Chapter 2: Analytical aspects

I - The law of large numbers: Existence of asymptotic shapes

The result is that upon observing a specific direction $(x,y) \in \mathbb{R}_+^2$, $L(\lfloor Nx \rfloor, \lfloor Ny \rfloor)$ grows linearly in N.

The main thm we shall prove is

<u>Thm</u>: [LLN - (weak) Existence of asymptotic shape] Assume $\mathbb{E}|\omega| < +\infty$.

There exists a <u>deterministic</u> $s : \mathbb{R}_+^2 \longrightarrow \mathbb{R} \sqcup \{\infty\}$

s.t. $\forall (x,y) \in \mathbb{R}_+^2$, $\lim_{N \to +\infty} \frac{L(\lfloor Nx \rfloor, \lfloor Ny \rfloor)}{N} = s(x,y)$ a.s

$$\boxed{S = S_{\mathcal{L}(\omega)}}$$

where s is

- (Homogenous) : $s(cx, cy) = c\,s(x,y)$
- (Superadditivity): $s(x_1, y_1) + s(x_2, y_2) \leq s(x_1 + x_2, y_1 + y_2)$
- (Symmetry) : $s(x,y) = s(y,x)$
- (Concave) : $\forall t \in [0,1]$, $t\, s(z_1) + (1-t)\, s(z_2) \leq s(t z_1 + (1-t) z_2)$
- (Monotone - upon correcting by mean):
  $s(x+h, y) \geq s(x,y) + h\, \mathbb{E}(\omega)$

<u>Rmk</u>: $- s$ is convex, symmetric & subadditive on $\mathbb{R}^2$
$\simeq$ Distance on $\mathbb{R}^2$ (if ≥ 0!)

Exercise :

1/ If s homogenous, s superadditive $\Leftrightarrow$ concave

2/ Prove that if $\mathcal{L}(\omega)$ is an environment

& $\omega' = \dfrac{\omega - \mathbb{E}(\omega)}{\sigma}$, $\sigma = \sqrt{\mathrm{Var}(\omega)}$

then
$$\begin{cases} s_{\mathcal{L}(\omega)}(x,y) = \mathbb{E}(\omega)(x+y) + \sigma\, s_{\mathcal{L}(\omega')}(x,y) \\ s_{\mathcal{L}'(\omega)} \geq 0 \quad \text{\& monotone in each coordinate} \end{cases}$$

3/ Show that $s(x+h, y) \geq s(x,y) + h\,\mathbb{E}(\omega)$

At this point, let us make two remarks

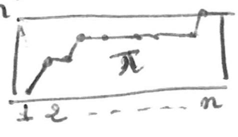
↳ For most ω, the shape $s_{\mathcal{L}(\omega)} = s$ is UNKNOWN since the existence comes from a non-constructive superadditive argument.

The few cases that are known are qualified as "integrable" because of the underlying algebraic structure (Chaper 3).

- $\omega \overset{\mathcal{L}}{=} \mathcal{E}$: $s(x,y) = x+y+2\sqrt{xy} = (\sqrt{x}+\sqrt{y})^2$
- $\omega \overset{\mathcal{L}}{=} \mathrm{Geom}(q)$: $s(x,y) = \dfrac{1}{p}\left(qx + qy + 2\sqrt{qxy}\right)$ (Johansson (2000))

  $p+q=1$
- $\omega \overset{\mathcal{L}}{=} \mathrm{Bernoulli}(p)$: $s(x,y) = \begin{cases} p(x-y) + 2\sqrt{xy}\sqrt{p(1-p)} & \text{if } p \leq \dfrac{x}{x+y} \\ x & \text{if } p \geq \dfrac{x}{x+y} \end{cases}$

By Seppäläinen (1998), for a ~~different~~ geometry of paths different from $\rightarrow \uparrow$ called "strict weak"



$\pi \in \widetilde{\Pi}^{n,m} = \{$ paths of the form $(0,y_1), (1,y_2), \dots, (m, y_m)\}$

y_i weakly increasing.

From the two first examples, one could conjecture

$$s(x,y) = m(x+y) + 2\sigma\sqrt{xy}$$

which is true on the boundary, but false globally...
A simulation quickly shows that the formula is unlikely

$\hookrightarrow$ In order to have an asymptotic shape for $\frac{\mathcal{B}(t)}{t}$, one needs some uniformity in the direction (x,y)... This will be a consequence of concentration of measure (Chapter 2 - II).

1 Kingman's superadditive ergodic theorem.

Originally stated in the suadditive form (Kingman 1968) [redacted] and this version with two indices is due to Liggett (1985).

We state it in the general form, however our proof works only for DLPP or models with similar structure.

Thm [Kingman's superadditive ergodic thm]
Suppose $(X_{m,n}\ ;\ 0 \leq m \leq n)$ is a random sequence in L^1 with the following properties

(i) Superadditive:

$$X_{0,m} + X_{m,n} \leqslant X_{0,n}$$

(ii) $\forall\, m \in \mathbb{N}$, $(X_{nm,\,(n+1)m}\ ;\ n \geqslant 1)$ stationary & ergodic (we will only need iid)

(iii) $\mathcal{L}(X_{m,m+k}\ ;\ k \geqslant 1)$ does not depend on m.

Then $\lim_{n \to +\infty} \frac{X_{0,n}}{n} = \gamma$ exists a.s. and in L^1. **IV**

Moreover $\gamma = \sup_{n \in \mathbb{N}} \frac{\mathbb{E}\, X_{0,n}}{n} \in \mathbb{R} \cup \{+\infty\}$ ⚠

Proof: Step 1: Let us start by proving the superadditive thm for constants (for e.g. $a_n = \mathbb{E}(X_{0,n})$) which satisfy $a_n + a_m \leqslant a_{n+m}$ ($\mathbb{E}(X_{0,n})$ does by (ii) & (iii))

$$* \quad \frac{a_n}{n} \leqslant \frac{a_{n+m}}{n} - \frac{a_m}{n} = \frac{n+m}{n}\,\frac{a_{n+m}}{n+m} - \frac{a_m}{n} \leqslant \frac{n+m}{n} \sup_{k \in \mathbb{N}^*} \frac{a_k}{k}$$

$$\limsup_{n \to +\infty} \frac{a_n}{n} \leqslant \sup_{k \in \mathbb{N}^*} \frac{a_k}{k}$$

* Now write $n = kq + \ell$ for fixed q ($k = k_n$)

$$\frac{a_n}{n} = \frac{a_{kq+\ell}}{kq+\ell} \geqslant \frac{1}{kq+\ell}\left(a_q + a_{k(q-1)+\ell}\right) \geqslant \dots \geqslant \frac{1}{kq+\ell}\left(k a_q + a_\ell\right) = \frac{k}{kq+\ell}\, a_q + \frac{a_\ell}{kq+\ell}$$

superadditivity k times

Then $\liminf_{n \to +\infty} \frac{a_n}{n} \geqslant \frac{a_q}{q} \quad \forall q$ fixed

Then $\liminf \frac{a_n}{n} \geqslant \sup_{k \in \mathbb{N}^*} \frac{a_k}{k}$ ∎

Step 2: This very nice idea for controlling liminf carries verbatim for the random case!
We write again $n = kq + \ell$

$$X_{0,n} \geqslant X_{0,q} + X_{q,n} \geqslant \dots \geqslant X_{0,q} + X_{q,2q} + \dots + X_{(k-1)q,kq} + X_{kq,kq+\ell}$$

$\boxed{V}$

Then $$\frac{X_{0,n}}{n} \geqslant \underbrace{\frac{k}{kq+l}}_{\substack{n \to +\infty \\ (k \to +\infty) \\ \frac{1}{q}}} \cdot \underbrace{\frac{\sum_{j=0}^{k-1} X_{jq,(j+1)q}}{k}}_{\text{Hypothesis (ii)},\ n \to +\infty} + \underbrace{\frac{X_{kq,kq+l}}{kq+l}}_{\substack{n \to +\infty \\ (k \to +\infty) \\ 0}}$$

Either by the LLN (if iid) we have convergence to $\mathbb{E}(X_{0,q})$

Either by Birkhoff's ergodic thm $\mathbb{E}(X_{0,q} \mid \mathcal{I}_q) = \mathbb{E}(X_{0,m})$

$\mathcal{I}_q$: Invariant σ-algebra for shifts by q

Via Borel-Cantelli.

Indeed, $\forall \varepsilon > 0$

$$\sum_{k=1}^{+\infty} \mathbb{P}\left(\left| \frac{X_{kq,kq+l}}{kq+l} \right| \geqslant \varepsilon \right) = \sum_{k=1}^{+\infty} \mathbb{P}\left(|X_{0,l}| \geqslant \varepsilon \frac{(kq+l)}{k} \times k \right)$$

$$\leqslant \sum_{k=1}^{+\infty} \mathbb{P}(|X_{0,l}| \geqslant \varepsilon' k)$$

$$\leqslant \int_0^{+\infty} \mathbb{P}(|X_{0,l}| \geqslant \varepsilon' t)\, dt < +\infty$$

as $\mathbb{E}|X| = \int_0^{+\infty} \mathbb{P}(|X| > t)\, dt$

In any case $$\liminf_{n \to +\infty} \frac{X_{0,n}}{n} \geqslant \frac{1}{q} \mathbb{E}(X_{0,n})$$

then $$\liminf_{n \to +\infty} \frac{X_{0,n}}{n} \geqslant \sup_{k \in \mathbb{N}^*} \frac{\mathbb{E}(X_{0,k})}{k} = \gamma$$

Step 3: $$\overline{X} = \limsup_{n \to +\infty} \frac{X_{0,n}}{n} \leqslant \gamma$$

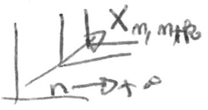
* Let $$\overline{X}_m = \limsup_{n \to +\infty} \frac{X_{m,m+n}}{n} \overset{\mathcal{L}}{=} \overline{X} \quad \text{by Hypothesis (iii)}$$

Moreover $$\frac{X_{0,m+n}}{n} \geqslant \frac{X_{0,m}}{n} + \frac{X_{m,m+n}}{n}$$

$$\rightsquigarrow \overline{X} \geqslant \overline{X}_m$$

Then $\overline{X} = \overline{X}_m$: We cannot have $\overline{X} \overset{\mathcal{L}}{=} \overline{X}_m$ and $\overline{X} > \overline{X}_m$ on a set of >0 probability!

* This is where the proof works only for DLPP, where $(X_{m,m+k} ;\ k \geqslant 0)$ depend on the environment for $(w_{ij} ;\ i,j \gg f(m))$



$\boxed{VI}$

$\overline{X}$ is in the tail σ-algebra of the ω_{ij} independent

$\rightsquigarrow$ Kolmogorov 0-1 law:

$$\overline{X} = \mathbb{E}(\overline{X})$$

$$\forall M>0, \quad \overline{X}\wedge M = \mathbb{E}(\overline{X}\wedge M)$$

$$= \mathbb{E}\left(\limsup_{n\to+\infty} \frac{X_{0,n}}{n} \wedge M\right)$$

$$\underset{\text{Reverse Fatou lemma}}{\leq} \limsup_{n\to+\infty} \mathbb{E}\left(\frac{X_{0,n}}{n}\wedge M\right)$$

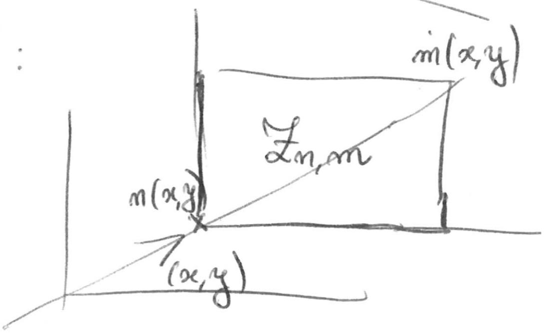
$$\leq \sup_{n\in\mathbb{N}^*} \mathbb{E}\left(\frac{X_{0,n}}{n}\right)$$

In the end $\overline{X} \leq \underline{\underline{\gamma}}$.

⚠ We have not proved L^1 convergence. $\boxed{VII}$ For details & examples we refer to the book of Durrett.

$\boxed{2}$ Proof of LLN:

The idea is to use Kingman's superadditive ergodic thm for different $(x,y)\in\mathbb{R}_+^2$:



$*$ $(x,y)\in\mathbb{N}_*^2$:

$$Z_{n,m} := L_{nx,ny}^{mx,my} \quad \text{superadditive}$$

$$\overset{\text{Kingman}}{\Longrightarrow} \quad \frac{L(nx,ny)}{n} = \frac{Z_{0,n}}{n} \xrightarrow[n\to+\infty]{} s(x,y) \text{ in } L^1 \text{ \& a.s}$$

Superadditivity is obvious. Homogeneity: $s(kx,ky) = \lim\limits_{n\to+\infty} \frac{L^{nkx,nky}}{n\times k}\times k$

$$= k\, s(x,y):$$

* $(x,y) \in \mathbb{Q}_+^2$:

Let $\begin{cases} k \in \mathbb{N} \text{ s.t } k(x,y) \in \mathbb{N}^2 \\ n = q_n k + r \text{ Euclidean division} \quad 0 \leqslant r < k \end{cases}$

$$\frac{L^{\lfloor nx \rfloor, \lfloor ny \rfloor}}{n} = \frac{L(qkx + \lfloor rx \rfloor, qky + \lfloor ry \rfloor)}{kq+r}$$

$$= \frac{L(qkx, qky)}{kq+r} + \underbrace{\frac{L(qkx + \lfloor rx \rfloor, qky + \lfloor ry \rfloor) - L(qkx, qky)}{kq+r}}$$

$$\frac{L(qkx,qky)}{kq+r} \xrightarrow{\text{a.s } \& \ L^1} \frac{s(kx, ky)}{k}$$

$$| \quad | \leqslant \frac{1}{n} \left(\lfloor rx \rfloor + \lfloor ry \rfloor \right) \max_{\substack{0 \leqslant i \leqslant \lfloor rx \rfloor \\ 0 \leqslant j \leqslant \lfloor ry \rfloor}} |\omega_{qkx+i,\, qky+j}|$$

$\xrightarrow[n \to +\infty]{} 0$ in L^1 (easy) & a.s (Borel-Cantelli)

* At this stage, recall that we can shift ω by its mean:

$$L^{n,m} = \underbrace{\max_{\pi : (1,1) \to (n,m)} \left(\sum_{(i,j) \in \pi} \omega_{ij} - \mathbb{E}(\omega) \right)}_{(\tilde{L}^{n,m})^{*} \text{ where } \omega_{ij} \leftrightarrow \omega_{ij} - \mathbb{E}(\omega)} + (n+m)\,\mathbb{E}(\omega)$$

This shifts a shape function s to $\tilde{s}(x,y) = s(x,y) - (x+y)\mathbb{E}(\omega)$

Assuming, without loss of generality $\mathbb{E}(\omega) = 0$,

we have $s(x, y+h) = \lim\limits_{n \to +\infty} \dfrac{\max\limits^{E(\pi)}_{\pi : (1,1) \to (\lfloor nx \rfloor, \lfloor n(y+h) \rfloor)}}{n}$ only one path!

$$\geqslant \lim_{n \to +\infty} \frac{\max\limits^{E(\pi)}_{\pi : (1,1) \to (\lfloor nx \rfloor, \lfloor ny \rfloor)} \longrightarrow (\lfloor nx \rfloor, \lfloor n(y+h) \rfloor)}{n}$$

$$= s(x,y) + \lim_{n \to +\infty} \frac{1}{n} \sum_{j = \lfloor ny \rfloor + 1}^{\lfloor n(y+h) \rfloor} \omega_{\lfloor nx \rfloor, j} \to 0 \text{ by LLN}$$

Hence we have

$$s : \mathbb{Q}_+^2 \longrightarrow \mathbb{R} \sqcup \{+\infty\}$$

$$\begin{cases} \text{monotone in every coordinate} \\ \text{homogenous + superadditive.} \end{cases}$$

* Let us prove that either $s < +\infty$

or $s = +\infty$ on $(\mathbb{Q}_+^*)^2$

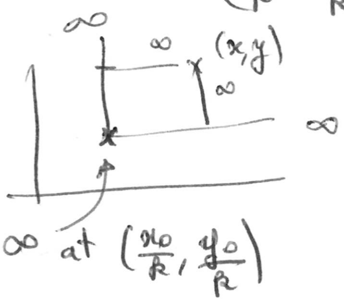
either $s < +\infty$

or $\exists\ (x_0, y_0) \in \mathbb{Q}_+^2$ s.t $s(x_0, y_0) = +\infty$

Let $(x,y) \in \mathbb{Q}_+^2$. Pick $k \in \mathbb{N}$ s.t $\left(\frac{x_0}{k}, \frac{y_0}{k}\right) < (x,y)$ coordinate wise

By coordinate-wise monotoniaty

$$s(x,y) = +\infty.$$



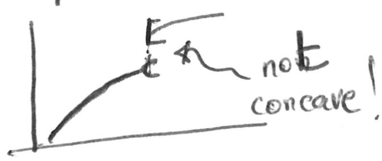
* <u>Conclusion:</u>

Either $s = +\infty$ èverywhre

or $s : \mathbb{Q}_+^2 \longrightarrow \mathbb{R}$

$$\begin{cases} \text{monotone in every coordinate} \\ \text{homogenous + superadditive} \end{cases} \Longrightarrow \text{concave}$$

In this case, because of monotoniaty, unique extension to $\mathbb{R}_+^2$, except that there can be jumps.



Concavity excludes jumps!

A finite concave function on an open set, is continuous.

As such, there is a unique <u>continuous</u> $s : \mathbb{R}_+^2 \to \mathbb{R}$

s.t $\frac{L(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} \longrightarrow s(x,y)$ for rationals.

By monotonicity, convergence holds for reals. ☒