Introduction to mathematical finance

Sheet 4 HS 2014

Hand-in your solutions until 15.10.2014, in Martina Dal Borgo's mailbox on K floor (or give the sheet in person on the 15.10 in class).

Exercise 1

Find an example of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ and an adapted process $(M_t)_{t\in[0,T]}$ with $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ for all $t = 0, \dots, T$ such that $(M_t)_{t\in[0,T]}$ is not a martingale.

Exercise 2

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables. Set $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and define $S_n := X_1 + \dots + X_n$, $n \in \mathbb{N}$.

i) Assume that for some $\alpha \in [0, \frac{1}{2}]$,

$$\mathbb{P}(X_1 = 1) = \frac{1}{2} + \alpha, \qquad \mathbb{P}(X_1 = -1) = \frac{1}{2} - \alpha.$$

Show that for all $k, n \in \mathbb{N}$, with $k \leq n$,

$$\mathbb{E}\left[S_n | \mathcal{F}_k\right] = S_k + 2\alpha(n-k).$$

ii) Assume that $X_1 \sim \mathcal{N}_{(0,\sigma^2)}$, with positive variance $0 < \sigma^2 < \infty$. Set

$$Z_n^u := \exp\left(uS_n - \frac{nu^2\sigma^2}{2}\right).$$

Show that for every $u \in \mathbb{R}$, the process $(Z_n^u)_{n\in\mathbb{N}}$ is an $(\mathcal{F}_n)$-martingale

Note: The relation $\mathbb{E}\left(XY|\mathcal{G}\right) = X\mathbb{E}\left(Y|\mathcal{G}\right)$ holds for X $\mathcal{G}$-measurable, as soon as both XY and Y are integrable.

Exercise 3

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Assume that $(X_t)_{t\in[0,T]}$ is an $\mathbb{R}^d$-valued adapted stochastic process and $\phi = (\phi_t)_{t\in[1,T]}$ is an $\mathbb{R}^d$-valued predictable process. We set

$$V_0 = \phi_1 \cdot X_0, \quad V_t = \phi_t \cdot X_t, \quad t = 1, \dots, T$$

and

$$G_0 = 0, \quad G_t = \sum_{k=1}^{t} \phi_k \cdot (X_k - X_{k-1}), \quad t = 1, \dots, T.$$

In the following assume that X is a martingale. Show that

i) If $(\phi_t)_{1\leq t\leq T}$ is bounded, then $(G_t)_{0\leq t\leq T}$ is a martingale.

ii) If $(\phi_t)_{1\leq t\leq T}$ is bounded, then

$$\phi_t \cdot X_t = \phi_{t+1} \cdot X_t, \quad \forall t = 1, \dots, T-1. \tag{1}$$

implies that $(V_t)_{0\leq t\leq T}$ is a martingale.

Note: This exercise has the following economical interpretation: a market model consists of d risky assets whose discounted prices are denoted by X. Moreover, ϕ is a trading strategy with associated value process V and a gain process G, that is the sum of the variation of the portfolio's values

$$G_t = \sum_{k=1}^{t} V_t - V_{t-1}.$$

Condition (1) is the self-financing condition which means that no funds are added or removed after time $t = 0$. If the probability measure is risk neutral, then X will be a martingale and the above exercise shows that also the gain and value process will be martingales under the self-financing condition.

Exercise 4

Let $(Y_t)_{1 \le t \le T}$ be a sequence of i.i.d. *normal* random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[Y_t] = 0$, $\mathrm{Var}(Y_t) = \sigma^2 > 0$. For the sake of simplicity, interest rate is zero. We define the filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_t = \sigma(Y_1, \ldots, Y_t), \quad t = 1, \ldots, T$$

and the (discounted) price process

$$S_0 = 1, \qquad S_t = S_0 + \sum_{k=1}^{t} Y_k.$$

We ignore the fact that S can take negative values. Furthermore we define another filtration $(\mathbb{G}_t)_{0 \le t \le T}$ by including the *insider* information about the final stock price S_T, i.e. $\mathcal{G}_t = \sigma(\mathcal{F}_t, S_T)$, $t = 0, \ldots, T$.

i) Show that (S_t) is a $(\mathcal{F}_t)$-martingale, but not a $(\mathcal{G}_t)$-martingale. Moreover prove that the process

$$S_t^* := S_t - \sum_{k=0}^{t-1} \frac{S_T - S_k}{T - k}, \quad t = 0, \ldots, T,$$

is a $(\mathcal{G}_t)$-martingale.
Hint: Use properties of the Gaussian to see that $Y_{t+1} - \alpha (S_T - S_t)$ is independent from $\mathcal{G}_t$ if and only if $\alpha = \frac{1}{T-t}$.

ii) Having additional information about S_T allows one to create portfolios with positive expected profit. Compute a trading strategy $(\tilde{\phi}_t)_{t=1,\ldots,T}$ which maximizes the expected discounted profit $\mathbb{E}[G_T]$ for

$$G_T := \sum_{k=1}^{T} \phi_k (S_k - S_{k-1}),$$

in the class of all $(\mathcal{G}_t)$-predictable trading strategy ϕ (describing the holdings in the stock) satisfying $|\phi_t| \le 1$ for all $t = 1, ..., T$.
Hint: begin using the Exercise 3 to deduce that

$$\mathbb{E}\left[\sum_{k=1}^{T} \phi_k (S_k - S_{k-1})\right] = \mathbb{E}\left[\sum_{k=1}^{T} \phi_k \cdot \frac{S_T - S_{k-1}}{T - (k-1)}\right].$$