The space $BV(S^2, S^1)$: minimal connection and optimal lifting.

Radu Ignat

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Abstract

We show that topological singularities of maps in $BV(S^2, S^1)$ can be detected by its distributional Jacobian. As an application, we construct an optimal lifting and we compute its total variation.

Résumé

On montre que le jacobien d'une fonction $u \in BV(S^2, S^1)$ permet de localiser les singularités topologiques de u. On applique ce résultat à la construction d'un relèvement optimal et on calcule sa variation totale.

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1 Introduction

Let $u \in BV(S^2, S^1)$, i.e. $u = (u_1, u_2) \in L^1(S^2, \mathbb{R}^2)$, $|u(x)| = 1$ for a.e. $x \in S^2$ and the derivative of u (in the sense of the distributions) is a finite $2 \times 2-$matrix Radon measure

$$\int_{S^2} |Du| = \sup \left\{ \int_{S^2} \sum_{k=1}^{2} u_k \operatorname{div} \zeta_k \, \mathrm{d}\mathcal{H}^2 \,:\, \zeta_k \in C^1(S^2, \mathbb{R}^2), \sum_{k=1}^{2} |\zeta_k(x)|^2 \leq 1, \forall x \in S^2 \right\} < \infty \,,$$

where the norm in $\mathbb{R}^2$ is the Euclidean norm. Observe that the total variation of Du is independent of the choice of the orthonormal frame (x, y) on S^2; a frame (x, y) is always taken such that (x, y, e) is direct, where e is the outward normal to the sphere S^2.

We begin with the notion of minimal connection between point singularities of u. The concept of a minimal connection associated to a function from $\mathbb{R}^3$ into S^2 was originally introduced by Brezis, Coron and Lieb [3]. Following the ideas in [3] and [6], Brezis, Mironescu and Ponce [4] studied the topological singularities of functions $g \in W^{1,1}(S^2, S^1)$. They show that the distributional Jacobian of g describes the location and the topological charge of the singular set of g. More precisely, let $T(g) \in \mathcal{D}'(S^2, \mathbb{R})$ be defined as

$$T(g) = 2 \det(\nabla g) = -(g \wedge g_x)_y + (g \wedge g_y)_x;$$

then there exist two sequences of points $(p_k), (n_k)$ in S^2 such that

$$\sum_k |p_k - n_k| < \infty \quad \text{and} \quad T(g) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}).$$

Our aim is to extend these notions for functions $u \in BV(S^2, S^1)$. In this case, the difficulty of the analysis of the singular set arises from the existence of more than one type of singularity: besides the point singularities carrying a degree, the jump singularities of u should be taken into account.

We start by introducing some notation. Write the finite Radon 2×2-matrix measure Du as

$$Du = D^a u + D^c u + D^j u,$$

where $D^a u, D^c u$ and $D^j u$ are the absolutely continuous part, the Cantor part and the jump part of Du (see e.g. [1]). We recall that $D^j u$ can be written as

$$D^j u = (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^1 \llcorner S(u),$$

where $S(u)$ denotes the set of jump points of u; $S(u)$ is a countably $\mathcal{H}^1$-rectifiable set on S^2 oriented by the Borel map $\nu_u : S(u) \to S^1$. The Borel functions $u^+, u^- : S(u) \to S^1$ are the traces of u on the jump set $S(u)$ with respect to the orientation ν_u. Throughout the paper we identify u by its precise representative that is defined $\mathcal{H}^1$-a.e. on $S^2 \setminus S(u)$.

We now introduce the distribution $T(u) \in \mathcal{D}'(S^2, \mathbb{R})$ as

$$\langle T(u), \zeta \rangle = \int_{S^2} \nabla^\perp \zeta \cdot (u \wedge (D^a u + D^c u)) + \int_{S(u)} \rho(u^+, u^-) \, \nu_u \cdot \nabla^\perp \zeta \, d\mathcal{H}^1, \ \forall \zeta \in C^1(S^2, \mathbb{R}). \quad (1)$$

Here, $\nabla^\perp \zeta = (\zeta_y, -\zeta_x)$,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \wedge \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = (u \wedge a, u \wedge b) = (u_1 a_2 - u_2 a_1, u_1 b_2 - u_2 b_1)$$

where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. The function $\rho(\cdot, \cdot) : S^1 \times S^1 \to [-\pi, \pi]$ is the signed geodesic distance on S^1 defined as

$$\rho(\omega_1, \omega_2) = \begin{cases} \mathrm{Arg}\left(\frac{\omega_1}{\omega_2}\right) & \text{if } \frac{\omega_1}{\omega_2} \neq -1 \\ \mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2) & \text{if } \frac{\omega_1}{\omega_2} = -1 \end{cases}, \ \forall \omega_1, \omega_2 \in S^1$$

where $\mathrm{Arg}\,(\omega) \in (-\pi, \pi]$ stands for the argument of the unit complex number $\omega \in S^1$. $T(u)$ represents the distributional determinant of the absolutely continuous part and the Cantor part of Du which is adjusted on $S(u)$ by the tangential derivative of $\rho(u^+, u^-)$. The second term in the RHS of (1) is motivated by the study of $BV(S^1, S^1)$ functions (see [9]): we defined there a similar quantity that represents a pseudo-degree for $BV(S^1, S^1)$ functions.

Remark 1 i) The integrand in (1) is computed pointwise in any orthonormal frame (x, y) and the corresponding quantity is frame-invariant.

ii) The 2-vector measure

$$\mu = (\mu_1, \mu_2) = u \wedge (D^a u + D^c u) = (u \wedge (D^a u_x + D^c u_x), u \wedge (D^a u_y + D^c u_y))$$

is well-defined since $D^a u + D^c u$ vanishes on sets which are σ-finite with respect to $\mathcal{H}^1$.

iii) Notice that the function ρ is antisymmetric, i.e.

$$\rho(\omega_1, \omega_2) = -\rho(\omega_2, \omega_1), \ \forall \omega_1, \omega_2 \in S^1$$

and therefore, $T(u)$ does not depend of the choice of the orientation ν_u on the jump set $S(u)$. By Lemma 5 (see below), we obtain

$$|\langle T(u), \zeta \rangle| \leq |u|_{BV\,S^1}, \ \forall \zeta \in C^1(S^2, \mathbb{R}) \text{ with } |\nabla \zeta| \leq 1$$

where $|u|_{BV\,S^1} = \int_{S^2} \Big(|D^a u| + |D^c u| \Big) + \int_{S(u)} d_{S^1}(u^+, u^-) \, d\mathcal{H}^1$ and d_{S^1} stands for the geodesic distance on S^1. Therefore, $T(u)$ is indeed a distribution (of order 1) on S^2.

For a compact Riemannian manifold X with the induced distance d, define

$$\mathcal{Z}(X) = \left\{ \Lambda \in \left[C^1(X)\right]^* \,:\, \exists (p_k), (n_k) \subset X, \sum_k d(p_k, n_k) < \infty \text{ and } \Lambda = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}) \right\}.$$

$\mathcal{Z}(X)$ is the set of distributions that can be written as a countable sum of dipoles.

Remark 2 i) In general, $\Lambda \in \mathcal{Z}(X)$ is not a measure. In fact, it can be shown that Λ is a measure if and only if Λ is a finite sum of dipoles (see Smets [11] and also Ponce [10]).

ii) $\Lambda \in \mathcal{Z}(X)$ has always infinitely many representations as a sum of dipoles and these representations need not be equivalent modulo a permutation of points. For example, a dipole $\delta_p - \delta_n$ may be represented as $\delta_p - \delta_{n_1} + \sum_{k \geq 1} (\delta_{n_k} - \delta_{n_{k+1}})$ for any sequence $(n_k)_k$ rapidly converging to n.

For each $\Lambda \in \mathcal{Z}(X)$, the length of a minimal connection between the singularities is defined as

$$\|\Lambda\| = \sup_{\substack{\zeta \in C^1(X) \\ |\nabla \zeta| \leq 1}} \langle \Lambda, \zeta \rangle.$$

For example, when $\Lambda = 2\pi \sum_{k=1}^{m} (\delta_{p_k} - \delta_{n_k})$ is a finite sum of dipoles, Brezis, Coron and Lieb [3] showed that

$$\|\Lambda\| = 2\pi \min_{\sigma \in S_m} \sum_{k=1}^{m} d(p_k, n_{\sigma(k)})$$

where S_m denotes the group of permutation of $\{1, 2, \ldots, m\}$. In general, for an arbitrary $\Lambda \in \mathcal{Z}(X)$, Bourgain, Brezis and Mironescu [2] proved the following characterization of the length of a minimal connection:

$$\|\Lambda\| = \inf_{(p_k),(n_k)} \left\{ 2\pi \sum_k d(p_k, n_k) \,:\, \Lambda = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}) \text{ and } \sum_k d(p_k, n_k) < \infty \right\}. \quad (2)$$

From (2), one can deduce that $\mathcal{Z}(X)$ is a complete metric space with respect to the distance induced by $\|\cdot\|$ (see e.g. [10]).

Our first theorem states that $T(u)$ is a countable sum of dipoles. It is the extension to the BV case of the result in [4] mentioned in the beginning.

Theorem 1 *For every $u \in BV(S^2; S^1)$, we have $T(u) \in \mathcal{Z}(S^2)$, i.e. there exist $(p_k), (n_k)$ in S^2 such that*

$$\sum_k |p_k - n_k| < \infty \quad \textit{and} \quad T(u) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}).$$

The proof relies on the fact that the derivative (in the sense of distributions) of the characteristic function of a bounded measurable set in $\mathbb{R}$ can be written as a sum of differences between Dirac masses:

Lemma 1 *Let $I \subset \mathbb{R}$ be a compact interval and $f : I \to 2\pi\mathbb{Z}$ be an integrable function. Define*

$$\langle \frac{\mathrm{d}f}{\mathrm{d}t}, \zeta \rangle := -\int_I f(t)\, \zeta'(t)\, \mathrm{d}t\,,\ \forall \zeta \in C^1(I).$$

Then

$$\frac{\mathrm{d}f}{\mathrm{d}t} \in \mathcal{Z}(I) \quad \textit{and} \quad \|\frac{\mathrm{d}f}{\mathrm{d}t}\| = \int_I |f|\, \mathrm{d}t.$$

The same property is valid for the distributional tangential derivative of an integrable function taking values in $2\pi\mathbb{Z}$ and defined on a C^1 1-graph (see Remark 3). Since every countably $\mathcal{H}^1$-rectifiable set $S \subset S^2$ can be covered $\mathcal{H}^1$-a.e. by a sequence of C^1 1-graphs, it makes sense to define for every $\Lambda \in \mathcal{Z}(S^2)$ the set

$$\mathcal{J}(\Lambda) = \left\{ (f, S, \nu) \,:\, \begin{array}{c} S \text{ is a countably } \mathcal{H}^1\text{- rectifiable set in } S^2,\ \nu \text{ is an orientation on } S, \\ f \in L^1(S, 2\pi\mathbb{Z}) \text{ is such that } \displaystyle\int_S f\,\nu \cdot \nabla^\perp \zeta \,\mathrm{d}\mathcal{H}^1 = \langle \Lambda, \zeta \rangle,\ \forall \zeta \in C^1(S^2) \end{array} \right\}.$$

We have the following reformulation of (2):

Lemma 2 *For every $\Lambda \in \mathcal{Z}(S^2)$, we have*

$$\|\Lambda\| = \min_{(f,S,\nu)\in\mathcal{J}(\Lambda)} \int_S |f| \,\mathrm{d}\mathcal{H}^1.$$

It is known that the infimum in (2) is not achieved in general (see [10]); the advantage of the above formula is that the minimum is always attained. It means that the length of Λ represents the minimal mass that an $\mathcal{H}^1$-integrable function with values into $2\pi\mathbb{Z}$ could carry between the dipoles of Λ.

In the sequel we are concerned with the lifting of $u \in BV(S^2, S^1)$. We call BV lifting of u every function $\varphi \in BV(S^2, \mathbb{R})$ such that

$$u = e^{i\varphi} \quad \text{a.e. on } S^2.$$

The existence of a BV lifting for functions $u \in BV(S^2, S^1)$ was initially shown by Giaquinta, Modica and Souček [8]. Later, Dávila and Ignat [5] proved the existence of a lifting $\varphi \in BV \cap L^\infty(S^2, \mathbb{R})$ such that

$$\int_{S^2} |D\varphi| \le 2 \int_{S^2} |Du|; \tag{3}$$

moreover, the constant 2 in (3) is the best constant (see Example 1 and Proposition 3 below).

We give the following characterization for a lifting of u:

Lemma 3 *Let $u \in BV(S^2, S^1)$. For every lifting $\varphi \in BV(S^2, \mathbb{R})$ of u, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that*

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u\, \mathcal{H}^1 \llcorner S(u) - f\nu\, \mathcal{H}^1 \llcorner S. \tag{4}$$

Conversely, for every triple $(f, S, \nu) \in \mathcal{J}(T(u))$ there exists a lifting $\varphi \in BV(S^2, \mathbb{R})$ of u such that (4) *holds.*

In this framework, it is natural to investigate the quantity

$$E(u) = \inf \left\{ \int_{S^2} |D\varphi| \,:\, \varphi \in BV(S^2, \mathbb{R}),\ e^{i\varphi} = u \text{ a.e. on } S^2 \right\}. \tag{5}$$

The infimum from above is achieved and it is equal to the relaxed energy

$$E_{\mathrm{rel}}(u) = \inf \left\{ \liminf_{k\to\infty} \int_{S^2} |\nabla u_k| \,\mathrm{d}\mathcal{H}^2 \,:\, u_k \in C^\infty(S^2, S^1),\ u_k \to u \text{ a.e. on } S^2 \right\} \tag{6}$$

(see Remark 4). A lifting $\varphi \in BV(S^2, \mathbb{R})$ of u is called optimal if

$$E(u) = \int_{S^2} |D\varphi|.$$

An optimal lifting need not be unique (see Proposition 3). Remark also that for $u \in BV(S^2, S^1)$, there could be no optimal BV lifting of u that belongs to L^∞ (see Example 3).

Our aim is to compute the total variation $E(u)$ of an optimal lifting and to construct an optimal lifting. Theorem 2 establishes the formula for $E(u)$ using the distribution $T(u)$.

Theorem 2 *For every $u \in BV(S^2, S^1)$, we have*

$$E(u) = \int_{S^2} (|D^a u| + |D^c u|) + \min_{(f,S,\nu)\in\mathcal{J}(T(u))} \int_{S\cup S(u)} \Big| f\nu\, \chi_S - \rho(u^+, u^-)\nu_u\, \chi_{S(u)} \Big| \, \mathrm{d}\mathcal{H}^1. \tag{7}$$

We refer the reader to [8] for related results in terms of cartesian currents.

As a consequence of Theorem 2, we recover the result of Brezis, Mironescu and Ponce [4] about the total variation of an optimal BV lifting for functions $g \in W^{1,1}(S^2, S^1)$: the gap

$$E(g) - \int_{S^2} |\nabla g| \, \mathrm{d}\mathcal{H}^2$$

is equal to the length of a minimal connection connecting the topological singularities of g.

Corollary 1 *For every $g \in W^{1,1}(S^2, S^1)$, we have*

$$E(g) = \int_{S^2} |\nabla g| \, \mathrm{d}\mathcal{H}^2 + \|T(g)\|.$$

From (7), we deduce an estimate for $E(u)$ (which is a weaker form of inequality (3)):

Corollary 2 *For every $u \in BV(S^2, S^1)$, we have*

$$E(u) \leq 2|u|_{BV\, S^1}.$$

In the spirit of [4], we have the following interpretation of $\|T(u)\|$ as a distance:

Theorem 3 *For every $u \in BV(S^2, S^1)$, we have*

$$\|T(u)\| = \min_{\psi\in BV(S^2,\mathbb{R})} \int_{S^2} \Big| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u\, \mathcal{H}^1 \llcorner S(u) - D\psi \Big|. \tag{8}$$

Moreover, there is at least one minimizer $\psi \in BV(S^2, \mathbb{R})$ of (8) that is a lifting of u.

Remark that in general, $\|T(u)\|$ is not the distance of the measure

$$u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u\, \mathcal{H}^1 \llcorner S(u)$$

to the class of gradient maps. In Example 4, we construct a function $u \in BV(S^2, S^1)$ such that

$$\|T(u)\| < \inf_{\psi\in C^\infty(S^2,\mathbb{R})} \int_{S^2} \Big| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u\, \mathcal{H}^1 \llcorner S(u) - D\psi \Big|.$$

In Section 2, we present the proofs of Lemmas 1, 2 and 3, Theorems 1, 2 and 3 and Corollaries 1 and 2. Some examples and interesting properties of $T(u)$ are given in Section 3. Among other things, we show that $T : BV(S^2, S^1) \to \mathcal{Z}(S^2)$ is discontinuous and we analyze some algebraic properties of $T(u)$. We also discuss the meaning of the point singularities of $T(u)$ and about their location on S^2.

All the results included here can be easily adapted for functions in $BV(\Omega, S^1)$ where Ω is a more general simply connected Riemannian manifold of dimension 2.

2 Remarks and proofs of the main results

We start by proving Lemma 1:

Proof of Lemma 1. Firstly, let us suppose that $f = 2\pi\chi_A$ where $A \subset I$ is an open set. Write $A = \bigcup_{j\in\mathbb{N}} (a_j, b_j)$ as a countable reunion of disjoint intervals. It is clear that

$$\langle \frac{\mathrm{d}\chi_A}{\mathrm{d}t}, \zeta\rangle = \sum_{j\in\mathbb{N}} (\zeta(a_j) - \zeta(b_j)),\ \forall \zeta \in C^1(I)$$

and $\sum_{j\in\mathbb{N}} (b_j - a_j) = \mathcal{H}^1(A)$. Thus $2\pi \frac{\mathrm{d}\chi_A}{\mathrm{d}t} \in \mathcal{Z}(I)$ and

$$\|\frac{\mathrm{d}f}{\mathrm{d}t}\| = 2\pi \sup_{\substack{\zeta\in C^1(I)\\ |\zeta'|\le 1}} \int_I \chi_A\, \zeta'\, \mathrm{d}t = 2\pi \sup_{\substack{\psi\in C(I)\\ |\psi|\le 1}} \int_I \chi_A\, \psi\, \mathrm{d}t = 2\pi\mathcal{H}^1(A).$$

Moreover, let $A \subset I$ be a Lebesgue measurable set and $f = 2\pi\chi_A$. Using the regularity of the Lebesgue measure, there exists a decreasing sequence of open sets $A \subset A_{k+1} \subset A_k \subset I$, $k \in \mathbb{N}$ such that $\lim_{k\to\infty} \mathcal{H}^1(A_k) = \mathcal{H}^1(A)$. Observe that $\frac{\mathrm{d}\chi_{A_k}}{\mathrm{d}t} \to \frac{\mathrm{d}\chi_A}{\mathrm{d}t}$ in $\left[C^1(I)\right]^*$. Since $\mathcal{Z}(I)$ is a complete metric space, we conclude that $2\pi\frac{\mathrm{d}\chi_A}{\mathrm{d}t} \in \mathcal{Z}(I)$ and $\|2\pi\frac{\mathrm{d}\chi_A}{\mathrm{d}t}\| = 2\pi\mathcal{H}^1(A)$. In the general case of an integrable function $f : I \to 2\pi\mathbb{Z}$, write

$$f = 2\pi \sum_{k\in\mathbb{Z}} k\, \chi_{E_k} \ \text{ in } \ L^1, \tag{9}$$

where $E_k = \{x \in I \,:\, f(x) = 2\pi k\}$. Notice that $2\pi\frac{\mathrm{d}\,(k\,\chi_{E_k})}{\mathrm{d}t} \in \mathcal{Z}(I)$ and the series $\sum_{k\in\mathbb{Z}} 2\pi\frac{\mathrm{d}\,(k\,\chi_{E_k})}{\mathrm{d}t}$ converges absolutely; indeed, we have

$$\sum_{k\in\mathbb{Z}} \|2\pi\frac{\mathrm{d}\,(k\,\chi_{E_k})}{\mathrm{d}t}\| = 2\pi\sum_{k\in\mathbb{Z}} |k|\mathcal{H}^1(E_k) = \int_I |f|\,\mathrm{d}t < \infty.$$

By (9), we conclude that $\frac{\mathrm{d}f}{\mathrm{d}t} \in \mathcal{Z}(I)$ and

$$\|\frac{\mathrm{d}f}{\mathrm{d}t}\| = \sup_{\substack{\zeta\in C^1(I)\\ |\zeta'|\le 1}} \int_I f\, \zeta'\, \mathrm{d}t = \sup_{\substack{\psi\in C(I)\\ |\psi|\le 1}} \int_I f\, \psi\, \mathrm{d}t = \int_I |f|\,\mathrm{d}t.$$

□

Remark 3 The conclusion of Lemma 1 is also true for $\mathcal{H}^1$-integrable functions with values in $2\pi\mathbb{Z}$ that are defined on C^1 1-graphs. For simplicity, we restrict to C^1 1-graphs in S^2, i.e. for an orthonormal frame (x, y) on S^2, we consider the set

$$\Gamma = \{(x, y) \,:\, \phi(x) = y\}$$

where ϕ is a C^1 function. Suppose $c : [0, 1] \to \Gamma$ is a parameterization of Γ and set $\tau(c(t)) = \frac{c'(t)}{|c'(t)|}$ the tangent unit vector to the curve Γ at $c(t)$, $\forall t \in (0, 1)$. Let $f : \Gamma \to 2\pi\mathbb{Z}$ be an $\mathcal{H}^1$-integrable function on Γ. Define

$$\langle\frac{\partial f}{\partial\tau},\zeta\rangle := -\int_0^1 f\circ c(t)\,(\zeta\circ c)'(t)\,\mathrm{d}t\,,\ \forall\zeta\in C^1(\Gamma).$$

By Lemma 1, we have

$$\frac{\partial f}{\partial\tau}\in\mathcal{Z}(\Gamma)\quad\text{and}\quad\|\frac{\partial f}{\partial\tau}\|=\int_0^1|f|(c(t))\,|c'(t)|\,\mathrm{d}t.$$

Before proving Lemma 3, we give the following result:

Lemma 4 *For every $u\in BV(S^2,S^1)$, we have*

$$u\wedge(D^a u+D^c u)=\frac{1}{i}\bar{u}(D^a u+D^c u)$$
$$\textit{and}\quad|u\wedge(D^a u+D^c u)|=|D^a u|+|D^c u|.$$

Proof. Write $u=(u_1,u_2)=u_1+i\,u_2$. We can consider the 2×2 matrix of real measures Du as a 2-vector of complex measures, i.e. $Du=Du_1+i\,Du_2$. Since $u_1^2+u_2^2=1$, it results $D(u_1^2+u_2^2)=0$. By the chain rule (see e.g. [1]), we obtain

$$u_1(D^a u_1+D^c u_1)+u_2(D^a u_2+D^c u_2)=0,$$

i.e. the real part of the $\mathbb{C}^2$-measure $\bar{u}(D^a u+D^c u)$ vanishes. Therefore,

$$u\wedge(D^a u+D^c u)=\frac{1}{i}\bar{u}(D^a u+D^c u).$$

Hence, using the fact that the absolutely continuous part and the Cantor part of Du are mutually singular, we conclude that

$$|u\wedge(D^a u+D^c u)|=|u|(|D^a u|+|D^c u|)=|D^a u|+|D^c u|.$$

$\square$

Proof of Lemma 3. Let $\varphi\in BV(S^2,\mathbb{R})$ be a lifting of u. Write

$$D\varphi=D^a\varphi+D^c\varphi+(\varphi^+-\varphi^-)\nu_\varphi\mathcal{H}^1\llcorner S(\varphi).$$

By the chain rule and Lemma 4, we obtain

$$D^a\varphi+D^c\varphi=\frac{1}{i}\bar{u}(D^a u+D^c u)=u\wedge(D^a u+D^c u).$$

Since $u=e^{i\varphi}$ a.e. on S^2, we have that $S(u)\subset S(\varphi)$ and by changing the orientation ν_φ, we may assume

$$\begin{cases}\nu_\varphi=\nu_u\\ e^{i\varphi+}=u^+\quad\mathcal{H}^1\text{-a.e. on }S(u).\\ e^{i\varphi-}=u^-\end{cases}$$

Therefore,

$$\varphi^+-\varphi^-\equiv\rho(u^+,u^-)\pmod{2\pi}\ \ \mathcal{H}^1\text{-a.e. on }S(u)$$
$$\text{and}\quad\varphi^+-\varphi^-\equiv 0\pmod{2\pi}\ \ \mathcal{H}^1\text{-a.e. on }S(\varphi)\setminus S(u).$$

Hence, there exists $f_\varphi : S(\varphi) \to 2\pi\mathbb{Z}$ a measurable function such that

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \, \mathcal{H}^1 \llcorner S(u) - f_\varphi \nu_\varphi \, \mathcal{H}^1 \llcorner S(\varphi). \tag{10}$$

Observe that f_φ is an $\mathcal{H}^1$-integrable function since

$$|\rho(u^+, u^-)| = d_{S^1}(u^+, u^-) \leq \frac{\pi}{2}|u^+ - u^-|.$$

Since $D\varphi$ is a measure, we have

$$\operatorname{curl} D\varphi = 0 \text{ in } \mathcal{D}',$$

i.e. for every $\zeta \in C^1(S^2, \mathbb{R})$,

$$\int_{S^2} \nabla^\perp \zeta \, D\varphi = 0.$$

By (10), it yields

$$\langle T(u), \zeta \rangle = \int_{S(\varphi)} f_\varphi \, \nabla^\perp \zeta \cdot \nu_\varphi \, \mathrm{d}\mathcal{H}^1, \ \forall \zeta \in C^1(S^2)$$

and therefore, $(f_\varphi, S(\varphi), \nu_\varphi) \in \mathcal{J}(T(u))$.

Conversely, take $(f, S, \nu) \in \mathcal{J}(T(u))$. Without loss of generality, we may consider $S = \{f \neq 0\}$. Consider the finite Radon $\mathbb{R}^2$-valued measure

$$\mu = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \, \mathcal{H}^1 \llcorner S(u) - f \, \nu \, \mathcal{H}^1 \llcorner S.$$

We check that $\operatorname{curl} \mu = 0$ in $\mathcal{D}'(S^2)$. Indeed, for every $\zeta \in C^1(S^2, \mathbb{R})$,

$$-\langle \operatorname{curl} \mu, \zeta \rangle = \int_{S^2} \nabla^\perp \zeta \, \mathrm{d}\mu = \langle T(u), \zeta \rangle - \int_S f \, \nabla^\perp \zeta \cdot \nu \, \mathrm{d}\mathcal{H}^1 = 0.$$

By the BV version of Poincare's lemma, there exists $\varphi \in BV(S^2, \mathbb{R})$ such that $D\varphi = \mu$ in $\mathcal{D}'(S^2, \mathbb{R}^2)$. Here, $S \cup S(u)$ is the jump set of φ. On the set $S \cup S(u)$, we choose an orientation ν_φ such that $\nu_\varphi = \nu_u$ on $S(u)$. We have

$$\begin{cases} D^a \varphi + D^c \varphi = u \wedge (D^a u + D^c u) = \frac{1}{i}\bar{u}(D^a u + D^c u) \\ \varphi^+ - \varphi^- \equiv \rho(u^+, u^-) \pmod{2\pi} \ \ \mathcal{H}^1\text{- a.e. on } S(u) \, . \\ \varphi^+ - \varphi^- \equiv 0 \pmod{2\pi} \ \ \mathcal{H}^1\text{- a.e. on } S \setminus S(u) \end{cases}$$

We now show that

$$D(u \, e^{-i\varphi}) = 0.$$

By the chain rule, we get

$$\begin{aligned} D(e^{-i\varphi}) &= -ie^{-i\varphi}(D^a \varphi + D^c \varphi) + (e^{-i\varphi^+} - e^{-i\varphi^-}) \otimes \nu_u \, \mathcal{H}^1 \llcorner S(u) \\ &= -e^{-i\varphi}\bar{u}(D^a u + D^c u) + (e^{-i\varphi^+} - e^{-i\varphi^-}) \otimes \nu_u \, \mathcal{H}^1 \llcorner S(u). \end{aligned}$$

Remark that the space $BV(S^2, \mathbb{C}) \cap L^\infty$ is an algebra. Differentiating the product $u \, e^{-i\varphi}$, we obtain

$$D(u \, e^{-i\varphi}) = e^{-i\varphi}(D^a u + D^c u) - u \, e^{-i\varphi}\bar{u}(D^a u + D^c u) + (u^+ \, e^{-i\varphi^+} - u^- \, e^{-i\varphi^-}) \otimes \nu_u \, \mathcal{H}^1 \llcorner S(u) = 0.$$

Thus, up to an additive constant, φ is a BV lifting of u and (4) is fulfilled. □

Proof of Theorem 1. Let $\varphi \in BV(S^2, \mathbb{R})$ be a lifting of u. By Lemma 3, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that (4) holds. Denote by $\tau : S \to S^1$ the tangent vector in $\mathcal{H}^1$-a.e. point of S such that (ν, τ, e) is direct. By (4),

$$\begin{aligned}\langle T(u), \zeta\rangle &= \int_S f\, \nabla^{\perp}\zeta \cdot \nu \, \mathrm{d}\mathcal{H}^1 \\ &= \int_S f\, \frac{\partial \zeta}{\partial \tau}\, \mathrm{d}\mathcal{H}^1 \\ &= \sum_{k\in\mathbb{N}} \int_{I_k} \chi_S f\, \frac{\partial \zeta}{\partial \tau}\, \mathrm{d}\mathcal{H}^1, \ \forall \zeta \in C^1(S^2)\end{aligned}$$

where $\{I_k\}_{k\in\mathbb{N}}$ is a family of disjoint compact C^1 1-graphs that covers $\mathcal{H}^1$-almost all of the countably rectifiable set S, i.e.

$$\mathcal{H}^1\left(S \setminus \bigcup_{k\in\mathbb{N}} I_k\right) = 0.$$

According to Lemma 1 and Remark 3, we conclude $T(u) \in \mathcal{Z}(S^2)$ and $\|T(u)\| \le \int_S |f|\, d\mathcal{H}^1$. □

Before proving Theorem 2, let us make some remarks about $E(u)$ and $E_{\mathrm{rel}}(u)$ for $u \in BV(S^2, S^1)$ (see also [4]):

Remark 4 i) $E(u) < \infty$ and $E_{\mathrm{rel}}(u) < \infty$ (the existence of a BV lifting of u was shown in [5] and [8]);

ii) The infimum in (5) is achieved; indeed, let $\varphi_k \in BV(S^2, \mathbb{R})$, $e^{i\varphi_k} = u$ a.e. on S^2, be such that

$$\lim_{k\to\infty} \int_{S^2} |D\varphi_k| = E(u) < \infty.$$

By Poincaré's inequality, there exists a universal constant $C > 0$ such that

$$\int_{S^2} \left|\varphi_k - \fint_{S^2} \varphi_k\right| \mathrm{d}\mathcal{H}^2 \le C \int_{S^2} |D\varphi_k|, \, \forall k \in \mathbb{N}$$

(where $\fint_{S^2}$ stands for the average). Therefore, by subtracting a suitable integer multiple of 2π, we may assume that $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $BV(S^2, \mathbb{R})$. After passing to a subsequence if necessary, we may assume that $\varphi_k \to \varphi$ a.e. and L^1 for some $\varphi \in BV(S^2, \mathbb{R})$. It follows that φ is a lifting of u on S^2 and

$$E(u) = \lim_{k\to\infty} \int_{S^2} |D\varphi_k| \ge \int_{S^2} |D\varphi| \ge E(u);$$

iii) The infimum in (6) is also achieved; take $u_k^m \in C^\infty(S^2, S^1)$ such that for each $k \in \mathbb{N}$,

$$u_k^m \to u \text{ a.e. on } S^2 \text{ and } \int_{S^2} |\nabla u_k^m|\, \mathrm{d}\mathcal{H}^2 \searrow a_k \in \mathbb{R} \text{ as } m \to \infty$$

and $\lim_{k\to\infty} a_k = E_{\mathrm{rel}}(u)$. Subtracting a subsequence, we may assume that for each $k \in \mathbb{N}$,

$$\int_{S^2} |u_k^m - u|\, \mathrm{d}\mathcal{H}^2 < \frac{1}{k} \text{ and } \int_{S^2} |\nabla u_k^m|\, \mathrm{d}\mathcal{H}^2 - a_k < \frac{1}{k}, \, \forall m \ge 1.$$

Therefore, $u_k^k \to u$ in L^1 and

$$\lim_{k\to\infty} \int_{S^2} |\nabla u_k^k|\, \mathrm{d}\mathcal{H}^2 = E_{\mathrm{rel}}(u).$$

iv) $E(u) = E_{\rm rel}(u)$. For "$\le$", take $u_k \in C^\infty(S^2, S^1), \forall k \in \mathbb{N}$ such that $u_k \to u$ a.e. on S^2 and $\sup_{k\in\mathbb{N}} \int_{S^2} |\nabla u_k| \, d\mathcal{H}^2 < \infty$. Since S^2 is simply connected, there exists $\varphi_k \in C^\infty(S^2, \mathbb{R})$ such that $e^{i\varphi_k} = u_k$. Moreover, $\int_{S^2} |\nabla \varphi_k| \, d\mathcal{H}^2 = \int_{S^2} |\nabla u_k| \, d\mathcal{H}^2$. Using the same argument as in ii), we may assume that $\varphi_k \to \varphi$ a.e. and L^1 for some $\varphi \in BV(S^2, \mathbb{R})$. Therefore, $e^{i\varphi} = u$ a.e. on S^2 and

$$E(u) \le \int_{S^2} |D\varphi| \le \liminf_{k\to\infty} \int_{S^2} |\nabla \varphi_k| \, d\mathcal{H}^2 = \liminf_{k\to\infty} \int_{S^2} |\nabla u_k| \, d\mathcal{H}^2.$$

For "$\ge$", consider a BV lifting φ of u and take an approximating sequence $\varphi_k \in C^\infty(S^2, \mathbb{R})$ such that $\varphi_k \to \varphi$ a.e. and $|D\varphi|(S^2) = \lim_{k\to\infty} \int_{S^2} |\nabla \varphi_k| \, d\mathcal{H}^2$. With $u_k = e^{i\varphi_k} \in C^\infty(S^2, S^1)$, we have $u_k \to u$ a.e. on S^2 and

$$E_{\rm rel}(u) \le \lim_{k\to\infty} \int_{S^2} |\nabla u_k| \, d\mathcal{H}^2 = \lim_{k\to\infty} \int_{S^2} |\nabla \varphi_k| \, d\mathcal{H}^2 = \int_{S^2} |D\varphi|.$$

□

Proof of Theorem 2. For "$\le$", take $(f, S, \nu) \in \mathcal{J}(T(u))$. By Lemma 3, there exists a lifting $\varphi \in BV(S^2, \mathbb{R})$ of u such that (4) holds. It follows that

$$E(u) \le \int_{S^2} |D\varphi| = \int_{S^2} (|D^a u| + |D^c u|) + \int_{S \cup S(u)} \Big| f\nu\chi_S - \rho(u^+, u^-)\nu_u \chi_{S(u)} \Big| \, d\mathcal{H}^1.$$

Let us prove now "$\ge$". By Remark 4, there is an optimal BV lifting φ of u, i.e. $E(u) = \int_{S^2} |D\varphi|$. By Lemma 3, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that (4) holds. It results that

$$E(u) = \int_{S^2} |D\varphi| = \int_{S^2} (|D^a u| + |D^c u|) + \int_{S \cup S(u)} \Big| f\nu\chi_S - \rho(u^+, u^-)\nu_u \chi_{S(u)} \Big| \, d\mathcal{H}^1.$$

From here, we also deduce that the minimum inside the RHS of (7) is achieved. □

Remark 5 (Construction of an optimal lifting) Take $(f, S, \nu) \in \mathcal{J}(T(u))$ that achieves the minimum

$$\min_{(f,S,\nu)\in\mathcal{J}(T(u))} \int_{S\cup S(u)} \Big| f\nu\,\chi_S - \rho(u^+, u^-)\nu_u\,\chi_{S(u)} \Big| \, d\mathcal{H}^1. \tag{11}$$

By Lemma 3, there exists a lifting $\varphi \in BV(S^2, \mathbb{R})$ of u such that (4) holds. Then

$$\int_{S^2} |D\varphi| = \int_{S^2} (|D^a u| + |D^c u|) + \int_{S\cup S(u)} \Big| f\nu\chi_S - \rho(u^+, u^-)\nu_u\chi_{S(u)} \Big| \, d\mathcal{H}^1 = E(u)$$

and therefore, φ is an optimal lifting of u. □

Proof of Lemma 2. For "$\le$", it is easy to see that if $(f, S, \nu) \in \mathcal{J}(\Lambda)$ then for every $\zeta \in C^1(S^2)$ with $|\nabla\zeta| \le 1$,

$$\langle \Lambda, \zeta \rangle = \int_S f\,\nu \cdot \nabla^\perp \zeta \, d\mathcal{H}^1 \le \int_S |f| \, d\mathcal{H}^1.$$

For "$\ge$", we use characterization (2) of the distribution $\Lambda \in \mathcal{Z}(S^2)$. We denote by d_{S^2} the geodesic distance on S^2. Let $\Lambda = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k})$ where $(p_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ belong to S^2 such that

$\sum_k d_{S^2}(p_k, n_k) < \infty$. For every $k \in \mathbb{N}$, consider $\widehat{n_k p_k}$ a geodesic arc on S^2 oriented from n_k to p_k. Take ν_k the normal vector to $\widehat{n_k p_k}$ in the frame (x, y). Set $S = \bigcup_k \widehat{n_k p_k}$. Since $\sum_k d_{S^2}(p_k, n_k) < \infty$, there exist an orientation $\nu : S \to S^1$ on S and an $\mathcal{H}^1$-integrable function $f : S \to 2\pi\mathbb{Z}$ such that

$$f\nu\chi_S = \sum_k 2\pi\nu_k\chi_{\widehat{n_k p_k}} \quad \text{in } L^1(S, \mathbb{R}^2). \tag{12}$$

Then

$$\int_S f\nu \cdot \nabla^\perp\zeta \, \mathrm{d}\mathcal{H}^1 = 2\pi \sum_k \int_{\widehat{n_k p_k}} \nu_k \cdot \nabla^\perp\zeta \, \mathrm{d}\mathcal{H}^1 = 2\pi \sum_k (\zeta(p_k) - \zeta(n_k)) = \langle \Lambda, \zeta \rangle, \, \forall \zeta \in C^1(S^2).$$

It follows that $(f, S, \nu) \in \mathcal{J}(\Lambda)$ and by (12),

$$\int_S |f| \, \mathrm{d}\mathcal{H}^1 \leq \sum_k 2\pi d_{S^2}(n_k, p_k).$$

Minimizing after all suitable pairs $(p_k, n_k)_{k\in\mathbb{N}}$, it follows

$$\|\Lambda\| = \inf_{(f,S,\nu)\in\mathcal{J}(\Lambda)} \int_S |f| \, \mathrm{d}\mathcal{H}^1. \tag{13}$$

We now show that the infimum in (13) is indeed achieved. By a dipole construction (see [2], Lemma 16), there exists $u \in W^{1,1}(S^2, S^1)$ such that $\Lambda = T(u)$. We choose $(f_k, S_k, \nu_k) \in \mathcal{J}(T(u))$ such that

$$\|T(u)\| = \lim_k \int_{S_k} |f_k| \, \mathrm{d}\mathcal{H}^1.$$

By Lemma 3, we construct a lifting $\varphi_k \in BV(S^2, \mathbb{R})$ of u such that

$$D\varphi_k = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \, \mathcal{H}^1 \llcorner S(u) - f_k \, \nu_k \, \mathcal{H}^1 \llcorner S_k.$$

Remark that

$$\int_{S^2} |D\varphi_k| \leq \int_{S^2} (|D^a u| + |D^c u|) + \int_{S(u)} |\rho(u^+, u^-)| \, \mathrm{d}\mathcal{H}^1 + \int_{S_k} |f_k| \, \mathrm{d}\mathcal{H}^1.$$

Subtracting a suitable number in $2\pi\mathbb{Z}$, we may assume that $(\varphi_k)_k$ is a bounded sequence in $BV(S^2, \mathbb{R})$. Up to a subsequence, we find $\varphi \in BV(S^2, \mathbb{R})$ such that

$$\varphi_k \to \varphi \text{ a.e. in } S^2 \text{ and } D\varphi_k \overset{*}{\rightharpoonup} D\varphi \text{ in the measure sense.}$$

Therefore, φ is a BV lifting of u and by Lemma 3, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \, \mathcal{H}^1 \llcorner S(u) - f \, \nu \, \mathcal{H}^1 \llcorner S.$$

We conclude

$$\begin{aligned}
\int_S |f| \, \mathrm{d}\mathcal{H}^1 &= \int_{S^2} \left| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\varphi \right| \\
&\leq \liminf_k \int_{S^2} \left| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\varphi_k \right| \\
&= \lim_k \int_{S_k} |f_k| \, \mathrm{d}\mathcal{H}^1 \\
&= \|T(u)\|.
\end{aligned}$$

□

Proof of Theorem 3. Let $\psi \in BV(S^2, \mathbb{R})$ and $\zeta \in C^1(S^2)$ be such that $|\nabla\zeta| \leq 1$. Then

$$\int_{S^2} \Big| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\psi \Big| \geq \langle T(u), \zeta \rangle - \int_{S^2} D\psi \cdot \nabla^\perp \zeta = \langle T(u), \zeta \rangle.$$

By taking the supremum over ζ, we obtain

$$\int_{S^2} \Big| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\psi \Big| \geq \|T(u)\|.$$

We now show that there is a lifting $\varphi \in BV(S^2, \mathbb{R})$ of u such that the minimum in (8) is achieved. By Lemma 2, choose $(f, S, \nu) \in \mathcal{J}(T(u))$ such that

$$\|T(u)\| = \int_S |f| \, \mathrm{d}\mathcal{H}^1.$$

Using Lemma 3, we construct a lifting $\varphi \in BV(S^2, \mathbb{R})$ such that (4) holds. Thus,

$$\|T(u)\| = \int_S |f| \, \mathrm{d}\mathcal{H}^1 = \int_{S^2} \Big| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\varphi \Big|.$$

□

Proof of Corollary 1. The result is a straightforward consequence of Theorem 2 and Lemma 2. □

In order to prove Corollary 2, we need the following estimation of $\|T(u)\|$ in terms of the seminorm $|u|_{BVS^1}$:

Lemma 5 *We have* $\|T(u)\| \leq |u|_{BVS^1}$, $\forall u \in BV(S^2, S^1)$.

Proof. By Lemma 4, it results that for every $\zeta \in C^1(S^2)$ with $|\nabla\zeta| \leq 1$,

$$\begin{aligned} |\langle T(u), \zeta \rangle| &\leq \int_{S^2} |u \wedge (D^a u + D^c u)| + \int_{S(u)} |\rho(u^+, u^-)| \, \mathrm{d}\mathcal{H}^1 \\ &= \int_{S^2} (|D^a u| + |D^c u|) + \int_{S(u)} d_{S^1}(u^+, u^-) \, \mathrm{d}\mathcal{H}^1; \end{aligned}$$

therefore

$$\|T(u)\| \leq |u|_{BVS^1}.$$

□

Proof of Corollary 2. By Theorem 2, Lemmas 2 and 5, we conclude that

$$\begin{aligned} E(u) &\leq \int_{S^2} (|D^a u| + |D^c u|) + \int_{S(u)} |\rho(u^+, u^-)| \, \mathrm{d}\mathcal{H}^1 + \min_{(f,S,\nu) \in \mathcal{J}(T(u))} \int_S |f| \, \mathrm{d}\mathcal{H}^1 \\ &= |u|_{BVS^1} + \|T(u)\| \\ &\leq 2|u|_{BVS^1}. \end{aligned}$$

□

Let $|u|_{BV} = \int_{S^2} |Du| = \int_{S^2} (|D^a u| + |D^c u|) + \int_{S(u)} |u^+ - u^-| \, \mathrm{d}\mathcal{H}^1$; we deduce that

$$|u|_{BV} \leq |u|_{BV\,S^1} \leq \frac{\pi}{2} |u|_{BV}, \forall u \in BV(S^2, S^1).$$

Therefore, Corollary 2 is a weaker estimate of $E(u)$ than inequality (3) obtained in [5].

3 Some other properties of the distribution T

We start by observing that $T : BV(S^2,S^1) \to \mathcal{D}'(S^2,\mathbb{R})$ is not continuous, i.e. there exists a sequence of functions $u_k \in BV(S^2,S^1)$ such that $u_k \to u$ strongly in $BV(S^2,S^1)$ and $T(u_k) \nrightarrow T(u)$ in $\mathcal{D}'(S^2,\mathbb{R})$. The reason for that is the discontinuity of the function ρ that enters in the definition of T.

Proposition 1 *The map $T : BV(S^2,S^1) \to \mathcal{D}'(S^2,\mathbb{R})$ is discontinuous.*

Proof. Write

$$S^2 = \{(\cos\theta\sin\alpha, \sin\theta\sin\alpha, \cos\alpha) \,:\, \alpha \in [0,\pi],\, \theta \in (0,2\pi]\}\,.$$

In the spherical coordinates $(\alpha,\theta) \in [0,\pi]\times[0,2\pi]$, consider the BV functions φ and u defined as

$$\varphi(\alpha,\theta) = \begin{cases} -2\theta & \text{if } \theta \in (0,\frac{\pi}{2}),\ \alpha \in (0,\frac{\pi}{2}) \\ -\pi & \text{if } \theta \in (\frac{\pi}{2},\frac{3\pi}{2}),\ \alpha \in (0,\frac{\pi}{2}) \\ 2(\theta-2\pi) & \text{if } \theta \in (\frac{3\pi}{2},2\pi),\ \alpha \in (0,\frac{\pi}{2}) \\ 0 & \text{if } \theta \in (0,2\pi),\ \alpha \in (\frac{\pi}{2},\pi) \end{cases} \quad \text{and} \quad u = e^{i\varphi}. \tag{14}$$

We have that the jump set of u and φ is concentrated on the equator $\{\alpha = \frac{\pi}{2}\}$ of the sphere S^2, i.e.

$$S(\varphi) = S(u) = \{\alpha = \frac{\pi}{2}\}.$$

On the equator we choose the orientation given by the normal vector $\vec{\alpha}$ oriented from the north to the south; so $(\vec{\alpha},\vec{\theta},\vec{e})$ is direct. We show that

$$T(u) = 2\pi(\delta_p - \delta_n) \tag{15}$$

where $n = (\frac{\pi}{2},\frac{3\pi}{2})$ and $p = (\frac{\pi}{2},\frac{\pi}{2})$ in the frame (α,θ). Indeed, we remark that

$$\varphi^+ - \varphi^- = \rho(u^+,u^-) + 2\pi\chi_{\widehat{np}} \quad \text{on } S(u);$$

by Lemma 3, we obtain

$$D\varphi = u \wedge \nabla u\,\mathcal{H}^2 + \rho(u^+,u^-)\vec{\alpha}\,\mathcal{H}^1 \llcorner S(u) + 2\pi\vec{\alpha}\,\mathcal{H}^1 \llcorner \widehat{np}$$

and it yields

$$\langle T(u),\zeta\rangle = -2\pi\int_{\widehat{np}} \vec{\alpha}\cdot\nabla^\perp\zeta\,\mathrm{d}\mathcal{H}^1 = -2\pi\int_p^n \frac{\partial\zeta}{\partial\theta}\,\mathrm{d}\mathcal{H}^1 = 2\pi(\zeta(p)-\zeta(n)),\ \forall\zeta \in C^1(S^2,\mathbb{R}).$$

Construct the approximation sequence $\varphi_\varepsilon \in BV(S^2,\mathbb{R})$, $\varepsilon \in (0,1)$ defined (in the spherical coordinates) as

$$\varphi_\varepsilon(\alpha,\theta) = \begin{cases} -2\theta & \text{if } \theta \in (0,\frac{\pi-\varepsilon}{2}),\ \alpha \in (0,\frac{\pi}{2}) \\ -\pi+\varepsilon & \text{if } \theta \in (\frac{\pi-\varepsilon}{2},\frac{3\pi+\varepsilon}{2}),\ \alpha \in (0,\frac{\pi}{2}) \\ 2(\theta-2\pi) & \text{if } \theta \in (\frac{3\pi+\varepsilon}{2},2\pi),\ \alpha \in (0,\frac{\pi}{2}) \\ 0 & \text{if } \theta \in (0,2\pi),\ \alpha \in (\frac{\pi}{2},\pi) \end{cases}.$$

and set $u_\varepsilon = e^{i\varphi_\varepsilon}$. An easy computation shows that $\varphi_\varepsilon \to \varphi$ strongly in BV; therefore, $u_\varepsilon \to u$ strongly in BV as $\varepsilon \to 0$. As before, we have

$$S(\varphi_\varepsilon) = S(u_\varepsilon) = \{\alpha = \frac{\pi}{2}\} \text{ and } \varphi_\varepsilon^+ - \varphi_\varepsilon^- = \rho(u_\varepsilon^+,u_\varepsilon^-) \ \text{ on } \{\alpha = \frac{\pi}{2}\}.$$

It follows that $T(u_\varepsilon) = 0$ and we conclude

$$T(u_\varepsilon) \not\rightharpoonup T(u) \text{ in } \mathcal{D}'(S^2, \mathbb{R}).$$

□

As Brezis, Mironescu and Ponce proved in [4], if we restrict ourselves to $W^{1,1}(S^2, S^1)$, then the map $T\big|_{W^{1,1}(S^2,S^1)} : W^{1,1}(S^2, S^1) \to \mathcal{Z}(S^2)$ is continuous, i.e. if $g, g_k \in W^{1,1}(S^2, S^1)$ such that $g_k \to g$ in $W^{1,1}$ then $\|T(g_k) - T(g)\| \to 0$ as $k \to \infty$. It is natural to ask if one could change the antisymmetric function ρ in order that the corresponding map T become continuous. The answer is negative:

Proposition 2 *There is no antisymmetric function $\gamma : S^1 \times S^1 \to \mathbb{R}$ such that the map $T_\gamma : BV(S^2, S^1) \to \mathcal{Z}(S^2)$ given for every $u \in BV(S^2, S^1)$ as*

$$\langle T_\gamma(u), \zeta\rangle = \int_{S^2} \nabla^\perp \zeta \cdot (u \wedge (D^a u + D^c u)) + \int_{S(u)} \gamma(u^+, u^-)\, \nu_u \cdot \nabla^\perp \zeta \,\mathrm{d}\mathcal{H}^1, \ \forall \zeta \in C^1(S^2, \mathbb{R})$$

is well-defined and continuous.

Proof. By contradiction, suppose that there exists such a function γ. First we show that

$$\gamma(\omega_1, \omega_2) \equiv \mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2) \pmod{2\pi}, \forall \omega_1, \omega_2 \in S^1. \tag{16}$$

Indeed, fix $\omega_1, \omega_2 \in S^1$. Take $f : [0, 2\pi] \to \mathbb{R}$ the linear function satisfying $f(0) = \mathrm{Arg}\,(\omega_1)$ and $f(2\pi) = \mathrm{Arg}\,(\omega_2)$; define $u \in BV(S^2, S^1)$ as

$$u(\alpha, \theta) = e^{if(\theta)}, \ \forall \alpha \in (0, \pi), \theta \in (0, 2\pi).$$

Consider the lifting $\varphi \in BV(S^2, \mathbb{R})$ of u given by

$$\varphi(\alpha, \theta) = f(\theta), \ \forall \alpha \in (0, \pi), \theta \in (0, 2\pi).$$

If $\omega_1 \neq \omega_2$, the jump set of u and φ is concentrated on the meridian $\{\theta = 0\}$ orientated counter-clockwise by the unit vector $\vec{\theta}$. We have that

$$D\varphi = u \wedge \nabla u \mathcal{H}^2 + (\mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2))\vec{\theta}\,\mathcal{H}^1 \llcorner \{\theta = 0\}.$$

Since $\mathrm{curl}\, D\varphi = 0$ in $\mathcal{D}'$, it yields

$$\begin{aligned}
\int_{S^2} u \wedge \nabla u \cdot \nabla^\perp \zeta \,\mathrm{d}\mathcal{H}^2 &= -\int_{\{\theta=0\}} (\mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2))\vec{\theta} \cdot \nabla^\perp \zeta \,\mathrm{d}\mathcal{H}^1 \\
&= (\mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2)) \int_p^n \frac{\partial \zeta}{\partial \alpha} \,\mathrm{d}\mathcal{H}^1 \\
&= (\mathrm{Arg}\,(\omega_2) - \mathrm{Arg}\,(\omega_1))(\zeta(p) - \zeta(n)), \ \forall \zeta \in C^1(S^2)
\end{aligned}$$

where $p = (0, 0)$ and $n = (\pi, 0)$ (in the spherical coordinates) are the north and the south pole of S^2. We obtain that

$$\begin{aligned}
\langle T_\gamma(u), \zeta\rangle &= \int_{S^2} \nabla^\perp \zeta \cdot (u \wedge \nabla u) \,\mathrm{d}\mathcal{H}^2 + \gamma(\omega_1, \omega_2) \int_{\{\theta=0\}} \vec{\theta} \cdot \nabla^\perp \zeta \,\mathrm{d}\mathcal{H}^1 \\
&= (\mathrm{Arg}\,(\omega_2) - \mathrm{Arg}\,(\omega_1) + \gamma(\omega_1, \omega_2))(\zeta(p) - \zeta(n)), \ \forall \zeta \in C^1(S^2, \mathbb{R}).
\end{aligned}$$

From the definition we know that $T_\gamma(u) \in \mathcal{Z}(S^2)$ and therefore, (16) holds. If $\omega_1 = \omega_2$, by the antisymmetry of γ, we have $\gamma(\omega_1, \omega_2) = 0$ and so, (16) is obvious.

Second we prove that the continuity of T_γ implies that γ is continuous on $S^1 \times S^1$. Indeed, let $(\omega_1^\varepsilon)_\varepsilon$ and $(\omega_2^\varepsilon)_\varepsilon$ be two sequences in S^1 such that $\omega_1^\varepsilon \to \omega_1$ and $\omega_2^\varepsilon \to \omega_2$. We want that

$$\gamma(\omega_1^\varepsilon, \omega_2^\varepsilon) \to \gamma(\omega_1, \omega_2). \tag{17}$$

Take $\beta \in [0, 2\pi)$ such that $e^{i\beta}$ is different from ω_1 and ω_2. For each $\omega \in S^1$ denote by $\arg_\beta(\omega) \in (\beta - 2\pi, \beta]$ the argument of ω, i.e.

$$e^{i \arg_\beta(\omega)} = \omega. \tag{18}$$

As above, define $f_\varepsilon : [0, 2\pi] \to \mathbb{R}$ as the linear function satisfying $f_\varepsilon(0) = \arg_\beta(\omega_1^\varepsilon)$ and $f_\varepsilon(2\pi) = \arg_\beta(\omega_2^\varepsilon)$ and consider $u_\varepsilon \in BV(S^2, S^1)$ such that

$$u_\varepsilon(\alpha, \theta) = e^{if_\varepsilon(\theta)}, \ \forall \alpha \in (0, \pi), \theta \in (0, 2\pi).$$

It's easy to check that $u_\varepsilon \to u$ strongly in BV, where $u(\alpha, \theta) = e^{if(\theta)}$ and f is the linear function satisfying $f(0) = \arg_\beta(\omega_1)$ and $f(2\pi) = \arg_\beta(\omega_2)$. As before, we obtain

$$\begin{aligned} T_\gamma(u_\varepsilon) &= (\arg_\beta(\omega_2^\varepsilon) - \arg_\beta(\omega_1^\varepsilon) + \gamma(\omega_1^\varepsilon, \omega_2^\varepsilon))(\delta_p - \delta_n) \\ \text{and} \quad T_\gamma(u) &= (\arg_\beta(\omega_2) - \arg_\beta(\omega_1) + \gamma(\omega_1, \omega_2))(\delta_p - \delta_n). \end{aligned}$$

Since T_γ and $\arg_\beta$ are continuous on $BV(S^2, S^1)$, respectively on $S^1 \setminus \{e^{i\beta}\}$, we deduce that (17) holds.

Observe now that the function

$$(\omega_1, \omega_2) \mapsto \gamma(\omega_1, \omega_2) - \mathrm{Arg}\,(\omega_1) + \mathrm{Arg}\,(\omega_2)$$

is continuous on the connected set $S^1 \setminus \{-1\} \times S^1 \setminus \{-1\}$ and takes values in $2\pi\mathbb{Z}$. Therefore, there exists $k \in \mathbb{Z}$ such that

$$\gamma(\omega_1, \omega_2) = \mathrm{Arg}\,(\omega_1) - \mathrm{Arg}\,(\omega_2) - 2\pi k \text{ in } S^1 \setminus \{-1\} \times S^1 \setminus \{-1\}.$$

In fact, $k = 0$ if one takes $\omega_1 = \omega_2$. But $\mathrm{Arg}\,(\cdot)$ is not a continuous map on S^1 which is a contradiction with the continuity of γ on $S^1 \times S^1$. □

The algebraic properties of T restricted to $W^{1,1}(S^2, S^1)$ (see [4], Lemma 1) do not hold in general for $BV(S^2, S^1)$ functions.

Remark 6 a) There exists $u \in BV(S^2, S^1)$ such that $T(\bar{u}) \neq -T(u)$. Indeed, take the function u defined in (14). A similar computation gives us that $T(\bar{u}) = 0 \neq -T(u)$.
b) The relation $T(gh) = T(g) + T(h)$, $\forall g, h \in W^{1,1}(S^2, S^1)$ need not hold for $BV(S^2, S^1)$ functions. As before, consider the function u in (14). Then $T(-u) = 0$. Since $T(-1) = 0$, we conclude $T(-u) \neq T(u) + T(-1)$. □

In the following we discuss the nature of the singularities of the distribution $T(u)$. As it was mentioned in the beginning, we deal with two types of singularity:
i) topological singularities carrying a degree which are created by the absolutely continuous part and the Cantor part of the distributional determinant of u;
ii) point singularities coming from the jump part of the derivative Du.

We give some examples in order to point out these two different kind of singularity. In Example 1, $T(u)$ is a dipole made up by two vortices of degree 1 and -1; these two vortices are generated by the absolutely continuous part of $\det(\nabla u)$ in a), respectively by the Cantor part of the distributional Jacobian of u in b).

Example 1 a) Let us analyze the function $g \in W^{1,1}(S^2, S^1)$,

$$g(\alpha, \theta) = e^{i\theta}, \ \forall \alpha \in (0, \pi), \theta \in [0, 2\pi).$$

Denote p and n the north and respectively the south pole of the unit sphere. We consider the lifting $\varphi \in BV(S^2, \mathbb{R})$ of u given by $\varphi(\alpha, \theta) = \theta$ for every $\alpha \in (0, \pi), \theta \in (0, 2\pi)$. Then the jump set of φ is concentrated on the meridian $\{\theta = 0\}$ oriented counterclockwise by the unit vector $\vec{\theta}$. We have

$$D\varphi = g \wedge \nabla g \, \mathcal{H}^2 - 2\pi \vec{\theta} \, \mathcal{H}^1 \llcorner \widehat{np}.$$

Therefore, $T(g) = 2\pi(\delta_p - \delta_n)$. The two poles are the vortices of the function g.

b) The same situation may occur for some purely Cantor functions. Let us consider the standard Cantor function $f : [0, 1] \to [0, 1]$; f is a continuous, nondecreasing function with $f(0) = 0$, $f(1) = 1$ and $f'(x) = 0$ a.e. $x \in (0, 1)$. Take $v \in BV(S^2, S^1)$ defined as

$$v(\alpha, \theta) = e^{2\pi i f(\theta/2\pi)}, \ \forall \alpha \in (0, \pi), \theta \in [0, 2\pi).$$

The lifting $\varphi \in BV(S^2, \mathbb{R})$ given by $\varphi(\alpha, \theta) = 2\pi f(\theta/2\pi)$ for every $\alpha \in (0, \pi), \theta \in (0, 2\pi)$ has the jump set concentrated on the meridian $\{\theta = 0\}$ and

$$D\varphi = v \wedge D^c v - 2\pi \vec{\theta} \, \mathcal{H}^1 \llcorner \widehat{np}.$$

As before, we obtain that $T(v) = 2\pi(\delta_p - \delta_n)$ where p and n are the poles of S^2.

Remark also that for the two functions constructed in Example 1, the constant 2 in inequality (3) is optimal and we have a specific structure for an optimal lifting:

Proposition 3 *Let $u \in BV(S^2, S^1)$ be one of the two functions defined in Example 1. Then for every lifting $\varphi \in BV(S^2, \mathbb{R})$ of u we have*

$$\int_{S^2} |D\varphi| \geq 2 \int_{S^2} |Du|.$$

Moreover, the set of all optimal liftings of u is given by

$$\{\arg_\beta(u) + 2\pi k \, : \, \beta \in [0, 2\pi), \, k \in \mathbb{Z}\}$$

where $\arg_\beta(\omega) \in (\beta - 2\pi, \beta]$ stands for the argument of $\omega \in S^1$ (as in (18)).

Proof. First remark that

$$\int_{S^2} |Du| = 2\pi^2 \quad \text{and} \quad \|T(u)\| = 2\pi d_{S^2}(n, p) = 2\pi^2$$

where n and p are the two poles of S^2.

Let $\varphi \in BV(S^2, \mathbb{R})$ be a lifting of u. By Theorem 2 and Lemma 2, we obtain

$$\int_{S^2} |D\varphi| \geq E(u) = \int_{S^2} |Du| + \|T(u)\| = 4\pi^2 = 2 \int_{S^2} |Du|.$$

Take now $\varphi \in BV(S^2, \mathbb{R})$ an optimal lifting of u. By Lemma 3, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ that achieves the minimum in (11) and satisfies

$$D\varphi = u \wedge Du - f\nu \, \mathcal{H}^1 \llcorner S.$$

That means

$$D^j\varphi = -f\nu\,\mathcal{H}^1 \llcorner S \quad \text{and} \quad \int_S |f| = 2\pi d_{S^2}(n,p). \tag{19}$$

We may assume here that $S = \{f \neq 0\}$. For every $\alpha \in (0,\pi)$ we denote L_α the latitude on S^2 corresponding to α and $\varphi_\alpha : L_\alpha \to \mathbb{R}$ the restriction of φ to L_α. Using the Characterization Theorem of BV functions by sections and Theorem 3.108 in [1], it results that for a.e. $\alpha \in (0,\pi)$, $\varphi_\alpha \in BV(L_\alpha;\mathbb{R})$ and the discontinuity set of φ_α is $S \cap L_\alpha$. Remark that $\deg(u;L_\alpha) = 1$ for every $\alpha \in (0,\pi)$. Thus, for a.e. $\alpha \in (0,\pi)$, φ_α will have at least one jump on L_α and the length of a jump is not less than 2π. It yields $\mathcal{H}^1(S) \geq \pi$ and $|f| \geq 2\pi\ \mathcal{H}^1 -$ a.e. on S. By (19), we deduce that

$$|f| = 2\pi\ \mathcal{H}^1 - \text{a.e. on } S \quad \text{and} \quad \mathcal{H}^1(S) = \pi.$$

We know that

$$\int_S \frac{f}{2\pi}\nu \cdot \nabla^\perp\zeta \,\mathrm{d}\mathcal{H}^1 = \zeta(p) - \zeta(n),\ \forall\zeta \in C^1(S^2).$$

By [7](Section 4.2.25), it results that S covers $\mathcal{H}^1$-almost all of a Lipschitz univalent path c between the two poles. Since $\mathcal{H}^1(S) = d_{S^2}(n,p)$ we deduce that S is a geodesic arc on S^2 between n and p and $\frac{f}{2\pi}\nu$ is the normal unit vector to the curve c. Take $\beta \in [0,2\pi)$ such that $S = \{\theta = \beta\}$ in the spherical coordinates. We have that $\varphi - \arg_\beta(u) : S^2 \setminus S \to 2\pi\mathbb{Z}$ is continuous on the connected set $S^2 \setminus S$. Therefore, there exists $k \in \mathbb{Z}$ such that

$$\varphi = \arg_\beta(u) + 2\pi k$$

and the conclusion follows. □

The appearance of non-topological singularities in the writing of $T(u)$ for $u \in BV(S^2,S^1)$ was already seen in the example (14); there the distribution $T(u)$ is a dipole even if the function u does not have any vortex. One should notice that the dipole (15) is created on the jump set of u by the discontinuity of the chosen argument Arg . In Remark 7, we will see that a dipole could disappear if we change the choice of the argument.

Remark 7 Let $\beta \in [0,2\pi)$. Define the antisymmetric function $\gamma_\beta(\cdot,\cdot) : S^1 \times S^1 \to [-\pi,\pi]$ as

$$\gamma_\beta(\omega_1,\omega_2) = \begin{cases} \mathrm{Arg}\left(\frac{\omega_1}{\omega_2}\right) & \text{if } \frac{\omega_1}{\omega_2} \neq -1 \\ \arg_\beta(\omega_1) - \arg_\beta(\omega_2) & \text{if } \frac{\omega_1}{\omega_2} = -1 \end{cases},\ \forall\omega_1,\omega_2 \in S^1.$$

Consider now the distribution $T_{\gamma_\beta}(u) \in \mathcal{D}'(S^2,\mathbb{R})$ given as in Proposition 2:

$$\langle T_{\gamma_\beta}(u),\zeta\rangle = \int_{S^2} \nabla^\perp\zeta \cdot (u \wedge (D^a u + D^c u)) + \int_{S(u)} \gamma_\beta(u^+,u^-)\nu_u \cdot \nabla^\perp\zeta \,\mathrm{d}\mathcal{H}^1,\ \forall\zeta \in C^1(S^2,\mathbb{R}).$$

Observe that T_{γ_β} inherits the properties of T given in Theorems 1, 2 and 3. However, the structure of the singularities of $T_{\gamma_\beta}(u)$ may be different from $T(u)$. Indeed, consider $u \in BV(S^2,S^1)$ the function constructed in (14). We saw that $T(u) = 2\pi(\delta_p - \delta_n)$ where $n = (\frac{\pi}{2},\frac{3\pi}{2})$ and $p = (\frac{\pi}{2},\frac{\pi}{2})$ (in the spherical coordinates). The same computation gives us $T_{\gamma_{\pi/2}}(u) = 0$. The difference between $T(u)$ and $T_{\gamma_{\pi/2}}(u)$ arises from the choice of the argument.

An interesting phenomenon is observed in Example 2 where the two types of singularity are mixed: some topological vortices may be located on the jump set of u.

Example 2 a) An example that points out the mixture of the two type of singularity is given by functions with pseudo-vortices: define $u \in BV(S^2, S^1)$ as

$$u(\alpha, \theta) = e^{3i\theta/2}, \forall \alpha \in (0, \pi), \theta \in (0, 2\pi).$$

The jump set of u is the meridian $\{\theta = 0\}$. We have

$$T(u) = 2\pi(\delta_p - \delta_n) \text{ and } T_{\gamma_{\pi/2}}(u) = 4\pi(\delta_p - \delta_n).$$

The two poles p and n arise on the jump set of u and behave like some pseudo-vortices, i.e. after a complete turn, the function u rotates $3/2$ times around the poles (with different signs: '+' around p and '−' around n). According to the choice of the argument in the definition of γ_β, the distribution $T_{\gamma_\beta}(u)$ will count once or twice the dipole.

b) A piecewise constant function $u \in BV(S^2, S^1)$ may create a dipole for $T(u)$. Indeed, let us define $\varphi \in BV(S^2, \mathbb{R})$ as

$$\varphi(\alpha, \theta) = \begin{cases} 0 & \text{if } \theta \in (0, 2\pi/3),\ \alpha \in (0, \pi) \\ 2\pi/3 & \text{if } \theta \in (2\pi/3, 4\pi/3),\ \alpha \in (0, \pi) \\ 4\pi/3 & \text{if } \theta \in (4\pi/3, 2\pi),\ \alpha \in (0, \pi) \end{cases}$$

and set $u = e^{i\varphi}$. The jump set of u and φ is the union of three meridians

$$S(u) = S(\varphi) = \{\theta = 0\} \cup \{\theta = 2\pi/3\} \cup \{\theta = 4\pi/3\}.$$

We have

$$\varphi^+ - \varphi^- = \rho(u^+, u^-) - 2\pi\chi_{\{\theta=0\}}.$$

We obtain $T(u) = 2\pi(\delta_p - \delta_n)$ where p and n are the two poles of the unit sphere. For every $\beta \in [0, 2\pi)$, T_{γ_β} has the same behavior, i.e. $T_{\gamma_\beta}(u) = 2\pi(\delta_p - \delta_n)$.

c) Let $u \in BV(S^2, S^1)$ be the function defined above in b) and take g the function constructed in Example 1 a). Set $w = gu \in BV(S^2, S^1)$. We have $S(w) = \{\theta = 0\} \cup \{\theta = 2\pi/3\} \cup \{\theta = 4\pi/3\}$. We show that $T(w) = 4\pi(\delta_p - \delta_n)$. Indeed, construct the lifting $\psi \in BV(S^2, \mathbb{R})$ of w as

$$\psi(\alpha, \theta) = \begin{cases} \theta & \text{if } \theta \in (0, 2\pi/3),\ \alpha \in (0, \pi) \\ \theta + 2\pi/3 & \text{if } \theta \in (2\pi/3, 4\pi/3),\ \alpha \in (0, \pi) \\ \theta - 2\pi/3 & \text{if } \theta \in (4\pi/3, 2\pi),\ \alpha \in (0, \pi) \end{cases}.$$

Observe that

$$\psi^+ - \psi^- = \rho(w^+, w^-) - 2\pi\chi_{\{\theta=0\}} - 2\pi\chi_{\{\theta=4\pi/3\}} \text{ on } S(w)$$

and conclude that $T(w) = 4\pi(\delta_p - \delta_n)$. So, the north pole p and the south pole n which are the vortices of g remain singularities for the function w; they appear now on the jump part of w. The same behavior happens to T_{γ_β} for every $\beta \in [0, 2\pi)$, i.e. $T_{\gamma_\beta}(w) = 4\pi(\delta_p - \delta_n)$.

As we mentioned before, for every $u \in BV(S^2, S^1)$ there exists a bounded lifting $\varphi \in BV \cap L^\infty(S^2, \mathbb{R})$ (see [5]). The striking fact is that we can construct functions $u \in BV(S^2, S^1)$ such that no optimal lifting belongs to L^∞. We give such an example in the following:

Example 3 On the interval $(0, 2\pi)$ we consider

$$p_1 = 1,\ n_k = p_k + \frac{1}{4^k} \text{ and } p_{k+1} = n_k + \frac{1}{2^k},\ \forall k \geq 1.$$

Suppose that this configuration of points lies on the equator $\{\frac{\pi}{2}\} \times [0, 2\pi]$ (in the spherical coordinates) of S^2 and we consider that each dipole (p_k, n_k) appears k times. Since $\sum_{k\geq 1} k d_{S^2}(p_k, n_k) < \infty$, set

$$\Lambda = 2\pi \sum_{k\geq 1} k(\delta_{p_k} - \delta_{n_k}) \in \mathcal{Z}(S^2).$$

By [2] (Lemma 16),

$$T\left(W^{1,1}(S^2, S^1)\right) = \mathcal{Z}(S^2).$$

Thus, take $g \in W^{1,1}(S^2, S^1)$ such that $T(g) = \Lambda$. Using (2), it follows that

$$\|T(g)\| = 2\pi \sum_{k\geq 1} k d_{S^2}(p_k, n_k).$$

Let $\varphi \in BV(S^2, \mathbb{R})$ be an optimal lifting of g. Then there is a triple $(f, S, \nu) \in \mathcal{J}(T(g))$ such that

$$D\varphi = g \wedge \nabla g \mathcal{H}^2 - f\, \nu\, \mathcal{H}^1 \llcorner S \quad \text{and} \quad \int_S |f| \, \mathrm{d}\mathcal{H}^1 = \|T(g)\|. \tag{20}$$

We may assume that $S = \{f \neq 0\}$.

We know that $\int_S f\nu \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1 = 2\pi \sum_{k\geq 1} k(\zeta(p_k) - \zeta(n_k))$, $\forall \zeta \in C^1(S^2)$. For each $k \geq 1$, we denote $V_k = (0, \pi) \times (p_k - \dfrac{1}{8^k}, n_k + \dfrac{1}{8^k})$. Then

$$\int_S f\nu \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1 = 2\pi k(\zeta(p_k) - \zeta(n_k)), \forall \zeta \in C^1(S^2) \text{ with } \operatorname{supp} \zeta \subset V_k.$$

By (20), it follows that

$$\int_{S\cap V_k} |f| \, \mathrm{d}\mathcal{H}^1 = 2\pi k \, d_{S^2}(p_k, n_k).$$

Using the same argument as in the proof of Proposition 3, we deduce that for each $k \in \mathbb{N}$,

$$S(\varphi) \cap V_k = S \cap V_k = \widehat{n_k p_k} \quad \text{and} \quad |\varphi^+ - \varphi^-| = |f| = 2k\pi \ \mathcal{H}^1\text{-a.e. on } \widehat{n_k p_k}$$

where $\widehat{n_k p_k}$ is the geodesic arc connecting n_k and p_k. It yields that $\varphi \notin L^\infty$. So, every optimal BV lifting of g does not belong to L^∞.

In the next example, we show that Theorem 3 fails if we minimize the energy in (8) just over the class of gradient maps:

Example 4 Let $u \in BV(S^2, S^1)$ be defined as

$$u(\alpha, \theta) = e^{i\theta/3}, \forall \alpha \in (0, \pi), \theta \in (0, 2\pi).$$

The jump set of u is the meridian $\{\theta = 0\}$ orientated counterclockwise and $\rho(u^+, u^-) = -2\pi/3$ on $S(u)$. We have that $T(u) = 0$. On the other hand, for every $\psi \in C^\infty(S^2, \mathbb{R})$, we have

$$\begin{aligned}\int_{S^2} |u \wedge \nabla u \, \mathcal{H}^2 + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - \nabla\psi \, \mathcal{H}^2| &= \int_{S^2} |u \wedge \nabla u - \nabla\psi| \, \mathrm{d}\mathcal{H}^2 + \int_{S(u)} |\rho(u^+, u^-)| \, \mathrm{d}\mathcal{H}^1 \\ &\geq \int_{S(u)} 2\pi/3 \, \mathrm{d}\mathcal{H}^1 = 2\pi^2/3 > \|T(u)\|.\end{aligned}$$

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ECOLE NORMALE SUPERIEURE,
45, RUE D'ULM
75230 PARIS CEDEX 05, FRANCE
E-Mail address: `Radu.Ignat@ens.fr`

LABORATOIRE J.-L. LIONS
UNIVERSITE P. ET M. CURIE, B.C. 187
4 PL. JUSSIEU
75252 PARIS CEDEX 05
E-Mail address: `ignat@ann.jussieu.fr`