Introduction to mathematical finance

Sheet 2 HS 2014

Hand-in your solutions until 1.10.2014 *in Martina Dal Borgo's mailbox on K floor.*

The Binomial Model

The binomial model is a discrete time financial market model, composed of a non-risky asset B (bond), corresponding to an investment into a savings account in a bank and of a risky asset S, which is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The quotes of each asset change at times times $0 < 1 < \cdots < T$.

- The dynamic of the bond is deterministic given by

$$B_t = (1+r)^t, \quad t \in \{0, 1, .., T\}$$

  where r is the risk-free interest rate on $[0, T]$.

- The risky asset $(S_t)_{t=0,\ldots,T}$ has the following stochastic dynamic: when passing from time $t-1$ to time t the stock can only increase or decrease its value with constant increase and decrease rates denoted u for "up" and d for "down":

$$\begin{cases} S_0 \in \mathbb{R}_{>0} \\ S_t = S_{t-1}\xi_t, \quad t = 1, \ldots, T \end{cases}$$

  where $(\xi_t)_{t\in\{0,1,..,T\}}$ are i.i.d. Bernouilli random variables whose distribution is a combination of Dirac's Deltas: Let $p \in (0,1)$ be the probability that the stock increases. We have:

$$\mathbb{P}[S_t = uS_{t-1}] = \mathbb{P}[\xi_t = u] = p,$$
$$\mathbb{P}[S_t = dS_{t-1}] = \mathbb{P}[\xi_t = d] = 1 - p.$$

  that is

$$S_t = \begin{cases} uS_{t-1}, & \text{with probability } \ p, \\ dS_{t-1}, & \text{with probability } \ 1-p. \end{cases}$$

Hence a trajectory of the stock is a vector such as $(T = 4)$

$$(S_0, uS_0, udS_0, u^2dS_0, u^3dS_0)$$

which can be identified with the vector

$$(u, d, u, u) \in \{u, d\}^4$$

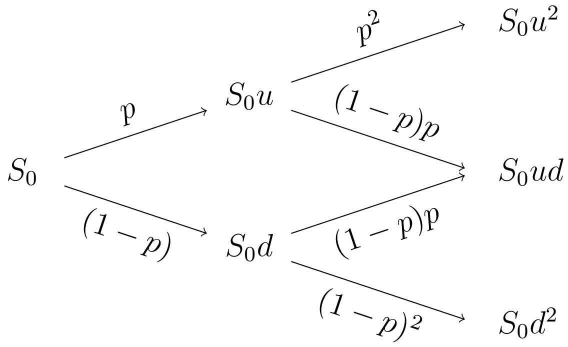
of the occurrence of the random vector $(\xi_1, \xi_2, \xi_3, \xi_4)$.Therefore we can assume that the sample space Ω is the family

$$\{u, d\}^T = \{(e_1, \ldots, e_N) \,|\, e_n = u \text{ or } e_n = d \},$$

containing 2^N elements and $\mathcal{F}$ is the σ-algebra of all subsets of Ω. Moreover we assume that $\mathbb{P}[\omega] > 0, \ \forall \omega \in \Omega$ and we endow this probability space with the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ defined by:

$$\mathcal{F}_0 = \{\emptyset, \Omega\} = \sigma(\emptyset), \tag{1}$$
$$\mathcal{F}_t = \sigma(\xi_1, .., \xi_t), \quad t \in \{1, .., T\}. \tag{2}$$

The σ-algebra $\mathcal{F}_t$ represents the amount of information available in the market at time t. Thanks to exercise 1 and formula (1), one sees that in the model the initial price S_0 of the asset is deterministic. The family of trajectories can be represented on a binomial tree $(T = 3)$



Exercise 1

Prove that if X is a random variable on $(\Omega, \mathbb{P})$, measurable with respect to the trivial σ-algebra $\mathcal{F} = \{\emptyset, \Omega\}$, then X is constant.

Exercise 2

Prove that in the binomial model the stock at time t, $t = 0, 1, \ldots, T$ is

$$S_t = S_0 u^{\mathcal{B}} d^{t-\mathcal{B}}$$

where $\mathcal{B} \sim$ Binomial (t, p) i.e.

$$\mathbb{P}[\mathcal{B} = k] = \binom{t}{k} p^k (1-p)^{t-k}.$$

Exercise 3

Consider a two-period (2 semesters) binomial market model (B, S) consisting of a bond, B, paying an annual risk-free rate $r = 4\%$, and a risky asset with current spot price $S_0 = 20$. The parameters of the model are $u = 1.25$, $d = 0.75$.
Verify whether the exercise of a European Call option written on S with strike $K = 18$ and with maturity $T = 6$ months, is more likely than the exercise of a European Call option as the one above but with maturity $T' = 1$ year.

Explanation: more likely means which of the two options has the bigger probability of being exercised by the buyer of the option.

Exercise 4

Consider a one-period (annual) financial market model (B, S) consisting of a bond, B, paying an annual risk-free rate $r \geq 0$, in one risky asset, S, whose final value depends on some random event (we assume that the event can assume only two possible states $\{\omega_1, \omega_2\}$, $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_1) = 1/2$ in which S_1 takes the values $S_1(\omega_1)$ and $S_1(\omega_2)$). We denote by C a Call option written on the risky asset, with maturity of 1 year and strike $K = 100$, depending on the same random event:

Time	0	$T = 1$ (year)
Bond	$B_0 = 1$	$B_1 = (1 + r)$
Risky asset	$S_0 = 100$	$S_1 = \begin{cases} 120 \text{ if } \omega_1 \\ 90 \text{ if } \omega_2 \end{cases}$
Call option	$C_0 = ?$	$C_1 = \begin{cases} (120 - 100)^+ = 20 \text{ if } \omega_1 \\ (90 - 100)^+ = 0 \text{ if } \omega_2 \end{cases}$

Notice that in one period, a self-financing strategy is given by a single vector $\phi = (\beta, \alpha)$ that is $\mathcal{F}_0$ measurable (hence deterministic). Find a trading strategy $\phi = (\beta, \alpha)$ such that the final value of this strategy $V_1(\phi)$ satisfies

$$C_1 = V_1(\phi) = \beta B_1 + \alpha S_1.$$

regardless of the outcome ω_1 or ω_2. Compute numerically the initial value of this strategy $V_0(\phi) = \alpha S_0 + \beta$.

Exercise 5

Consider a one-period (annual) financial market model (B, S) consisting of a bond, B, paying an annual risk-free rate $r \geq 0$, in one risky asset, S, whose final value depends on some random event (we assume that the event can assume three possible states $\{\omega_1, \omega_2, \omega_3\}$, with $\mathbb{P}(\omega_1) = p_1,\ \mathbb{P}(\omega_2) = p_2,\ \mathbb{P}(\omega_3) = 1 - p_1 - p_2$, in which S_1 takes the values $S_1(\omega_1)$ and $S_1(\omega_2)$). We denote by C a Call option written on the risky asset, with maturity of 1 year and strike $K = 100$, depending on the same random event:

Time	0	$T = 1$ (year)
Bond	$B_0 = 1$	$B_1 = (1 + r)$
Risky asset	$S_0 = 100$	$S_1 = \begin{cases} 150 & \text{if } \omega_1 \\ 110 & \text{if } \omega_2 \\ 40 & \text{if } \omega_3 \end{cases}$
Call option	$C_0 =?$	$C_1 = \begin{cases} (150 - 100)^+ = 50 & \text{if } \omega_1 \\ (110 - 100)^+ = 10 & \text{if } \omega_2 \\ (40 - 100)^+ = 0 & \text{if } \omega_3 \end{cases}$

In the same spirit as the previous exercise, try to find a trading strategy $\phi = (\beta, \alpha)$ such that the final value of this strategy $V_1(\phi)$ satisfies

$$C_1 = V_1(\phi) = \beta B_1 + \alpha S_1.$$