Introduction to mathematical finance

Sheet 8 HS 2014

Hand-in your solutions until 19.11.2014, in Martina Dal Borgo's mailbox on K floor.

Exercise 1

In a three-period binomial model, consider an American Put option with payoff $X_n = (\frac{1}{2} - S_n)^+$, $n = 0, 1, 2, 3$. We assume that $u = 2$, $d = r = \frac{1}{2}$ and the initial price of the underlying asset is $S_0 = 1$.

i) Represent the asset prices and the values of the payoff of the American Put on the binomial tree. Compute the arbitrage price process using the Snell envelope. What is the minimal capital requirement throughout which is necessary to perfectly hedge the American option?

Exercise 2

As we have seen in the last exercise sheet, in an arbitrage free T-period binomial model with $r \geq 0$, the prices of a European Call and an American Call option are the same at each time $t = 0, \dots, T$. The same does not hold in the case of a Put option (as you can see from Exercise 4 Sheet 6). Therefore we want now to study, qualitatively, the graph of the price of a European Put option (as a function of the underlying asset), and compare it with the one of an American Put.

Let P^E and P^A be the prices of a European, resp. an American Put option with strike K, on the underlying S. Setting $S_0 = x$, you know that

$$P^E(x) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\left(K - x\prod_{k=1}^T \xi_k\right)^+}{(1+r)^T}\right], \quad \text{or} \tag{1}$$

$$P^E(x) = \frac{1}{(1+r)^T}\sum_{h=0}^{T}\binom{T}{h} q^h (1-q)^{T-h}\left(K - u^h d^{N-h} x\right)^+, \tag{2}$$

and

$$P^A(x) = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^{\mathbb{Q}}\left[\frac{(K - x\prod_{k=1}^{\tau}\xi_k)^+}{(1+r)^{\tau}}\right], \quad \text{or} \tag{3}$$

$$\begin{cases} P_T^A(x) = X_T = \left(K - x\prod_{k=1}^T \xi_k\right)^+ \\ P_t^A(x) = \max\left\{\left(K - x\prod_{k=1}^t \xi_k\right)^+, \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[P_{t+1}^A(x)|\mathcal{F}_t\right]\right\} \end{cases} \tag{4}$$

Assume $d < 1$.

The function $x \to P^E(x)$ is continuous (being linear combination of continuous functions (2)). Prove that

i) it is convex and decreasing for all $x \geq 0$;

ii) using the representation (2) that

$$P^E(0) = \frac{K}{(1+r)^T}, \quad P^E(x) = 0 \ \ \forall x \in \left[\frac{K}{d^T}, +\infty\right[;$$

iii) there exists $\bar{x} \in]0, K[$ (assume after having proved that it exists that is unique), such that

$$P^E(x) < (K-x)^+ \ \text{ for } \ x \in [0, \bar{x}[\quad \text{and} \quad P^E(x) > (K-x)^+ \ \text{ for } \ x \in]\bar{x}, K/d^T[.$$

Hint:

- *For the point iii) consider the convex (as a sum of convex functions) continuous function*

$$g(x) := P^E(x) - (K - x), \quad x \in [0, K].$$

- *Remember that the* max *of two convex functions and the positive weighted sum of convex functions is again a convex function.*

The function $x \to P^A(x)$ is continuous (being recursively defined as the composition of continuous functions (4)).The facts that the price function is also convex and monotone decreasing follow from (3) since the functions

$$x \to \mathbb{E}^{\mathbb{Q}}\left[\frac{(K - x\prod_{k=1}^{\tau}\xi_k)^+}{(1+r)^\tau}\right]$$

are convex and decreasing and their least upper bound, when τ varies, preserves such properties. Prove that

i) using the representation (3) that

$$P^A(0) = K, \quad P^A(x) = 0 \;\; \forall\, x \in \left[\frac{K}{d^T}, +\infty\right[;$$

ii) there exists $x^* \in \,]0, K[$, such that

$$P^A(x) = (K-x)^+ \;\text{ for }\; x \in [0, x^*] \quad \text{and} \quad P^A(x) > (K-x)^+ \;\text{ for }\; x \in]x^*, K/d^T[.$$

Hint: For the point ii) consider the convex (as a sum of convex functions) continuous function

$$f(x) := P^A(x) - (K-x)^+, \quad x \in [0, K].$$