Introduction to mathematical finance

Sheet 7 HS 2014

Hand-in your solutions on the 12.11.2014 in class.

Introduction

As an optimal stopping problem, we consider the situation where a recruiter has to pick the best candidate among N applicants. To that effect, one can associate to each candidate $i \in \{1, 2, \ldots, N\}$ a random variable X_i that models his or her "worth". The random variables are independent and identically distributed. There are however two limitations:

- the candidates are revealed one at a time.
- the recruiter's decision for every candidate is final. There is no going back, even if the recruiter realises later that he missed on an exceptional candidate.

It is also known as the "secretaries problem" if one has to hire secretaries, or as the "princess problem" if we consider a princess trying to find the best among N suitors. There are two takes on the problem, which we will adress in Part I and Part II respectively. The choice of filtration $\mathbb{F} = \{\mathcal{F}_0 \subset \mathcal{F}_1 \cdots \subset \mathcal{F}_N\}$ will be crucial. Let $\mathcal{T}$ be the set of $\mathbb{F}$-stopping times, with integer values between 1 and N.

After finding optimal solutions, we are interested in the asymptotics as $N \to \infty$ of the optimal solutions.

Part I: Maximising expected utility

In this part, we consider that the recruiter can exactly evaluate the worth of each candidate. Therefore, the natural choice of filtration $\mathbb{F}$ is:

$$\forall t, \mathcal{F}_t := \sigma\left(X_1, X_2, \ldots, X_t\right)$$

We consider that the recruiter is not interested in making the best pick, and maximises the expected utility. The optimal stopping problem at hand is:

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left(X_\tau\right)$$

- Formulate the problem thanks to the Snell envelope of $(X_1, \ldots, X_N)$.

$$U_N^N = X_N$$

$$U_N^t = \max\left(X_t, \mathbb{E}\left(U_{t+1}^N \mid \mathcal{F}_t\right)\right)$$

- Prove $\mathbb{E}\left(U_t^N \mid \mathcal{F}_{t-1}\right)$ is in fact a constant random variable. If denoted by u_t^N, prove that it is only a function of $N - t$.
- Prove that it satisfies the recurrence:

$$u_t^N = u_{t+1}^N \mathbb{P}\left(X_t < u_{t+1}^N\right) + \mathbb{E}\left(X_t \mathbb{1}_{\{X_t \geq u_{t+1}^N\}}\right)$$

  Notice it is monotone.

- Starting now, we suppose that the X_i's are uniformly distributed on $[0,1]$. You are urged to pick your own distribution and adapt the next arguments. Using the distribution:

$$u_t^N - u_{t+1}^N = \frac{1}{2}\left(1 - u_{t+1}^N\right)^2$$

  And then, the rest boils down to the analysis of this recursive sequence. Prove that as $N - t \to \infty$ (and hence $N \to \infty$), we have the equivalent:

$$u_t^N = 1 - \frac{2\,(1 + o(1))}{N - t}$$

  Hint: First prove that u_t^N converges to a value that is necessarily 1. Second, form $v_t^N = 1 - u_t^N$ and show that $\frac{1}{v_t^N} - \frac{1}{v_{t+1}^N} \sim \frac{1}{2}$

- The earliest optimal stopping time is:

$$\tau_0^N = \inf\left\{t \geq 1 \mid U_t^N = X_t^N\right\}$$

  By computing the asymptotics of $\mathbb{P}\left(\frac{\tau_0^N}{N} \geq t\right)$, find a limiting distribution of the optimal stopping time. This limiting random variable τ can be understood as the portion of time one needs to make the optimal decision. Prove that, on average, the recruiter takes his decision after seeing two thirds of the candidates.

Part II: Maximising the probability of making the best pick

Because it is refreshing to change perspective, we will now make the recruiter into a princess. And because princesses settle for nothing less than what is best, we will maximise the probability of making the best pick in this second part. The optimal stopping problem at hand is:

$$\sup_{\tau \in \mathcal{T}} \mathbb{P}\left(X_\tau = X\right)$$

where $X = \max_i X_i$. A beautiful answer resides in the "$\frac{1}{e}$ law" (e is the Euler constant), which states that the princess should wait until she meets approximately a third of the suitors, then pick the best one that shows up among the two thirds left.

As discussed in class, it is possible to make the problem distribution independent, unlike the previous case. To that endeavor, one has to postulate that the princess can rank the candidates, but is not able to tell their exact worth. Therefore, a natural choice is the filtration generated by the events:

$$\left\{X_{s_n(1)} > X_{s_n(2)} > \cdots > X_{s_n(n)}\right\}$$

for $s_n \in S_n$ a permutation of $\{1, 2, \ldots, n\}$. Hence:

$$\mathcal{F}_n := \sigma\left(\left\{X_{s_n(1)} > X_{s_n(2)} > \cdots > X_{s_n(n)}\right\},\ s_n \in S_n\right)$$

.

Also, let σ_n be the random permutation of S_n that ranks the first n candidates. Notice that $\sigma_n = s_n$ if and only if the event $\left\{X_{s_n(1)} > X_{s_n(2)} > \cdots > X_{s_n(n)}\right\}$ happens.

- Explain why $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, making $\mathbb{F}$ into a filtration.
- Prove that

$$\mathbb{P}\left(\sigma_n = s_n\right) = \frac{1}{n!}$$

- Let A_k be the event "At step k, the suitor k is the best so far". Clearly, $A_k \in \mathcal{F}_k$. Prove that:

$$\mathbb{P}\left(X_k = X \mid \mathcal{F}_k\right) = \mathbb{1}_{A_k} \frac{k}{N}$$

and deduce that:

$$\mathbb{P}\left(X_\tau = X\right) = \mathbb{E}\left(\mathbb{1}_{A_\tau} \frac{k}{N}\right)$$

- Thanks to the previous result, give the Snell envelope formulation of the optimal stopping problem:

$$U_N^N = \mathbb{1}_{A_N}$$

$$U_t^N = \max\left(\mathbb{1}_{A_t} \frac{t}{N}, \mathbb{E}\left(U_{t+1}^N \mid \mathcal{F}_t\right)\right)$$

- Prove that U_t^N is $\sigma\left(A_t\right)$-measurable. Deduce that there are two sequences a_t and b_t such that:

$$U_t^N = a_t \mathbb{1}_{A_N} + b_t \mathbb{1}_{A_N^c}$$

In fact, prove that:

$$\frac{a_t}{t} = \max\left(\frac{1}{N}, \frac{b_t}{t}\right)$$

$$\frac{b_t}{t} = \frac{b_{t+1}}{t+1} + \frac{a_{t+1}}{t+1} \frac{1}{t}$$

- Let

$$t_0^N = \inf\left\{t \geq 0 \mid \frac{b_t}{t} \leq \frac{1}{N}\right\}$$

And let τ_0^N be the earliest optimal stopping time. Prove that:

$$\tau_0^N = \inf\left\{t \geq t_0^N \mid A_t \text{ or } t = N\right\}$$

- Deduce the "$\frac{1}{e}$ law" by proving that, as $N \to \infty$:

$$\frac{t_0^N}{N} \to \frac{1}{e}$$