Introduction to mathematical finance

Sheet 6 HS 2014

Hand-in your solutions on the 05.11.2014 in class.

Exercise 1

Let $(X_t)_{t=0,..,T}$ a stochastic process on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ and let ν be a $(\mathcal{F}_t)_t$-stopping time. The stopped process of X by ν is defined as

$$X_t^\nu(\omega) = X_{\min\{\nu(\omega),t\}}(\omega)$$

i.e., on the set $\{\nu = \tau\}$ we have

$$X_t^\nu = \begin{cases} X_\tau & \text{if } \tau \le t, \\ X_t & \text{if } \tau > t. \end{cases}$$

Prove that

i) If X is adapted, then X^ν is adapted.

ii) If X is a martingale, then X^ν is a martingale.

Exercise 2

Let $\mathbb{Q}$ be the risk neutral measure, the context of the usual binomial model. Let $0 < q < 1$ be the underlying probability. Show that the price at time t of an American Put option on S with maturity T and strike K can be expressed as $P_t = P(t, S_t)$, where : $\mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function, satisfying the recursive backward equation

$$\begin{cases} P(T, x) = (K - x)^+ \\ P(t, x) = \max\left\{(K - x)^+, \frac{q \cdot P(t+1,xu) + (1-q) \cdot P(t+1,xd)}{1+r}\right\}, \quad t = 0, \dots, t_{N-1}. \end{cases}$$

Show that the (super) replicating strategy of the American Put is characterized by a quantity $\phi_t = \Delta(t, S_{t-1})$ at time t, where Δ can be expressed in terms of the function P.

Exercise 3

Let $(X_t)_{t\in\{0,..,T\}}$ be an adapted integrable process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\{0,..,T\}}, \mathbb{P})$ and let $(H_t)_{t\in\{0,..,T\}}$ be its Snell envelope. Consider also the process $V_t = \mathbb{E}\left[X_T | \mathcal{F}_t\right], \ t \in \{0, .., T\}$.

i) Show that $H_t \ge V_t$ a.s. for all $t = 0, ..., T$.

ii) Show that if $V_t \ge X_t$ for all $t = 0, ..., T$, then $H = V$ a.s.

iii) Show that if X is a submartingale, then $H = V$ a.s.

iv) Use *ii)* to conclude that the prices of an American Call and a European Call in the arbitrage-free binomial model with $r \ge 0$ are equal at each time $t = 0, ..., T$.

Exercise4

Consider an American Put option in a one-period binomial model with $r > 0$, $q = \frac{1}{2}$, strike $K > uS_0$ (S_0 current spot price of the underlying) . When is it profitable to exercise the option?