Introduction to mathematical finance

Sheet 12 HS 2014

Hand-in your solutions until 17.12.2014, *in Martina Dal Borgo's mailbox on K floor.*

Exercise 1 is perhaps more technical than the other two. You can start by exercise 2 and 3.

Exercise 1

In what follows W is a real Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where the usual hypotheses hold. Recall that a stochastic process Y belongs to $\mathbb{L}^p$ if

- Y is $(\mathcal{F}_t)$-adapted,
- $Y \in \mathbb{L}^p([0,T]\times\Omega)$ that is

$$\mathbb{E}\left[\int_0^T |Y_s|^p ds\right] = \int_0^T \mathbb{E}\left[|Y_s|^p\right] ds < \infty.$$

Consider an Itô process, that is a stochastic process X of the form

$$X_t = X_0 + \int_0^t \nu_s\, ds + \int_0^t Y_s\, dW_s, \quad t \in [0,T],$$

where X_0 is a $\mathcal{F}_0$-measurable random variable, $\nu \in \mathbb{L}^1,\ Y \in \mathbb{L}^2$.

We want to prove that:

$$X_t \text{ is a martingale } \Leftrightarrow \nu = 0 \text{ a.s}$$

The implication from right to left is obvious from the construction of the stochastic integral seen in class. Indeed, if $\nu = 0$, X_t is only a stochastic integral against Brownian motion, which is a martingale by construction. The implication from left to right requires a little more work.

- Prove that if X is a martingale, then for all times t and s:

$$\mathbb{E}\left(\int_t^{t+s} \nu_u du | \mathcal{F}_t\right) = 0$$

- Deduce that:

$$\nu_t = \mathbb{E}\left(\frac{1}{s}\int_t^{t+s} (\nu_t - \nu_s) du | \mathcal{F}_t\right) = \frac{1}{s}\int_t^{t+s} \mathbb{E}\left(\nu_t - \nu_s | \mathcal{F}_t\right) du$$

- Assuming for example that ν is $a.s$ continuous and bounded, show that $\nu = 0$ a.s. If possible, you can replace the path-wise continuity and boundedness hypotheses by $s \mapsto \mathbb{E}(|\nu_s|)$ continuous on $[0,T]$.

Exercise 2[Polarisation of Ito isometry]. We know from class that if $Y \in \mathbb{L}^2$, the stochastic integral $t \mapsto \int_0^t Y_s\, dW_s$ is an L^2-martingale satisfying the Ito isometry:

$$\mathbb{E}\left[\left(\int_0^t Y_s\, dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t Y_s^2\, ds\right],$$

Now consider two processes $Y, Z \in \mathbb{L}^2$. Prove the polarised version:

$$\mathbb{E}\left[\int_0^t Y_s\, dW_s \times \int_0^t Z_s\, dW_s\right] = \mathbb{E}\left[\int_0^t Y_s Z_s\, ds\right], \tag{1}$$

Hint: Apply the Ito isometry to $\int_0^t (Y_s + Z_s)\, dW_s$ *and* $\int_0^t (Y_s - Z_s)\, dW_s$*, make the difference then develop.*

Exercise 3. Let W be a real Brownian motion on a filtered (natural filtration) probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where the usual hypotheses hold.

- Apply the Itô formula to

  i) $X_t = W_t,\ f(x) = x^n,\ n \in \mathbb{N};$

  ii) $X_t = W_t,\ f(x) = e^{\sigma x},\ \sigma \in \mathbb{R};$

  ii) $X_t = W_t,\ f(t,x) = xt.$

  to give the explicit decomposition of $f(t, X_t)$ as an Ito process.

- Recall from exercise 1 and the class, that the martingale part has zero expectation. Using the Itô formula and recurrence, compute $\mathbb{E}[W_t^4]$, $\mathbb{E}[W_t^6]$. Prove that $\mathbb{E}[W_t^n] = 0$ if n is odd and

$$\mathbb{E}[W_t^n] = \left(\frac{t}{2}\right)^{\frac{n}{2}} \frac{n!}{(n/2)!}.$$