Introduction to mathematical finance

Hand-in your solutions on the 22.10.2014 in class.

Exercise 1
Consider a two period (1 period = 4 months) binomial model. The current spot price of a risky asset is $S_0 = 40$ and the parameters are $u = 1.04$ and $d = 0.98$. The annual risk free interest rate is $r = 0.04$. Compute the initial price of an option with maturity $T = 8$ months and payoff given by

$$X = \left(S_T - \frac{(60 - S_0)^2}{10}\right)^+ .$$

Exercise 2
Consider a two period (1 period = 1 year) binomial model. The current spot price of a risky asset is $S_0 = 40$ and the parameters are $u = 1.5$ and $d = 0.5$. The annual risk free interest rate is $r = 0.04$. Also a European Call option written on S with maturity $T = 2$ years and strike $K = 25$ is given.

Time	0	$t = 1$ year	$T = 2$ year
Bond	$B_0 = 1$	$B_1 = (1 + r)$	$B_2 = (1 + r)^2$
Risky asset	$S_0 = 40$	$S_1 = \begin{cases} 40 \cdot 1.5 & p \\ 40 \cdot 0.5 & 1 - p \end{cases}$	$S_2 = \begin{cases} 40 \cdot (1.5)^2 & p^2 \\ 40 \cdot (1.5)(0.5) & 2p(1-p) \\ 40 \cdot (0.5)^2 & (1-p)^2 \end{cases}$

i) Find a trading strategy (ϕ_0, ϕ_1) replicating the option, and deduce the price of the call. Note that $\phi_i = (\alpha_i, \beta_i)_{i=0,1}$ is the strategy built at time i.

ii) If the risk free interest rate was $r = 6\%$, would the replicating strategy and the price of the option change?

Exercise 3 (Binomial model considered)

i) Let $\mathbb{Q}$ be some probability measure on $(\Omega, \mathcal{F})$ and $\tilde{S}_t = \frac{S_t}{(1+r)^t}$, $t = 0, .., T$ the discounted price process. Show that

(a) $\tilde{S}$ is a $\mathbb{Q}$-martingale if and only if

$$\mathbb{E}^{\mathbb{Q}}\left[1 + \mu_t | \mathcal{F}_{t-1}\right] = (1 + r),\ \forall\, t = t_2, ..., T.$$

(b) $\tilde{S}$ is a $\mathbb{Q}$-martingale if and only if the random variables $\mu_0, \ldots, \mu_T$ are i.i.d. under the probability measure $\mathbb{Q}$, with

$$\mathbb{Q}(1 + \mu_1 = u) = \frac{1 + r - d}{u - d}.$$

Here $1 + \mu_t = \xi_t$ (ξ_t r.v. introduced in the binomial model).

ii) Now let $\mathbb{Q}$ be the risk-neutral measure.

Show that the price at time t of a European Call option on S with maturity T and strike K can be expressed as $C_t = c(t, S_t)$, where $c : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function, satisfying the recursive backward equation

$$\begin{cases} c(T, x) = F(x) & \text{(payoff for } S_N = x \in \mathbb{R}_+) \\ c(t, x) = \frac{1}{1+r}\Big(q \cdot c(t+1, xu) + (1-q) \cdot c(t+1, xd)\Big), & t = 0, \dots, t_{N-1}. \end{cases}$$

Show that the replicating strategy is characterized by a quantity $\phi_t = \Delta(t, S_{t-1})$ at time t, where Δ can be expressed in terms of the function c.

iii) Determine the hedging strategy also in the case of a European Put option with same strike K and maturity and show that the hedging strategy always invest a non positive amount in the stock and therefore involves only short sales.