{r_\varepsilon}$ for $i\in I_R^+\cup I_R^-$, we have $a(p_i)\ll a_0$ and we deduce that for ε small enough,

$$\pi\sum_{i\in I_R^+\cup I_R^-}a(p_i)|d_i|\big(|\ln\varepsilon|-\mathcal{K}_0\ln|\ln\varepsilon|\big)-\frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I_R^+\cup I_R^-}\big(a^2(p_i)-\nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i\geq 0$$

which leads to

$$\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) \geq \mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon)\, D_j + o_R(1). \tag{3.17}$$

Combining (3.8) and (3.17), we obtain (3.12). Similarly, the inequality (3.17) applied with $R=\sqrt{a_0}/2$, and (3.9) yield (3.13). ■

4 Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1 in terms of the map v_ε. We start by proving that vortices must be of degree one. This yields a fundamental improvement of the estimates obtained in the previous section.

4.1 Vortices have degree one

Lemma 4.1. *Whenever ε is small enough, $D_j=+1$ for $j=1,\dots,n_\varepsilon$.*

Proof. By the results in [14], we have $\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)\leq o(1)$. According to (3.13), it yields

$$\pi\sum_{j=1}^{n_\varepsilon}|D_j|\,a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} - \frac{\pi a_0\Omega}{1+\Lambda^2}\sum_{D_j>0} a(x_j^\varepsilon)\,D_j \leq \pi\sum_{j=1}^{n_\varepsilon}|D_j|\,a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon)\,D_j \leq O(1).$$

From (1.7), we derive that

$$\sum_{j=1}^{n_\varepsilon}|D_j|\,a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} \leq \sum_{D_j>0} D_j\,a(x_j^\varepsilon)|\ln\varepsilon| + o(|\ln\varepsilon|).$$

Since $\rho\geq\varepsilon^\mu$, it leads to (we recall that $D_j\neq 0$)

$$(1-\mu)\sum_{D_j<0}|D_j|\,a(x_j^\varepsilon)|\ln\varepsilon| \leq \mu\sum_{D_j>0}|D_j|\,a(x_j^\varepsilon)|\ln\varepsilon| + o(|\ln\varepsilon|).$$

By Theorem 2.1, $a(x_j^\varepsilon)\geq a_0/2$ and consequently,

$$\sum_{D_j<0}|D_j| \leq \frac{2\mu}{1-\mu}\sum_{D_j>0}|D_j| + o(1) \leq \frac{C\mu}{1-\mu} + o(1).$$

Choosing μ sufficiently small, it yields $D_j>0$ for $j=1,\dots,n_\varepsilon$ whenever ε is small enough. Since $|x_j^\varepsilon|\leq C$ and $D_j>0$, we may now assert that

$$-\pi\sum_{i\neq j} D_iD_j\,a(x_j^\varepsilon)\ln|x_i^\varepsilon - x_j^\varepsilon| \geq O(1)$$

and thus $W_{\frac{\sqrt{a_0}}{2},\varepsilon} \geq -\pi\sum_{j=1}^{n_\varepsilon} D_j \Psi_{\frac{\sqrt{a_0}}{2},\varepsilon}(x_j^\varepsilon) = O(1)$. Hence the inequality (3.12) (applied with $R = \sqrt{a_0}/2$) together with $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq o(1)$ leads us to

$$\pi\sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon)|\ln\rho| + \pi\sum_{j=1}^{n_\varepsilon} D_j\, a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon)\, D_j \leq O(1).$$

As previously, we derive from (1.7), $\sum_{j=1}^{n_\varepsilon}(D_j^2 - D_j)\, a(x_j^\varepsilon)|\ln\rho| \leq o(|\ln\varepsilon|)$. Since $\rho \leq \varepsilon^{\overline{\mu}}$ and $a(x_j^\varepsilon) \geq a_0/2$, we conclude that

$$\frac{\overline{\mu}\, a_0}{2}\sum_{j=1}^{n_\varepsilon}(D_j^2 - D_j) \leq o(1)$$

which yields $D_j = +1$ whenever ε is small enough. ∎

As a direct consequence of Lemma 4.1, we obtain the following improvement of Lemma 3.2:

Corollary 4.1. *For any* $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$*, we have*

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \pi\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) + W_{R,\varepsilon}\big((x_1^\varepsilon, +1), \ldots, (x_{n_\varepsilon}^\varepsilon, +1)\big) + O_R(1).$$

Proof. It follows directly from (3.12) and Lemma 4.1 that for any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$,

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) + W_{R,\varepsilon}\big((x_1^\varepsilon, +1), \ldots, (x_{n_\varepsilon}^\varepsilon, +1)\big) + O_R(1).$$

On the other hand, we have proved in [14], that $|\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) - \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)| = o(1)$ and $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathbb{R}^2 \setminus \mathcal{D}_\varepsilon) \geq o(1)$. Hence we have $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + o(1)$ and the conclusion follows. ∎

4.2 The subcritical case

We are now able to prove *(i)* in Theorem 1.1. By the results in [14], it suffices to show the following proposition.

Proposition 4.1. *Assume that* (1.7) *holds with* $\omega_1 < 0$*. Then for* ε *sufficiently small, we have that*

$$|v_\varepsilon| \to 1 \quad \text{in } L^\infty_{\mathrm{loc}}(\mathcal{D}) \text{ as } \varepsilon \to 0. \tag{4.1}$$

Moreover,

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = o(1) \quad \text{and} \quad \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) = o(1). \tag{4.2}$$

.

Proof. We fix some $\frac{\sqrt{a_0}}{2} < R_0 < \sqrt{a_0}$. In [14], we have proved that $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq o(1)$ so that Corollary 4.1 applied with $R = \frac{\sqrt{a_0}}{2}$ leads to

$$\pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi a_0 \Omega}{1+\Lambda^2} \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon) \leq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) \leq O(1).$$

Since $a(x_j^\varepsilon) \geq a_0/2$ and $\omega_1 < 0$, we deduce that

$$\frac{a_0|\omega_1|\, n_\varepsilon}{2} \ln|\ln\varepsilon| \leq -\omega_1 \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon) \ln|\ln\varepsilon| \leq O(1)$$

and then $n_\varepsilon \leq o(1)$ which implies that $n_\varepsilon \equiv 0$ whenever ε is small enough. Using the notation (3.14), we derive from (3.16) that

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{i \in I_{R_0}^+ \cup I_{R_0}^-} a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0 \ln|\ln\varepsilon|\big) - \frac{\pi\Omega}{1+\Lambda^2} \sum_{i \in I_{R_0}^+ \cup I_{R_0}^-} \big(a^2(p_i) - \nu_\varepsilon^2 |\ln\varepsilon|^{-3}\big) d_i$$

By the results in [14], we have $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq O(|\ln\varepsilon|^{-1})$. Since $a(p_i) \ll a_0$ for $i \in I_{R_0}^+ \cup I_{R_0}^-$, we infer that exists $c > 0$ independent of ε such that

$$\begin{aligned} c \sum_{i \in I_{R_0}^+ \cup I_{R_0}^-} a(p_i)|d_i||\ln\varepsilon| \leq{}& \pi \sum_{i \in I_{R_0}^+ \cup I_{R_0}^-} a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0 \ln|\ln\varepsilon|\big) \\ & - \frac{\pi\Omega}{1+\Lambda^2} \sum_{i \in I_{R_0}^+ \cup I_{R_0}^-} \big(a^2(p_i) - \nu_\varepsilon^2 |\ln\varepsilon|^{-3}\big) d_i \leq O(|\ln\varepsilon|^{-1}). \end{aligned}$$

Since $a(x) \geq |\ln\varepsilon|^{-3/2}$ in $\mathcal{D}_\varepsilon$, we finally obtain

$$\sum_{i \in I_{R_0}^+ \cup I_{R_0}^-} |d_i| \leq O(|\ln\varepsilon|^{-1/2}).$$

Hence $\sum_{i \in I_{R_0}^+ \cup I_{R_0}^-} |d_i| = 0$ for ε sufficiently small and we conclude from (3.15),

$$\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i \in I_{R_0}^+ \cup I_{R_0}^-} B_i) = o(1).$$

By [14], we also have $\mathcal{R}_\varepsilon(v_\varepsilon, \cup_{i \in I_{R_0}^+ \cup I_{R_0}^-} B_i) = o(1)$ so that $\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = o(1)$. Consequently,

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + o(1) \leq o(1).$$

Then the rest of the proof follows as in [14]. ∎

4.3 The supercritical case

In this section, we will prove *(ii)* in Theorem 1.1. Writing

$$\Omega = \frac{1+\Lambda^2}{a_0}\big(|\ln\varepsilon| + \omega(\varepsilon)\ln|\ln\varepsilon|\big), \tag{4.3}$$

we assume that

$$(d-1)+\delta \le \omega(\varepsilon) \le d-\delta \tag{4.4}$$

for some integer $d \ge 1$ and some positive number $\delta \ll 1$ independent of ε. We start by proving that, in this regime, v_ε has vortices :

Proposition 4.2. *Assume that* (4.4) *holds. Then, for ε sufficiently small, v_ε has exactly d vortices of degree one, i.e. $n_\varepsilon \equiv d$, and*

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = -\pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| + O(1). \tag{4.5}$$

Proof. Step 1. We start by proving that $n_\varepsilon \ge 1$ for ε sufficiently small. By Theorem 5.1 in Section 5 (with $d=1$), there exists $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$F_\varepsilon(\tilde{u}_\varepsilon) \le F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \pi a_0\omega(\varepsilon)\ln|\ln\varepsilon| + O(1).$$

By the minimizing property of u_ε and (1.9), we have

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\eta_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \le F_\varepsilon(\tilde{u}_\varepsilon)$$

and since $|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)| = o(1)$ (see [14]), we deduce that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \le -\pi a_0\omega(\varepsilon)\ln|\ln\varepsilon| + O(1).$$

From here, it turns out by Corollary 4.1 applied with $R = \frac{\sqrt{a_0}}{2}$ (recall that $W_{\frac{\sqrt{a_0}}{2},\varepsilon} \ge O(1)$),

$$\begin{aligned}
-\pi a_0\omega(\varepsilon)\ln|\ln\varepsilon| + O(1) \ge \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) &\ge \pi\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) + O(1)\\
&\ge \pi\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\left(-\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\Omega|x_j^\varepsilon|_\Lambda^2}{1+\Lambda^2}\right) + O(1)\\
&\ge -\pi a_0\omega(\varepsilon)n_\varepsilon\ln|\ln\varepsilon| + O(1).
\end{aligned}$$

Hence $n_\varepsilon \ge 1 + o(1)$ and the conclusion follows.

Step 2. Now we show that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \ge -\pi a_0\, n_\varepsilon\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(n_\varepsilon^2 - n_\varepsilon)\ln|\ln\varepsilon| + O(1). \tag{4.6}$$

In the case $n_\varepsilon = 1$, we have already proved the result in the previous step. Then we may assume that $n_\varepsilon \geq 2$. Since $\|\Psi_{\frac{\sqrt{a_0}}{2},\varepsilon}\|_\infty = O(1)$, we get from Corollary 4.1 applied with $R = \frac{\sqrt{a_0}}{2}$,

$$\begin{aligned}\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) &\geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\Big(|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon} \ln|x_i^\varepsilon - x_j^\varepsilon| - \frac{\Omega a(x_j^\varepsilon)}{1+\Lambda^2}\Big) + O(1)\\ &\geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\Big(-\omega(\varepsilon)\ln|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega|x_j^\varepsilon|^2_\Lambda}{1+\Lambda^2}\Big) + O(1) \qquad (4.7)\end{aligned}$$

Since $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq o(1)$, we derive that

$$-\sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} |x_j^\varepsilon|^2_\Lambda \leq C\ln|\ln\varepsilon|.$$

On the other hand $-\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| \geq O(1)$ so that $|x_j^\varepsilon|^2 \leq C\big(\ln|\ln\varepsilon|\big)|\ln\varepsilon|^{-1}$ and hence

$$\begin{aligned}\pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\Big(-\omega(\varepsilon)\ln|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega|x_j^\varepsilon|^2_\Lambda}{1+\Lambda^2}\Big) = \qquad (4.8)\\ = -\pi a_0\, n_\varepsilon\omega(\varepsilon)\ln|\ln\varepsilon| - \pi a_0 \sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi a_0\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}|x_j^\varepsilon|^2_\Lambda + o(1)\end{aligned}$$

Setting $r = \max_j |x_j^\varepsilon|$, we remark that

$$-\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}|x_j^\varepsilon|^2_\Lambda \geq -(n_\varepsilon^2 - n_\varepsilon)\ln 2r + \frac{\Omega\Lambda^2 r^2}{1+\Lambda^2} \geq \frac{n_\varepsilon^2 - n_\varepsilon}{2}\ln|\ln\varepsilon| + O(1). \qquad (4.9)$$

Combining (4.7), (4.8) and (4.9), we obtain (4.6).

Step 3. We start by proving that $n_\varepsilon \geq d$. The case $d = 1$ is proved in Step 1 so that we may assume that $d \geq 2$. By Theorem 5.1 in Section 5, there exists for ε small enough, $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$F_\varepsilon(\tilde{u}_\varepsilon) \leq F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + O(1).$$

As in Step 1, $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$ yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq -\pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + O(1) \qquad (4.10)$$

Matching (4.6) with (4.10), we deduce that

$$-\omega(\varepsilon)n_\varepsilon + \frac{n_\varepsilon^2 - n_\varepsilon}{2} \leq -\omega(\varepsilon)d + \frac{d^2 - d}{2} + o(1)$$

and it yields

$$\omega(\varepsilon)(d - n_\varepsilon) \leq \frac{(d - n_\varepsilon)(d + n_\varepsilon - 1)}{2} + o(1). \qquad (4.11)$$

If assume that $n_\varepsilon \le d-1$, it would lead to

$$(d-1)+\delta \le \frac{d+n_\varepsilon-1}{2}+o(1) \le d-1+o(1)$$

which is impossible for ε small enough.

Assume now that $n_\varepsilon \ge d+1$. As previously we infer that (4.11) holds and therefore

$$d-\delta \ge \frac{d+n_\varepsilon-1}{2}+o(1) \ge d+o(1)$$

which is also impossible for ε small. Hence $n_\varepsilon \equiv d$ whenever ε is small enough which leads to (4.5) by (4.6) and (4.10). ■

By Proposition 4.2, we may now assume that v_ε has exactly d vortices. We follow with a first information on their location:

Lemma 4.2. *We have*

$$|x_j^\varepsilon| \le C|\ln\varepsilon|^{-1/2} \quad \textit{for } j=1,\dots,d \quad \textit{and if } d\ge 2, \quad |x_i^\varepsilon - x_j^\varepsilon| \ge C|\ln\varepsilon|^{-1/2} \quad \textit{for } i\neq j.$$

Proof. Matching (4.5) with (4.7) and (4.8) and using that $n_\varepsilon = d$, we deduce that

$$-\pi a_0 \sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi a_0 \Omega}{1+\Lambda^2}\sum_{j=1}^d |x_j^\varepsilon|_\Lambda^2 \le \pi a_0(d^2-d)\ln\left(|\ln\varepsilon|^{1/2}\right)+O(1).$$

Hence

$$\sum_{j=1}^d \left(-\sum_{i\neq j} \ln\left(\sqrt{|\ln\varepsilon|}\,|x_i^\varepsilon - x_j^\varepsilon|\right) + \frac{\Omega|x_j^\varepsilon|^2}{2}\right) \le O(1)$$

and the conclusion follows. ■

Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ by Lemma 4.2, we may now improve the lower estimates obtained in Lemma 3.1 following the method in [20, 21], proof of Proposition 5.2.

Lemma 4.3. *For any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$, we have*

$$\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \ge \pi a_0 \sum_{j=1}^d a(x_j^\varepsilon)|\ln\varepsilon| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1)$$

where γ_0 is an absolute constant.

Proof. Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ and $D_j = 1$, Proposition 3.1 yields

$$\frac{1}{2}\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \ge \pi \sum_{j=1}^d a(x_j^\varepsilon)|\ln\rho| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + o_R(1) \tag{4.12}$$

and it remains to estimate $\mathcal{E}_\varepsilon(v_\varepsilon, B(x_j^\varepsilon,\rho))$ for $j=1,\dots,d$. We proceed as follows. Since $D_j=1$, we may write on $\partial B(x_j^\varepsilon,\rho)$ in polar coordinates with center x_j^ε,

$$v_\varepsilon(x) = |v_\varepsilon(x)|\, e^{i(\theta+\psi_j(\theta))}, \quad \theta\in[0,2\pi]$$

where $\psi_j \in H^1([0,2\pi],\mathbb{R})$ and $\psi_j(0)=\psi_j(2\pi)=0$. Then in each disc $B(x_j^\varepsilon,2\rho)$, we consider the map $\hat{v}_\varepsilon$ defined by

$$\hat{v}_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & \text{if } x\in B(x_j^\varepsilon,\rho),\\ \big(\frac{r-\rho}{\rho} + \frac{2\rho-r}{\rho}|v_\varepsilon(x_j^\varepsilon+\rho\, e^{i\theta})|\big)\exp i\left(\theta+\psi_j(\theta)\frac{2\rho-r}{\rho}+\psi_j(0)\frac{\rho-r}{\rho}\right) & \text{if } x\in B(x_j^\varepsilon,2\rho)\setminus B(x_j^\varepsilon,\rho). \end{cases}$$

Then $\hat{v}_\varepsilon = \exp i(\theta+\psi_j(0))$ on $\partial B(x_j^\varepsilon,2\rho)$. Exactly as in the proof of Proposition 5.2 in [20, 21], we prove that

$$\big|\mathcal{E}_\varepsilon(\hat{v}_\varepsilon, B(x_j^\varepsilon,2\rho)\setminus B(x_j^\varepsilon,\rho)) - \pi a(x_j^\varepsilon)\ln 2\big| = o(1). \tag{4.13}$$

Since $|a(x)-a(x_j^\varepsilon)| = O(\rho)$ in $B(x_j^\varepsilon,2\rho)$, we may write

$$\mathcal{E}_\varepsilon(\hat{v}_\varepsilon, B(x_j^\varepsilon,2\rho)) = \frac{a(x_j^\varepsilon)}{2}\int_{B(x_j^\varepsilon,2\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 + o(1). \tag{4.14}$$

Now we shall recall a result in [8]. For $\tilde{\varepsilon}>0$, we consider

$$I(\tilde{\varepsilon}) = \operatorname*{Min}_{u\in\mathcal{C}} \frac{1}{2}\int_{B(0,1)} |\nabla u|^2 + \frac{1}{2\tilde{\varepsilon}^2}(1-|u|^2)^2$$

where

$$\mathcal{C} = \left\{ u\in H^1(B(0,1),\mathbb{C}),\ u(x)=\frac{x}{|x|} \text{ on } \partial B(0,1)\right\}.$$

Then we have

$$\lim_{\tilde{\varepsilon}\to 0}\big(I(\tilde{\varepsilon})+\pi\ln\tilde{\varepsilon}\big) = \gamma_0. \tag{4.15}$$

Since $\hat{v}_\varepsilon(x) = \frac{x-x_j^\varepsilon}{|x-x_j^\varepsilon|}\, e^{i\psi_j(0)}$ on $\partial B(x_j^\varepsilon,2\rho)$, we obtain by scaling

$$\frac{1}{2}\int_{B(x_j^\varepsilon,2\rho)} |\nabla\hat{v}_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 \geq I\left(\frac{\varepsilon}{2\rho\sqrt{a(x_j^\varepsilon)}}\right) = \pi\ln\frac{\rho}{\varepsilon} + \pi\ln 2 + \frac{\pi}{2}\ln a(x_j^\varepsilon) + \gamma_0 + o(1).$$

With (4.13) and (4.14), we derive that for $j=1,\dots,d$,

$$\begin{aligned}\mathcal{E}_\varepsilon(v_\varepsilon, B(x_j^\varepsilon,\rho)) &\geq \pi a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + \frac{\pi a(x_j^\varepsilon)}{2}\ln a(x_j^\varepsilon) + a(x_j^\varepsilon)\gamma_0 + o(1)\\ &\geq \pi a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + \frac{\pi a_0}{2}\ln a_0 + a_0\gamma_0 + o(1).\end{aligned}$$

Combining this estimate with (4.12), we get the result. ■

We are now able to give the asymptotic expansion of $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon)$ which will allow us to locate precisely the vortices. This concludes the proof of Theorem 1.1.

Proposition 4.3. *Setting* $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\, x_j^\varepsilon$ *for* $j = 1, \dots, d$, *as* $\varepsilon \to 0$ *the* $\tilde{x}_j^\varepsilon$*'s tend to minimize the renormalized energy* $w : \mathbb{R}^{2d} \to \mathbb{R}$ *given by*

$$w(b_1, \dots, b_d) = -\pi a_0 \sum_{i \neq j} \ln|b_i - b_j| + \frac{\pi a_0}{1+\Lambda^2} \sum_{j=1}^d |b_j|^2_\Lambda .$$

Moreover, we have

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = -\pi a_0 d\, \omega(\varepsilon) \ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + \operatorname*{Min}_{b \in \mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + o(1) \tag{4.16}$$

where $Q_{\Lambda,d} = \dfrac{\pi a_0}{2}(d^2 - d)\ln(1+\Lambda^2) + \pi a_0 d \ln a_0 - \dfrac{\pi a_0 d^2}{2} \ln a_0 + a_0 d \gamma_0 - \pi a_0 d^2 \ell(\Lambda)$ *and* $\ell(\Lambda)$ *is given by* (A.2).

Proof. From Lemma 4.3 and (3.17), we infer that for any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$,

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^d a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^d a^2(x_j^\varepsilon) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1).$$

As in the proof of Corollary 4.1, this estimate implies

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \pi \sum_{j=1}^d a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^d a^2(x_j^\varepsilon) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1).$$

Expanding Ω and $a(x_j^\varepsilon)$, we derive that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \pi \sum_{j=1}^d a(x_j^\varepsilon)\bigg(-\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\Omega|x_j^\varepsilon|^2_\Lambda}{1+\Lambda^2}\bigg) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1)$$

and by Lemma 4.2, it yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq -\pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{1+\Lambda^2} \sum_{j=1}^d \Omega|x_j^\varepsilon|^2_\Lambda + W_{R,\varepsilon} + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1). \tag{4.17}$$

By Lemma 4.2, we also have

$$W_{R,\varepsilon} = -\pi a_0 \sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| - \pi \sum_{j=1}^d \Psi_{R,\varepsilon}(x_j^\varepsilon) + o(1). \tag{4.18}$$

Since $D_j = 1$ for all j, the function $\Psi_{R,\varepsilon}$ satisfies the equation

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla\Psi_{R,\varepsilon}\right) = -\displaystyle\sum_{j=1}^d a(x_j^\varepsilon)\,\nabla\left(\frac{1}{a}\right)\cdot\nabla\left(\ln|x - x_j^\varepsilon|\right) & \text{in } B_R^\Lambda, \\ \Psi_{R,\varepsilon} = -\displaystyle\sum_{j=1}^d a(x_j^\varepsilon)\ln|x - x_j^\varepsilon| & \text{on } \partial B_R^\Lambda. \end{cases} \tag{4.19}$$

We infer from Lemma 4.2 that for $j = 1, \dots, d$,

$$a(x_j^\varepsilon)\,\nabla\left(\frac{1}{a}\right)\cdot\nabla\left(\ln|x - x_j^\varepsilon|\right) = \frac{-2a_0|x|_\Lambda^2}{a^2(x)|x|^2} + f_\varepsilon^j(x).$$

where f_ε^j satisfies $\|f_\varepsilon^j\|_{L^p(B_R^\Lambda)} = o_R(1)$ for any $p \in [1,2)$ and $\big\|a_0\ln|x| - a(x_j^\varepsilon)\ln|x - x_j^\varepsilon|\,\big\|_{C^1(\partial B_R^\Lambda)} = o(1)$. Letting Ψ_R to be the solution of the equation

$$\begin{cases}\operatorname{div}\left(\dfrac{1}{a}\nabla\Psi_R\right) = \dfrac{-2|x|_\Lambda^2}{a^2(x)|x|^2} & \text{in } B_R^\Lambda,\\ \Psi_R = -\ln|x| & \text{on } \partial B_R^\Lambda,\end{cases} \tag{4.20}$$

it follows by classical results that $\|\Psi_{R,\varepsilon} - a_0 d\Psi_R\|_{L^\infty(B_R^\Lambda)} = o_R(1)$. Hence we obtain from (4.18),

$$\lim_{\varepsilon\to 0}\left\{W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + \pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon|\right\} = -\pi a_0 d^2\Psi_R(0). \tag{4.21}$$

Combining (4.17) and (4.21), we are led to

$$\liminf_{\varepsilon\to 0}\left\{\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| + \pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| - \frac{\pi a_0}{1+\Lambda^2}\sum_{j=1}^d \Omega|x_j^\varepsilon|_\Lambda^2\right\} \geq$$
$$\geq \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 - \pi a_0 d^2\Psi_R(0).$$

Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\,x_j^\varepsilon$, it yields

$$\liminf_{\varepsilon\to 0}\left\{\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| - w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon)\right\} \geq$$
$$\geq \frac{\pi a_0}{2}(d^2 - d)\ln(1+\Lambda^2) + \pi a_0 d\ln a_0 - \frac{\pi a_0 d^2}{2}\ln a_0 + a_0 d\gamma_0 - \pi a_0 d^2\Psi_R(0).$$

Since $\Psi_R(0) \to \ell(\Lambda)$ as $R \to \sqrt{a_0}$ by Lemma A.1 in Appendix A, we conclude that

$$\liminf_{\varepsilon\to 0}\left\{\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0\omega(\varepsilon) d\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| - w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon)\right\} \geq Q_{\Lambda,d} \tag{4.22}$$

and hence

$$\liminf_{\varepsilon\to 0}\left\{\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0\omega(\varepsilon) d\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon|\right\} \geq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d}. \tag{4.23}$$

By Theorem 5.1 in Section 5, for any $\delta' > 0$, there exists $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$\limsup_{\varepsilon\to 0}\left\{F_\varepsilon(\tilde{u}_\varepsilon) - F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon|\right\} \leq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + \delta'$$

As in the proof of Proposition 4.2, $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$ implies

$$\limsup_{\varepsilon\to 0}\left\{\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon|\right\} \leq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + \delta'. \tag{4.24}$$

Matching (4.23) with (4.24), we conclude that

$$\lim_{\varepsilon\to 0}\left\{\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon)+\pi a_0 d\,\omega(\varepsilon)\ln|\ln\varepsilon|-\frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\right\}=\operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b)+Q_{\Lambda,d}$$

since δ' is arbitrarily small. Coming back to (4.22), we are led to

$$\operatorname{Min} w(b)+Q_{\Lambda,d}-\limsup_{\varepsilon\to 0} w(x_1^\varepsilon,\dots,x_d^\varepsilon)\geq Q_{\Lambda,d}$$

and therefore $\lim_{\varepsilon\to 0} w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon)=\operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b)$ which ends the proof. ■

Remark 4.1. In the case $d=1$, the expansion of the energy takes the simpler form

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon)=-\pi a_0\omega(\varepsilon)\ln|\ln\varepsilon|+Q_{\Lambda,1}+o(1)$$

and the renormalized energy $w(\cdot)$ reduces to $w(b)=(\pi a_0|b|_\Lambda^2)/(1+\Lambda^2)$. In particular, if x^ε denotes the single vortex of v_ε, we have $\sqrt{\Omega}\,x^\varepsilon\to 0$ as ε goes to 0.

5 Upper bound of the energy

In this section, we give the construction of the test functions used in the previous sections. We assume that (1.7) holds. Using notation (4.3), the result can be stated as follows:

Theorem 5.1. *Let $d\geq 1$ be an integer. For any $\delta>0$, there exists $(\tilde{u}_\varepsilon)_{\varepsilon>0}\subset\mathcal{H}$ verifying $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)}=1$ and*

$$\limsup_{\varepsilon\to 0}\left\{F_\varepsilon(\tilde{u}_\varepsilon)-F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S})+\pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon|-\frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\right\}\leq\operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b)+Q_{\Lambda,d}+\delta$$

where the constant $Q_{\Lambda,d}$ is defined in Proposition 4.3.

The proof of Theorem 5.1 is based on a first construction given by the following proposition. Here, some of the main ingredients are taken from André and Shafrir [5].

Proposition 5.1. *Let $d\geq 1$ be an integer. For any $\delta>0$, there exists $(\hat{v}_\varepsilon)_{\varepsilon>0}$ such that $\tilde{\eta}_\varepsilon\hat{v}_\varepsilon\in\mathcal{H}$ and*

$$\limsup_{\varepsilon\to 0}\left\{\tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon)+\pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon|-\frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\right\}\leq\operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b)+Q_{\Lambda,d}+\delta.$$

Proof. Step 1. Let $\sigma>0$ and $\kappa>0$ be two small parameters that we will choose later. We consider the function $a_\sigma:\overline{\mathcal{D}}\to\mathbb{R}$ given by

$$a_\sigma(x)=\begin{cases}a(x) & \text{if } |x|_\Lambda\leq\sqrt{a_0-\sigma},\\ -2\sqrt{a_0-\sigma}\,|x|_\Lambda+2a_0-\sigma & \text{otherwise}\end{cases}$$

It turns out that $a_\sigma \in C^1(\overline{\mathcal{D}})$, $a_\sigma \geq a$ and $a_\sigma \geq C\sigma^2$ in $\overline{\mathcal{D}}$ for some positive constant C. Since a_σ does not vanish in $\overline{\mathcal{D}}$, we may define $\Phi_\sigma : \mathcal{D} \to \mathbb{R}$ the solution of the equation

$$\begin{cases} \operatorname{div}(\dfrac{1}{a_\sigma}\nabla\Phi_\sigma) = 2\pi d\,\delta_0 & \text{in } \mathcal{D}, \\ \Phi_\sigma = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{5.1}$$

By the results in Chap. I of [8], we may find a map $v_0^\sigma \in C^2(\overline{\mathcal{D}} \setminus \{0\}, S^1)$ satisfying

$$v_0^\sigma \wedge \nabla v_0^\sigma = \frac{1}{a_\sigma}\nabla^\perp\Phi_\sigma \quad \text{in } \mathcal{D}\setminus\{0\}. \tag{5.2}$$

Set $\Theta_{\kappa,\varepsilon} = \mathcal{D} \setminus B(0, \kappa^{-1}\Omega^{-1/2})$. By (5.1) and (5.2), we have for ε small enough,

$$\begin{aligned} \int_{\Theta_{\kappa,\varepsilon}} a_\sigma|\nabla v_0^\sigma|^2 = \int_{\Theta_{\kappa,\varepsilon}} \frac{1}{a_\sigma}|\nabla\Phi_\sigma|^2 &= -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{1}{a}\frac{\partial\Phi_\sigma}{\partial\nu}\Phi_\sigma \\ &= -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{a_0^2 d^2}{a}\Big(\frac{\partial\Psi_\sigma}{\partial\nu} + \frac{1}{|x|}\Big)\big(\Psi_\sigma + \ln|x|\big) \end{aligned} \tag{5.3}$$

where $\Psi_\sigma(x) = (a_0 d)^{-1}\Phi_\sigma(x) - \ln|x|$. Notice that $\Psi_\sigma \in C^{1,\alpha}(\overline{\mathcal{D}})$ for any $0 < \alpha < 1$, since it satisfies the equation

$$\begin{cases} \operatorname{div}\big(\dfrac{1}{a_\sigma}\nabla\Psi_\sigma\big) = f_\sigma(x) & \text{in } \mathcal{D}, \\ \Psi_\sigma = -\ln|x| & \text{on } \partial\mathcal{D} \end{cases} \tag{5.4}$$

with

$$f_\sigma(x) = -\nabla\Big(\frac{1}{a_\sigma(x)}\Big)\cdot\frac{x}{|x|^2} = \begin{cases} \dfrac{-2|x|_\Lambda^2}{a_\sigma^2(x)|x|^2} & \text{if } |x| \leq \sqrt{a_0 - \sigma}, \\ \dfrac{-2\sqrt{a_0-\sigma}\,|x|_\Lambda}{a_\sigma^2(x)|x|^2} & \text{otherwise.} \end{cases}$$

From (5.3), we derive that

$$\begin{aligned} \limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}} a|\nabla v_0^\sigma|^2 - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} &\leq \lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}} a_\sigma|\nabla v_0^\sigma|^2 - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} \\ &\leq -\pi a_0 d^2\Psi_\sigma(0). \end{aligned}$$

By Lemma A.1 in Appendix A, $\Psi_\sigma(0) \to \ell(\Lambda)$ as $\sigma \to 0$ where the constant $\ell(\Lambda)$ is defined in (A.2). Consequently, we may choose σ small such that

$$\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}} a|\nabla v_0^\sigma|^2 - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} \leq -\pi a_0 d^2\ell(\Lambda) + \frac{\delta}{2}. \tag{5.5}$$

In $\mathbb{R}^2 \setminus B(0, \kappa^{-1}\Omega^{-1/2})$, we define

$$\hat{v}_\varepsilon(x) = \begin{cases} v_0^\sigma(x) & \text{if } x \in \Theta_\kappa, \\ v_0^\sigma\big(\dfrac{\sqrt{a_0}\,x}{|x|_\Lambda}\big) & \text{if } x \in \mathbb{R}^2\setminus\mathcal{D}. \end{cases}$$

By the results in [14], we have $\|\tilde\eta_\varepsilon^2\|_{L^\infty(\mathbb{R}^2\setminus\mathcal{D}_\varepsilon)} = o(1)$. Since $\hat v_\varepsilon$ does not depend on ε in $\mathbb{R}^2 \setminus \mathcal{D}_\varepsilon$ and $|\hat v_\varepsilon| = 1$ in $\mathbb{R}^2 \setminus \mathcal{D}_\varepsilon$, we derive that

$$\lim_{\varepsilon\to 0} \tilde{\mathcal{E}}_\varepsilon(\hat v_\varepsilon, \mathbb{R}^2 \setminus \mathcal{D}_\varepsilon) = 0. \tag{5.6}$$

We also proved in [14],

$$\left\| \frac{a - \tilde\eta_\varepsilon^2}{\tilde\eta_\varepsilon^2} \right\|_{L^\infty(\mathcal{D}_\varepsilon)} \le C\varepsilon^{1/3} \tag{5.7}$$

and hence (5.5) remains valid if one replaces a by $\tilde\eta_\varepsilon^2$ in the left hand side. Since v_0^σ is S^1-valued, we deduce that

$$\limsup_{\varepsilon\to 0} \Big\{ \tilde{\mathcal{E}}_\varepsilon(\hat v_\varepsilon, \mathbb{R}^2 \setminus B(0, \kappa^{-1}\Omega^{-1/2})) - \pi a_0 d^2 \ln(\kappa\Omega^{1/2}) \Big\} \le -\pi a_0 d^2 \ell(\Lambda) + \frac{\delta}{2}. \tag{5.8}$$

Step 2. We are going to extend $\hat v_\varepsilon$ to $B(0,\kappa^{-1}\Omega^{-1/2})$. As in [8], we may write in a neighborhood of 0 (using polar coordinates),

$$v_0^\sigma(x) = \exp\big(i(d\theta + \psi_\sigma(x))\big)$$

where ψ_σ is a smooth function in that neighborhood. Let $(b_1, \dots, b_d) \in \mathbb{R}^{2d}$ be a minimizing configuration for $w(\cdot)$, i.e.,

$$w(b_1, \dots, b_d) = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) \tag{5.9}$$

(note that we necesarily have $b_i \ne b_j$ for $i \ne j$). We choose κ sufficiently small such that $\max |b_j| \le 1/4\kappa$ and we set $b_j^{(\varepsilon)} = \Omega^{-1/2}\, b_j$. Following the proof of Lemma 2.6 in [5], we write

$$e^{i\psi_\sigma(0)} \prod_{j=1}^d \frac{x - b_j^{(\varepsilon)}}{|x - b_j^{(\varepsilon)}|} = \exp\big(i(d\theta + \phi_\varepsilon(x))\big) \quad \text{for } x \in A_{\kappa,\varepsilon} = B(0,\kappa^{-1}\Omega^{-1/2}) \setminus B(0,(2\kappa)^{-1}\Omega^{-1/2})$$

where ϕ_ε is a smooth function satisfying $|\nabla\phi_\varepsilon(x)| \le C_\sigma\, \kappa^2\Omega^{1/2})$ and $|\phi_\varepsilon(x) - \psi_\sigma(0)| = C_\sigma\, \kappa^2$ for $x \in A_{\kappa,\varepsilon}$. We define in $A_{\kappa,\varepsilon}$,

$$\hat v_\varepsilon(x) = \exp\big(i(d\theta + \hat\psi_\varepsilon(x))\big)$$

with

$$\hat\psi_\varepsilon(x) = \big(2 - 2\kappa\Omega^{1/2}|x|\big)\phi_\varepsilon(x) + \big(2\kappa\Omega^{1/2}|x| - 1\big)\psi_\sigma(x).$$

As in [5], we get that (using (5.7))

$$\limsup_{\varepsilon\to 0} \Big\{ \tilde{\mathcal{E}}_\varepsilon(\hat v_\varepsilon, A_{\kappa,\varepsilon}) - \pi a_0 d^2 \ln 2 \Big\} \le \limsup_{\varepsilon\to 0} \left\{ \frac{1}{2}\int_{A_{\kappa,\varepsilon}} a_\sigma |\nabla \hat v_\varepsilon|^2 - \pi a_0 d^2 \ln 2 \right\} \le C_\sigma\, \kappa^2. \tag{5.10}$$

Next we define $\hat v_\varepsilon$ in $\Xi_{\kappa,\varepsilon} = B(0,(2\kappa)^{-1}\Omega^{-1/2}) \setminus \cup_{j=1}^d B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$ by

$$\hat v_\varepsilon(x) = e^{i\psi_\sigma(0)} \prod_{j=1}^d \frac{x - b_j^{(\varepsilon)}}{|x - b_j^{(\varepsilon)}|}.$$

Once more as in [5], we have (using (5.7))

$$\limsup_{\varepsilon\to 0} \tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, \Xi_{\kappa,\varepsilon}) \leq \limsup_{\varepsilon\to 0} \frac{1}{2}\int_{\Xi_{\kappa,\varepsilon}} a_\sigma |\nabla \hat{v}_\varepsilon|^2 \leq \pi a_0 (d^2+d)\ln\frac{1}{2\kappa} - \pi a_0 \sum_{i\neq j} \ln|b_i - b_j| + C_\sigma\,\kappa. \quad (5.11)$$

Finally, in each $B_j^{(\varepsilon)} := B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$, we set

$$\hat{v}_\varepsilon(x) = e^{i\psi_\sigma(0)}\tilde{w}_\varepsilon^j\left(\frac{x-b_j^{(\varepsilon)}}{2\kappa\Omega^{-1/2}}\right) \quad (5.12)$$

where $\tilde{w}_\varepsilon^j$ realizes

$$\operatorname{Min}\left\{\frac{1}{2}\int_{B(0,1)} |\nabla v|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|v|^2)^2\,,\ v(y) = \prod_{i=1}^d \frac{2\kappa y + b_j - b_i}{|2\kappa y + b_j - b_i|}\ \text{ on } \partial B(0,1)\right\} \quad (5.13)$$

with

$$\hat{\varepsilon} = \frac{\varepsilon}{2\kappa\sqrt{a_0}\,\Omega^{-1/2}}.$$

As in the proof of Lemma 2.3 in [5], we derive

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B(0,1)} |\nabla \tilde{w}_\varepsilon^j|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|\tilde{w}_\varepsilon^j|^2)^2 - \pi|\ln\hat{\varepsilon}|\right\} = \gamma_0 + X(\kappa)$$

where γ_0 is defined in (4.15) and $X(\kappa)$ denotes a quantity satisfying $X(\kappa)\to 0$ as $\kappa\to 0$. By scaling, we obtain

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B_j^{(\varepsilon)}} |\nabla \hat{v}_\varepsilon|^2 + \frac{a_0}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 - \pi\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\} = \frac{\pi}{2}\ln a_0 + \gamma_0 + X(\kappa).$$

Notice that in $B_j^{(\varepsilon)}$,

$$a_\sigma(x) = a(x) \leq a_0 - (|\ln\varepsilon| + \omega_1\ln|\ln\varepsilon|)^{-1}\min_{y\in B(b_j,2\kappa)}\frac{a_0|y|_\Lambda^2}{1+\Lambda^2}$$

and consequently,

$$\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B_j^{(\varepsilon)}} a_\sigma|\nabla \hat{v}_\varepsilon|^2 + \frac{a_0 a_\sigma}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 - \pi a_0\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\} \leq$$

$$\leq \frac{\pi a_0}{2}\ln a_0 + a_0\gamma_0 - \frac{\pi a_0|b_j|_\Lambda^2}{1+\Lambda^2} + X(\kappa).$$

By (5.7), it yields

$$\limsup_{\varepsilon\to 0}\left\{\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, B_j^{(\varepsilon)}) - \pi a_0\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\} \leq \frac{\pi a_0}{2}\ln a_0 + a_0\gamma_0 - \frac{\pi a_0|b_j|_\Lambda^2}{1+\Lambda^2} + X(\kappa). \quad (5.14)$$

Combining (5.8), (5.10), (5.11) and (5.14), we conclude that for κ small enough,

$$\limsup_{\varepsilon\to 0}\left\{\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon)-\pi a_0 d|\ln\varepsilon|-\frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\right\}\leq \tag{5.15}$$

$$\leq -\pi a_0\sum_{i\neq j}\ln|b_i-b_j|-\frac{\pi a_0}{1+\Lambda^2}\sum_{j=1}^d|b_j|^2_\Lambda+Q_{\Lambda,d}+\delta.$$

Step 3. Now it remains to estimate $\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)$. Cauchy-Schwartz inequality yields

$$|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,\mathbb{R}^2\setminus\mathcal{D}_\varepsilon)|\leq C\Omega\left(\int_{\mathbb{R}^2\setminus\mathcal{D}_\varepsilon}|x|^2\tilde{\eta}_\varepsilon^2\right)^{1/2}\left(\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,\mathbb{R}^2\setminus\mathcal{D}_\varepsilon)\right)^{1/2}. \tag{5.16}$$

By the results in [14], $\Omega^2\int_{\mathbb{R}^2\setminus\mathcal{D}_\varepsilon}|x|^2\tilde{\eta}_\varepsilon^2\to 0$ as $\varepsilon\to 0$ and according to (5.6), it leads to

$$\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)-\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)\right|=0. \tag{5.17}$$

By the results in Chap. IX in [8], for $\hat{\varepsilon}$ sufficiently small and each $j=1,\ldots,d$, there exists exactly one disc $\hat{D}^j_\varepsilon\subset B(0,1)$ with $\mathrm{diam}(\hat{D}^j_\varepsilon)\leq C\hat{\varepsilon}$ such that $|\tilde{w}^j_\varepsilon|\geq 1/2$ in $B(0,1)\setminus\hat{D}^j_\varepsilon$. By scaling, we infer that exist exactly d discs $D^1_\varepsilon,\ldots,D^d_\varepsilon$ with $D^j_\varepsilon\subset B^{(\varepsilon)}_j$ and $\mathrm{diam}(D^j_\varepsilon)\leq C\varepsilon$ such that

$$|\hat{v}_\varepsilon|\geq\frac{1}{2}\quad\text{in }\ \mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D^j_\varepsilon.$$

We derive from (5.14) that

$$\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,\cup_{j=1}^d D^j_\varepsilon)\right|\leq C\Omega\,\varepsilon\sum_{j=1}^d\left(\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,B^{(\varepsilon)}_j)\right)^{1/2}\underset{\varepsilon\to 0}{\longrightarrow}0,$$

and by (5.17), it leads to $\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)-\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D^j_\varepsilon)\right|=0$. From (5.7), we infer that

$$\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D^j_\varepsilon)-\mathcal{R}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D^j_\varepsilon)\right|=0$$

and hence

$$\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)-\mathcal{R}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D^j_\varepsilon)\right|=0. \tag{5.18}$$

To compute $\mathcal{R}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}\setminus\cup_{j=1}^d D^j_\varepsilon)$, we proceed as in [14] (here we use that $\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon)\leq C|\ln\varepsilon|$ by (5.15)). It yields

$$\lim_{\varepsilon\to 0}\left(\mathcal{R}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D^j_\varepsilon)+\frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^d a^2(b^{(\varepsilon)}_j)\right)=0$$

since $\deg(\hat{v}_\varepsilon/|\hat{v}_\varepsilon|,\partial D^j_\varepsilon)=+1$ for $j=1,\ldots,d$. Expanding $a^2(b^{(\varepsilon)}_j)$ and Ω, we deduce from (5.18) that

$$\lim_{\varepsilon\to 0}\left(\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)+\pi a_0 d|\ln\varepsilon|+\pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon|\right)=\frac{2\pi a_0}{1+\Lambda^2}\sum_{j=1}^d|b_j|^2_\Lambda. \tag{5.19}$$

Combining (5.9), (5.15) and (5.19), we obtain the announced result. ■

Proof of Theorem 5.1. We consider the map $\hat{v}_\varepsilon$ given in Proposition 5.1 and we set

$$\tilde{v}_\varepsilon = m_\varepsilon^{-1}\hat{v}_\varepsilon \quad \text{and} \quad \tilde{u}_\varepsilon = \tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon \quad \text{with} \quad m_\varepsilon = \|\tilde{\eta}_\varepsilon \hat{v}_\varepsilon\|_{L^2(\mathbb{R}^2)}.$$

We are going to prove that the map $\tilde{u}_\varepsilon$ satisfies the required property. By Lemma 3.2 in [14], we have

$$F_\varepsilon(\tilde{u}_\varepsilon) = F(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon).$$

In view of Proposition 5.1, it suffices to prove that $\left|\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) - \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon)\right| \to 0$ and $\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon) \to 0$ as $\varepsilon \to 0$. We first estimate m_ε. Since $|\hat{v}_\varepsilon| = 1$ in $\mathbb{R}^2 \setminus \cup_{j=1}^d B_j^{(\varepsilon)}$ and $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we have

$$m_\varepsilon^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 + \int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1) = 1 + \int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1).$$

Using Cauchy-Schwarz inequality, we derive from (5.12), (5.13) and Theorem III.2 in [8] that

$$\Big|\int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1)\Big| \leq C|\ln\varepsilon|^{-1/2}\left(\int_{\cup_{j=1}^d B_j^{(\varepsilon)}} (|\hat{v}_\varepsilon|^2 - 1)^2\right)^{1/2} \leq C\varepsilon|\ln\varepsilon|^{-1/2} \tag{5.20}$$

and thus

$$m_\varepsilon^2 = 1 + O(\varepsilon|\ln\varepsilon|^{-1/2}). \tag{5.21}$$

Using $|\hat{v}_\varepsilon| = 1$ in $\mathbb{R}^2 \setminus \cup_{j=1}^d B_j^{(\varepsilon)}$, $|\nabla S| \leq C|x|$, $|k_\varepsilon| \leq C|\ln\varepsilon|$, (5.20) and (5.21), we derive that

$$\begin{aligned}\left|\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\right| &\leq C|\ln\varepsilon|^2\left(|1 - m_\varepsilon^{-2}|\int_{\mathbb{R}^2}(1 + |x|^2)\tilde{\eta}_\varepsilon^2 + \int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^2\big(|1 - m_\varepsilon^{-2}||\hat{v}_\varepsilon|^2 + (1 - |\hat{v}_\varepsilon|^2)\big)\right)\\ &\leq C\varepsilon|\ln\varepsilon|^{3/2}.\end{aligned}$$

Now we may estimate using (5.15), (5.19) and (5.21),

$$\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\tilde{v}_\varepsilon|^2 = m_\varepsilon^{-2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 + O(\varepsilon|\ln\varepsilon|^{1/2}), \tag{5.22}$$

and

$$\tilde{\mathcal{R}}_\varepsilon(\tilde{v}_\varepsilon) = m_\varepsilon^{-2}\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) = \tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) + O(\varepsilon|\ln\varepsilon|^{1/2}). \tag{5.23}$$

We write

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1 - |\tilde{v}_\varepsilon|^2)^2 = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1 - |\hat{v}_\varepsilon|^2)^2 + \frac{2(1 - m_\varepsilon^{-2})}{\varepsilon^2}\int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^4(1 - |\hat{v}_\varepsilon|^2)|\hat{v}_\varepsilon|^2\\ + \frac{(1 - m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4|\hat{v}_\varepsilon|^4.\end{aligned} \tag{5.24}$$

We infer from (5.15) and (5.21) that

$$\frac{(1 - m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4|\hat{v}_\varepsilon|^4 \leq C|\ln\varepsilon|^{-1}, \tag{5.25}$$

and from (5.20) and (5.21),

$$\frac{|1-m_\varepsilon^{-2}|}{\varepsilon^2}\int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde\eta_\varepsilon^4|\hat v_\varepsilon|^2\big|1-|\hat v_\varepsilon|^2\big| \le C|\ln\varepsilon|^{-1}. \tag{5.26}$$

Combining (5.22), (5.23), (5.24), (5.25) and (5.26), we finally obtain that $\tilde{\mathcal F}_\varepsilon(\tilde v_\varepsilon) = \tilde{\mathcal F}_\varepsilon(\hat v_\varepsilon)+o(1)$ and the proof is complete. ∎

A Appendix

In this appendix, we prove that the functions Ψ_R and Ψ_σ defined by (4.20) and respectively (5.4) converge to the same limiting function as $R\to\sqrt{a_0}$ and $\sigma\to 0$.

Lemma A.1. *For any $0<R<\sqrt{a_0}$, respectively any $\sigma>0$, let Ψ_R be the solution of equation* (4.20)*, respectively Ψ_σ the solution of* (5.4)*. Then $\Psi_R\to\Psi_\star$ as $R\to\sqrt{a_0}$, respectively $\Psi_\sigma\to\Psi_\star$ as $\sigma\to 0$, in $C^1_{\rm loc}(\mathcal D)$ where $\Psi_\star$ is the unique solution in $C^0(\overline{\mathcal D})$ of*

$$\begin{cases}\operatorname{div}\big(\dfrac{1}{a}\nabla\Psi_\star\big) = \dfrac{-2|x|_\Lambda^2}{a^2(x)|x|^2} & \text{in } \mathcal D,\\ \Psi_\star = -\ln|x| & \text{on } \partial\mathcal D.\end{cases} \tag{A.1}$$

In particular,

$$\lim_{R\to\sqrt{a_0}}\Psi_R(0) = \lim_{\sigma\to 0}\Psi_\sigma(0) = \Psi_\star(0) =: \ell(\Lambda). \tag{A.2}$$

Proof. Step 1: Uniqueness of $\Psi_\star$. Assume that (A.1) admits two solutions $\Psi_\star^1$ and $\Psi_\star^2$ in $C^0(\overline{\mathcal D})$. Then the difference $\Psi_\star^1-\Psi_\star^2$ satisfies $\operatorname{div}(\frac1a\nabla(\Psi_\star^1-\Psi_\star^2))=0$ in $\mathcal D$ and $\Psi_\star^1-\Psi_\star^2=0$ on $\partial\mathcal D$. By elliptic regularity, we infer that $\Psi_\star^1-\Psi_\star^2\in C^2(\mathcal D)\cap C^0(\overline{\mathcal D})$. Hence it follows $\Psi_\star^1-\Psi_\star^2\equiv 0$ by the classical maximum principle.

Step 2: Existence of $\Psi_\star$. We set for $y\in\mathcal D$,

$$\Upsilon_R(y) = \Psi_R\bigg(\frac{Ry}{\sqrt{a_0}}\bigg) - \zeta(y) + \ln(R/\sqrt{a_0})$$

where ζ is the solution of

$$\begin{cases}\Delta\zeta = 0 & \text{in } \mathcal D,\\ \zeta = -\ln|y| & \text{on } \partial\mathcal D.\end{cases}$$

Since Ψ_R solves (4.20), we deduce that Υ_R is the unique solution of

$$\begin{cases}-\operatorname{div}\bigg(\dfrac{1}{a_R(y)}\nabla\Upsilon_R\bigg) = \dfrac{f(y)}{a_R^2(y)} & \text{in } \mathcal D,\\ \Upsilon_R = 0 & \text{on } \partial\mathcal D.\end{cases} \tag{A.3}$$

where $a_R(y) = a_0^2/R^2 - |y|_\Lambda^2$ and

$$f(y) = \frac{2|y|_\Lambda^2}{|y|^2} + 2(y_1, \Lambda^2 y_2) \cdot \nabla\zeta(y).$$

We easily check that $y \mapsto K a_R(y)$, respectively $y \mapsto -K a_R(y)$, defines a supersolution, resp. a subsolution, of (A.3) whenever the constant K satisfies $K \geq \|f\|_{L^\infty(\mathcal{D})}/(\Lambda^2 a_0)$. Hence

$$|\Upsilon_R| \leq C a_R \quad \text{in } \mathcal{D} \tag{A.4}$$

for a constant C independent of R. By elliptic regularity, we deduce that Υ_R remains bounded in $W^{2,p}_{\text{loc}}(\mathcal{D})$ as $R \to \sqrt{a_0}$ for any $1 \leq p < \infty$. Therefore, from any sequence $R_n \to \sqrt{a_0}$, we may extract a subsequence, still denoted by (R_n), such that $\Upsilon_{R_n} \to \Upsilon_\star$ in $C^1_{\text{loc}}(\mathcal{D})$ where $\Upsilon_\star$ satisfies

$$-\operatorname{div}\left(\frac{1}{a(y)}\nabla\Upsilon_\star\right) = \frac{f}{a^2(y)} \quad \text{in } \mathcal{D}.$$

We infer from (A.4) that $|\Upsilon_\star(y)| \leq C a(y)$ for any $y \in \mathcal{D}$ and hence $\Upsilon_\star \in C^0(\overline{\mathcal{D}})$ with $\Upsilon_{\star|\partial\mathcal{D}} = 0$. Consequently, the function $\Psi_\star := \Upsilon_\star + \zeta$ defines a solution of (A.1) which is continuous in $\overline{\mathcal{D}}$.

Step 3. By the uniqueness of $\Psi_\star$, we have that $\Upsilon_R \to \Psi_\star - \zeta$ in $C^1_{\text{loc}}(\mathcal{D})$ as $R \to \sqrt{a_0}$ which clearly implies $\Psi_R \to \Psi_\star$ in $C^1_{\text{loc}}(\mathcal{D})$ as $R \to \sqrt{a_0}$. To prove that $\Psi_\sigma \to \Psi_\star$ in $C^1_{\text{loc}}(\mathcal{D})$ as $\sigma \to 0$, we may proceed as in Step 2. Indeed, we may show as in Step 2, that $|\Psi_\sigma - \zeta| \leq C a_\sigma$ in $\mathcal{D}$ for a constant C independent of σ. ■

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