Introduction to mathematical finance

Sheet 3 HS 2014

Hand-in your solutions until 8.10.2014, in Martina Dal Borgo's mailbox on K floor (or give the sheet in person on the 8.10.2014 in class).

Exercise 1

Consider the model of the Exercise 4 Sheet 2 .

i) Determine the probability measure $\mathbb{Q}$, $\mathbb{Q}(\omega_i) > 0$, $i = 1, 2$ such that

$$S_0 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1].$$

ii) Does the initial value of the strategy (found in Sheet 2) equal the risk-neutral price $(1+r)^{-1}\mathbb{E}^{\mathbb{Q}}[C_1]$?

iii) Show that there are arbitrage strategies in the model if one prices the Call option at time $t = 0$ as the discounted expectation under the real-world (historic) probability measure, i.e. $(1+r)^{-1}\mathbb{E}^{\mathbb{P}}[C_1] \neq (1+r)^{-1}\mathbb{E}^{\mathbb{Q}}[C_1]$ and determine an arbitrage strategy which yelds a risk-free profit of

$$\left|\mathbb{E}^{\mathbb{P}}[C_1] - \mathbb{E}^{\mathbb{Q}}[C_1]\right| > 0. \tag{1}$$

You are required to give numerical answers in the case of $1 + r = 1$ first and, then $1 + r = 1.1$. Specify in each case if the call priced under $\mathbb{P}$ is too expensive or too cheap.

Hint: If the call option for the price $(1+r)^{-1}\mathbb{E}^{\mathbb{P}}[C_1]$ is too expensive, sell it and produce a replicating strategy which will leave you with $\mathbb{E}^{\mathbb{P}}[C_1] - \mathbb{E}^{\mathbb{Q}}[C_1] > 0$ as left over money. If it is too cheap, buy one and use the opposite of the replication strategy for calls . One can also resort to call-put parity in this second case and use the replication strategy for the put: If the call is too cheap, then the put is too expensive.

Exercise 2

Consider a one-period (1 year) binomial market model (B, S) consisting of a bond, paying an annual risk-free rate of $r = 4\%$ and a risky asset, with current spot price $S_0 = 100$. In the next year the price of the risky asset can increase, with increase rate $u = 1.5$, with probability $p = 50\%$ and decrease, with decrease rate $d = 0.6$, with probability $1 - p = 50\%$.
Now, consider a European call option on S, with strike 18 and maturity of 1 year.

i) Determine the initial price of the Call option.

ii) Would you buy this option?

iii) What if I told you now that the dynamics of the stock are given by $p = 99\%$ and $1 - p = 1\%$? Does the price change? Does your position regarding buying the option change?

Exercise 3

Consider a one period (annual) binomial model consisting of a bond, B, paying an annual risk-free rate of 25% and in one risky asset S, whose price can only increase or decrease with constant increase and decrease rates $u = 2$, $d = 0.5$ in the next year:

Time	0	$T = 1$ year
Bond	$B_0 = 1$	$B_1 = (1 + 0.25)$
Risky asset	$S_0 = 50$	$S_1 = \begin{cases} 100 \\ 25 \end{cases}$

i) Find, via the probability measure $\mathbb{Q}$, the price of a European Call option on one share of the stock S, with strike $K = 50$ and maturity $T = 1$ (year).

ii) Suppose that the option in *i)* is initially priced 1 above the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.

iii) What is the arbitrage free price for a European Put with the same strike and maturity as the Call in *i)* ?

Exercise 4

Consider a one-period (annual) financial market model (B, S^1, S^2) consisting of a bond, B, paying an annual risk-free rate of 5% and in two risky assets S^1 and S^2, given by:

Time	0	$T = 1$ (year)
Bond	$B_0 = 1$	$B_1 = (1 + r)$
Risky asset 1	$S_0^1 = 10$	$S_1^1 = \begin{cases} 12 & \text{if } \omega_1 \\ 10 & \text{if } \omega_2 \\ 6 & \text{if } \omega_3 \end{cases}$
Risky asset 2	$S_0^2 = 10$	$S_1^2 = \begin{cases} 15 & \text{if } \omega_1 \\ 8 & \text{if } \{\omega_2, \omega_3\} \end{cases}$

with $\mathbb{P}(\omega_i) > 0$, $i = 1, 2, 3$ and $\sum_{i=1}^3 \mathbb{P}(\omega_i) = 1$.

i) Prove that there exists a unique probability measure $\mathbb{Q}$, with $\mathbb{Q}(\omega_i) > 0$, $i = 1, 2, 3$, such that

$$S_0^1 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1^1] \qquad \text{and} \qquad S_0^2 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1^2].$$

ii) Consider two derivatives A and B with maturity 1 year and payoffs

$$X_A = \left(\frac{S_1^1 + S_1^2}{2} - 8\right)^+, \qquad X_B = \left(S_1^1 - S_1^2\right)^+.$$

Find two strategies ϕ^A (resp ϕ^B) such that

$$X_A = V_1(\phi^A) \quad (\text{ resp } X_B = V_1(\phi^B)\ )$$

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iii) Discuss whether in the market formed only by (B, S^1) there exists a probability measure $\mathbb{Q}$, with $\mathbb{Q}(\omega_i) > 0$, $i = 1, 2, 3$, such that

$$S_0^1 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1^1]$$

and if this probability measure is unique.